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INSTABILITY ANALYSIS OF CYLINDRICAL SHELLS UNDER HYDROSTATIC PRESSURE

By George Herrmann

Northwestern University

SUMMARY

To determine the elastic buckling pressure of simply supported cylindrical shells subjected to lateral and axial hydrostatic forces, various versions of linear bending theories have been employed in the past. For certain shell dimensions, however, the expressions commonly used may yield substantially differing results. In what follows, recent work on this problem by A. E. Armenakas and the writer is briefly reviewed. This work consisted primarily in employing a general bending theory of circular cylindrical shells under the influence of initial stress, developed earlier by the same authors, to re-examine the problem mentioned, and compare the results with those of previous investigations. The outcome was the establishment of a simple but accurate expression for the buckling pressure applicable to a wide range of shell dimensions.

INTRODUCTION

In the past, various versions of linear bending theories have been employed in establishing the buckling value of the external pressure acting on circular cylindrical shells. One of the earliest investigations by von Mises (ref. 1) resulted in a simple expression for the critical uniform lateral pressure (no axial stress) that has been utilized extensively. Timoshenko (ref. 2) indicated that this formula is in close agreement with a more intricate formula obtained by Flügge (ref. 3). Much later, von Mises also considered the case of a shell under all around pressure (axial stress equals one half of the circumferential stress) and presented a formula for the critical pressure (see ref. 2, page 498).

More recently, Batdorf (ref. 4) reinvestigated this problem and using the Donnell equations (ref. 5) obtained the same expression as von Mises for the case of all around pressure, and a somewhat different expression for the case of only lateral pressure. As is well known, however, for shells whose dimensions conduce to buckling in modes with

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a small number of circumferential waves, the assumptions made in deriving the Donnell equations are not valid and, concomitantly, for such shells the Batdorf results are inaccurate. Mushtari and Galimov (ref. 6) present an analysis of the same problem also using the Donnell equations and their results are thus subject to the same limitations. Loo (ref. 7) attempted to use, in establishing the buckling pressure, an extended Donnell equation which included the effect of transverse shear forces on the equilibrium in the circumferential direction. Calculations based on this modification appear, for long shells, to be in reasonable agreement with the results obtained by von Mises.

Some of the aforementioned formulas have been referred to frequently in the literature. For certain ranges of shell dimensions, however, they yield substantially differing results. Thus, it is evident that a systematic re-examination of the buckling of circular cylindrical shells under hydrostatic pressure was desirable. For this purpose, the bending theory of cylindrical shells, under the influence of a general state of initial stress, presented recently by Herrmann and Armenakas (ref. 8), was used to establish the value of the critical pressure acting on a cylindrical shell. The results were compared with those of previous investigations and a simplified but accurate expression for the buckling pressure, applicable to a wide range of shell dimensions, was evolved, valid for simple supports. A detailed account of this work, including considerations concerning the character of the external pressure, is given in ref. 9.

SYMBOLS

L	shell length
R	shell radius
h	shell thickness
E	Young's modulus
ν	Poisson's ratio
D	flexural stiffness, $\frac{Eh^3}{12(1 - \nu^2)}$
n	number of circumferential waves (lobes)
m	number of axial half-waves

s	thickness parameter, $\frac{h}{2R}$
β	length parameter, $\frac{Ln}{\pi R}$
Z	curvature parameter, $\frac{L^2\sqrt{1 - \nu^2}}{Rh}$
p	external pressure
N	circumferential shell stress
T	axial shell stress
K	buckling coefficient

Subscript:

cr	critical

CYLINDRICAL SHELLS UNDER HYDROSTATIC PRESSURE

Exact Expression for Buckling Pressure

Hydrostatic pressure (as distinct, for example, from constant-directional pressure) is defined as a pressure acting always perpendicular to the area element and having during the process of deformation a constant magnitude per unit actual area. It can act on the lateral surface of the shell, inducing a stress N, or at the end of the shell, inducing a stress T.

$\frac{T}{N} = \frac{1}{2}$ corresponds to all around pressure,

$\frac{T}{N} = 0$ to lateral pressure only.

If the shell is assumed to be simply supported, i. e. the circumferential displacement, the radial displacement, the axial force and the axial bending moment are all zero, then the following expression for the critical hydrostatic pressure p_{cr} is obtained, on the basis of the equations presented in ref. 8.

$$p_{cr} = K \frac{\pi^2 D}{L^2 R}$$

$$K = \frac{G}{H}$$

$$G = n^4(\beta^2 + m^2)^4 + 12Z^2n^4m^2\pi^{-4} + \beta^8(1 - 2n^2)$$
$$+ 2m^2\beta^6 \left[2(1 - 2n^2) + \nu(n^2 - 1) \right] \quad - 6m^4n^2\beta^4 - 2m^6n^2\beta^2\nu$$

$$H = \beta^2n^4(\beta^2 + m^2) + \frac{T}{N}(1 + s)m^2n^4(\beta^2 + m^2)^2 + \frac{T}{N}(1 + s)m^2n^2\beta^2 \left[\beta^2 \right.$$
$$\left. + m^2(3 + 2\nu) \right] \quad - \beta^4n^2(\beta^2 + 3m^2) + m^2n^2s \left[\beta^4 + n^2(\beta^2 + m^2) \right]$$

Terms involving s result from taking into account the fact that the pressure is not acting on the middle surface of the shell. The above expression for p_{cr} is similar, but not identical to the one obtained by Flügge (ref. 3). For thin shells this expression is unnecessarily complicated and may be simplified considerably.

Approximate expression for Buckling Pressure

It is well known that shells having a small curvature parameter Z buckle into a large number of circumferential waves, and therefore $\frac{1}{n^2}$ may be disregarded as compared to unity. The buckling coefficient K may then be simplified to, with m = 1,

$$K = \frac{(1 + \beta^2)^2}{\frac{T}{N} + \beta^2} + \frac{12Z^2}{\pi^4(1 + \beta^2)^2\ (\frac{T}{N} + \beta^2)}$$

For $\frac{T}{N} = \frac{1}{2}$, this expression is identical to that obtained by von Mises (ref. 2, p. 498) and Batdorf (ref. 4). For the range of shell dimensions for which the above equation is not valid, that is for shells buckling into a small number of circumferential waves n, β^2 is large as compared to unity. Therefore, the exact expression for K may be simplified by disregarding in it terms not included in the approximate expression above, whose order of magnitude is $\frac{1}{\beta^2}$ times that of terms retained. The following expression for the buckling coefficient is then obtained, valid for a wide range of shell dimensions

$$K = \frac{(1 + \beta^2)^2\ (n^2 - 1)}{(\frac{T}{N} + \beta^2)n^2} + \frac{12Z^2n^2}{\pi^4(1 + \beta^2)^2\ (\frac{T}{N} + \beta^2)\ (n^2 - 1)}$$

In order to evaluate this buckling coefficient for a ahell of specified dimensions, it is necessary to establish first the buckling mode (number of circumferential waves n) which yields the smallest value of K for given $\frac{T}{N}$, β and Z. This can be done by repeated calculations, or by using the chart given in ref. 9, which is an extension of curves presented in ref. 10.

Concluding Remarks

An evaluation of earlier work is presented in ref. 9 and the conclusion was reached that neither von Mises' nor Loo's expressions are based on consistent, admissible shell theories, and that they yield, for certain shell dimensions, incorrect results. Donnell-type equations, such as used by Batdorf (ref. 4) cannot be expected to yield good results for long shells.

Further, it is of interest to point out that the new expression for the buckling pressure cannot be derived directly from a consistent simplified shell theory. It appears that this expression can be obtained only on the basis of the complete bending theory, followed by a simplification of the expression for the buckling coefficient. In future work it might be worthwhile to attempt constructiong a similar expression for the buckling coefficient valid for more realistic boundary conditions.

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