ON THE BUCKLING OF THIN ELASTIC SHELLS

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SUMMARY

The differential equations which must be solved to predict the buckling load are reviewed. The different possible physical interpretations of these equations are discussed.

INTRODUCTION

This article is concerned with the prediction of the response to arbitrary disturbances of a structure initially at rest in a deformed state under known static loads. The response to loads varying in time is not considered. This problem will be called the elastic buckling problem. In the elastic buckling problem, certain general conclusions can be drawn about linear systems (ref. 2). But shells are characterized by the importance of non-linearities (ref. 1). So far, no similar comprehensive study of the possible instabilities of non-linear elastic systems has been made.

There are several possible versions of the elastic buckling problem for non-linear systems[1]. Any of the following questions might be asked. (1) Are there loads for which two infinitesimally different equilibrium states exist? (2) Are there loads for which the second variation of strain energy ceases to be positive definite? (3) Are there loads for which infinitesimal oscillations diverge? (4) Will the stiffness decrease greatly at some load so that intolerable deflections occur? (5) Will a dynamic jump from one equilibrium configuration to another occur at some load due to a given magnitude of disturbance? (6) Will a limit cycle develop as a result of finite disturbances? The fundamental question here is the equivalence of these problems.

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[1]ref. 1, pg. 123
[2]ref. 3, pg. 54

STATIC EQUILIBRIUM

The exact equations governing the deformations of linearly elastic solids are geometrically non-linear. The restriction to linear elasticity implies, for most materials, a restriction to small stretches and small shears. For such deformations, the displacement gradients will be small and therefore the non-linearities negligible unless one or two dimensions of the body are small compared to the others[2]. Thin elastic shells are three dimensional linearly elastic solids for which one dimension, the thickness, is much smaller than the other two. For shell problems, the geometric non-linearities may therefore be important. However, because of the two-dimensional nature of the shell, not all of the non-linear terms can be of equal importance[3]. Some simplifications result because of the thinness and because of the flatness. Equations which systematically introduce simplifications appropriate to the relative magnitude of these two parameters for various possible modes of deformation have been accomplished (ref. 4). However, much further work remains to be done. In particular, the strain-displacement and curvature-displacement relations corresponding to each family of basic equations must be determined before application to special problems and comparison with existing special theories is possible.

Since the general theory of linearly elastic solids can be formulated as the stationary condition of the potential energy, each set of shell equations will also provide the Euler equations for the stationary value of a functional, the potential energy, W. The static equilibrium states are thus determined by

$$\delta W = 0 \quad . \tag{1}$$

Unlike the linear theory, non-uniqueness of equilibrium states is to be expected for a wide variety of loads (refs. 5,6,7).

The usual problem is concerned with the stability of static equilibrium positions which differ only slightly from the undeformed configuration. The non-linear terms might, therefore, be of negligible importance. No exact solutions are available which clearly resolve this question, and test results on spherical caps (ref. 9) indicate that

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[2] ref. 3, pg. 54
[3] ref. 3, pg. 182

the non-linear terms can not be neglected in determining the equilibrium state even before buckling.

STABILITY

Suppose the shell is subject to loads derivable from a potential, then the rest position is determined by a set of non-linear differential equations and boundary conditions corresponding to the stationary condition for the potential energy. If a disturbance in the form of finite displacements and velocities is introduced, the equations which describe the motion will be non-linear.

The motion may subside, diverge, or reach a limit cycle. For a given rest state, the motion will depend upon the magnitude of the initial disturbances. Should there exist other static equilibrium positions (at the same loads), disturbances of sufficient magnitude could cause the structure to jump from its initial static equilibrium position and come to rest in a second static equilibrium position. An important question, not yet solved, is the relation between such motions and the von Karman-Tsien energy criterion (ref. 10). With disturbances of sufficient magnitude it would appear possible to have a jump from one static equilibrium position to another at loads less than those predicted by this criterion. No general results are known about the motion of non-linear structures subjected to finite disturbances.

If only infinitesimal vibrations are superposed on the deformed system, then. as for linear systems, the motion will be harmonic if $\delta^2 W > 0$ and diverging if $\delta^2 W < 0$. The first case will be called stable. All other cases (including "neutral equilibrium") will be called unstable and the corresponding loads will be called the buckling loads.

It is generally not possible to directly investigate a static equilibrium state to determine the sign of the second variation for all possible variations from it. Instead the function which minimizes the second variation is determined. The differential equations governing such a function are the Euler equations of

$$\delta(\delta^2 W) = 0 \quad . \tag{2}$$

The minimum is then investigated to see whether it is positive or negative. Generally $\delta^2 W > 0$ for small loads. As the load is increased, the least value of the second variation becomes smaller and finally becomes zero and then negative. The lowest value of the load for which the second variation is non-negative is thus the buckling load. The differential equations determined by Eq. 2 are linear equations with variable coefficients. When the loads are derivable from a potential, they will be the same equations as those found by considering the differential equations satisfied by the difference between two infinitesimally different static equilibrium states[4] (corresponding to the same load). Thus problems (4), (5), (6) as stated above are equivalent when the loads are derivable from a potential.

A DONNELL TYPE THEORY

The notation followed is that of ref. 11 wherever possible: Greek letters will have the range (1,2). Curvilinear (material) coordinates in the middle surface will be denoted by x^α and are chosen so that $x^1, x^2, \bar{n}$ form a right-handed system when $\bar{n}$ is the unit, inward normal. The components of the metric tensor of the surface are $a_{\alpha\beta}$ The coefficients of the second quadratic form are $b_{\alpha\beta}$ The $e_{\alpha\beta}$ system is skew-symmetric and $e_{12} = \sqrt{a}$ where a is the determinant of the metric tensor. A bar denotes covariant differentiation based on the $a_{\alpha\beta}$. The components of tangential displacement are u^α. The normal displacement, positive inward is denoted by w.

The Donnell type theory for large deflections (ref. 12) leads to the set of equations:

$$n^{\alpha\beta}|_\beta + p^\alpha = 0 \quad , \tag{3}$$

$$m^{\alpha\beta}|_{\alpha\beta} + b_{\alpha\beta}\, n^{\alpha\beta} + n^{\alpha\beta}\, w|_{\alpha\beta} - p^\alpha\, w|_\alpha + p = 0 \quad , \tag{4}$$

$$n^{\alpha\beta} = B\, H^{\alpha\beta\nu\lambda}\, \alpha_{\nu\lambda} \quad , \tag{5}$$

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[4]ref. 3, Chap. 5

$$m^{\alpha\beta} = -D\, H^{\alpha\beta\nu\lambda}\, w|_{\nu\lambda} \quad , \tag{6}$$

$$B = \frac{E\,t}{1-\nu^2} \quad , \quad D = \frac{E\,t^3}{12(1-\nu^2)} \quad ,$$

$$\alpha_{\nu\lambda} = \frac{1}{2}\,(u_\nu|_\lambda + u_\lambda|_\nu - 2\,b_{\nu\lambda}\,w + w|_\nu\, w|_\lambda) \quad , \tag{7}$$

$$H^{\alpha\beta\nu\lambda} = \frac{1}{2}\,(a^{\alpha\nu}\,a^{\beta\lambda} + a^{\alpha\lambda}\,a^{\beta\nu} + \nu[e^{\alpha\nu}\,e^{\beta\lambda} + e^{\alpha\lambda}\,e^{\beta\nu}]) \quad .$$

The $n^{\alpha\beta}$ is a symmetric tensor that may be interpreted as the membrane stress, $m^{\alpha\beta}$ is a symmetric tensor that may be interpreted as the bending moment, and $\alpha_{\nu\lambda}$ is the membrane strain tensor. The tangential surface loads are p^α and the normal pressure is p , positive inward. The boundary conditions to be satisfied on the edge with unit, outward normal in the middle surface m^α consist of specifying the quantities

$$u_\nu \quad \text{or} \quad n^{\nu\lambda}\, m_\lambda \quad , \tag{8}$$

$$w \quad \text{or} \quad m^{\alpha\beta}|_\alpha\, m_\beta + n^{\alpha\beta}\, w|_\alpha\, m_\beta + \frac{\partial}{\partial s}(e_{\nu\alpha}\, m_\beta\, m^\nu\, m^{\alpha\beta}) \quad , \tag{9}$$

$$\frac{\partial w}{\partial n} \quad \text{or} \quad m^{\alpha\beta}\, m_\alpha\, m_\beta \quad , \tag{10}$$

where n and s are distances normal and tangential to the boundary.

These equations are derived from the necessary conditions for the stationary of the potential energy

$$W = \frac{1}{2} \iint H^{\alpha\beta\nu\lambda}\,(B\,\alpha_{\alpha\beta}\,\alpha_{\nu\lambda} + D\,w|_{\alpha\beta}\,w|_{\nu\lambda})dS - \iint (p^\alpha\,u_\alpha + p\,w)dS \tag{11}$$

when the surface loads are fixed in magnitude and direction and either displacements are given on the edges or the given edge forces are specified to be zero.

The equations governing an equilibrium state corresponding to the same loads and differing only by an infinitesimal amount can be derived[5] in the manner of ref. 3. Denoting the difference between the quantities, defining the

[5] Report 62-1, Feb. 1962, Dept. of Aeronautical Engineering, University of Washington, by E. H. Dill: Stability of Thin Elastic Shells.

two states, by superposed bars, these equations are

$$\bar{n}^{\alpha\beta}|_{\beta} = 0 \quad . \tag{12}$$

$$\bar{m}^{\alpha\beta}|_{\alpha\beta} + b_{\alpha\beta}\,\bar{n}^{\alpha\beta} + n^{\alpha\beta}\,\bar{w}|_{\alpha\beta} + \bar{n}^{\alpha\beta}\,w|_{\alpha\beta} - p^{\alpha}\,\bar{w}|_{\alpha} = 0 \quad , \tag{13}$$

$$\bar{n}^{\alpha\beta} = B\,H^{\alpha\beta\nu\lambda}\,\bar{\alpha}_{\nu\lambda} \quad , \tag{14}$$

$$\bar{m}^{\alpha\beta} = -\,D\,H^{\alpha\beta\nu\lambda}\,\bar{w}|_{\nu\lambda} \quad , \tag{15}$$

$$\bar{\alpha}_{\nu\lambda} = \frac{1}{2}\left(\bar{u}_{\nu}|_{\lambda} + \bar{u}_{\lambda}|_{\nu} - 2\,b_{\nu\lambda}\,\bar{w} + \bar{w}|_{\nu}\,w|_{\lambda} + w|_{\nu}\,\bar{w}|_{\lambda} \right) \quad . \tag{16}$$

The quantities

$$\bar{u}_{\nu} \quad \text{or} \quad \bar{n}^{\nu\beta}\,m_{\beta} \; , \tag{17}$$

$$\bar{w} \quad \text{or} \quad \bar{m}^{\alpha\beta}|_{\alpha}\,m_{\beta} + \bar{n}^{\alpha\beta}\,m_{\beta}\,w|_{\alpha} + n^{\alpha\beta}\,m_{\beta}\,w|_{\alpha} + \frac{\partial}{\partial s}\left(e_{\nu\lambda}\,m_{\beta}\,m^{\lambda}\,\bar{m}^{\nu\beta}\right) \; , \tag{18}$$

$$\frac{\partial w}{\partial n} \quad \text{or} \quad \bar{m}^{\alpha\beta}\,m_{\alpha}\,m_{\beta} \quad . \tag{19}$$

must be given on the edge.

It can be shown that equations (12) to (19) result from Eq. (2) when the direction and magnitude of the surface loads are fixed.

SYMMETRICAL DEFORMATIONS OF CYLINDRICAL SHELLS

For a cylindrical shell of radius R , let $x^1 = x$ be the distance along the generator, and $x^2 = R\,\theta$ be the circumferential distance from a given generator. The coordinates x^1, x^2 and the inward normal form a right-hand system. Consider an axially compressed cylinder undergoing only axially symmetric deformations. Then

$$u_2 = 0 \; , \; n^{12} = 0 \; , \; m^{12} = 0 \; , \; p^{\alpha} = 0 \; , \; p = 0 \; , \tag{20}$$

and all derivatives with respect to x^2 are zero.

For free ends, the boundary conditions are

$$n^{11} = -P \quad , \quad w|_{11} = 0 \quad , \quad -D\,w|_{111} + n^{11}\,w|_{1} = 0 \quad . \tag{21}$$

A solution of Eqs. (3) to (7) is easily seen to be

$$w = -\frac{\nu P R}{E t} \quad , \quad n^{11} = -P \quad , \quad n^{22} = 0 \quad , \text{etc.} \; . \tag{22}$$

However, if the ends are restrained the result is different. For simply supported ends, the boundary conditions are

$$n^{11} = -P \quad , \quad w = 0 \, , \quad w|_{11} = 0 \; . \tag{23}$$

The solution for

$$P < P_c \; , \quad P_c = E\,t\,\left(\frac{d}{R}\right)^2 \quad , \quad d^2 = \frac{R\,t}{[3(1-\nu^2)]} \tag{24}$$

is

$$n^{11} = -P \quad , \quad n^{22} = -w\,\frac{Et}{R} - \nu P \quad , \tag{25}$$

$$w = -\nu\,\frac{PR}{Et} + e^{\beta x}(C_1 \cos \alpha x + C_2 \sin \alpha x) + e^{-\beta x}(C_3 \cos \alpha x + C_4 \sin \alpha x)$$

$$\beta^2 = \frac{1-\lambda}{d^2} \quad , \quad \alpha^2 = \frac{1+\lambda}{d^2} \quad , \quad \lambda = \frac{P}{P_c} \tag{26}$$

The constants can be adjusted to satisfy the boundary conditions. The axially symmetric solution for $P \geq P_c$ • can be written in terms of similar terms.

With restrained ends there will be lateral deflections, moments, and circumferential stress whose magnitude and distribution depend upon the load but is always confined to a narrow edge zone. Usually the membrane solution, which is the same as Eq. (22), is assumed to be a sufficiently accurate estimate of the state before buckling even for restrained ends.

STABILITY OF CYLINDERS

For the axially compressed cylinder, the stability of the axially symmetric initial state is determined by seeking a solution of equations (12) to (19). For the free ends, the solution[6] shows that an axially symmetric buckling mode exists, localized near the ends, for which the buckling load is $p = \frac{1}{2} p_c$.

It may be shown that no axially-symmetric buckling mode exists for the simply supported ends for $P < P_c$. Therefore no axially-symmetric buckling mode is possible. This does not imply that the axially-symmetric deformations are stable; that question remains to be answered by solving equations (12) to (19). This solution has not been accomplished exactly.

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