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BUCKLING OF CONICAL SHELLS UNDER EXTERNAL PRESSURE

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SUMMARY

It is shown that the initial and the buckled shapes of a certain element of a conical shell can be considered as similar to those of an element of a cylindrical shell of which the radius and length are conservatively determined. It is concluded that therefore the buckling pressure of the conical shell is equal to that of the comparable cylindrical shell. A simple method for finding the buckling pressure if it varies along a generatrix is also given.

INTRODUCTION

The buckling pressure of simply supported conical shells has been the subject of several recent papers on theoretical as well as experimental investigations. Several of these papers (refs. 1 through 6) refer to an unpublished company report of 1953 (ref. 7) in which the present author conservatively derived the buckling pressure of complete or truncated conical shells under uniform or non-uniform external pressure. The background of this derivation will be more elaborately explained here.

BUCKLING UNDER UNIFORM EXTERNAL PRESSURE

For design purposes a reliable estimate of the buckling pressure of simply supported conical shells was required. At that time (1952) insufficient information was available. The problem had been dealt with in references 8 and 9, both assuming uniform pressure and using small deflection theory. However, also within the limitations of this theory the results obtained in these references were open to question. The differential equation derived in reference 8 could be solved only by assuming radial deflection

$$w = C\, r^{m} \cos n\theta \qquad (1)$$

and at the same time assuming the thickness t to be proportional to the radii r of latitudinal cross sections (Figs. 1 and 2). The angle θ is measured in the circumferential direction. For a minimum buckling load m has to be equal to 2. It is evident that at edges where r differs from zero, w from equation (1) is not zero, so that it does not satisfy the geometric boundary conditions of simple support. From this relaxation of restraints, reference 8 will underestimate the buckling stresses for simply supported shells.

In reference 9 an energy method was used, assuming radial displacements

$$w = C \sin\lambda x \cos n\theta \tag{2}$$

where $\lambda = \pi/\ell$, so that w is assumed to vary as a half sine wave in the direction of a generatrix. Obviously, for a complete conical shell this is far from reality, since for that case at the apex equation (1) satisfies the geometrical boundary condition $w = 0$ for a simply supported edge, so that the deflection of a generatrix near the apex actually approximates that of a clamped beam. Hence assumption of a deflection as in equation (2), that differs substantially from the real one, will lead to a too high buckling pressure. Apart from that, several approximations were made in reference 9 of which the effect was difficult to assess.

In order to find a more reliable solution it was reasoned as follows: One can imagine that for a cylindrical shell, where for free edges w is constant with varying axial coordinate x, the simply supported buckling mode, as given by equation (2), is obtained by multiplication of this constant deflection with $\sin\lambda x$. Hence, for a conical shell, of which for free edges the buckling mode is given by equation (1), with $m = 2$, that for simply supported edges can be approximately obtained by multiplying w from equation (1) with $\sin\lambda x$, whence

$$w = C\, r^2 \sin\lambda x \cos n\theta. \tag{3}$$

This deflection is shown by the solid curves in Figs. 1b and 2b. It may be pointed out that equation (1) and therefore equation (3) actually applies for a shell of which the wall thickness t is proportional to r. Therefore, for uniform wall thickness, as considered here, the deflection w for smaller r values will be relatively somewhat smaller than would follow from equations (1) or (3). As explained later on, this will make the results obtained by using equation (3) somewhat conservative.

Using the energy method, equation (3) and accessory displacements in the other two directions, could be expected to yield lower buckling pressures than equation (2), since the former may be expected to approximate the actual buckling mode better. However, since equation (3) could not be expected to be the real mode, it would overestimate the real buckling pressure by an unknown amount, which was undesirable. Therefore, since no time was available for other lengthy methods, a simple reasoning was used, which led more directly and much quicker to a result and also afforded an opportunity to remain sufficiently at the safe side to account for the lowering of the buckling stress due to snap-through, which was estimated at not more than 25% (see ref. 10). This method will now be described.

If an equivalent cylindrical shell can be found for which, at a given point, the initial shape and loading and also the deflection function and its derivatives are the same as those at a point of the conical shell, it is evident that for elements at these corresponding points the same equilibrium equations apply. Therefore the critical pressures for these corresponding elements and hence for the entire shells will be equal.

From Fig. 1b, an element near the lower edge of the conical shell will have the same initial shape and loading as an element of a cylindrical shell with equal thickness t and with radius $\rho = r_2/\sin\alpha$. Indeed, with the same all sided pressure p, at the lower edge also the compressive membrane stresses σ_θ and $\sigma_x = \sigma_\theta/2$ are equal. The simply supported length of the equivalent cylindrical shell, buckling in a half sine wave along a generatrix, can be chosen such that near the lower edge this half sine wave coincides with the buckling deflection of the conical shell. From Fig. 1b, where $r_1/r_2 = 0$, this simply supported length is

$$(\ell_{eq})_{r_1/r_2 = 0} = (0.5 \text{ to } 0.55)\ell \tag{4}$$

Hence, for both shells the buckling pressure will be the same function of the number of lobes. For a frustrum of a conical shell, where $r_1/r_2 = 0.5$ (Fig. 2), the length of the equivalent cylinder is, from Fig. 2b,

$$(\ell_{eq})_{r_1/r_2 = 0.5} = (0.75 \text{ to } 0.80)\ell \tag{5}$$

Conservatively, using the higher values in equations (4) and (5), this leads to an equivalent cylindrical shell with length

$$\ell_{eq} = \frac{r_1+1.2r_2}{2.2r_2}\ell \tag{6}$$

and radius $\rho = r_2/\sin\alpha$, with, of course, the same thickness t as the conical shell. It should be pointed out that by assuming the radius of the equivalent cylindrical shell as $r_2/\sin\alpha$ again some conservatism is introduced. Actually the point where the buckling modes will coincide will be above the lower edge, since equation (3) only satisfies the geometric boundary condition that $w = 0$ for $x = \ell$, but not the natural one that the second derivative with respect to x is zero, which the half sine wave does. Hence the radius ρ of the equivalent cylinder is actually smaller than $r_2/\sin\alpha$. Moreover, as stated in the foregoing, the actual buckling deflection of the conical shell for smaller values of x will be relatively somewhat smaller as compared with that for larger values of x. From Figs. 1b and 2b this would slightly decrease ℓ_{eq}, so that also the assumed buckling mode tends to lead to conservative results if this method is used, although it would be unconservative if using the energy method, since it need not be the real buckling mode. In connection with all this built-in conservatism it was judged that the actual buckling pressure, if equated to that of a cylindrical shell with length ℓ_{eq} and radius $\rho = r_2/\sin\alpha$, could be considered to include the influence of snap-through and thus would be reliable for design purposes. Hence, using a formula given in reference 11, the buckling pressure is

$$p_b = \frac{0.92E(t/\rho)^2}{(\ell_{eq}/\rho)(\rho/t)^{1/2}-0.636} \tag{7}$$

As derived in reference 12 the same formula applies for a clamped cylindrical shell of length ℓ and radius ρ, with $\ell_{eq} = (2/3)\ell$.

BUCKLING UNDER NON-UNIFORM EXTERNAL PRESSURE

To find the effect of non-uniform external pressure, it is observed that the curvature changes of the conical shell depend mainly on the radial deflection w and can be expressed as

$$\chi_\theta = (w/r^2)\ \sin^2\alpha + \partial^2 w/(r^2\,\partial\theta^2) \qquad \text{and } \chi_x = \partial^2 w/\partial x^2 \tag{8}$$

The radial deflecting forces dD acting upon an element $r\,d\theta\,dx$ exert an amount of work upon a ring of the conical shell of length dx that can be expressed as

$$dV = \frac{1}{2}\int_0^{2\pi} w\ dD = -\frac{1}{2}\,t\ dx\int_0^{2\pi}(\sigma_\theta\chi_\theta + \sigma_x\chi_x)wr\,d\theta \tag{9}$$

Neglecting the relatively small influence of $\sigma_x\chi_x$, with uniform external pressure p, where $t\sigma_\theta = pr/\sin\alpha$, using equations (3) and (8) in

(9) yields

$$dV = \frac{\pi}{2} p \left(\frac{n^2}{\sin\alpha} - \sin\alpha\right) C^2 r^4 \sin^2 \lambda\, x\, dx = C_1 r^4 \sin^2 \lambda\, x\, dx \qquad (10)$$

where C_1 is proportional to p and independent of x. Presenting this graphically for ratios r_1/r_2 of 0 and 0.5, it is observed that if p varies linearly with x, from p_1 at the upper edge $(r = r_1)$ to p_2 at the lower edge $(r = r_2)$, where $2 > p_2/p_1 > 0.5$ as happened to be the case, the deflection surfaces will not differ appreciably from that of equation (3). For example, for a simply supported long plate, subjected to compressive stresses in its plane that vary from σ_1 at one edge to $\sigma_2 = 2\sigma_1$ at the other edge, from page 173 of reference 13 the buckling stress coefficient k for σ_2 is 5.32, while, assuming the same buckling mode as for constant compressive stress, one would find k = 4 for the average stress and hence, k = 5.33 for σ_2, so practically no difference. Hence, from equation (10) , where C_1^2 is proportional to p, the work done by the variable pressure p is found by multiplying the ordinates of the curves presenting dV from equation (10) by a constant times p. From a simple calculation it followed that the result can be approximated very well by assuming that the varying pressure is equivalent to a constant pressure p_{eq} equal to the pressure p at the center of the length ℓ_{eq} of the equivalent half sine wave, so that

$$p_{eq} = p_2 + \frac{\ell_{eq}}{2\ell}(p_1 - p_2) \qquad (11)$$

CONCLUDING REMARKS

When these results were reported in reference 7 no tests were available to check them. In the meantime, however, several experimental results were published. These were compiled for complete conical shells in reference 2, from which Fig. 3 has been copied. It shows that the method of reference 7 as reviewed here leads indeed to a reliable design formula, since it forms the lower bound to the test results. Fig. 4 was copied from Fig. 11 of reference 4, adding the curve according to reference 7 and the present note, and gives the results for truncated shells, where p_N is the buckling pressure from reference 9. Figs. 3 and 4 also present several theoretical results. For low taper ratios (nearly cylinders) where equation (7) was not meant for, the conservatism in determining ℓ_{eq} and ρ vanishes, so that it is understandable that there in Fig. 4 some test results are below the line according to reference 7. As stated in the foregoing, equation (2), used in reference 9, could be expected to overestimate the buckling pressure. Assuming that reference 2 gave the correct buckling pressure from small deflection theory this is not revealed in Fig. 3. Apparently the lower buckling stress found in reference 9 is due to additional approximations. Reference 6 also uses equation (2) and indeed gives higher values than reference 2. It is interesting that, as mentioned in reference 2, reference 14 later used the same mode, according to equation (3), as reference 7. From Fig. 3 its results are lower than those from reference 6, so that indeed equation (3) gives a better approximation than equation (2). It also shows that the method of reference 7 using the same equation (3), although very simple, served better for attaining its aim, which was not a formal computation, but a reliable design formula. Using the energy method would have led, with much more effort, to the unconservative results of reference 14.

After reference 7 was distributed the author was informed about an earlier paper that compares a conical shell to an equivalent cylindrical one (reference 15). From a partial translation only the case of a complete conical shell is considered there. Strips along a generatrix are considered as beams, clamped at the apex and simply supported at the base, so that their maximum deflection occurs at about 0.6ℓ from the top. Therefore the radius of the equivalent shell was assumed as the radius of curvature of the conical shell at 0.6ℓ from the top, that is, in the present notation, equal to $0.6\ r_2/\sin\alpha$, with a length equal to the total slant length ℓ of the cone. This leads to smaller buckling stresses than the present method and is not based on the same principles.

It should be realized that for many problems, that require extremely elaborate computations for exact or even approximate solutions, often good results can be obtained by a simple reasoning. In several cases this will even yield exact solutions (see for example refs. 12, 16, 17, and 18).

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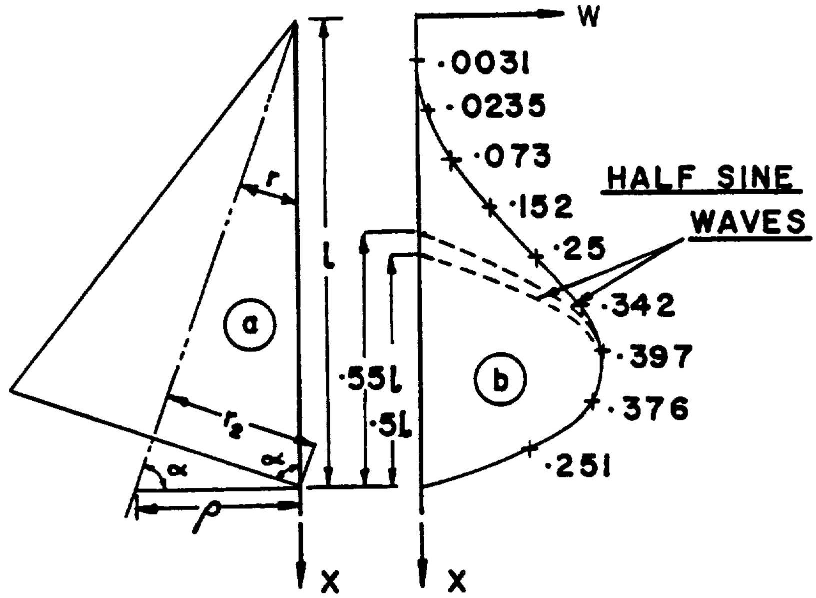
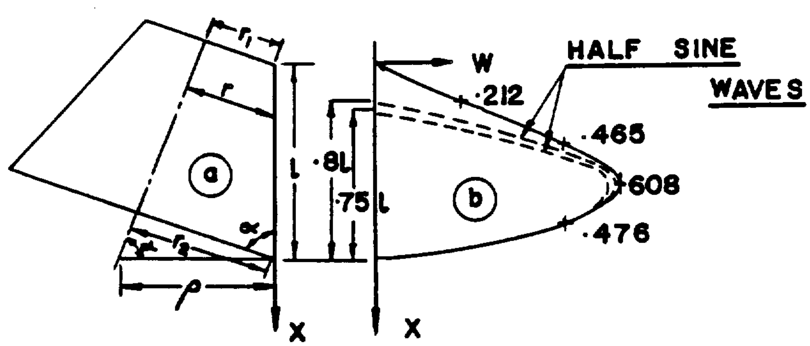


(a) Complete conical shell. (b) Buckling mode of shell according to eq. (3).

Figure 1.



(a) Frustrum of conical shell. (b) Buckling mode of frustrum according to eq. (3).

Figure 2.

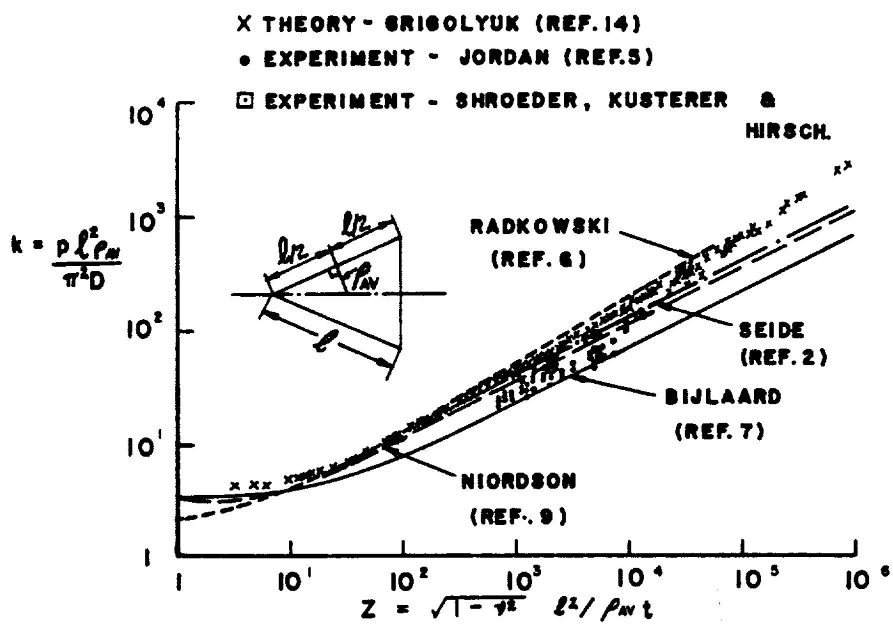


Figure 3.- Comparison of theoretical buckling pressures (curves and crosses) of complete conical shells with experiments (from 2nd ref. 2).

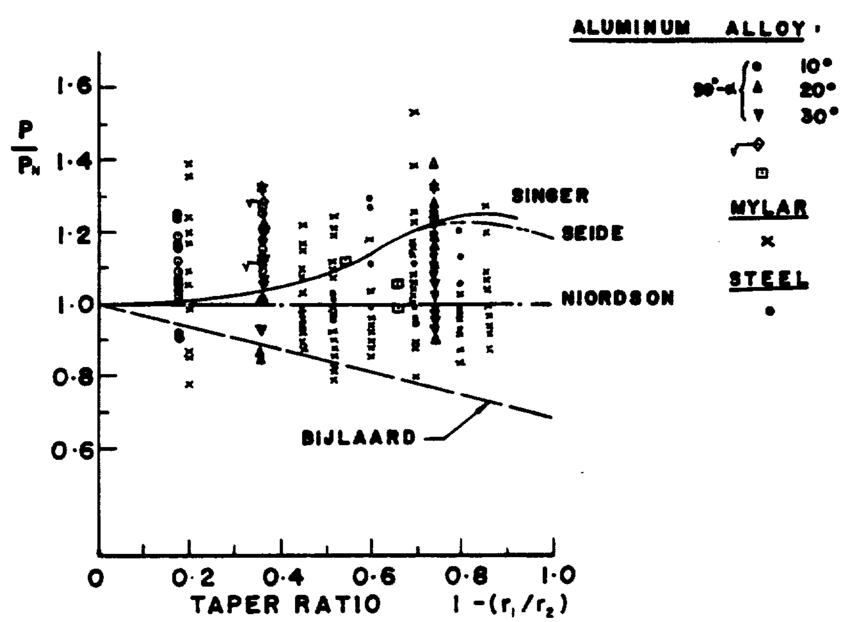


Figure 4.- Comparison of theoretical buckling pressures (curves) of truncated conical shells with experiments (from ref. 4).