BUCKLING AND POST-BUCKLING OF ELASTIC SHELLS

by
H. Langhaar and A. Boresi
University of Illinois

SUMMARY

Some theoretical investigations of buckling of elastic shells are surveyed in this report. Only geometrically perfect shells are considered; initial dents and out-of-roundness are not taken into account. Several questions raised by the studies are: (a) Under what conditions is the infinitesimal theory of buckling of shells adequate? (b) How does the energy theory of buckling of shells correlate with the method based on equilibrium equations for bending moments, tensions, and shears in a buckled configuration? (c) How important are nonlinear terms in the tangential displacements u, v in the strain-displacement relations for buckling and post-buckling studies? (d) How important are the boundary conditions for u, v in affecting stability? (e) If a condition of snap-through is approached, how much external work is required to push the shell "over the hump" into the buckled configuration? (f) How reliable are mathematical approximations used previously in the infinitesimal theory of buckling of shells? Tentative and incomplete answers to some of these questions are suggested.

INFINITESIMAL THEORY OF BUCKLING

One of the objectives of nonlinear theories of shells is to show how good (or bad) the linear eigenvalue theory of buckling is. It is to be expected that a shell that is heavily reinforced by stringers approximates roughly the behavior of a set of parallel columns. The linear eigenvalue theory of buckling is known to be satisfactory for columns; consequently, it may be expected to determine the buckling load of a shell that derives its strength mainly from stringers. This conjecture is supported by an analysis of buckling of cylindrical stringer-reinforced sheet panels subjected to nonuniform axial compression (1). Cylinders of arbitrary cross-sectional form were considered. A panel was considered to be supported by bulkheads at its ends and by spars along its longitudinal edges. The bulkheads prevented normal and circumferential displacements of the sheet at the ends, whereas only normal displacements were prevented by the spars. Elastic rotational restraints of the bulkhead chord members were taken into account. A panel was considered to be so short that the wave form of a buckled stringer was a single loop. However, several loops were admitted in the wave form of a buckled cross section transverse to the stringers. The analysis was based on the principle that the work of the external forces equals the increment of strain energy when an infinitesimal buckling deformation occurs. The equation $w = w_0 \sin m\xi \sin\eta$ was adopted for the normal displacement due to buckling, where ξ and η are respectively dimensionless circumferential and axial coordinates. Euler's equation of the calculus of variations was used to determine the circumferential displacement u to minimize the buckling load. Results of the theory were compared with experimental data from five box beams having cambered stringer-reinforced compression surfaces. The beams were subjected to pure bending. Although ideal buckling did not occur, the deflection curves of the test panels relative to the spars showed well-defined knees. In all cases, the computed buckling loads fell nicely on the knees. When the buckling loads were represented by the Euler column formula, the plate-stringer column-fixity factors ranged from 1 to 3. Consequently, the specification of a single constant fixity factor (or reduced column length) for computing allowable compression stresses in cambered plate-stringer panels may be either dangerous or exceedingly conservative.

Another investigation treated the infinitesimal theory of buckling of a circular cylindrical shell reinforced by rings and subjected to uniform external hydrostatic pressure and uniform axial compression (2). The axial compression was considered to be so small that the typical fluted buckling pattern was not impaired. The end plates were considered to be so flexible that no restraints were imposed on the axial displacement u at the ends. The Kirchhoff assumption that radial line elements remain straight and normal to the middle surface was used. Also, the stresses σ_r, $\tau_{r\theta}$, τ_{rx} were neglected. Without loss of generality, the length of the shell was set equal to π. The axial, circumferential, and radial displacements were assumed to be represented respectively by $u = u_0 + x_1 \sin x \cos n\theta$, $v = y_1 \cdot \cos x \sin n\theta$, $w = (z_0 + z_1 \cos n\theta) \cos x$. Here the axial coordinate x is measured from the center cross section; u_0 is a function of x, and x_1, y_1, z_0, z_1 are constants. The term $z_0 \cos x$ is an approximation for the deflection before buckling and the other terms represent the infinitesimal deformation that results from buckling. By the calculus of variations, u_0 was chosen to minimize the total potential energy V. The previous equations differ from those adopted by von Mises only by the introduction of u_0. After elimination of u_0, V was approximated as a cubic polynomial in the generalized coordinates x_1, y_1, z_0, z_1; this degree of approximation is adequate for the infinitesimal theory of buckling, since higher degree terms would introduce nonlinear forms in x_1, y_1, z_0, z_1 into the second variation of V. The second variation of V is a quadratic form in the virtual increments of the generalized coordinates x_1, y_1, z_0, z_1; the buckling criterion is that the determinant of the coefficients in this quadratic form be zero.

Results of the analysis agree very well with von Mises' theory if the ratio of length to radius L/a is greater than 1. However, if $L/a < 1$, the computed buckling pressures for unreinforced shells are considerably less than those given by von Mises' theory; in some cases the difference is as great as 25%. In the range $L/a < 1$ the infinitesimal theory has been reported to give buckling pressures larger than the experimental values; consequently, from a practical standpoint, the new theory introduces an improvement. A comparison with test data for a machined ring-reinforced cylindrical shell also shows better agreement with the new theory than with other infinitesimal theories of buckling of perfect shells. The results suggest that discrepancies between theory and experiment for buckling of hydrostatically loaded cylindrical shells may derive partly from inadmissible mathematical approximations in the infinitesimal theories, rather than from weak stability preceding snap-through. The reason for the lowered buckling pressure given by the new theory is not easy to trace, since other investigators have not approached the problem by way of the second variation of potential energy. It is hoped that further investigations will disclose the cause of the difference.

SNAP-THROUGH OF HYDROSTATICALLY LOADED CYLINDRICAL SHELLS.

Despite the fact that the infinitesimal theory of buckling yields results in fair agreement with tests of cylindrical shells subjected to external hydrostatic pressure, snap-through is a definite possibility. This fact is disclosed by some theoretical studies of post-buckling behavior of unreinforced elastic cylindrical shells loaded by external normal pressure (3). The analysis was based on the principle of minimum potential energy. When the Ritz method is used in this type of analysis, it is imperative that the assumed deflection pattern shall not impose excessive membrane strains, for then the membrane strain energy is far too large. Approximations concerning the strain energy of bending may be rather rough, but the membrane strain energy is a delicate matter. As Lord Rayleigh remarked, "We can bend a piece of sheet metal easily with our fingers, but we can not stretch it noticeably."

The displacement pattern due to buckling was assumed to be represented by $u = u_0 + u_1 \cos n\theta + u_2 \cos 2n\theta + u_3 \cos 3n\theta$, $v = v_1 \sin n\theta + v_2 \sin 2n\theta + v_3 \cdot \sin 3n\theta$, $w = w_0 + w_1 \cos n\theta + w_2 \cos 2n\theta + w_3 \cos 3n\theta$, where u, v, w are axial, circumferential, and radial displacement components of the middle surface. The coefficients u_i, v_i, w_i are functions of x. Excessive membrane strains were avoided by the assumption that there is no increment of the membrane hoop strain ϵ_θ caused by buckling. This assumption undoubtedly causes ϵ_x to be too large in some regions, and it consequently leads to Euler buckling pressures that are too large in the case of short thick shells. However, it yields a comparatively simple theory that readily provides numerical results. The assumption $\Delta\epsilon_\theta = 0$ leads to explicit formulas for v_1, v_2, v_3, w_0, w_2, w_3, in terms of w_1. After the strain energy was linearized in u, the functions u_0, u_1, u_2, u_3 were obtained with the aid of Euler's equation of the calculus of variations so that the potential energy V was minimized. Observations of buckled shells usually enable us to estimate approximate functions for w quite accurately, but much greater difficulties are encountered with u and v. Consequently, it is desirable to determine u and v by means of the exact formulas of the calculus of variations, rather than by the Rayleigh-Ritz procedure, whenever possible.

The remaining unknown function w_1 was assumed to be given by $w_1 = (W_0 \cos \pi x/L) / (n - \frac{1}{n})$ where W_0 is a constant and n is the number of waves in the cross section of the buckled cylinder. Thus, the shell was effectively reduced to a system with one degree of freedom, the generalized coordinate being W_0. The results exhibit the typical snap-through behavior. Figure 1 illustrates qualitatively the nature of the load-deflection curves that were derived. The falling part of the curve (dotted in Fig. 1) represents unstable equilibrium configurations. Also, the continuation of line OE (dotted) represents unstable unbuckled configurations. Actually, the shell snaps from some configuration A to another configuration B, as indicated by the dashed line. Theoretically, point A coincides with the Euler critical pressure E, but initial imperfections, residual stresses, or accidental shocks may prevent the shell from reaching point E. In any case, point A is higher than the minimum point C. The pressure at point C is the smallest pressure at which a buckled form can persist; when the pressure drops below this value, the shell snaps back to the unbuckled form.

An analysis of the post-buckling behavior of a structure determines the buckling load automatically. For example, an analysis of the form of a buckled column reveals that there is no real nonzero solution unless the load exceeds a certain value, the Euler critical load. Accordingly, in principle, the nonlinear theory of equilibrium eliminates the need for a special theory of buckling. However, in practice, it is usually easier to determine the Euler buckling load of a structure by solving a linear eigenvalue problem than by determining a bifurcated curve in configuration space that represents all equilibrium configurations.

Since the shell was reduced to a system with a single degree of freedom the theory provides an equation which expresses the increment of potential energy ΔV due to buckling as a function of the deflection parameter W_0. Thus, for any given value of the external pressure p, a curve of ΔV versus W_0 may be plotted. The forms of the graphs corresponding to several values of p are illustrated by Fig. 2. The pressures indicated on the curves are such that $p_1 < p_2 < p_3 < p_4$. The minima on the curves represent configurations of stable equilibrium, and the maxima represent configurations of unstable equilibrium. If $p < p_4$, the unbuckled state is stable, since the configuration $W_0 = 0$ then provides a relative minimum to the potential energy. However, if $p \geq p_4$, the unbuckled state becomes a configuration of maximum potential energy; hence, it is unstable. Accordingly, p_4 is the Euler critical pressure. The curve corresponding to p_1 has an inflection point at which the tangent is horizontal; hence, p_1 corresponds to point C on Fig. 1; it is the smallest pressure at which a buckled form can persist. For any pressure greater than p_1 there is a state of minimum potential energy with $W_0 > 0$; hence snap-through is possible. Let us take, for example, the curve corresponding to p_2, for which the value of ΔV at the minimum is zero. In other words, for $p = p_2$, the potential energies of the buckled and unbuckled configurations are equal. Tsien suggested that this condition be taken as a criterion for buckling; accordingly, p_2 may be called the Tsien critical pressure. The maximum on the curve for p_2 represents a potential energy barrier that the shell must cross to reach the buckled form. If this maximum is high, there is little danger of snap-through, since a large amount of external work must be provided to carry the shell "over the hump." However, if the potential-energy hill is low, snap-through is imminent. Of course, this argument is not linked to the Tsien hypothesis; it applies for any of the curves in the range $p_1 < p < p_4$. It appears that the snap-through theory of ideal shells would be enhanced by further studies of the potential-energy barriers separating the buckled and unbuckled forms. Such studies might enable us to evade the extremely complicated problems of initially dented shells, since accidental shocks may be used as a criterion for design instead of initial dents or out-of-roundness. If anticipated shocks will not provide enough energy to carry a shell over the hill to the buckled form, the design may be considered to be safe.

One numerical calculation was performed for a shell with $L/a = 0.60$, $a/h = 1000$, $E = 30,000,000$ lb/in.^{2}, $a = 20$ in., where L is the length of the shell, a is the radius, and h is the thickness. It was found that, with the pressure equal to the Tsien critical value, 0.23 ft lb of work would carry the shell over the hill to the buckled form. Since this small amount of work might easily come from accidental disturbances, we see why the Euler critical pressure is practically unattainable in some cases.

Another interesting conclusion arose from a study of end constraints. In all cases, the end plates were considered to provide simple support to the cylindrical wall, so that the bending moment M_x vanished at the ends. Also, the end plates imposed the boundary conditions $v = w = 0$. However, with regard to the axial displacement u, two different conditions were considered. In one case, the end plates were free to warp, so that no restraints were imposed on u at the ends. In the second case, the end plates were rigid, so that u had a constant value at either end. Surprisingly, the constraint imposed upon u by a rigid end plate raised the buckling pressure significantly. In general, the number n of waves in the periphery of a buckled cylinder is greater for rigid end plates than for flexible end plates. The results are illustrated by Fig. 3, which shows computed load-deflection curves for $a/h = 100$ with both flexible and rigid end plates. Besides the pressure p on the lateral surface, the cylinder was subjected to an axial compression force $F = \pi a^2 p$, which would result from uniform hydrostatic pressure on the end plates. Instead of the pressure p, the ordinate in Fig. 3 is a dimensionless coefficient K, defined by $p = K E h/a$. The Euler critical pressures represented by the intercepts of the curves with the K-axis in Fig. 3 are too high, since the present theory employed the assumption $\Delta\epsilon_\theta = 0$. The improved infinitesimal theory of buckling (2), discussed earlier, yields Euler critical pressures that are considerably lower for short shells, and therefore the effect of snap-through would not be so pronounced as one might infer from the steep curves in Fig. 3.

A generalization of the nonlinear theory of Ref. 3 has been given by the authors. (4). In this generalization, the shell was reduced to a system of 21 degrees of freedom. However, it was found unfeasible to handle the nonlinear equilibrium problem for a system with 21 degrees of freedom. Consequently, for the numerical work, some higher harmonics were discarded so that the system was reduced to 7 degrees of freedom. Calculations were confined principally to the determination of the minimum point C on the post-buckling curve (Fig. 1), that is, to the determination of the pressure at which a buckled form can exist. It was found that the ordinate of point C, as determined in (3), was somewhat too high. The numerical studies indicated that the theory of (4) may be used effectively with an electronic digital computer. However, for cursory studies of post-buckling behavior, the theory and the tables of (3) are recommended.

BUCKLING OF UNREINFORCED PRESSURIZED CYLINDRICAL SHELLS SUBJECTED TO AXIAL COMPRESSION

The strain-energy formulas developed in (3) apply without modification for a cylindrical shell that is subjected to internal pressure and uniform axial compression. Again, a snap-through condition is anticipated.

In an analysis of post-buckling behavior of an axially compressed cylindrical shell (5), the strain formulas are linearized in the axial displacement u. For the time being, consideration of end effects is avoided by the supposition that the shell is infinitely long. Then, as is well known, the buckling pattern consists of diamond-shaped lobes (Fig. 4). This pattern signifies that the radial deflection w is doubly periodic, with its fundamental region in the form of a rhombus. In any rhombus (e.g., the cross-hatched rhombus in Fig. 4), the function w assumes all its values; the function is merely duplicated in any other rhombus. A rhombus subtends the angle 2β from the axis of the cylinder; hence, $\beta = \pi/n$, where n is an integer representing the number of rhombuses in the circumference of the cylinder. The function w is symmetrical about the diagonals of a rhombus. Consequently, if the origin of the (x,θ) plane is taken at the center of a rhombus, the function w is even in x and θ (Fig. 4).

The rhomboidal pattern and the associated symmetry properties of u, v, w require that these functions be represented by series of the following forms* if the origin for x and θ is taken at the center of a rhombus:

$$u = c_0 + c_1 x + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} a_{ij} \cos i\theta \sin \frac{j\pi x}{\lambda}, \quad a_{ij} = 0 \text{ if } i+j \text{ is odd}$$

$$v = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \sin i\theta \cos \frac{j\pi x}{\lambda}, \quad b_{ij} = 0 \text{ if } i+j \text{ is odd}$$

$$w = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \cos i\theta \cos \frac{j\pi x}{\lambda}, \quad c_{ij} = 0 \text{ if } i+j \text{ is odd}$$

Here c_0, c_1, a_{ij}, b_{ij}, c_{ij} are constants, and λ is half the length of a diagonal of a rhombus in the x-direction (Fig. 4). Writing these equations in expanded form as far as second harmonics, we obtain

$$u = c + u_0 x + u_1 \cos n\theta \sin \frac{\pi x}{\lambda} + u_2 \sin \frac{2\pi x}{\lambda} + u_3 \cos 2n\theta \sin \frac{2\pi x}{\lambda}$$

$$v = v_0 \sin n\theta \cos \frac{\pi x}{\lambda} + v_1 \sin 2n\theta + v_2 \sin 2n\theta \cos \frac{2\pi x}{\lambda}$$

* By error, some additional terms were included in the formula for u in Ref. 5.

$$w = w_0 + w_1 \cos n\theta \cos \frac{\pi x}{\lambda} + w_2 \cos \frac{2\pi x}{\lambda} + w_3 \cos 2n\theta + w_4 \cos 2n\theta \cos \frac{2\pi x}{\lambda}$$

The coefficients u_i, v_i, w_i are constants. The additive constant c is irrelevant, since it represents a translation.

The preceding equations were developed for a shell of infinite length. Experiments indicate that the number of half waves in the length of a finite shell is an integer. Therefore, the length L of the shell is assumed to be a multiple of λ; that is, $\lambda = L/m$, where m is an integer. To satisfy the boundary conditions $v = w = 0$ at the ends $x = 0$ and $x = L$, the factor sin $\pi x/L$ was introduced in the formulas for v and w in the last equations. The formula for u is unchanged if the end plates are flexible.

The preceding equations are somewhat more general than the buckling patterns that have been used to study this problem. For correlation with the buckling patterns assumed by some other investigators, a translation of the origin from the center of a rhombus to the midpoint of an edge of a rhombus may be required. In studying the infinitesimal theory of buckling, Timoshenko discarded all terms except u_1, v_0, w_1. Von Karman and Tsien adopted a function w that is equivalent to the preceding if $w_4 = 0$ and $w_2 = w_3$. Donnell and Wan concluded that the relation $w_2 = w_3$ is not plausible on the basis of observed buckling patterns.

If all coefficients are retained in the expansion to second harmonics, there are 12 degrees of freedom. It is highly desirable to retain all 12 of these generalized coordinates, but the problem of minimization of the potential energy then becomes exceedingly complicated. A part of the trouble lies in the quadratic terms in v in the strain-displacement relations. Kempner and other investigators have neglected this quadratic term, and workable results have been obtained.

SLENDER CIRCULAR CYLINDERS (RINGS)

Since Levy (6) first published his classical theory on the buckling of rings, several theories on the buckling of rings and cylinders have appeared. Levy obtained the result

$$P_{cr} = K_{cr}\, EI/r_0^3, \quad K_{cr} = 3$$

as the critical pressure for a uniformly loaded ring, where r_0 is the radius of the centroidal axis, I is the moment of inertia of the cross section and E is the modulus of elasticity of the material. Levy did not consider the effects of ring thickness h and of Poisson's ratio ν. The load was considered to remain normal to the ring surface throughout the deformation process. In 1914, R. v. Mises (7) starting with the general differential equations of equilibrium and considering only linear terms in the strain tensor, developed a theory which yielded a buckling pressure of $Eh^3/[4(1-\nu^2)\, r_0^3]$ for the uniformly loaded infinitely long circular cylinder. In 1933, Donnell (8), by making several simplifications of the general shell equations, arrived at a theory for the buckling of thin cylindrical shells. For the infinitely long circular cylinder (for which Donnell's theory is not strictly valid), Donnell's theory predicts a critical pressure of $Eh^3/[3(1-\nu^2)\, r_0^{\,3}]$. The theoretical result that is generally accepted for a long circular cylinder is $Eh^3/[4(1-\nu^2)\, r_0^{\,3}]$ (provided the load remains directed perpendicular to the cylinder's surface).

Several authors have noted that the critical pressure depends strongly upon the post-buckling direction of the load (9, 10, 11). For example, for a ring subjected to a uniform pressure that remains directed toward the center of the ring $K_{cr} = 4.5$, (9), and if the load remains constant in direction $K_{cr} = 4$ (11). The effects of ring thickness h and of Poisson's ratio upon p_{cr} for uniformly loaded rings have been studied in (9), the general result being a decrease in K_{cr} with an increase in h. The sway buckling of a semi-circular arch loaded by vertical point load P at the midsection of the arch has been studied in (12), where the result $P_{cr} = 6.54\ EI/r_0^{\,2}$ was obtained. The sway buckling of a semi-circular arch loaded by vertical inertia forces (or by dead weight of the arch) has been discussed in (13), where it was shown that $w_{cr} = 2.68\ EI/r_0^{\,3}$, where $w_{cr} = \rho a$, ρ = mass density per unit length of arc and a = vertical acceleration of the supports of the arch. The problem of stability and large deflection of rings, including the effects of ring thickness and of Poisson's ratio, subjected to nonuniform loads is relatively unexplored.

EQUILIBRIUM APPROACH TO THE NONLINEAR THEORY OF SHELLS

The middle surface of any shell is defined by $\bar{r} = \bar{r}(x,y)$, where $\bar{r}$ is a position vector and (x, y) are parameters called "surface coordinates." Attention will here be restricted to orthogonal surface coordinates. Then, since the derivative vectors $\bar{r}_x$ and $\bar{r}_y$ are tangent respectively to the x and y coordinate lines, $\bar{r}_x \cdot \bar{r}_y = 0$. In this case, the distance ds between neighboring points on the middle surface S is given by

$$ds^2 = A^2\, dx^2 + B^2\, dy^2 \tag{1}$$

where $A^2 = \bar{r}_x \cdot \bar{r}_x$ and $B^2 = \bar{r}_y \cdot \bar{r}_y$. Accordingly, the unit tangent vectors to the x and y coordinate lines are $\bar{r}_x/A$ and $\bar{r}_y/B$, respectively. The unit normal $\hat{n}$ to the middle surface S is accordingly,

$$\hat{n} = \frac{\bar{r}_x \times \bar{r}_y}{A\,B} \tag{2}$$

The coefficients of the second differential quadratic form of S are

$$e = \hat{n} \cdot \bar{r}_{xx}\ , \qquad f = \hat{n} \cdot \bar{r}_{xy}\ , \qquad g = \hat{n} \cdot \bar{r}_{yy} \tag{3}$$

If the coordinate lines on S are the lines of principal curvature, $f = 0$, and the principal curvatures of S are $1/r_1 = e/A^2$, $1/r_2 = g/B^2$. However, we shall not make the restriction $f = 0$.

The tensions N_x, N_y, the shears N_{xy}, N_{yx}, Q_x, Q_y, the bending moments M_x, M_y, and the twisting moments M_{xy}, M_{yx}, referred to an element of the middle surface of the shell, are shown with their positive senses in Fig. 5. The normals to the element dS of the middle surface generate a volume element of the shell. The external force on this element is denoted by $\bar{P}\, dS = \bar{P}\, A\, B\, dx\, dy$. This force ordinarily results from the weight of the volume element and from loads applied to the external faces of the shell. The force $\bar{P}\, dS$ is specified to act at a point in dS. Then there may be an external couple $\bar{L}\, dS$ that acts on the volume element; usually it results from tangential external distributed loads applied to the faces of the shell. The components of the vectors $\bar{P}$ and $\bar{L}$ in the directions of the orthogonal vectors $\bar{r}_x$, $\bar{r}_y$, $\hat{n}$ are denoted by P_x, P_y, P_z and L_x, L_y. The vector $\bar{L}$ has no component in the direction $\hat{n}$.

The equilibrium equations for the quantities N_x, N_y, etc. may be derived in an elementary manner by Gibbs' vector theory; they were derived more than twenty years ago in tensor form by Synge and Chien. In the present notations, their equilibrium equations are

$$\frac{\partial}{\partial x}(B N_x) + \frac{\partial}{\partial y}(A N_{yx}) + A_y N_{xy} - B_x N_y - \frac{B e}{A} Q_x - f Q_y + A B P_x = 0 \qquad (4)$$

$$\frac{\partial}{\partial x}(B N_{xy}) + \frac{\partial}{\partial y}(A N_y) - A_y N_x + B_x N_{yx} - f Q_x - \frac{A g}{B} Q_y + A B P_y = 0 \qquad (5)$$

$$\frac{\partial}{\partial x}(B Q_x) + \frac{\partial}{\partial y}(A Q_y) + \frac{B e}{A} N_x + \frac{A g}{B} N_y + (N_{xy} + N_{yx}) f + A B P_z = 0 \qquad (6)$$

$$\frac{\partial}{\partial x}(B M_x) + \frac{\partial}{\partial y}(A M_{yx}) + A_y M_{xy} - B_x M_y + A B L_y = A B Q_x \qquad (7)$$

$$\frac{\partial}{\partial x}(B M_{xy}) + \frac{\partial}{\partial y}(A M_y) - A_y M_x + B_x M_{yx} - A B L_x = A B Q_y \qquad (8)$$

$$N_{xy} - N_{yx} - \frac{e}{A^2} M_{xy} + \frac{f}{A B}(M_x - M_y) + \frac{g}{B^2} M_{yx} = 0 \qquad (9)$$

Equations (7) and (8) may be used to eliminate Q_x and Q_y from Eq. (6); thus, the moment equilibrium equation is obtained. The shears Q_x and Q_y are usually discarded from Eqs. (4) and (5), since they are small compared to N_x, N_y, N_{xy}. Furthermore, the terms Q_x and Q_y that occur in Eqs. (4) and (5) represent only components of the transverse shears tangent to surface S which arise because the volume element is slightly tapered. For a flat plate, these terms drop from Eqs. (4) and (5) automatically, since then $e = f = g = 0$. Also, the approximations $N_{xy} = N_{yx}$ and $M_{xy} = M_{yx}$ are nearly always legitimate; they are exactly true for a flat plate. When the approximation $N_{xy} = N_{yx}$ is used, Eq. (9) is disregarded.

The weight of the material often contributes to the tangential loads (P_x, P_y), but frequently the effect of the weight on the stresses is negligible. Then, if no tangential distributed loads are applied to the exterior faces of the shell, the terms (P_x, P_y) are practically zero. In this case, if the terms Q_x and Q_y are discarded from Eqs. (4) and (5), and if the approximation $N_{xy} = N_{yx}$ is introduced, the general solution of Eqs. (4) and (5) may be expressed as follows in terms of a generalized Airy stress function $H(x, y)$ for the case in which the Gaussian curvature K of surface S is constant (14):

$$N_x = B^{-2} H_{yy} + A^{-2} B^{-1} B_x H_x - B^{-3} B_y H_y + K H$$

$$N_y = A^{-2} H_{xx} - A^{-3} A_x H_x + A^{-1} B^{-2} A_y H_y + K H \qquad (10)$$

$$N_{xy} = N_{yx} = -A^{-1} B^{-1} H_{xy} + A^{-2} B^{-1} A_y H_x + A^{-1} B^{-2} B_x H_y$$

The case K = constant, to which Eq. (10) is restricted, includes flat plates, spheres, cylinders, and cones, as well as many less common surfaces. For example, any surface that can be obtained by bending a thin flat piece of sheet metal or a piece of a thin spherical shell without stretching it has constant K, since the Gaussian curvature is a bending invariant. There are also surfaces of constant negative Gaussian curvature, called pseudospheres. For flat plates, $K = 0$, and H is a generalized Airy stress function that is applicable to any orthogonal coordinates in the middle plane of the plate.

The equilibrium equations naturally refer to the stressed state. Sometimes the deformation caused by stressing alters the geometry of the shell appreciably. For example, a plate that is initially flat becomes a curved shell when it is loaded, and the curvature may have important effects on the equilibrium conditions. Likewise, a shell that is initially rotationally symmetric may lose its symmetry because of elastic or plastic deformations. Accordingly, if (e, f, g) are the components of the curvature tensor of the undeformed reference surface S, some modifications of these coefficients may be required to account for the effects of the deformation on the equilibrium conditions. This is always true in problems of post-buckling behavior.

When the shell is deformed, the reference surface S passes into another surface S*. The asterisk or star will be used generally to denote the stressed state. The displacement vector of S is denoted by $\bar{q}(x, y)$; that is, the point $\bar{R}$ on S* corresponding to point $\bar{r}$ on S is $\bar{R} = \bar{r} + \bar{q}$. Evidently, if the vector function $\bar{q}(x, y)$ is known, the position vector $\bar{R}$ is a known function of (x, y). Thus, the same coordinates (x, y) serve for surfaces S and S*. The metric coefficients of S* are $E^* = \bar{R}_x \cdot \bar{R}_x$, $F^* = \bar{R}_x \cdot \bar{R}_y$, $G^* = \bar{R}_y \cdot \bar{R}_y$. The distance between two neighboring points on S* is determined by $(ds^*)^2 = E^* dx^2 + 2 F^* dx\, dy + G^* dy^2$. If surface S is bent without straining, $E^* = A^2$, $F^* = 0$, $G^* = B^2$, since $ds = ds^*$. If the strains of surface S are small, as usually happens, ds is very nearly equal to ds^*. Consequently, even in the large-deflection theories of shells, changes of A and B caused by straining of surface S need not be taken into account in the equilibrium equations. Seemingly, this approximation is not generally appreciated, for some of the modern investigations of buckling of shells introduce undue complications into the equilibrium equations as a consequence of incremental changes in the metric coefficients due to straining. In particular, if the coordinates (x, y) on S are orthogonal, as is here assumed they are orthogonal on S* too, for the equation $F = 0$ signifies that F^* is nearly zero. Since the Gaussian curvature K and the Christoffel symbols Γ^i_{jk}

are determined by the metric tensor alone, they may also be considered to be unaffected by the deformation, insofar as the equilibrium are concerned. However, in nonlinear theories of shells, the quantities (e, f, g) must be replaced by the corresponding quantities (e^*, f^*, g^*) in Eq. (6). It is not important to make this change in Eqs. (4) and (5), since the terms Q_x and Q_y are ordinarily dropped from these equations anyway.

The components of the displacement vector $\bar{q}$ in the directions of the vectors $\bar{r}_x$, $\bar{r}_y$, $\hat{n}$ are denoted by (u, v, w), respectively. The effects of the tangential displacements (u, v) on the changes of curvature are usually small. Consequently, for computation of e^*, f^*, g^*, the equation $\bar{R} = \bar{r} + \bar{q}$ is approximated by $\bar{R} = \bar{r} + \hat{n}\, w$. The curvature tensor for S^* is $e^* = \hat{n}^* \cdot \bar{R}_{xx}$, $f^* = \hat{n}^* \cdot \bar{R}_{xy}$, $g^* = \hat{n}^* \cdot \bar{R}_{yy}$. The following notations are introduced:

$$K_x = e^* - e\ , \qquad K_y = g^* - g\ , \qquad K_{xy} = f^* - f \tag{11}$$

It is a routine problem of differential geometry to derive K_x, K_y, K_{xy}. The results have been obtained by Koiter, with (u, v) terms included (15). Dropping the (u, v) terms, we get

$$K_x = -\frac{A_x}{A} w_x + \frac{A A_y}{B^2} w_y + w_{xx}\ , \quad K_y = \frac{B B_x}{A^2} w_x - \frac{B_y}{B} w_y + w_{yy}$$

$$K_{xy} = -\frac{A_y}{A} w_x - \frac{B_x}{B} w_y + w_{xy} \tag{12}$$

These equations are merely first-degree approximations; nonlinear terms in derivatives of w have been neglected. Also, a linear undifferentiated w-term has been dropped. This term may be interpreted by considering the case, w= constant. For example, if w is constant for a circular cylindrical shell, there is a small change of curvature because the radius is changed. However, since A, B have been retained instead of E^*, F^*, G^*, it would be inconsistent to retain the undifferentiated w-term, since the effect of this term is about the same as effects of changes of the metric tensor due to straining of surface S.

Equations (11) and (12) determine the quantities (e^*, f^*, g^*) that are to be introduced into Eq. (6) instead of (e, f, g) when large deflections or buckling problems are considered. To obtain this generalization, the term f must be retained in Eq. (6), for, even though coordinates are chosen so that $f = 0$, f^* is not generally zero. In other words, in nonlinear theories, the lines of principal curvature on S are altered significantly by the deformation. For example, for a flat plate referred to rectangular coordinates, $A = B = 1$ and $e = f = g = 0$. Equations (11) and (12) give $e^* = w_{xx}$, $f^* = w_{xy}$, $g^* = w_{yy}$. Introducing these quantities into Eq. (6) instead of (e, f, g), supposing that $L_x = L_y = 0$, and eliminating Q_x and Q_y by means of Eqs. (7) and (8), we obtain a well-known equation in the theory of buckling of flat plates:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x\, w_{xx} + N_y\, w_{yy} + 2\, N_{xy}\, w_{xy} + P_z = 0$$

STRAINS OF A SURFACE

The strains of surface S due to the displacement vector (u,v,w) have been derived by Love and many others. If only quadratic terms in w_x, w_y are retained, these equations are

$$\epsilon_x = \frac{u_x}{A} + \frac{A_y\, v}{A\,B} - \frac{e\, w}{A^2} + \frac{w_x^2}{2\,A^2}\,, \qquad \epsilon_y = \frac{v_y}{B} + \frac{B_x u}{A\,B} - \frac{g\, w}{B^2} + \frac{w_y^2}{2\,B^2}$$

$$\gamma_{xy} = \frac{u_y}{B} + \frac{v_x}{A} - \frac{A_y\, u}{A\,B} - \frac{B_x\, v}{A\,B} - \frac{2\, f\, w}{A\,B} + \frac{w_x\, w_y}{A\,B} \tag{13}$$

The metric tensor of the deformed surface S^* is given by $E^* = A^2\,(1 + 2\epsilon_x)$, $G^* = B^2\,(1 + 2\epsilon_y)$, $F^* = A\,B\,\gamma_{xy}$. Since, by differential geometry, the Gaussian curvature K^* of surface S^* can be expressed in terms of E^*, F^*, G^* alone, the increment of K due to the deformation can accordingly be expressed in terms of ϵ_x, ϵ_y, γ_{xy}. The result of this calculation is

$$\begin{aligned} A\,B\,(K^* - K) = {} & \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} - \frac{A}{B}\,\frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{B}{A}\,\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{B_x}{A}\,\frac{\partial \epsilon_x}{\partial x} + \frac{A_y}{B}\,\frac{\partial \epsilon_y}{\partial y} \\ & + \left(\frac{A\,B_y}{B^2} - \frac{2\,A_y}{B}\right)\frac{\partial \epsilon_x}{\partial y} + \left(\frac{B\,A_x}{A^2} - \frac{2\,B_x}{A}\right)\frac{\partial \epsilon_y}{\partial x} + \frac{A_y}{A}\,\frac{\partial \gamma_{xy}}{\partial x} + \frac{B_x}{B}\,\frac{\partial \gamma_{xy}}{\partial y} \\ & + 2\left(\frac{B_{xx}}{A} - \frac{A_x\, B_x}{A^2}\right)\epsilon_x + 2\left(\frac{A_{yy}}{B} - \frac{A_y\, B_y}{B^2}\right)\epsilon_y \\ & + \left(\frac{A_{xy}}{A} + \frac{B_{xy}}{B} - \frac{A_x\, A_y}{A^2} - \frac{B_x\, B_y}{B^2}\right)\gamma_{xy} \end{aligned} \tag{14}$$

Substituting Eq. (13) into Eq. (14), and simplifying the result with the Gauss and Codazzi equations, we obtain

$$\begin{aligned} A\,B\,(K^* - K) = {} & B\,K_x\, u + A\,K_y\, v + 2\,A\,B\,K\,M\,w + \frac{1}{A^2 B}\left(\frac{B\,B_x\, e}{A} + 2\,A_y f - A_x g\right) w_x \\ & + \frac{1}{A\,B^2}\left(\frac{A\,A_y\, g}{B} + 2\,B_x\, f - B_y\, e\right) w_y + \frac{g}{A\,B}\, w_{xx} - \frac{2\,f}{A\,B}\, w_{xy} + \frac{e}{A\,B}\, w_{yy} \end{aligned}$$

$$- \frac{1}{A^3} \left(\frac{A_x B_x}{A} + \frac{A_y^2}{B} + K A^2 B \right) w_x^2 - \frac{1}{B^3} \left(\frac{A_y B_y}{B} + \frac{B_x^2}{A} + K A B^2 \right) w_y^2 + \frac{B_x}{A^3} w_x w_{xx}$$
$$+ \frac{2 A_y}{A^2 B} w_x w_{xy} - \frac{A_x}{A^2 B} w_x w_{yy} - \frac{B_y}{A B^2} w_y w_{xx} + \frac{2 B_x}{A B^2} w_y w_{xy} + \frac{A_y}{B^3} w_y w_{yy} \tag{15}$$
$$+ \frac{w_{xx} w_{yy} - w_{xy}^2}{A B}$$

In the linear theory of shells, all nonlinear terms are dropped from Eq. (15). The term M represents the mean curvature of S; i.e., $2M = 1/r_1 + 1/r_2$.

For brevity, Eqs. (14) and (15) are written as follows: $A B (K^* - K) = L(\epsilon_x, \epsilon_y, \gamma_{xy})$, $A B (K^* - K) = B K_x u + A K_y v + \psi (w)$. These equations yield

$$L (\epsilon_x , \epsilon_y , \gamma_{xy}) = B K_x u + A K_y v + \psi (w) \tag{16}$$

If K is constant, u and v disappear from Eq. (16). Then Eq. (16) is a compatibility equation for the strain components of surface S. If K is not constant, u and v can not be eliminated without the use of differential operators of order greater than 2. For example, if a flat plate is referred to rectangular coordinates, $A = B = 1$ and $e = f = g = K = M = 0$. Then Eq. (16) yields

$$\frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} - \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \epsilon_y}{\partial x^2} = w_{xx} w_{yy} - w_{xy}^2$$

This is an important equation in the large-deflection theory of plates.

Shell problems may be formulated in terms of the displacement components (u, v, w) or in terms of the stress function and the normal displacement (H, w). If (u, v, w) are regarded as the unknowns, only the equilibrium equations and the boundary conditions are needed; there is no need to consider compatibility equations. Certainly, this is the more general approach, since the stress function H is limited to shells of constant Gaussian curvature, although a generalization to cover all rotationally symmetric shells is possible (14). Furthermore, the boundary conditions can always be formulated in terms of (u, v, w), but they can not always be expressed in terms of stresses. The compatibility equation represented by Eq. (16) is useful only for shells of constant Gaussian curvature, since only then do the terms u and v drop out. However, it is only for shells of constant Gaussian curvature that a compatibility equation is needed, since the (u, v, w) formulation will be used for other shells.

Equations (1) to (16) do not involve stress-strain relations and they remain valid if shear deformation is significant. For a complete formulation of the shell problem, we must express (N_x , N_y , N_{xy}) in terms of $(\epsilon_x , \epsilon_y , \gamma_{xy})$ and also (M_x , M_y , M_{xy}) in terms of $(\mathcal{K}_x , \mathcal{K}_y , \mathcal{K}_{xy})$. Here, the simplest approximations to adopt are exactly the same as in flat-plate theory. Questions of nonlinear geometric relations do not enter here, since they arise only in the strain-displacement relations [Eq. (13)]. It is a moot point whether nonlinear terms in (u, v) should be retained in Eq. (13). A rigorous analysis of stress-strain relations and moment-curvature relations, with thermal effects included, was developed on the basis of the Kirchhoff assumption by the authors (16). Some investigators have concluded, on the basis of order-of-magnitude considerations, that, when the Kirchhoff assumption is used, consistency requires that the strains be linearized in the normal coordinate z. However, if this argument is applied to beams, it signifies that the Winkler theory of curved beams is no better than straight-beam theory. A striking example of an error that can be incurred by linearization with respect to z is provided by a curved cantilever beam of rectangular cross section with depth h and width b (Fig. 6). The load P is applied at the centroid of the end section. By Winkler's theory, the stress σ_θ at ordinate z is

$$\sigma_\theta = \frac{P}{bh} + \frac{P\,z\,(1 - \sin\theta)}{b\,(aC - h)\,(a + z)}, \qquad C = \log \frac{2a + h}{2a - h} \tag{a}$$

The net tension and the bending moment are

$$F = b \int_{-h/2}^{h/2} \sigma_\theta \, dz\,, \qquad M = b \int_{-h/2}^{h/2} z\,\sigma_\theta \, dz \tag{b}$$

Substituting Eq. (a) into Eq. (b), we get $F = P \sin\theta$, $M = P\,a\,(1 - \sin\theta)$. These relations agree with elementary statics. Suppose now that z is dropped from the denominator of Eq. (a), so that the equation is linearized in z. Then, the net tension calculated by $F = \int_{-h/2}^{h/2} \sigma_\theta \, dA$ is $F = P$.

This relation disagrees grossly with statics.

The sensitivity in this example comes from the fact that the stress σ_θ has a large negative value on the inside of the beam, and a large positive value on the outside. Therefore, σ_θ must be given quite accurately if $\int \sigma_\theta \, dA$ is to be evaluated correctly. We may conclude that linearization of the stresses or strains with respect to z is usually admissible if the objective is to ~~predict yielding or other types~~ of failure of the material. However, it is a questionable approximation if the stresses are to be integrated through the thickness for the purpose of determining N_x, M_x, etc.

If linearization with respect to z is admissible, we obtain from (7), when temperature terms and nonlinear terms in h are discarded,

$$N_x = \frac{E\,h}{1-\nu^2}(\epsilon_x + \nu\,\epsilon_y)\,, \qquad N_y = \frac{E\,h}{1-\nu^2}(\epsilon_y + \nu\,\epsilon_x) \tag{17}$$

$$N_{xy} = N_{yx} = G\,h\,\gamma_{xy}$$

where E is Young's modulus, ν is Poisson's ratio, G is the shear modulus, and h is the thickness of the shell. Naturally, these relations are restricted to isotropic elastic shells. Also, if effects of u and v on the bending moments are neglected, we obtain

$$M_x = -D\left[\frac{K_x}{A^2} + \frac{\nu K_y}{B^2} + w\,k_1(k_1 - k_2)\right]$$

$$M_y = -D\left[\frac{K_y}{B^2} + \frac{\nu K_x}{A^2} + w\,k_2(k_2 - k_1)\right] \tag{18}$$

$$M_{xy} = M_{yx} = -\frac{D(1-\nu)}{A\,B}K_{xy}\,, \qquad D = \frac{E\,h^3}{12(1-\nu^2)}$$

Here, k_1 and k_2 are the principal curvatures of the middle surface. The quantities K_x, K_y, K_{xy} are defined by Eq. (12). Most writers have dropped w from Eq. (18). However, in some cases, this term has a significant effect on computed buckling loads. For example, if a very long cylindrical shell of radius a is subjected to uniform external pressure, the buckling pressure is $p_{cr} = 3\,D/a^3$. This result is obtained if w is retained in Eq. (18), but we get $p_{cr} = 4\,D/a^3$ if w is dropped.

Some special applications of the preceding equations have been studied. For example, if w is dropped from Eq. (18), Donnell's equation for cylindrical shells is obtained readily. Also, the equations of Reiss, Greenberg, and Keller for snap-through of a shallow spherical cap are obtained immediately, although in a different form. For flat plates, von Kármán's equations are obtained. By further studies, the authors hope to get a better correlation between the equilibrium approach and the energy approach to problems of buckling and post-buckling.

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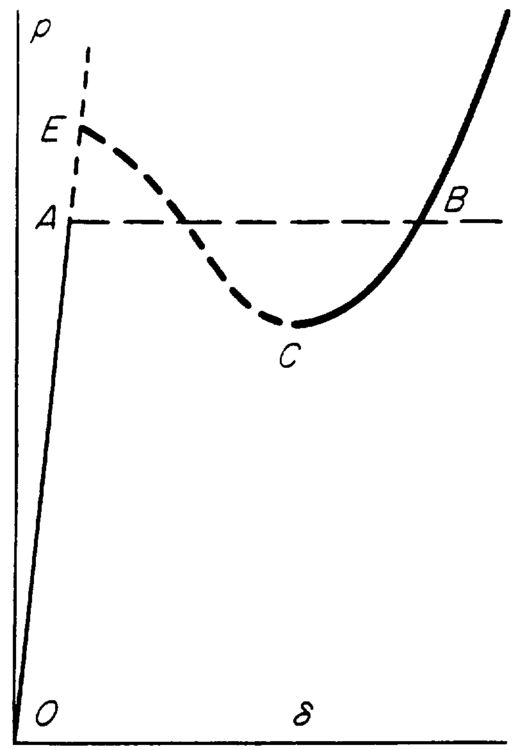

Figure 1.- Pressure-deflection curve.

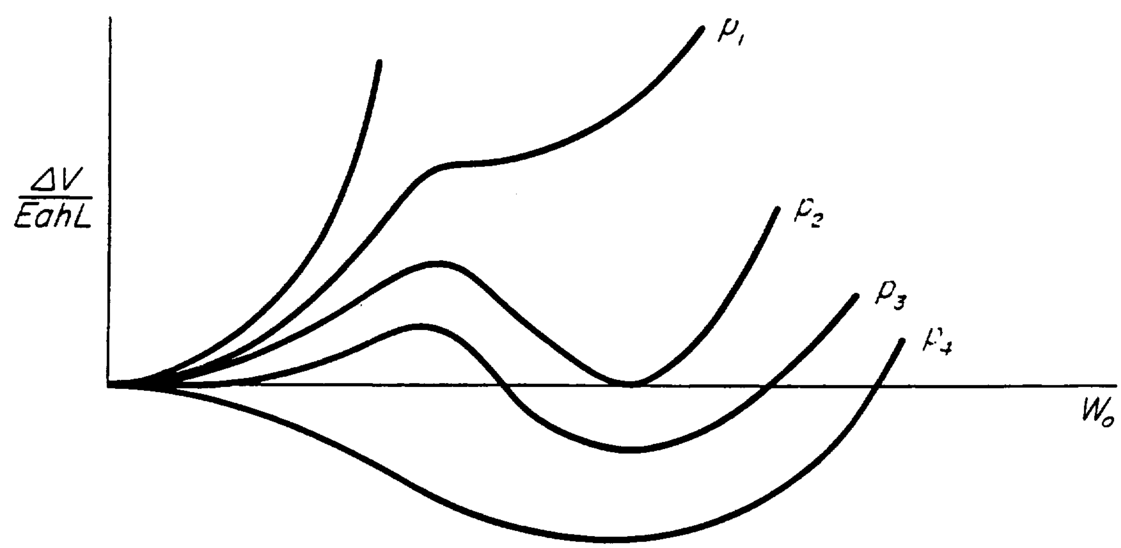

Figure 2.- Increment of potential energy versus deflection parameter.

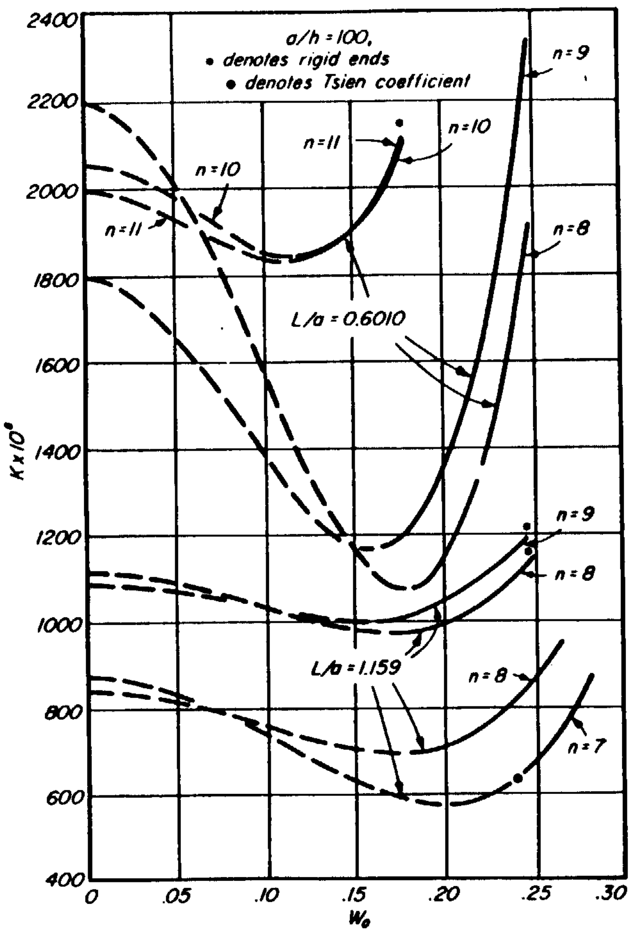

Figure 3.- Buckling coefficient K versus deflection parameter W_0.

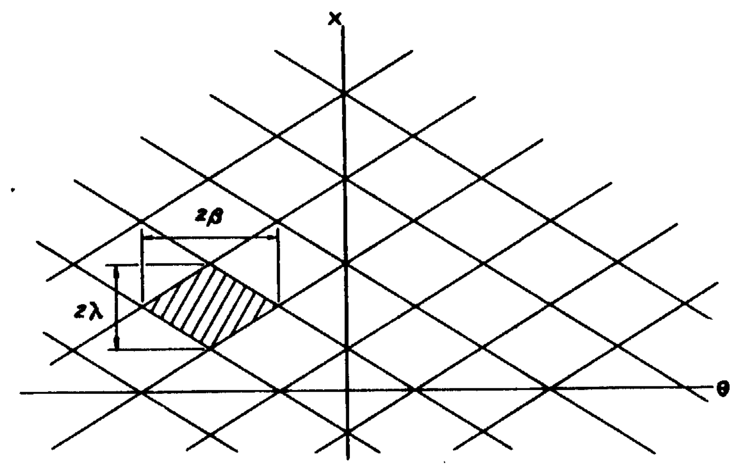

Figure 4.- Deflection pattern.

A

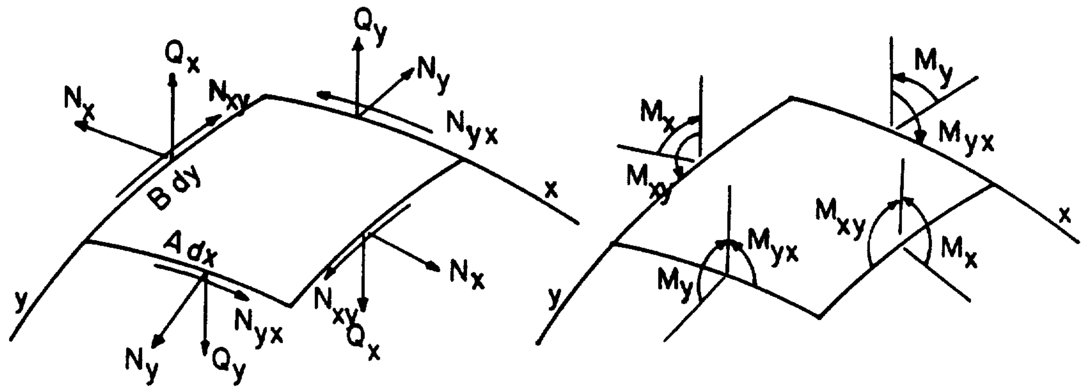

Figure 5.- Notation and sign convention for tractions, bending moments and twisting moments.

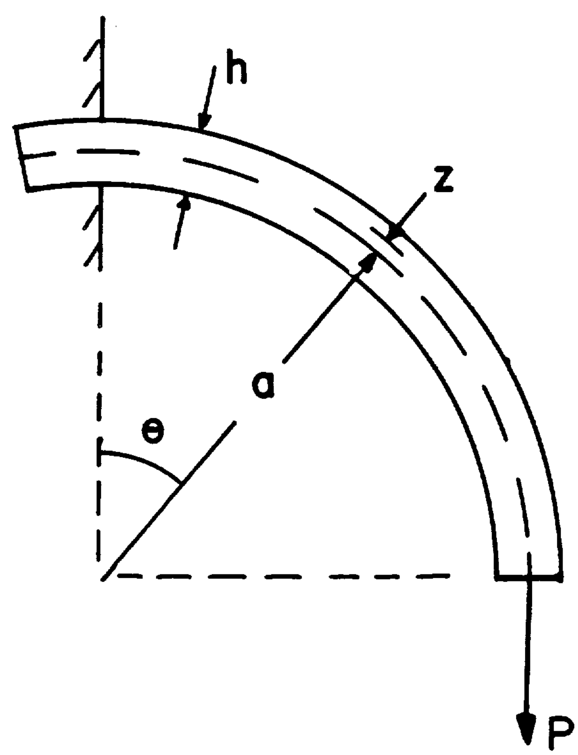

Figure 6.