ON THE POSTBUCKLING BEHAVIOR OF THIN CYLINDRICAL SHELLS

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SUMMARY

Results of tests on postbuckling equilibrium positions of isotropic thin-walled circular cylindrical shells under axial compression, external pressure and combined loading are reported and compared with available theoretical results.

INTRODUCTION

Since, in 1941, von Kármán and Tsien in their basic work on the postbuckling behavior of axially compressed thin-walled circular cylinders [1] gave an explanation for the discrepancy between experimental collapse loads and the buckling loads predicted by classical small-deflection theory, many investigators improved the results of von Kármán and Tsien and extended the methods used by these authors to other loading cases (references see [2]).

The non-linear problem of the postbuckling behavior of thin shells can be represented by two partial differential equations [2]. Their approximate solution by energy methods generally involves a large amount of numerical work. Therefore, in most of the investigations known the approximations to the "exact" solution are not yet fully satisfying.

The degree of approximation of the theoretical results should be measured by tests on cylindrical shells made in the postbuckling region. Unfortunately, tests in this region are difficult to perform since the large deflections of the walls of the shells may result in plastic deformations, the effect of which is generally not included in the theoretical investigations on the postbuckling behavior. However, cylinders made of Mylar, which have been used by several experimenters in the recent time, show elastic properties even at large deflections of the walls, so that by using this material a possibility is given to investigate experimentally the postbuckling behavior of thin shells, and to compare the test results with the known results of theoretical investigations on this problem.

In the following, some results of experimental investigations on the postbuckling behavior of thin isotropic cylindrical shells under axial compression, external pressure, and combinations of these loadings, made at the DFL at Braunschweig (Germany) will be reported and the agreement with available theoretical results will be discussed.

REMARKS ON THE THEORY OF POSTBUCKLING BEHAVIOR

In the investigations known on the postbuckling behavior of thin-walled shells, in most cases, the Ritz energy method has been used. In this method, the total potential energy of the system will be varied with respect to the free amplitudes of a trigonometric series approximating the buckling pattern of the shell.

At axisymmetrically loaded circular cylinders buckling patterns with periodic buckles (number of buckles n) in the circumferential direction can be observed in tests, whereas in longitudinal direction no periodicity of the buckles occurs (fig. 1). These buckling patterns may be generally described by the series:

$$w = \sum_{j=0}^{\infty} f_j(x) \cos j \frac{ny}{r} . \qquad (1)$$

$f_j(x)$ can be expanded to the Fourier series

$$f_j(x) = \sum_{i=0}^{\infty} \left(a_{ij} \cos i \frac{\pi x}{l} + b_{ij} \sin i \frac{\pi x}{l} \right) , \qquad (2)$$

where l is the length of the cylinder.

In spite of the fact that in tests on axially compressed cylinders local buckles are observed (see fig. 1a), which should be described by (1), in the work of von Kármán and Tsien and in the later works on this problem it was assumed that the buckles are periodically distributed over the entire length of the cylinder. The influence of the length of the cylinder and the boundary conditions on the buckling pattern was neglected. In this case, the buckling pattern can be described in the simplified form:

$$w = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{ij} \cos i \frac{\pi x}{l_x} \cos j \frac{ny}{r} ; \quad (i+j) \text{ even} , \qquad (3)$$

where l_x is the half-wave length of a buckle in axial direction.

Von Kármán and Tsien, in their investigation on the axially compressed cylinder [1], approximated the buckling pattern using three terms out of the series (3), and varied the potential energy with respect to the amplitudes of these terms. They kept the number n of the buckles in circumferential direction constant and also fixed the half-wave length l_x of the buckles. For constant values of the ratio $\mu = l_y/l_x$ ($l_y = \pi r/n$, half-wave length of buckles in circumferential direction), the states of equlibrium of the buckled shell were represented in a series of load-shortening curves with integral numbers n as parameter. For small values of μ, states of equilibrium of the buckled shell connected with axial tensile forces were found, a result which was regarded to be unlikely. In the later investigations [3], [4], the potential energy of the system was not only varied with respect to the amplitudes of the terms of the Fourier series but also with respect to the wave lenghts l_x and l_y. This procedure resulted in one single curve for the states of equilibrium in the post-buckling region, all these states of equilibrium being connected with compression forces.

However, the variation of the total potential energy with respect to l_y, i.e. to the number n of the buckles in circumferential direction, is not admissible since by applying the Ritz method according to Lagrange's principle only such displacements can be admitted which do not violate the condition of geometric compatibility of the system. It is obvious that a continuous variation of n would violate the condition of periodicity in the circumferential direction. Therefore, n should be kept constant in the investigation of the postbuckling behavior of shells.

The variation of the total potential energy with respect to the half-wave length l_x is only admissible, if the influence of the length of the cylinder on the buckling pattern is neglected, as assumed in the investigations mentioned above. For the description of actual buckling patterns of cylinders with finite length the series (1) and (2) should be used rather than the series (3). In this case the variation with respect to a half-wave length in axial direction will become meaningless.

The proposed procedure which is based on a more rigorous application of the principles of mechanics will result in a series of load-deflection curves with integral values of n as parameters. Consequently, at given values of load or deformation several equilibrium positions may exist with different numbers n of buckles in the circumferential direction, which is in agreement with the behavior of cylinders observed in tests (see fig. 2, 3 and 4).

These equilibrium positions are, of course, connected with different values of potential energy. The cylinder not necessarily adopts the equilibrium position with the lowest level of potential energy in the postbuckling region. The transition, at a given load or deformation, from a stable equilibrium position of higher energy level to another stable finitely adjacent equilibrium position with lower energy level generally requires the surmounting of an "energy hump", i.e. the input of additional external energy of finite magnitude into the system. At certain critical values of given load or deformation the transition from a position of higher energy level to another one with lower energy level occurs without input of additional external energy: the equilibrium at these critical values becomes indifferent. These critical loads shall be called "secondary buckling loads".

Unfortunately little can be said about the stability of the states of equilibrium by the results of the Ritz method. The theoretical determination of the secondary buckling loads involves more difficulties than the determination of the classical primary buckling load, since, in the latter case, a pure membrane state of stress is assumed in the prebuckling region, whereas in the former case the stability of a complicated combined membrane and bending state of stress has to be investigated.

CYLINDERS UNDER AXIAL COMPRESSION

Tests in the Postbuckling Region

The tests were made on circular cylinders manufactured of Mylar. Dimensions of the specimen: length $l = 400$ mm, radius $r = 200$ mm, and wall thickness $t = 0{,}254$ mm . Young's modulus of Mylar is approximately $E = 500$ kp/mm^2 and Poisson's ratio $\nu = 0{,}3$. The modulus of elasticity of each specimen has been measured. The edges of the cylinders were clamped by plastic cement to end plates. The cylinders were tested in a rigid test equipment allowing a control of the shortening. The loads and the deformations of the specimen in axial direction were measured and registered by an x-y-plotter. The loading of the specimen was performed continuously.

Fig. 2 shows the load-shortening curves of the cylinder in the postbuckling region obtained in the test. At the critical value of shortening of the specimen, according to an axial stress ratio of $\sigma / \sigma_{cl} = 0{,}63$ ($\sigma_{cl} = Et/r \sqrt{3(1 - \nu^2)}$, the initial form of the cylinder becomes unstable, and the cylinder jumps into a new stable equilibrium position, which is connected with a local buckling pattern of two staggered rows of buckles ($n = 14$) near the middle region of the cylinder (see fig. 1a).

With increasing shortening the equilibrium of the buckled cylinder also becomes unstable at a critical load, which has been called secondary buckling load in the preceding section. The specimen jumps into a new similar buckling pattern the number of buckles of which is reduced by one ($n = 13$). This process will be repeated with increasing shortening of the specimen. By loading and unloading the specimen a series of curves representing stable equilibrium positions in the postbuckling region can be obtained. It can be seen from fig. 2, that indeed several stable equlibrium positions exist for the same value of shortening of the specimen. The fact that the transition from one buckling pattern (n) to the other ($n - 1$) is connected with a jump indicates that the stable equilibrium position (n) near the critical secondary buckling load is connected with a higher energy level than the adjacent equilibrium position ($n - 1$),

A characteristic feature of the postbuckling behavior of cylinders under axial compression is the low load carrying capacity in the postbuckling region which is in the order of 10 - 25 per cent of the classical buckling load. No considerable increase of the axial load can be obtained with increasing shortening due to the low values of the secondary buckling loads.

Comparison with Theory

No theoretical results are known which could be directly compared with the test results, since no attempt has been made yet to describe the local buckling pattern (fig. 1a) by a Fourier series in the form of (1) and to use integral numbers of n as parameters in the investigation. But there are results from the theoretical work of Kempner [4] and of Almroth*). In both investigations it was assumed that the buckles are distributed over the entire length of the cylinder, and terms of Fourier series (3) were used to describe the buckling pattern. The potential energy was varied with respect to n, so that the postbuckling equilibrium positions are represented by a single load-deflection curve. Kempner used three terms of (3) with the amplitudes a_{20}, a_{11} and a_{02} while Almroth took into consideration nine terms of (3) with the amplitudes a_{20}, a_{40}, a_{60}, a_{11}, a_{31}, a_{02}, a_{22}, a_{13}, a_{33}. The results of the two investigations are compared

*) Private communication to the author of May 11, 1962

with the test results in fig. 2 . It is interesting to note that Almroth's more complete description of the buckling pattern results in a considerable improvement of Kempner's solution. The agreement of Almroth's solution with the test results is remarkable with regard to the fact, that the buckling pattern used in the theoretical investigation is considerably different from the local buckling pattern observed in tests.

CYLINDERS UNDER EXTERNAL PRESSURE

Tests in the Postbuckling Region

Tests were performed on two Mylar cylinders. One specimen had the same dimension as the specimen for axial loading, and the second one had a length of 200 mm, while the other dimensions remained unchanged. The edges of the cylinders were again clamped by plastic cement to end plates. The external pressure produced by decreasing the pressure in the cylinder was controlled. The pressure and the axial shortening of the cylinder were registered on an x-y-plotter.

Fig. 3 shows the stable equilibrium positions of the cylinders in the postbuckling region. At critical values of external pressure the initial form of the cylinder becomes unstable. The buckling pattern develops in a progressive fashion rather than with a sudden appearance of the buckles around the entire cylinder. But after a small increase of the external pressure the complete buckling pattern can be observed ($n_{cr} = 10$ and $n_{cr} = 14$, resp.) (see fig. 1c). This postbuckling configuration of the cylinder shows a remarkable degree of stability under increasing external pressure loading. Contrary to the case of axial loading no secondary buckling pressure could be observed. To avoid a final uncontrolled collaps of the specimen the increase of the loading was stopped at values of about 150 per cent of the critical value of external pressure. However, by applying additional local external loads to the surface of the cylinder transition to an adjacent buckling pattern with a number of buckles $n - 1$ can be achieved at loads below the secondary buckling pressure.

This process can be repeated, and a series of curves representing stable equilibrium positions in the postbuckling region for different values of n can be registered by loading and unloading the cylinder. It can again be noticed from fig. 3 that for a given external pressure several stable equilibrium positions of the cylinder exist in the postbuckling region related to different values of n .

During unloading the cylinder in a buckling configuration $n < n_{cr}$, at a critical value of external pressure, the cylinder jumps back into a buckling configuration with the number of buckles $n + 1$. Equilibrium positions with number of buckles n below this critical value of external pressure are unstable. By further decrease of the loading this process will be repeated, until the cylinder jumps into its initial unbuckled configuration. (Since during the snap-through process the pressure in the cylinder cannot be kept constant, the curves representing the jumps, are not horizontal straight lines as it should be expected for jumps at controlled external pressure.)

The test results indicate that due to the very high degree of stability of the buckling configuration, only the buckling pattern $n = n_{cr}$ can be expected to be observed in tests after buckling. Transitions to other configurations ($n < n_{cr}$) require a considerable increase of the external pressure loading beyond the critical pressure. This behavior is in contrast to the behavior of axially compressed cylinders. Since in the latter case the stability of the postbuckling configurations is lost at values far below the critical buckling load the cylinder may jump into buckling patterns with different numbers of n, depending on the stiffness of the test equipment.

Comparison with Theory

Direct comparison of the test results with results of theoretical investigations of the postbuckling behavior of cylinders under external pressure is difficult, since available results either do not take into account the actual boundary conditions of the test cylinders (clamped edges) or do not use a sufficient number of terms of a Fourier-series to describe the buckling pattern in a satisfying manner.

The influence of the boundary conditions on the buckling load and the postbuckling behavior of a cylinder under external pressure is considerably stronger than in the case of axial compression. In the latter case for cylinders with ratios $l/r \geq 1{,}5$ the influence of length and boundary conditions on the buckling load can be neglected [5], while in the former case the classical buckling load found for cylinders with simply supported edges (v. Mises' theory) will be increased for cylinders with clamped edges by a factor of approximately 1.3. This factor is not well established, since for clamped edges only approximate solutions [6] for the buckling load of the cylinder are known, the accuracy of the results depending on the degree of approximation of the buckling pattern by the Fourier series.

Under the assumption that the theoretical critical pressure p_{cr} of cylinders with clamped edges is 1.3 of the classical v. Mises buckling pressure for hinged edges the actual buckling pressures of the two test cylinders were 70 % and 92 % of the theoretical buckling load p_{cr}.

The buckling pattern in the postbuckling region of the cylinder with clamped edges can be described by the equations (1) and (2), which for hinged edges will be simplified to

$$w = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} a_{ij} \sin i \frac{\pi x}{l} \cos j \frac{ny}{r}; \quad i \text{ odd}, \qquad (4)$$

the origin of the coordinate-system being on the one edge of the cylinder.

Fig. 4 shows theoretical load deformation curves of simply supported cylinders under external pressure. For different length parameters $\lambda = \sqrt[4]{3(1-\nu^2)}\, l/\sqrt{rt}$ these curves have been calculated at the DFL using the Ritz method. Four terms with the amplitudes a_{10}, a_{11}, a_{30} and a_{31} out of the series (4) have been used to describe the buckling pattern approximately. A similar investigation has been performed by Kempner [7], who introduced two terms with the amplitudes a_{10} and a_{11} in his calculations. He restricted his investigation to a single load-deformation curve of the equilibrium states related to the buckling pattern with the number of buckles $n = n_{cl}$, where n_{cl} is the number of buckles received from linear buckling theory.

The theoretical load-deflection curves of fig. 4 show the same principal trend as the experimental curves of fig. 3. For cylinders with higner values of λ the approximate theoretical investigation indicates equilibrium positions related to negative values of external pressure (internal pressure). This result of the theory is not confirmed by the tests. The reason for this unsatisfactory theoretical result is without doubt - beside the assumption of hinged edges - the incomplete description of the buckling pattern by only four terms of the series (4), which, apparently, is not sufficient for long cylinders and large values of deformation. Under these conditions the buckling pattern of fig. 1c will be changed to a pattern, showing some kind of corrugation with approximately constant amplitude of the buckles in axial direction on a considerable part of the cylinder length. The transition from the corrugated form to the circular form at the ends of the cylinder is limited to a relatively narrow edge region (see fig. 1d). This buckling pattern, of course, cannot be described by the four terms used in the theoretical investigation with satisfying accuracy.

However, the buckling patterns with $n < n_{cr}$ are of more academic interest, as, due to the high degree of stability of the buckling pattern $n = n_{cr}$, only this pattern will be observed in tests without input of additional external energy.

In fig. 4 the minimum loads in the postbuckling region for the buckling pattern $n = n_{cr}$ are given as function of the length parameter λ. This value can be approximately regarded as a lower limit for the actual buckling load of the cylinder. Due to initial imperfections the actual buckling load may vary between the theoretical buckling load p_{cr} and the minimum load in the postbuckling region p_{min}.

In fig. 4 theoretical results for p_{min}/p_{cr} for cylinders with hinged edges from the investigations of Kempner and the DFL are compared with results obtained in tests for cylinders with clamped edges. Donnell [6] investigated theoretically the postbuckling behavior of circular cylinders under external pressure with clamped edges, using a two-terms expression for the buckling pattern, but did not succeed in finding a minimum load in the postbuckling region.

CYLINDER UNDER COMBINED LOADING

Experimental curves of postbuckling states of equilibrium for different numbers of buckles n are given for a cylinder tested under combined axial compression and external pressure as a further example for the postbuckling behavior of isotropic cylinders. The external pressure $p/p_{cr} = 0.7$ has been kept constant during the test while the shortening of the cylinder has been varied. As in the cases of pure axial loading and pure external pressure loading, it can be seen from fig. 5 that the postbuckling behavior of the cylinder is represented by a series of load-deflection curves rather than by a single curve. Fig. 1b shows the buckling pattern of this loading case.

Again the transition from one buckling pattern to the adjacent one occurs at secondary buckling loads. No theoretical investigation of the combined loading case is known, the results of which could be compared with the test results.

CONCLUSIONS

The test results discussed in this report indicate that cylindrical shells under axisymmetrical loading show in the postbuckling region for given values of end shortening or loading several stable equilibrium positions related to different integral circumferential wave numbers n . With increasing shortening or loading the equilibrium positions may become unstable and transition to an adjacent stable equilibrium position may occur. In theoretical investigations on the postbuckling behavior the variation of the potential energy with respect to n is not in agreement with a rigorous application of the principles of mechanics, and should be abandoned, as the results of such theoretical investigations may deviate considerably from the actual postbuckling behavior of the shell.

The theoretical determination of the secondary buckling loads involves more difficulties than the determination of the classical primary buckling load. For external pressure loading, such a theoretical investigation on secondary buckling loads has approximately been performed by Ivanov [8].

The results of the available theoretical investigations of the postbuckling behavior of shells under axisymmetric loading are not yet satisfactory. For axially loaded cylinders remarkable progress has been made by Almroth. In a future extension of the investigations equation (1) should be used for the description of the local buckling pattern of fig. 1a , taking into account the boundary conditions and the length of the cylinder.

For cylinders under external pressure loading the investigation on the postbuckling behavior of the, practically, important case of clamped edges should be extended to more complete descriptions of the buckling pattern by equation (1), in order to get better agreement of experimental and test results. Even the primary theoretical buckling load of cylinders with clamped edges should be better determined.

The experimental investigation of the DFL on the postbuckling behavior of isotropic circular cylindrical shells under axisymmetrical loading will be continued.

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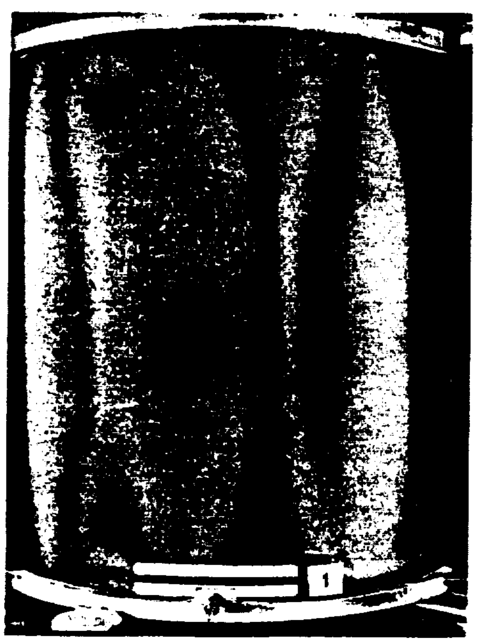
(a) Axial compression.

(b) Combined axial compression and external pressure.



(c) External pressure. $n = n_{cr} = 10$.

(d) External pressure. $n = 9$.

Figure 1.- Buckling patterns of axisymmetrically loaded cylinders.

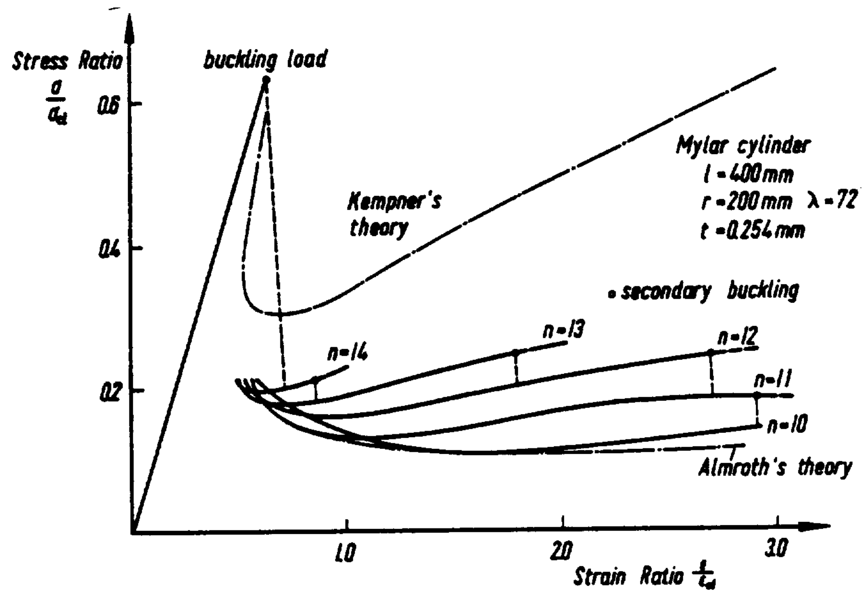


Figure 2.- Cylinder under axial compression. Postbuckling states of equilibrium - test results.

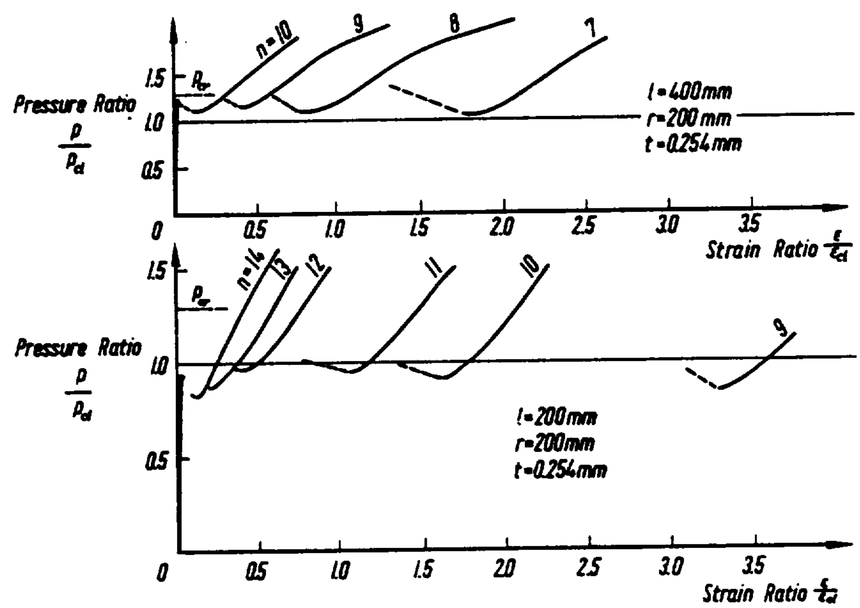


Figure 3.- Cylinders under external pressure. Test results.

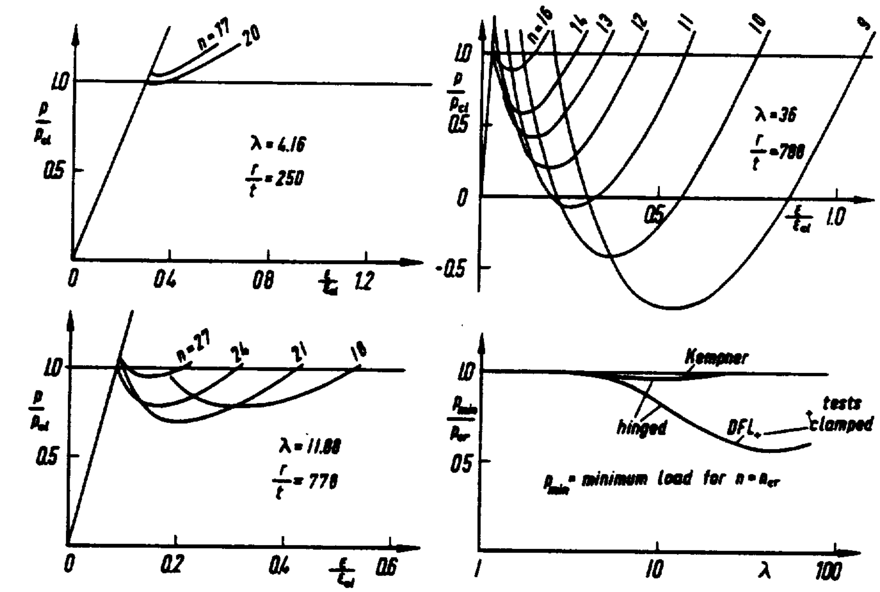


Figure 4.- Cylinders under external pressure. Theoretical results.

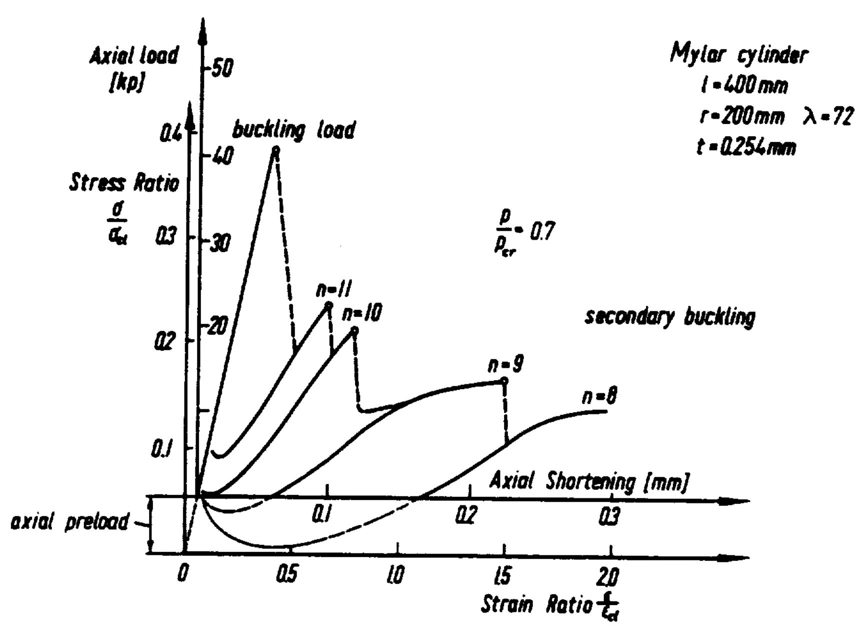


Figure 5.- Cylinder under axial compression and external pressure.
Postbuckling states of equilibrium - test results.