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AXISYMMETRIC SNAP BUCKLING OF CONICAL SHELLS

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SUMMARY

The authors give a brief account of some of their recent analytical and numerical studies of cone buckling, limiting the discussion to axisymmetric deformations.

Pertinent numerical results for the relaxation buckling of full cones subjected to uniform external pressure and Belleville springs deformed by axial edge loads are presented. In addition, bifurcation buckling problems are discussed. For a specific case, the existence of Friedrichs' intermediate buckling load, P_m, as applied to cones, is established. Upper and lower bounds for its value are given.

INTRODUCTION

The buckling of conical shells is, in many cases, characterized by a snapping phenomenon. Thus, at some critical load value, the shell suddenly jumps from a slightly deformed equilibrium state into a non-adjacent one with relatively large deformations. If the cone is initially shallow it may buckle axisymmetrically, as in spherical cap snapping. However, for slender cones experimentally observed buckling modes are asymmetrical and appear to be related to those of cylinder buckling.

The principal unresolved buckling problem for conical shells, as well as for cylindrical and spherical shells, is to determine the mechanism which "triggers" the sudden snapping and to estimate the load at which it occurs. Some investigators† have sought to determine this "critical" load by using classical linearized buckling theory or variants thereof. However, to obtain a deeper insight into the buckling phenomenon it is essential to employ a nonlinear theory.

In this paper we briefly describe some of our recent analytical and numerical investigations of axisymmetric cone buckling.‡ We first present the

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†See the review articles of refs. 1-2 and ref. 3 for detailed accounts of previous work.

‡The authors are currently preparing a paper which describes this work in greater detail.

results of a numerical study pertaining to the buckling of simply supported full cones subjected to uniform external pressure. We also indicate some numerical results obtained for the buckling of shallow truncated cones, sometimes called Belleville springs, deformed by axial edge loads. In both of these problems the shell undergoes small deformations from its initial conical shape immediately upon application of load. With increasing load, compressive membrane stresses are developed which essentially reduce the "stiffness" of the shell. Thus the shell "softens" with increasing load. After snapping, the membrane stresses tend to become tensile thereby increasing the "stiffness." Hence a "hardening" effect is observed yielding, at least for a limited range of parameters, deformations which increase with load at a decreasing rate (see fig. 1). In analogy to the discussion in reference 4 for spherical caps, we call the buckling phenomenon associated with this behavior relaxation buckling.

In our final example, a full cone subjected to external pressure is again considered. However, the pressure is no longer uniform. Instead, the pressure and boundary conditions are prescribed such that a membrane state of uniform compression is a solution (unbuckled) of the nonlinear problem. We then conjecture that bifurcation buckling will occur by branching from the unbuckled solution, yielding a load deflection characteristic similar to that shown in figure 2. For this problem, analytical studies are facilitated by our precise knowledge of the unbuckled solution. We are thus able to prove the existence of Friedrichs' intermediate buckling load, P_m (ref. 5), for which the potential energies of the buckled and unbuckled states are equal; we also establish upper and lower bounds on its value.

1. RELAXATION BUCKLING OF A COMPLETE CONE

Formulation of the Boundary Value Problem

We consider a complete conical shell of thickness t, base angle θ and slant length s_1 (fig. 3) subjected to a uniform external pressure p which is counted positive when directed inward. Assuming that the shell deforms axisymmetrically, the non-vanishing middle surface displacements u and w (see fig. 3) are functions of s only. Here s is the distance along a generator of the conical middle surface measured from the apex. The base of the cone is rigidly pinned, i.e., the meridional bending moment and horizontal displacement vanish at $s = s_1$.

We assume that the shell is constructed of a homogeneous, isotropic, elastic material for which Hooke's law is valid. Employing the usual assumptions of thin shell theory, we have derived the following nonlinear boundary value problem which describes the small finite deformations of the cone:

$$Ly(x) - Kz(x)\left[y(x) + 1\right] = Px^2 \qquad (1a)$$

$$Lz(x) = -\frac{K}{2}\left[y^2(x) + 2y(x)\right] \qquad (1b)$$

$$y(0) = z(0) = 0 \tag{2a}$$

$$\frac{dy(1)}{dx} + \nu y(1) = \frac{dz(1)}{dx} - \nu z(1) = 0. \tag{2b}$$

The differential operator L in (1) is defined by

$$L \equiv x \frac{d}{dx} \frac{1}{x} \frac{d}{dx} x$$

and the following dimensionless variables are employed:

$$x \equiv \frac{s}{s_1}, \quad y(x) \equiv \cot\theta \frac{dw(s)}{ds}$$

$$P \equiv \frac{6(1-\nu^2){s_1}^3}{t^3 \tan\theta} \frac{p}{E}, \quad K \equiv \left[12(1-\nu^2)\right]^{\frac{1}{2}} \frac{s_1}{t} \tan\theta.$$

Here E is Young's modulus and ν is Poisson's ratio which we henceforth take as $\nu = .30$. The stress function $z(x)$ is defined in terms of the membrane (or middle surface) stresses $\sigma_\theta^o(s)$ and $\sigma_\phi^o(s)$ by the relations

$$\frac{\left[12(1-\nu^2)\right]^{\frac{1}{2}} s_1}{Et \tan\theta} \sigma_\theta^o(s) = \frac{z(x)}{x},$$

$$\frac{\left[12(1-\nu^2)\right]^{\frac{1}{2}} s_1}{Et \tan\theta} \sigma_\phi^o(s) = \frac{dz(x)}{dx}.$$

Equations (1), in which we refer to K as the geometric parameter and P as the loading parameter, can also be obtained by specializing the work of previous authors (references 6 - 8). In addition, a special case of equations (1) has previously been given by Grigoliuk (reference 9) in connection with his work on shallow cones.

The conditions (2a) are obtained from the assumption of regularity at the apex. However, in a sufficiently small neighborhood of the apex the shell is not "thin" and hence equations (1) may be invalid in this neighborhood. To circumvent this difficulty we define our boundary value problem for the complete cone as the limit of a sequence of boundary value problems for truncated cones (with the same values of K and P) as the slant length approaches that of the complete cone.

Presentation and Analysis of Numerical Results

We suppose that for a limited range of parameters the relation between P and deflection is similar to that shown in figure 1. The indicated curve implies that for $P < P_L$ and $P > P_U$ only one equilibrium state is possible. In the former case the equilibrium state is represented by a point on the unbuckled branch IOU, while in the latter case the equilibrium state corresponds to a point on the buckled branch LN. For P in the range $P_L < P < P_U$ there are three equilibrium states, represented by points on the unbuckled and buckled branches and the unstable branch UL. Friedrichs' energy buckling criterion (reference 5) as applied to cones implies the existence of an intermediate load P_m in the range $P_L < P_m < P_U$. For $P < P_m$ the potential energy of the unbuckled state is less than that of the buckled state and conversely for $P > P_m$.

We have obtained, for a range of K and P, numerical solutions* of the nonlinear boundary value problem defined by eqs. (1) and (2). The numerical method employed consists in solving, by iteration, a finite difference approximation of the boundary value problem. Essentially, the technique is similar to that previously employed in studies of the nonlinear bending and buckling of circular plates (references 11 - 13) and spherical caps (reference 10). Details of the method and extensive results will appear in a subsequent paper.

In the present paper, we give some of the numerical results directly concerned with the evaluation of P_U, P_m, and P_L. Figure 4 shows the variation of dimensionless base slope $y(1)$ vs. load for several values of K. We note that for $K = 2$ and 3.5 the cones are nonbuckling since the base slopes are single valued functions of load. Buckled branches are first discernible at $K = 4$. Thus, the transition between nonbuckling and buckling cones occurs in the interval $3.5 < K < 4$.

In figure 5, the variations of P_U, P_m, and P_L with K are shown. We note that when $K > 7.5$, then $P_L < 0$. This indicates the existence of buckled equilibrium states for $P \leq 0$, i.e., for unpressurized or internally pressurized cones. The numerical results, however, indicate that for these pressures the buckled states possess greater potential energy than the unbuckled states. For $P = 0$ this can be proven analytically. Thus, in this sense, the buckled solutions for $P \leq 0$ are unstable.

The dashed curve in figure 5 is obtained from a linearized approximation. The "critical" load value, $P = P_0$, thus determined, gives an exceptionally close approximation to P_m for the range of K considered.

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*All computations were performed on the IBM 7090 computer at the Republic Aviation Corporation. The authors are indebted to B. Sackaroff and M. Gershinsky of the Applied Math. Section, Digital Computing Division for their aid in programming and running the computer code.

2. THE BELLEVILLE SPRING

Belleville springs are shallow truncated conical shells for which $x_0 \le x \le 1$, where x_0 is the dimensionless distance from the imagined apex to the plane of truncation.

We have applied our numerical procedure to a specific problem wherein the edges are subjected to compressive axisymmetric axial loads F. The edges are free to rotate and move radially. The differential equations describing the axisymmetric deformations are the same as equations (1) if the right side of equation (1a) is replaced by R, where

$$R \equiv \left[\frac{12(1-\nu^2)s_1}{\pi E t^3 \sin 2\theta} \right] F .$$

The boundary conditions are

$$\frac{dy(x_0)}{dx} + \frac{\nu}{x_0} y(x_0) = \frac{dy(1)}{dx} + \nu y(1) = 0, \tag{3a}$$

$$z(x_0) = z(1) = - \frac{\sin^2\theta}{K} R . \tag{3b}$$

In figures 6 and 7, some results of the numerical computations are given for two buckling cone configurations. These are compared with the experimental results of Almen and Laszlo (ref. 14). The graphs show fair agreement between the calculated and measured axial shortening. However, Almen and Laszlo do not give a description of their testing technique and boundary conditions.

Stresses and deflections for several other cone configurations have been calculated. These have been compared with the results of approximate formulas and computer calculations given by Wempner (refs. 15 - 16), and Schmidt and Wempner (ref. 17). The agreement of results for the cases considered was found to be good.

3. BIFURCATION BUCKLING

We now consider a full cone subjected to an external pressure distribution which varies inversely with x. The appropriate differential equations are the same as equations (1) if Px^2 is replaced by Px in equation (1a). The edge $x = 1$ is assumed to be restrained against rotation but free to expand or contract horizontally. It is then easy to show that

$$y(x) \equiv 0, \quad z(x) = - \frac{P}{K} x \tag{4}$$

is a solution (unbuckled) of the nonlinear boundary value problem for all K and P. We conjecture* that additional solutions (buckled) will appear by branching from equation (4) at an infinite number of discrete values $P = P_i$, $i = 1, 2, \cdots$. The P_i are the eigenvalues of the linearized shell buckling theory obtained by omitting nonlinear terms in the differential equations. The load deflection curves are then similar to those shown in figure 2.

It is assumed that for each K and P the potential energy functional possesses a minimum. We can then prove the existence of an intermediate buckling load P_m. Furthermore, we have obtained upper and lower bounds for P_m given by,

$$\omega^2 \leq P_m < \overline{P}(K) ,$$

where ω is the first zero of the Bessel function $J_1(x)$ and $\overline{P}(K) = \text{g.l.b. } P_i$. The quantity ω^2 is also the dimensionless buckling load, HR^2/D of a radially compressed clamped circular plate, where H, R, and D are the critical edge thrust, plate radius and flexural rigidity, respectively. As in the case of spherical caps (ref. 4) we refer to this as the "equivalent" flat plate problem.

Upper bounds for P_m which are lower than $\overline{P}$ have also been obtained by a minimization procedure.

CONCLUDING REMARKS

We are currently extending our numerical calculations for the relaxation buckling problems discussed in Sections 1 and 2 to include a larger range of parameter values. In addition, numerical solutions for other cone problems are being considered. In particular, we plan to obtain accurate numerical approximations of P_m for the bifurcation problem discussed in the previous section.

It appears likely that some of the analysis briefly outlined in Section 3 can be extended, with suitable modification, to unsymmetric bifurcation buckling of cones.

There are, to the authors' knowledge, no experimental results available for the problems and ranges of parameters considered in Sections 1 and 3. Carefully performed experiments for these cases may give valuable insight into the buckling mechanism.

*Similar conjectures have been proved for circular plates (ref. 18) and spherical caps (ref. 4)

REFERENCES

1. Nash, W.A., "Recent Advances in the Buckling of Thin Shells," Applied Mechanics Reviews, Vol. 13, No. 3, pp. 161-164, (1960).

2. Fung, Y.C., and Sechler, E.E., "Instability of Thin Elastic Shells," Proc. of the First Symposium on Naval Structural Mechanics, Stanford University, California, Pergamon Press, pp. 115-168, (1960).

3. Seide, P., "On the Buckling of Truncated Conical Shells Under Uniform Hydrostatic Pressure," Proc. of the Symposium on the Theory of Thin Elastic Shells, I.U.T.A.M., Delft, pp. 363-386, (1959).

4. Reiss, E.L., "Birurcation Buckling of Spherical Caps," to appear.

5. Friedrichs, K.O., "On the Minimum Buckling Load for Spherical Shells," von Kārmān Anniversary Volume, California Institute of Technology, Pasadena, pp. 258-272, (1941).

6. Chien, W.Z., "The Intrinsic Theory of Thin Shells and Plates," Quart. of Appl. Math., Vol. II, No. 2, pp. 120-135, (1944).

7. Reissner, E., "On Axisymmetrical Deformations of Thin Shells of Revolution," Proc. Symp. Appl. Math., Vol. III, pp. 27-52, (1950).

8. Mushtari, Kh.M., and Galimov, K.Z., Nonlinear Theory of Thin Elastic Shells, Tatknigoizdat, Kazan, p. 304, (1957). Now available in English as NASA-TT-F62, (1961).

9. Grigoliuk, E.I., "On the Instability with Large Deflections of a Closed Laminated Conical Shell Subject to Uniform Normal Surface Pressure," Institute of Mech. of the Academy of Sciences, USSR, Vol. XXII, pp. 111-119, (1955).

10. Keller, H.B., and Reiss, E.L., "Spherical Cap Snapping," Jour. Aero/Space Sciences, Vol. 26, No. 10, pp. 643-652, (1959).

11. Keller, H.B., and Reiss, E.L., "Iterative Solutions for the Non-Linear Bending of Circular Plates," Comm. on Pure and Appl. Math, Vol. XI, No. 3, pp. 273-292, (1958).

12. Keller, H.B. and Reiss, E.L., "Non-Linear Bending and Buckling of Circular Plates," Proc. Third U.S. Nat'l. Cong. of Appl. Mech., Providence, R.I., pp. 375-385, (1958).

13. Newman, M. and Forray, M., "Axisymmetric Large Deflections of Circular Plates Subjected to Thermal and Mechanical Load," Proc. of the Aerospace Forum I Session, S.M.F. Fund Paper No. FF-30, pp. 56-67, presented at the IAS 30th Annual Meeting, New York, (1962).

14. Almen, J.O., and Laszlo, A., "The Uniform-Section Disk Spring," Trans. ASME, Vol. 58, No. 4, pp. 305-314, (1936).

15. Wempner, G.A., "The Conical Disk Spring," Proc. Third U.S. Nat'd. Congress of Appl. Mech., Providence, R.I., pp. 473-478, (1958).

16. Wempner, G.A., "Axially Symmetric Deformations of a Shallow Conical Shell," Ph.D. Thesis, University of Illinois, (1957).

17. Schmidt, R., and Wempner, G.A., "The Nonlinear Conical Spring," Trans. ASME, 81E (J. Appl. Mech.), 4, pp. 681-682, (1959).

18. Keller, H.B., Keller, J.B., and Reiss, E.L., "Buckled States of Circular Plates," Quart. Appl. Math., Vol. XX, No. 1, pp. 55-65, (1962).

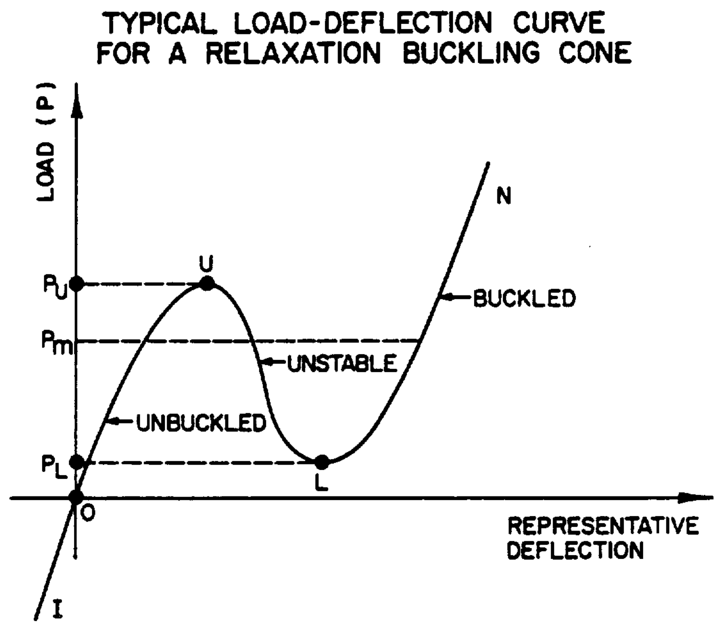

Figure 1

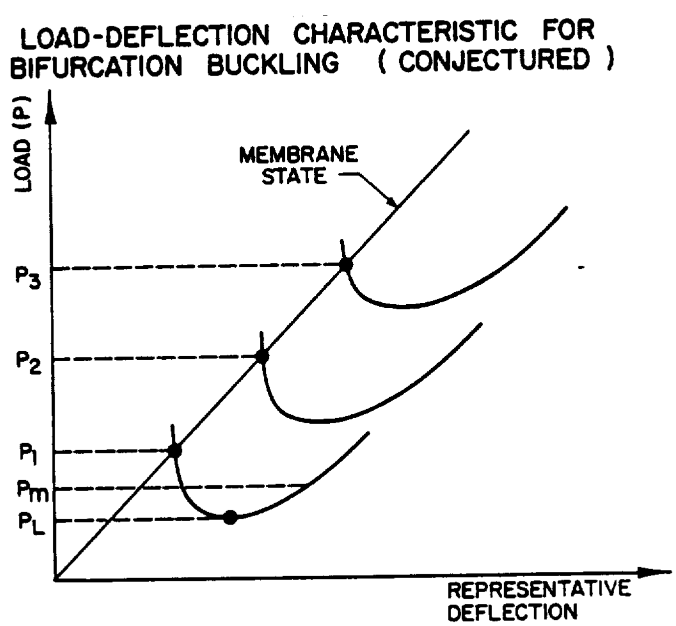

Figure 2

CONE GEOMETRY AND DISPLACEMENTS

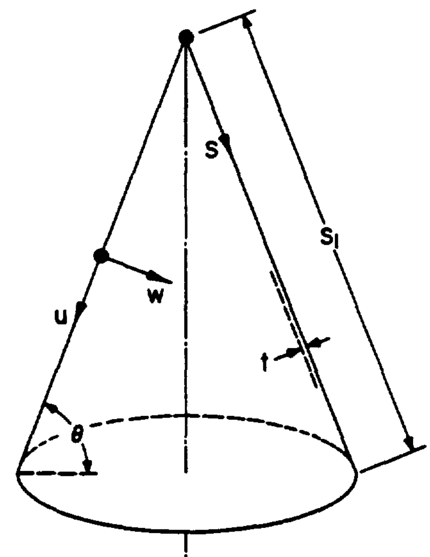

Figure 3

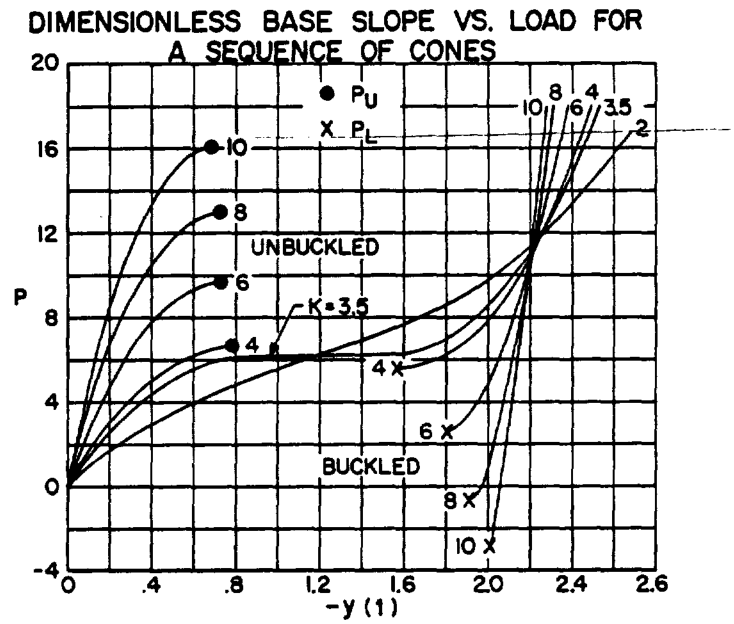

Figure 4

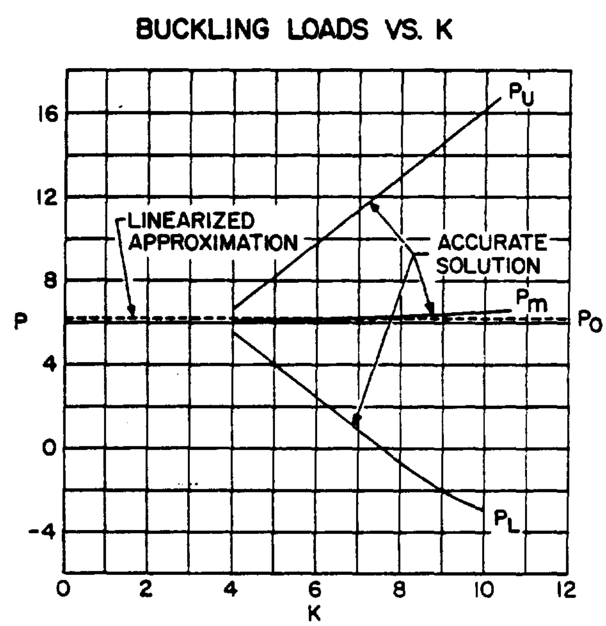

Figure 5

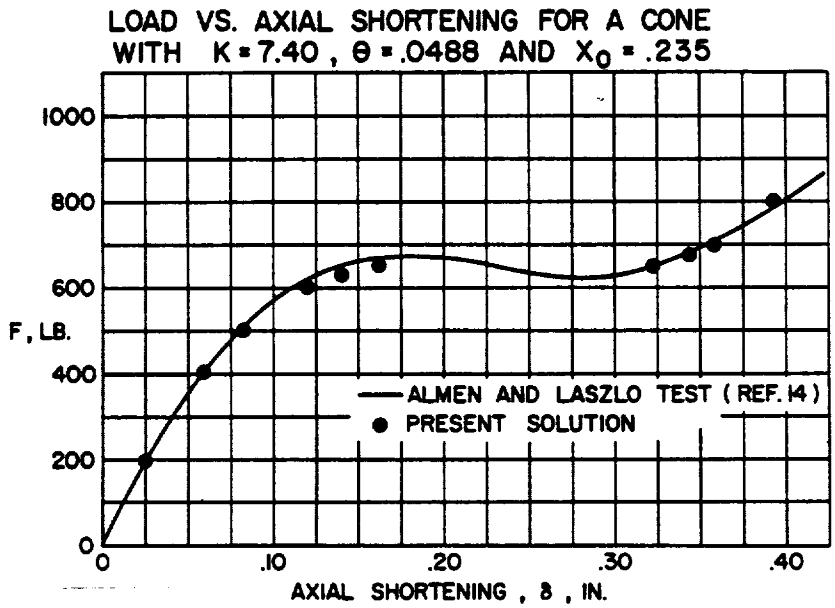

Figure 6

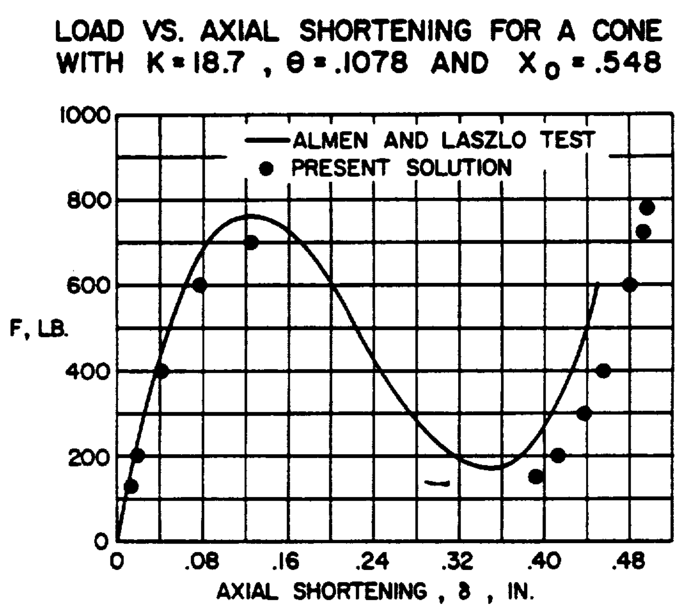

Figure 7