THE EFFECT ON THE BUCKLING OF PERFECT CYLINDERS OF PREBUCKLING DEFORMATIONS AND STRESSES INDUCED BY EDGE SUPPORT

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SUMMARY

Large deflection theory is used to compute buckling loads of simply supported initially perfect cylinders under axial compression, external hydrostatic pressure, and combinations of axial compression and internal or external pressure. Important results are obtained by taking into account prebuckling deformations and stresses induced by edge support. For example, the presence of these deformations and stresses can decrease the axial-compression buckling load of an unpressurized perfect cylinder by 50 percent or more.

INTRODUCTION

Classical theory and experiment are in good agreement for buckling of circular cylindrical shells under uniform external lateral pressure. (See ref. 1.) For external hydrostatic pressure there is similar agreement between experiment and theory except for the lower range of curvature parameter ($L^2/rt < 100$). For axial compression, however, severe disagreement exists; experiments have shown that the actual buckling stress is from 15 to 50 percent of that predicted by classical theory. (See ref. 2.) The disagreement found in hydrostatic pressure tests at low values of the curvature parameter is probably also due to the inability of classical theory to account for axial compression. (See ref. 1.)

Convincing arguments have been made that the occurrence of lower-than-expected buckling stresses for axial compression is due in part to initial imperfections. For example, the results of large deflection analysis (see ref. 3) have indicated that small initial imperfections can lead to large reductions in the buckling load. However, another potential reason for this disagreement of classical theory and experiment has, until recently, been unexplored. This potential reason for the disagreement is the inconsistent assumption of classical theory with regard to prebuckling and buckling-edge conditions. It is assumed that the prebuckling deflection and stress components are either constant or zero and imply that the edges of the shell are free until buckling occurs; however, during the buckling process the edges are assumed to be radially restrained (simply supported or clamped).

The effect of one deviation from the classical-edge conditions has already been investigated (see refs. 4 and 5) for buckling in axial compression by use of linear equations. The edges of the shell were allowed to remain free during the buckling process. The resulting buckling load was less than half the classical load; the result demonstrates effectively the importance of the edge conditions. In practice, however, the occurrence of free edges is rare; the edges of the shell are usually attached to a ring or pressed against the platens of a testing machine. Thus, it seems more realistic to take a consistent but opposite approach, wherein from the inception of loading through buckling the edges of the cylinder are radially restrained. Moreover, it is apparent that such restraint must lead to nonuniform prebuckling deflections and stresses throughout the cylinder, the importance of which should be determined. This approach to cylinder buckling analysis has been adopted in the present investigation.

A cylinder without initial imperfections is considered, and large deflection theory is used to determine the deformations and stresses prior to buckling and to determine the buckling equation. Results are obtained for buckling of simply supported cylinders under axial compression, external hydrostatic pressure, and combinations of axial compression and internal or external pressure.

SYMBOLS

n	number of waves in circumferential direction
p	pressure
r	radius of cylinder
t	thickness of cylinder wall
u,v,w	displacements in the x-, y-, and radial directions, respectively
x,y	axial and circumferential directions
D	plate stiffness, $Et^3/12\left(1 - \mu^2\right)$
E	Young's modulus for material

L	length of cylinder
M	number of stations in half length
P	applied axial midplane compressive force per unit length
U,V,W	functions of x which appear in the buckling displacements u_B, v_B, and w_B, respectively
Z	curvature parameter $\left(\frac{L^2}{rt}\sqrt{1-\mu^2}\right)$
μ	Poisson's ratio for material
u_A, w_A	prebuckling displacements (functions of x)
u_B, v_B, w_B	buckling displacements (functions of x and y)
N_x, N_y, N_{xy}	midplane forces per unit length
$\epsilon_x, \epsilon_y, \gamma_{xy}$	midplane strains

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

When the subscripts x and y follow a comma, they indicate partial differentiation of the principal symbol with respect to x and y.

ANALYSIS

In the large-deflection Donnell theory, the basic differential equations of equilibrium of a cylinder:

$$\left.\begin{aligned} &N_{x,x} + N_{xy,y} = 0 \\ &N_{y,y} + N_{xy,x} = 0 \\ &D\nabla^4 w + \frac{N_y}{r} - \left(N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy}\right) = p \end{aligned}\right\} \tag{1}$$

together with Hooke's law:

$$\left.\begin{aligned} N_x &= \frac{Et}{1 - \mu^2}\left(\epsilon_x + \mu\epsilon_y\right) \\ N_y &= \frac{Et}{1 - \mu^2}\left(\epsilon_y + \mu\epsilon_x\right) \\ N_{xy} &= \frac{Et}{2(1 + \mu)}\gamma_{xy} \end{aligned}\right\} \tag{2}$$

and the nonlinear strain-deformation relations:

$$\left.\begin{aligned} \epsilon_x &= u_{,x} + \frac{1}{2} w_{,x}^2 \\ \epsilon_y &= v_{,y} + \frac{w}{r} + \frac{1}{2} w_{,y}^2 \\ \gamma_{xy} &= u_{,y} + v_{,x} + w_{,x}w_{,y} \end{aligned}\right\} \tag{3}$$

provide a complete set of 9 equations in the 9 unknown forces, strains, and deformations which, together with boundary conditions, specify the problem.

It is to be expected that prebuckling deformations are axisymmetric; that is, they are functions of x only and may be obtained directly from equations (1) to (3) and boundary conditions which include zero radial displacement at the edges from the onset of loading. A solution to the axisymmetric problem was first obtained in reference 6 and is reported in reference 7.

Let the solution to the prebuckling axisymmetric problem just described be identified as u_A, w_A (and $v_A = 0$). To the prebuckling deformations are added infinitesimal nonaxisymmetric changes u_B, v_B, and w_B that occur at buckling:

$$\left.\begin{aligned} u &= u_A(x) + u_B(x,y) \\ v &= v_B(x,y) \\ w &= w_A(x) + w_B(x,y) \end{aligned}\right\} \tag{4}$$

The displacements u_B, v_B, and w_B also satisfy simple support boundary conditions consistent with the axisymmetric solution. The following buckling equations may now be obtained by substituting equations (4) into equations (1) to (3) and then neglecting nonlinear subscript B deformations (and subtracting out identities relating subscript A deformations):

$$\left.\begin{aligned}
&u_{B,xx} + \frac{1-\mu}{2} u_{B,yy} + \frac{1+\mu}{2} v_{B,xy} + \frac{\mu}{r} w_{B,x} + \left(w_{A,x} w_{B,x}\right)_{,x} \\
&\qquad + \frac{1-\mu}{2} w_{A,x} w_{B,yy} = 0 \\
&\frac{1+\mu}{2} u_{B,xy} + v_{B,yy} + \frac{1-\mu}{2} v_{B,xx} + \frac{1}{r} w_{B,y} + \frac{1-\mu}{2} w_{A,xx} w_{B,y} \\
&\qquad + \frac{1+\mu}{2} w_{A,x} w_{B,xy} = 0 \\
&D\nabla^4 w_B + \frac{1}{r} N_{yB} + P w_{B,xx} + \left(\mu P - \frac{Et}{r} w_A\right) w_{B,yy} - w_{A,xx} N_{xB} = 0
\end{aligned}\right\} \tag{5}$$

where

$$N_{xB} = \frac{Et}{1-\mu^2}\left[u_{B,x} + w_{A,x} w_{B,x} + \mu\left(v_{B,y} + \frac{w_B}{r}\right)\right]$$

$$N_{yB} = \frac{Et}{1-\mu^2}\left[v_{B,y} + \frac{w_B}{r} + \mu\left(u_{B,x} + w_{A,x} w_{B,x}\right)\right]$$

The physical conditions of continuity around the cylinder are satisfied if

$$\left.\begin{aligned}
u_B &= U(x) \sin \frac{ny}{r} \\
v_B &= V(x) \cos \frac{ny}{r} \\
w_B &= W(x) \sin \frac{ny}{r}
\end{aligned}\right\} \tag{6}$$

where n, the number of waves around the cylinder, is an integer. Equations (5) may now be converted to ordinary differential equations relating U, V, W, and the eigenvalues. Of course, the subscript A displacements introduce variable coefficients, and these equations are difficult to solve. Instead of solving the ordinary differential equations directly, equivalent numerical expressions for U, V, and W at M stations in a half length (W as an even function of x being assumed) were developed by using the energy method and solved for the necessary eigenvalue.

LIMITATIONS OF THE CALCULATIONS

In order to obtain accurate results, M was arbitrarily taken large enough so that there were at least five stations for each prebuckling (inward or outward) wrinkle. For $Z > 1,000$, this criterion led to equations involving determinants that were too large for economical application of the computing machine used. Hence, calculations have been limited to $Z \leq 1,000$.

The proper value of n is the integral value which yields the lowest buckling load with the physical restriction that n cannot be less than 2 (since $n = 1$ is simple translation and $n = 0$ is an axisymmetric form). Little accuracy is lost, however, if n is considered to be continuously variable for $n > 2$. It was found that $n = 2$ gave the condition for instability for almost every case except for the range of higher external pressures. The differential equations of equilibrium (eqs. (1)) are accurate for the $n = 2$ case only if there are present at least three wrinkles in every part of a length equal to the radius so that the deformations are extensional. (See ref. 8.) For this reason small values of the curvature parameter ($Z < 50$) could not be treated for axial compression and for combinations of axial compression and internal pressure.

RESULTS

In figures 1 and 2 interaction curves are presented for a low value (50) and a high value (500), respectively, of the curvature parameter Z. Each point on the curve presents a combination of axial compression and lateral pressure that causes buckling. Where the curves depend on r/t they correspond to $n = 2$; at the higher external pressures, the results were given by $n > 2$ with n assumed continuously variable. At the end points to the left the curves give the buckling pressures for cylinders under external lateral pressure alone. The hydrostatic pressure for buckling is given by the point on the curve marked by a cross. When the pressure is zero, note that the axial buckling stress is 50 percent or less of the classical value. With internal pressure present, the axial stress required for buckling increases until it approaches the classical value. Stress coefficients for external hydrostatic pressure alone and axial compression alone are presented in figures 3 and 4, respectively, for a wide range of curvature parameter Z (within the limitations specified in the previous section).

DISCUSSION

Experimental results are available in reference 9 for cylinders with combinations of internal pressure and axial compression and with curvature parameters about equal to those presented in figures 1 and 2. The experimentally obtained buckling stress coefficients are plotted along with the theoretical curves in figures 1 and 2. A comparison of the results shows that, although the experimental cylinders were ring supported and the theory was for simply supported cylinders, there is much better quantitative agreement of experiment with present theory than with classical theory. The evident disagreement in the shapes of the theoretical and experimental interaction curves has not been explained.

In figure 3 results of the present investigation for external hydrostatic pressure are plotted against the curvature parameter Z together with the corresponding classical curve and with experiment. Comparison of the theoretical results shows that prebuckling stresses and deformations serve to stiffen the cylinder by about 25 percent for higher values of Z. In the range of lower Z this stiffening effect disappears and prebuckling stresses and deformations serve to weaken the cylinder to about 80 percent of the classical value. Thus, whereas the classical theory agrees with experiment in the range of higher values of Z and disagrees for lower values of Z, the present results follow the trend of experiment and yield buckling pressures roughly 25 percent high in both regions.

For axial compression of unpressurized cylinders the present results are more than 50 percent below the classical values. (See fig. 4.) Thus, the axial buckling load is sensitive to the prebuckling deformations and stresses resulting from restraint of the edges. The value of the buckling load from the present theory is dependent on radius-thickness ratio whereas in the classical theory it is not. The dependence on radius-thickness ratio occurs because the critical wave form is determined to have two waves in the circumferential direction. It can be seen from figure 4 that the empirical curves of reference 2 - and therefore experimental results - are also dependent on radius-thickness ratio, and that agreement between theory and experiment is much better with present theory than with classical theory especially for low radius-thickness ratios.

Neither the results of the present theory nor the results of classical theory for buckling in axial compression indicate that the buckle wave form is of the diamond pattern as indicated by buckled experimental cylinders. The results of present theory which specify two waves in the circumferential direction at buckling deviate even further from the experimental buckled wave form than the results of classical theory. However, previous work has shown (see ref. 10) that the equilibrium configuration which is the mode at buckling need not be stable under usual conditions of loading. If the buckling mode is not stable, it might not necessarily resemble the final shape of a buckled experimental cylinder.

CONCLUDING REMARKS

The present paper has focused attention on a serious shortcoming of classical buckling theory. To avoid complicated prebuckling deformations and stresses, the classical approach is to relax completely the supports in the prebuckling range and thus assume that the prebuckling stresses are zero or constant and the prebuckling deformations are zero, constant, or linear. Prebuckling deformations and stresses due to edge support have been ignored even in studies of effects of initial imperfections. Yet, in every practical cylindrical shell structure, some measure of radial support is present from the beginning of loading so that, prior to buckling, complicated axisymmetric deformations and stresses are present to modify the load-shortening behavior of the cylinder and to influence its buckling load. This influence is especially notable for cylinders in axial compression and for short cylinders under external hydrostatic pressure, where it accounts for a large part of the disagreement between classical theory and experiment.

Further work needed in this field includes studies of cylinders with clamped edges and flexible rings at the edges. In order to study the behavior of longer cylinders (cylinders of larger Z) in axial compression, it would also be desirable to analyze the semi-infinite cylinder. Further, in future cylinder studies, it would be useful to extend this work by using a more exact theory that is valid for buckling into two circumferential waves for less than three wrinkles in the axial direction. It is also clear that the adoption of the present approach - seeking an infinitesimal nonaxisymmetric change at buckling from a symmetric prebuckling state which satisfies the boundary conditions - would probably clarify greatly the problem of buckling of a spherical cap under external pressure.

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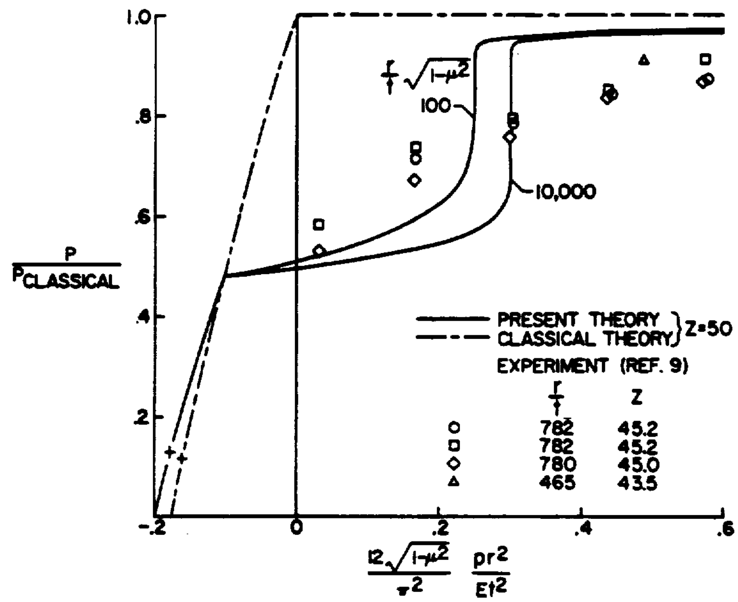

Figure 1.- Theoretical and experimental results for buckling of a cylinder of low Z under combinations of axial compression and internal pressure. Crosses indicate hydrostatic external pressure for buckling.

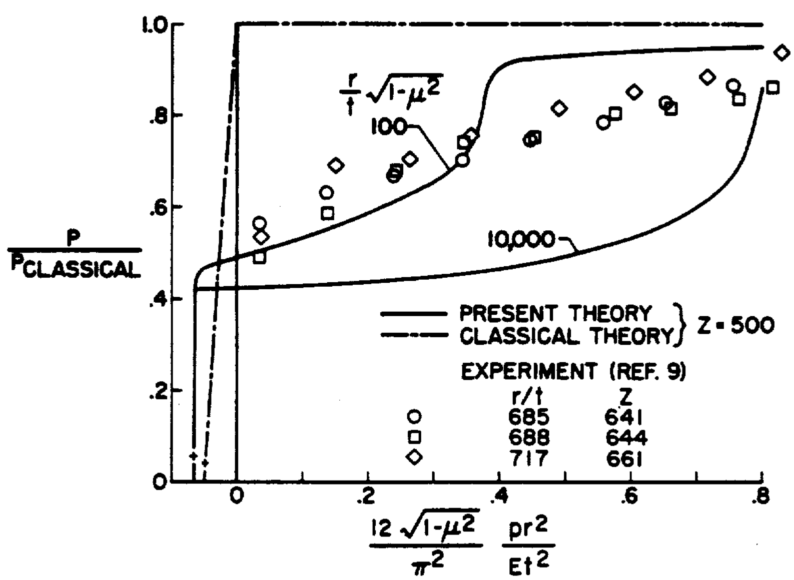

Figure 2.- Theoretical and experimental results for buckling of a cylinder of high Z under combinations of axial compression and internal pressure. Crosses indicate hydrostatic external pressure for buckling.

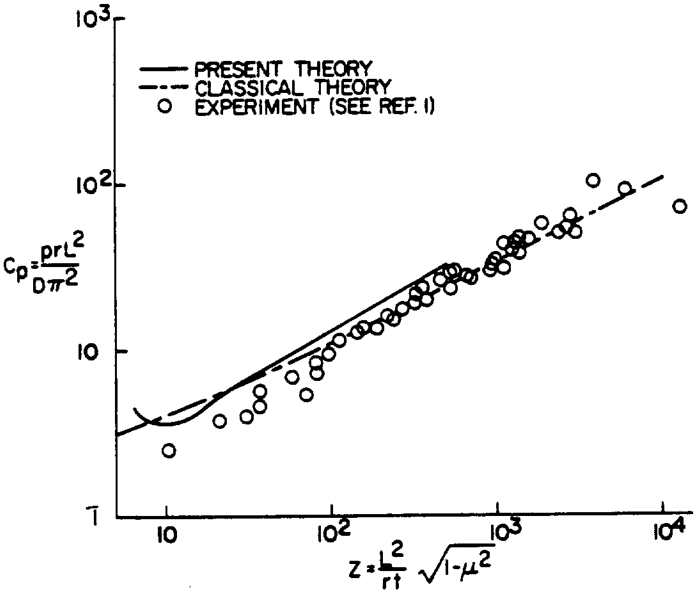

Figure 3.- Theoretical and experimental results for buckling of cylinders under hydrostatic pressure.

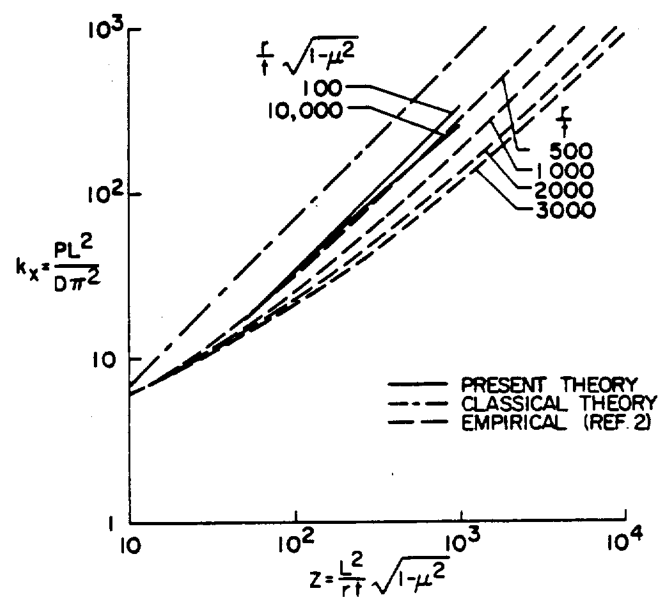

Figure 4.- Theoretical and empirical results for buckling of cylinders in axial compression.