BUCKLING OF INITIALLY IMPERFECT AXIALLY COMPRESSED CYLINDRICAL SHELLS

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SUMMARY

The effects of imperfections on the buckling strength of pressurized cylinders under axial compression are studied. The imperfection factor is considered in the finite-deflection compatibility equation, as well as in the equilibrium equation. A new relation to express the imperfection as an explicit exponential function of pressure and radius-thickness ratio is proposed. A method for finding the exponential function is described, and the solution for the critical stress is found in a fairly simple form. The relation of decrease of stability with respect to magnitude of imperfections is found.

INTRODUCTION

The elastic postbuckling behavior of initially perfect cylindrical shells subject to internal pressure together with axial compression or bending has been discussed in references 1, 2, 3, and 4. It was observed that the available test results (refs. 2, 3, and 4) were lower than the critical stress found by the analysis in reference 1. On the other hand, the ratio of the increment of critical stress, due to the internal pressure, to the critical stress in an unpressurized cylinder, found in the analysis in reference 1, was fairly close to the test data in references 2, 3, and 4. This indicates that practically no shell can be considered perfect and that the solution to the "perfect" shell can only serve as an upper bound. Therefore, an investigation of the effect of imperfections is necessary.

The influence of initial imperfections was first studied by Donnell in 1934 (ref. 5). The solution for cylindrical shells under axial compression was later carried out by Donnell and Wan (ref. 6). For shells under external pressure, the effects of imperfections and finite deflections were studied by Nash (ref. 7), while the case of fixed edges was discussed by Donnell in 1958 (ref. 8).

SYMBOLS

c	index in the exponential function defined in equation 11
D	flexural rigidity, $D = Et^3/\ 12(1 - \nu^2)$
E	Young's modulus
F	Airy stress function
R	radius of middle surface of shell
m, n	number of waves in axial and circumferential directions, respectively
p	internal pressure
t	wall thickness of shell
w	total normal deflection
w_i	initial deflection
x, s	co-ordinates of point in middle surface of shell
Γ	w_i/w, imperfection ratio
α	R/m^2t
η	$b_3/t\alpha$

μ	n^2/m^2
γ	imperfection coefficient defined in equation 24
ν	Poisson's ratio, $\nu = 0.3$
∇^4	$= \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial s^2} + \frac{\partial^4}{\partial s^4}$
φ	stress parameter defined in equation 18
σ	axial stress
o	superscript indicating perfect shell
cr	subscript indicating critical condition

BASIC EQUATIONS

Let w equal the total radial deflection; w_i, the initial imperfection in the radial direction; and F, the Airy stress function. The strain-displacement relations are:

$$\left.\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_i}{\partial x}\right)^2 \\
\epsilon_s &= \frac{\partial v}{\partial s} - \frac{(w - w_i)}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2 - \frac{1}{2}\left(\frac{\partial w_i}{\partial s}\right)^2 \\
\epsilon_{xs} &= \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial s} - \frac{\partial w_i}{\partial x}\frac{\partial w_i}{\partial s}
\end{aligned}\right\} \quad (1)$$

The strain-stress relations in the x-s plane are:

$$\left.\begin{aligned}
\epsilon_x &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial s^2} - \nu \frac{\partial^2 F}{\partial x^2}\right) \\
\epsilon_s &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial s^2}\right) \\
\epsilon_{xs} &= -\frac{2(1+\nu)}{E}\frac{\partial^2 F}{\partial x \partial s}
\end{aligned}\right\} \quad (2)$$

The compatibility equation derived from equation 2 is:

$$\nabla^4 F = E\left(\frac{\partial^2 \epsilon_x}{\partial s^2} + \frac{\partial^2 \epsilon_s}{\partial x^2} - \frac{\partial^2 \epsilon_{xs}}{\partial x \partial s}\right) \quad (3)$$

After equation 1 is substituted in equation 3, the finite-deflection compatibility equation has the following form:

$$\nabla^4 F = E\left[\left(\frac{\partial^2 w}{\partial x \partial s}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial s^2} - \frac{1}{R}\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial^2 w_i}{\partial x \partial s}\right)^2 \right.$$

$$\left. + \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial s^2} + \frac{1}{R}\frac{\partial^2 w_i}{\partial x^2}\right] \quad (4)$$

The equilibrium equation in the radial direction is:

$$D\nabla^4(w - w_i) = \frac{t}{R}\frac{\partial^2 F}{\partial x^2} + t\left[\frac{\partial^2 F}{\partial s^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial s}\frac{\partial^2 w}{\partial x \partial s} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial s^2}\right] - p \qquad (5)$$

When w_i is a known function, the solution may be found by solving for w and F in equations 4 and 5 simultaneously. However, w_i is, in general, unknown. For simplicity, w is assumed to be proportional to w_i as in references 5 through 8, that is,

$$w_i/w = \Gamma = \text{imperfection ratio} \qquad (6)$$

where Γ is independent of x and s. The expression for Γ will be discussed in the next section. The critical stress to be found is then a function of Γ.

With the relation from equation 6, the compatibility and equilibrium equations are expressed, respectively, as:

$$\left(\frac{1}{1-\Gamma}\right)\nabla^4 F = E(1+\Gamma)\left[\left(\frac{\partial^2 w}{\partial x \partial s}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial s^2}\right] - \frac{E}{R}\frac{\partial^2 w}{\partial x^2} \qquad (7)$$

and

$$D(1-\Gamma)\nabla^4 w = \frac{t}{R}\frac{\partial^2 F}{\partial x^2} + t\left[\frac{\partial^2 F}{\partial s^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial s}\frac{\partial^2 w}{\partial x \partial s} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial s^2}\right] - p \qquad (8)$$

DEFLECTION FUNCTION w AND IMPERFECTION RATIO Γ

In the present study, the deflection function for a cylindrical shell under internal pressure and axial compression is assumed to be:

$$w = b_1 + b_2 \cos\frac{mx}{R}\cos\frac{ns}{R} + b_3\cos\frac{2mx}{R} + b_3\cos\frac{2ns}{R} \tag{9}$$

The parameter b_1 is not independent but is used to satisfy the condition of periodicity of circumferential displacement.

The Airy stress function, accordingly, is assumed to be:

$$F = -\frac{\sigma}{2}s^2 + \frac{p}{2}\frac{R}{t}x^2 + a_{11}\cos\frac{mx}{R}\cos\frac{ns}{R} + a_{20}\cos\frac{2mx}{R}$$

$$+ a_{22}\cos\frac{2mx}{R}\cos\frac{2ns}{R} + a_{02}\cos\frac{2ns}{R} \tag{10}$$

Before the solution is discussed, the expression for the imperfection ratio, Γ, should be studied. Evidence from previous tests (refs. 2, 3, and 4) reveals that imperfections have the greatest effect on shells having the largest ratio of R/t. The simplest way to express this relation is to assume that Γ increases linearly with R/t and then to determine the proportionality constant by means of available test data. However, this method may fit only a certain range of values of R/t. The limiting physical conditions require that: (1) $\Gamma = 0$ when $R/t = 0$, and (2) $\Gamma = 1$ only when $R/t \to \infty$. A relation to express Γ as a function of both R/t and the internal pressure p is now proposed, such that:

$$\Gamma = 1 - \exp(-C \frac{R}{t}) \tag{11}$$

The index C in the above equation is always positive and is assumed to be a function of internal pressure only. The method for determining C will be discussed later.

SOLUTIONS

The Galerkin method was employed to solve equations 7 and 8 when the assumed forms of F and w in equations 10 and 9, respectively, were used. The coefficients of the stress function F are found first as follows:

$$\left.\begin{aligned}
\frac{1}{1-\Gamma}\frac{a_{11}}{Et^2} &= \frac{1}{(1+\mu)^2}\left[1 - 4\mu(1+\Gamma)\eta\right]\alpha\left(\frac{b_2}{t}\right) \\
\frac{1}{1-\Gamma}\frac{a_{22}}{Et^2} &= -\frac{1+\Gamma}{(1+\mu)^2}\mu\eta^2\alpha^2 \\
\frac{1}{1-\Gamma}\frac{a_{02}}{Et^2} &= -\frac{1+\Gamma}{32\mu}\left(\frac{b_2}{t}\right)^2 \\
\frac{1}{1-\Gamma}\frac{a_{20}}{Et^2} &= \frac{1}{16}\left[4\eta\alpha^2 - \frac{\mu}{2}(1+\Gamma)\left(\frac{b_2}{t}\right)^2\right]
\end{aligned}\right\} \tag{12}$$

where

$$\mu = \frac{n^2}{m^2} \tag{13}$$

$$\alpha = \frac{R}{m^2 t} \tag{14}$$

$$\eta = \frac{b_3/t}{\alpha} \tag{15}$$

From the integration of equation 8 and the relation in equation 12, the following two equations are established after simplification:

$$\frac{1}{1-\Gamma}\varphi = \frac{(1+\mu)^2}{12(1-\nu^2)}\frac{1}{\alpha} + \alpha\left\{\frac{1}{(1+\mu)^2} - \left[\frac{4(2+\Gamma)}{(1+\mu)^2} + \frac{1}{2}\right]\mu\eta + (1+\Gamma)\frac{16\mu^2\eta^2}{(1+\mu)^2}\right\} + \frac{(1+\Gamma)(1+\mu^2)}{16\alpha}\left(\frac{b_2}{t}\right)^2 \tag{16}$$

and

$$\frac{1}{1-\Gamma}\varphi = \frac{1+\mu^2}{3(1-\nu^2)}\frac{1}{\alpha} + \left[\frac{1}{4} + \frac{4(1+\Gamma)}{(1+\mu)^2}\mu^2\eta^2\right]\alpha + \frac{1}{\alpha}\left\{\frac{2(1+\Gamma)}{(1+\mu)^2}\mu^2 - \frac{1}{\eta}\left[\frac{\mu}{2(1+\mu)^2} + \frac{1+\Gamma}{32}\mu\right]\right\}\left(\frac{b_2}{t}\right)^2 \tag{17}$$

In the above equations, φ is a dimensionless stress parameter and

$$\varphi = \frac{\sigma R}{Et} - \mu\frac{pR^2}{Et^2} \tag{18}$$

Equations 16 and 17 can be rewritten as:

$$\frac{1}{1-T}\varphi = \frac{A_1}{\alpha} + \alpha\left(A_2 + A_3\eta + A_4\eta^2\right) + \frac{A_5}{\alpha}\left(\frac{b_2}{t}\right)^2 \qquad (16a)$$

$$\frac{1}{1-T}\varphi = \frac{B_1}{\alpha} + \alpha\left(B_2 + B_4\eta^2\right) + \left(\frac{B_5 + B_6/\eta}{\alpha}\right)\left(\frac{b_2}{t}\right)^2 \qquad (17a)$$

In the above equations, the following notations are used:

$$\left.\begin{aligned}
A_1 &= \frac{(1+\mu)^2}{12(1-\nu^2)} & B_1 &= \frac{1+\mu^2}{3(1-\nu^2)} \\
A_2 &= \frac{1}{(1+\mu)^2} & B_2 &= \frac{1}{4} \\
A_3 &= -\left[\frac{4(2+T)}{(1+\mu)^2} + \frac{1}{2}\right]\mu & B_4 &= \frac{4(1+T)\mu^2}{(1+\mu)^2} \\
A_4 &= \frac{16(1+T)\mu^2}{(1+\mu)^2} & B_5 &= \frac{2(1+T)\mu^2}{(1+\mu)^2} \\
A_5 &= \frac{(1+T)(1+\mu^2)}{16} & B_6 &= -\mu\left[\frac{1}{2(1+\mu)^2} + \frac{1+T}{32}\right]
\end{aligned}\right\} \qquad (19)$$

To eliminate b_2/t from equations 16a and 17a it has been found that:

$$\frac{1}{1-T}\varphi = \frac{C_1 A_1}{\alpha} + C_2 A_2 \alpha \qquad (20)$$

where

$$C_1 = \frac{\frac{A_5}{A_1} B_1 - B_5 - \frac{B_6}{\eta}}{A_5 - B_5 - \frac{B_6}{\eta}} \tag{21a}$$

$$C_2 = \frac{\left(A_5 B_2 - A_3 B_6 - A_2 B_5\right) - \frac{A_2 B_6}{\eta} - \left(A_3 B_5 + A_4 B_6\right)\eta + \left(A_5 B_4 - A_4 B_5\right)\eta^2}{A_2 \left(A_5 - B_5 - \frac{B_6}{\eta}\right)} \tag{21b}$$

When φ is assumed to be a continuous function of α, the minimization of φ with respect to α leads to

$$\frac{\partial \varphi}{\partial \alpha} = 0$$

Thus, from equation 20,

$$\alpha = \sqrt{\frac{C_1 A_1}{C_2 A_2}} \tag{22}$$

and we have

$$\varphi_\alpha = 2(1 - T)\sqrt{C_1 C_2}\sqrt{A_1 A_2} = (1 - T)\sqrt{C_1 C_2}\sqrt{\frac{1}{3(1 - \nu^2)}} \tag{23}$$

The notation φ_α represents the value of φ that has been minimized with respect to α. It is to be noted that $\sqrt{\frac{1}{3(1-\nu^2)}}$ is the classical value from the small-deflection solution for perfect shells. However, the expressions C_1 and C_2 are dependent on Γ and, therefore, are different from those used for the case of the perfect shell, which can be considered as a limiting case by taking $\Gamma = 0$ in the present solution. The solution reduces to the small-deflection solution by letting b_3 or η approach zero and letting $C_1 = C_2 = 1$ for either perfect or imperfect shells.

The next step is to find the minimum of φ_α versus η when μ and Γ are given. This minimized value is denoted as $\varphi_{\alpha,\eta}$, which is, therefore, a function of μ and Γ. The superscript o is hereafter used to indicate the parameters of perfect shells, for which $\Gamma = 0$ identically. The magnitudes of $\varphi^o_{\alpha,\eta}$ versus μ were found in reference 1.

Let us introduce the notation

$$\gamma = \frac{\varphi_{\alpha,\eta}}{\varphi^o_{\alpha,\eta}} \tag{24}$$

The ratio γ is here called an "imperfection coefficient." The variation of γ with Γ for different values of μ is shown in figure 1. Note that curve IV ($\mu = 1.5$) is for unpressurized shells, while curve I ($\mu = 0$) is for shells subject to rather high pressures, for example, $pR^2/Et^2 \geq 1$. Let us rewrite equation 18 as

$$\frac{\sigma R}{Et} = \gamma \varphi^o_{\alpha,\eta} + \mu \frac{pR^2}{Et^2} \tag{25}$$

At the critical condition,

$$\left(\frac{\sigma R}{Et}\right)_{cr} = \gamma \varphi^{o}_{\alpha,\eta} + \mu_{cr} \frac{pR^2}{Et^2} \tag{25a}$$

The notation μ_{cr} represents the value of μ at which the magnitude of σ is minimum.

The problem, now, is to determine the index C in equation 11. Since C is assumed to change only with p, it can, of course, be found by a series of tests on shells of the same R/t under different pressures. However, C is essentially an experimental constant, and the simple approach discussed in the next paragraph will reduce the necessary number of tests to one.

Previous experiences (refs. 1 and 2) indicate that μ decreases with increasing p. When pR^2/Et^2 increases to approximately unity, or even greater, μ_{cr} approaches zero. It is also physically true that Γ becomes smaller at greater values of p. The exact relations among these variables are, of course, unknown. In figure 1 it can be seen that γ is a decreasing function of Γ but an increasing function of μ. Therefore, the change of pressure should not significantly change the magnitude of γ due to the somewhat counter effect of Γ and μ. At the time C is determined, γ is here assumed to be independent of p. Hence, from equation 25, it can be assumed that at certain values of R/t:

$$\left(\frac{\sigma}{\sigma^{o}}\right)_{p=0} = \left(\frac{\varphi_{\alpha,\eta}}{\varphi^{o}_{\alpha,\eta}}\right)_{p=0} \approx \left(\frac{\varphi_{\alpha,\eta}}{\varphi^{o}_{\alpha,\eta}}\right)_{\frac{pR^2}{Et^2}\geq 1} = \left(\frac{\sigma}{\sigma^{o}}\right)_{\frac{pR^2}{Et^2}\geq 1} \approx \bar{\gamma} \tag{26}$$

In the above equation, $\bar{\gamma}$ is taken as an average value of γ for all pressures. Equation 26 is used only for the purpose of estimating C. When pR^2/Et^2 varies, the ratio $\frac{\sigma}{\sigma_0}$ is not expected to be much different from $\bar{\gamma}$. Tests can be made at any one value of pressure to find σ. The σ_0 is the corresponding critical stress in the perfect shell under the same pressure and can be found in reference 1. Then $\bar{\gamma}$ $(= \frac{\sigma}{\sigma_0})$ is determined.

After $\bar{\gamma}$ is chosen, the relation between Γ and μ is found from figure 1. With the value of R/t known in equation 11, C can be found in terms of Γ and, hence, in terms of μ. Also, μ can be found in terms of pR^2/Et^2 from equation 25a. The index C is thus determined as a function of p.

NUMERICAL EVALUATION

For the purpose of illustrating the method of finding the index C, assume $\bar{\gamma} = 0.6$ at $R/t = 1{,}500$. From figure 1, the change of Γ with μ can be found, and, from equation 11, C can be determined in terms of Γ. For the perfect shell, the relation between $\varphi^{o}_{\alpha,\eta}$ and μ has been found previously. From equation 25a the μ_{cr} at which the shell buckles can be evaluated at various values of p. Some of the numerical relations are listed in the following table.

μ, μ_{cr}	1.5	1.0	0.5	$\rightarrow 0$
Γ	0.63	0.565	0.49	0.4
$C \times 10^3$	0.662	0.555	0.448	0.34
$\varphi^{o}_{\alpha,\eta}$	0.161	0.183	0.29	0.605
$\frac{pR^2}{Et^2}$	0	0.06	0.19	$\geqq 1.0$

From the above table, C versus p is plotted in figure 2.

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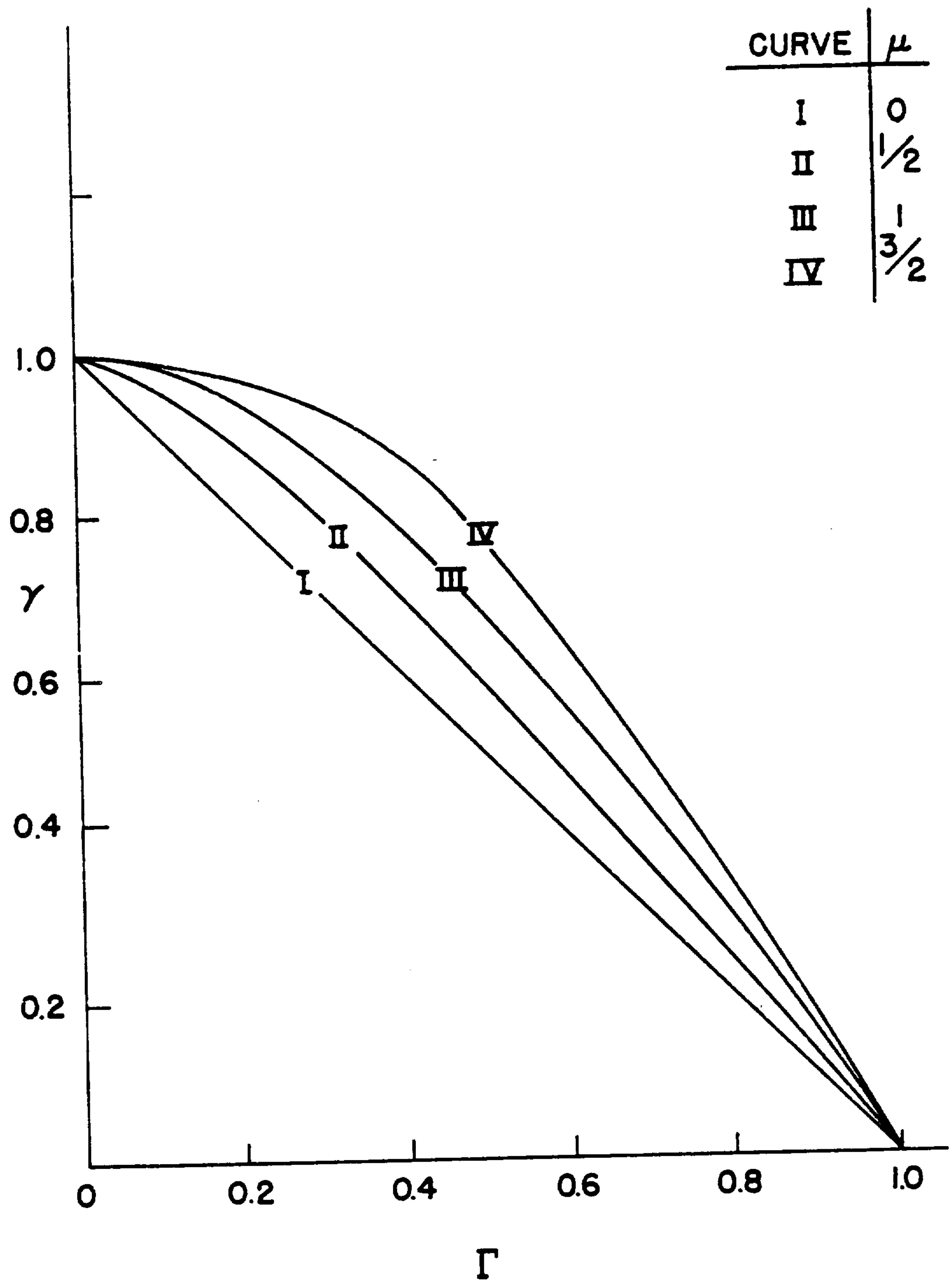

Figure 1.- Variation of imperfection coefficient with imperfection ratio.

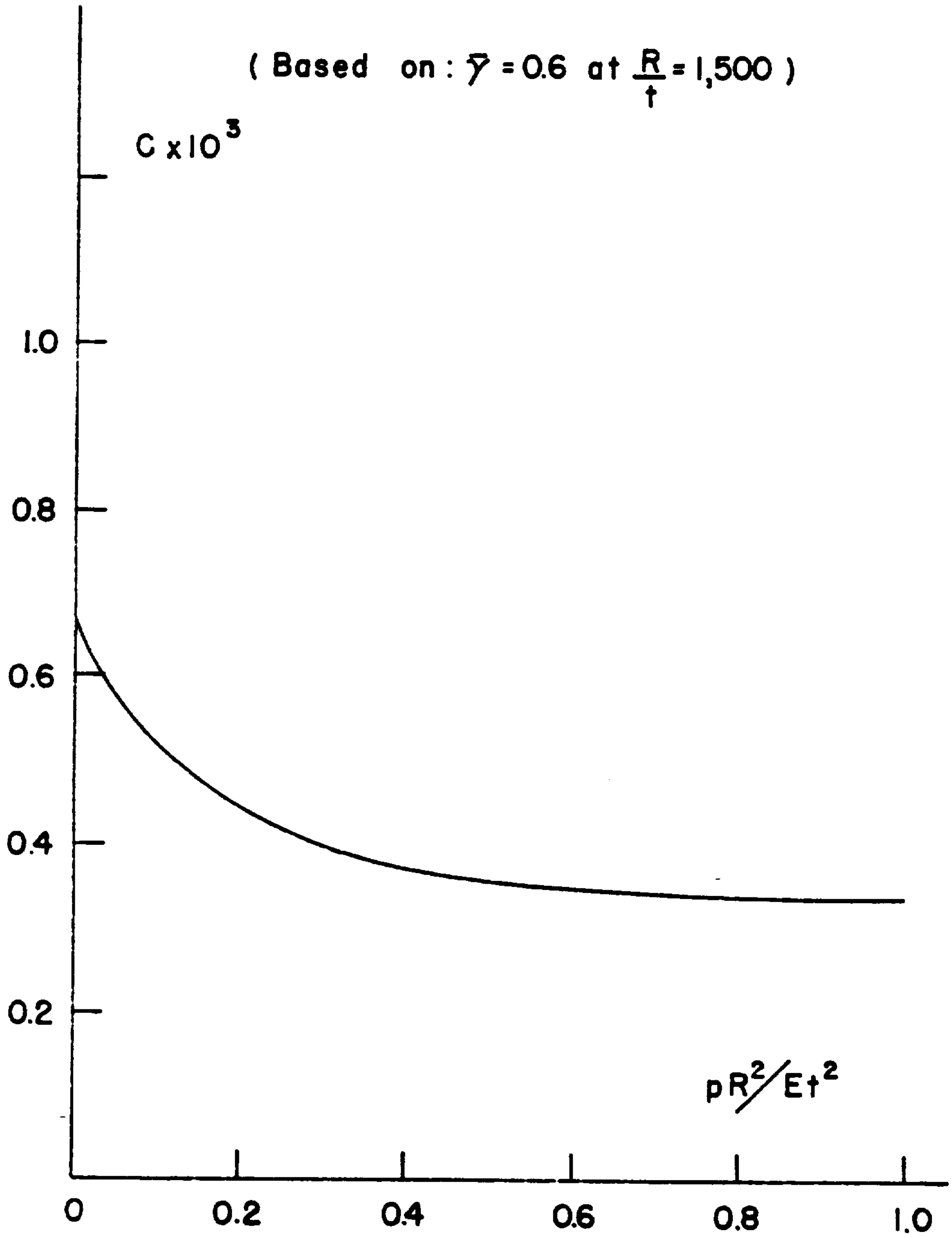

Figure 2.- Variation of index C with pressure.