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BUCKLING OF ORTHOTROPIC AND STIFFENED CONICAL SHELLS*

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SUMMARY

Donnell type stability equations for thin circular orthotropic conical shells are presented and solved for external pressure, axial compression and combined loading. The solution is likewise applied to stiffened conical shells. Correlation with equivalent cylindrical shells yields a simple approximate stability analysis for orthotropic or ring-stiffened conical shells under hydrostatic pressure. The general instability of stiffened conical shells under hydrostatic pressure is also analysed by a more accurate approach. Preliminary experimental results for buckling of ring-stiffened conical shells under hydrostatic pressure are presented and discussed.

INTRODUCTION

Most aerospace shell structures are orthotropic or stiffened shells. The increasing use of new constructional materials, such as reinforced plastics, fiber reinforced materials etc., which have orthotropic elastic properties, has focussed attention on orthotropic shell theory and the corresponding stability analysis. Buckling of orthotropic cylinders has been subject to extensive investigations (See refs. 1 - 4), and the general instability of stiffened cylindrical shells has likewise been analysed by consideration of an equivalent orthotropic shell (refs.5 - 6), as well as by other approaches (See, for example, ref. 7). In this report, the investigations are extended to orthotropic and stiffened conical shells.

The method developed in reference 8 for isotropic conical shells is applied to the solution of Donnell type stability equations for orthotropic conical shells, derived in reference 9, for external pressure loading. The solution is then used to analyse the general instability of ring-stiffened conical shells under external pressure by consideration of an equivalent orthotropic shell. Typical cases of orthotropic and ring stiffened conical shells are computed and correlated with equivalent cylindrical shells. The comparison brings out again the taper ratio as the most significant factor representing the conicity in

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the case of buckling under uniform external pressure, as shown for isotropic shells by Seide (ref. 11) and reconfirmed by the author for slightly different boundary conditions (ref. 8). A relatively simple approximate analysis for the buckling of any orthotropic or ring - stiffened conical shell under uniform external pressure is obtained from the correlation.

As for isotropic conical shells (ref. 12), the same linear analysis is extended to the case of axial compression and combined axial compression and external or internal pressure. On the basis of results for cylindrical shells (ref. 1), the linear orthotropic theory may be expected also to yield fairly realistic buckling loads for conical shells with closely spaced stiffeners.

The more accurate method of separate "distributed stiffness" of rings and stringers is then employed to show the effect of eccentricity of stiffeners on the general instability of stiffened conical shells under external pressure.

Preliminary experimental results for 3 machined ring-stiffened conical shells verify in general the theoretical analysis for buckling under hydrostatic pressure.

The analysis referred to is written in non-dimensional form, and the coordinates and displacements are non-dimensionalized through division by a, the distance along a generator of the top of a truncated cone from the vertex. (See fig. 1).

SYMBOLS

D	= $[Eh^3/12\ (1 - \nu^2)]$, in. lb.
$E_x, E_\varphi, E, E_1, E_2$	= moduli of elasticity of orthotropic shell, and of stiffened shell and its stringers and rings respectively, psi
G	= shear modulus, psi
h	= thickness of shell, in.
K^4	= $12\ \mu (a/h)^2$
p	= hydrostatic pressure or critical hydrostatic pressure, psi
$\bar{p}$	= critical hydrostatic pressure or equivalent cylindrical shell, psi
P	= axial compressive load, lb.
t	= number of circumferential waves

u	$= (u^{*}/a)$ = non-dimensional displacement of shell middle surface along a generator
v	$= (v^{*}/a)$ = non-dimensional circumferential displacement of shell middle surface
w	$= (w^{*}/a)$ = non-dimensional radial displacement of shell middle surface
x	$= (x^{*}/a)$ = non-dimensional axial coordinate, along a generator
x_2	= ratio of distance of the bottom of a truncated cone from the vertex to that of the top.
α	= cone angle
$\alpha_1 = (h\, E_x/\mu)$	, lb./in.
$\alpha_2 = (h\, E_\varphi/\mu)$	, lb./in.
$\alpha_3 = G\, h$	, lb./in.
γ	$= (1 - \nu_{\varphi x})/2$ for orthotropic, or $(1-\nu)/2$ for isotropic shells
μ	$= 1 - \nu_{x\varphi}\, \nu_{\varphi x}$
$\nu_{x\varphi}, \nu_{\varphi x}, \nu$	= Poisson's ratios for orthotropic and isotropic shells
$\bar{\sigma}_x, \bar{\sigma}_\varphi, \bar{\tau}_{x\varphi}$	= membrane stresses of prebuckling state, psi
φ	= circumferential coordinate

Subscripts following a comma indicate differentiation.

ORTHOTROPIC THEORY

Buckling of Orthotropic Conical Shells Under External Pressure

The stability equations for thin circular orthotropic conical shells employed in the analysis are presented (in non-dimensional form) in Appendix A. These equations are derived in reference 9 and reduce to Seide's equations (ref.10) for the case of isotropic shells or to Bodner's equations for orthotropic cylindrical shells (ref.5) when the cone angle approaches zero. The third equation, in the radial direction, is however a Batdorf type modified equation, instead of the usual eighth order equation, to facilitate its solution by the Galerkin method. It reduces therefore to the modified equation of reference 8 for the case of isotropic shells, instead of the corresponding equation of reference 10.

The problem is solved for a cone supported in a manner approximating conventional simple supports. The conditions for the radial displacement are therefore

$$w = 0 \quad \text{at} \quad x = 1,\ x_2 \tag{1}$$

and

$$w_{,xx} + (\nu_{\varphi x}/x) w_{,x} = 0 \quad \text{at} \quad x = 1, x_2 \tag{2}$$

The circumferential and axial displacements (along the generators) are assumed to be resisted by elastic supports instead of the usual requirements that $v = 0$ and u is unrestrained. These elastic supports, however, approximate the usual conditions fairly closely as in the case of isotropic shells (ref. 8).

Now, since an orthotropic shell may be expected to buckle in a mode similar to that of an isotropic one, the same solution is assumed for the displacement functions

$$\begin{aligned} u &= \mathrm{Im} \sum_{n=1}^{\infty} A_n x^s \sin t\varphi \\ v &= \mathrm{Im} \sum_{n=1}^{\infty} B_n x^s \cos t\varphi \\ w &= \mathrm{Im} \sum_{n=1}^{\infty} C_n x^s \sin t\varphi \end{aligned} \tag{3}$$

where C_n and t are real (t is the number of circumferential waves of the buckling deformations), s is the complex number

$$s = \gamma + in\beta \tag{4}$$

n is an integer and the symbol Im indicates the imaginary part of the solution.

The detailed analysis is carried out in reference 13, and only the salient features are given here. Substitution of the complex functions of equations (3) into the first two stability equations (eqs. (A1) and

(A2) of Appendix A) yields A_n and B_n in terms of C_n, and thence the spring constants representing the elastic restraints. Since these restraints arise from the non-compliance of the assumed solutions with the u and v boundary conditions, their effect may be expected to be of the same order of magnitude for isotropic and orthotropic conical shells, and this is confirmed by calculations for a typical shell (ref. 13). Hence the effect of the elastic restraints is very small and may be neglected (see refs. 14 and 8). The boundary conditions on w, equations (1) and (2), are enforced rigorously, and hence β and γ are determined as

$$\beta = \pi/\lg_e x_2 \tag{5}$$

and

$$\gamma = (1-\nu_{\varphi x})/2 \tag{6}$$

With the assumption that the membrane stresses represent the prebuckling stress state satisfactorily, the third stability equation, eq. (A3), is then solved by the Galerkin method, as in reference 8. The critical pressure is obtained from the resulting set of linear equations, which are for uniform hydrostatic pressure

$$\sum_{n=1}^{\infty} C_n \Big\{ [(-1)^{m+n} x_2^{2\gamma-2} - 1] G_1(n,m) + K^4(\alpha_2/\alpha_1)\cos^2\alpha [(-1)^{m+n} x_2^{2\gamma} - 1] G_2(n,m)$$

$$+ K^4[(-1)^{m+n} x_2^{2\gamma+1} - 1] G_3(n,m)(p/E_x)(a/h)\tan\alpha \Big\} = 0 \tag{7}$$

where the symbols G (n,m) denote values of the G functions (algebraic expressions given in ref. 13) for the particular n and m .

The critical pressures, for the case of uniform hydrostatic pressure loading, were computed for some typical orthotropic shells, and compared with those for similar isotropic shells (see table 1). The results confirm Hess's conclusions (ref. 2), about the desirability of $(E_x/E_\varphi) < 1$ and the general weight saving potential in the use of orthotropic material, also for conical shells.

The analysis can readily be applied to the case of external pressure varying in the axial direction. Since the orthotropy does not affect the load terms (the G_3 terms of equations (7) are identical to those of ref.8), one has only to replace them by the corresponding terms for axially varying external pressure derived for isotropic shells (ref. 8) and proceed as before.

General Instability of Stiffened Conical Shells Under External Pressure

The above analysis is now applied to the investigation of the general instability of stiffened conical shells under external pressure by consideration of an equivalent orthotropic shell. Though the method may be used for longitudinal stiffening (stringers) as well as for circumferential stiffening (rings or frames), the former is omitted on account of the marked inferiority of stringers as stiffeners against general instability under external pressure, and since the orthotropic approach would be limited only to stringers which increase in area, or number, in accordance with the cone diameter.

The ring-stiffened conical shell is correlated to an equivalent orthotropic one in the manner proposed by Bodner for cylindrical shells (ref. 5). Essentially, the equivalent orthotropic shell is an isotropic one with a larger effective thickness in the circumferential direction to account for the contribution of the rings to the circumferential extensional rigidity, and having also a larger bending rigidity in the circumferential direction due to the marked increase in the effective moment of inertia of the ring and shell combination. The increase in extensional rigidity is represented by the parameter

$$k = 1 + (A_2/a_o\ h) \tag{8}$$

where A_2 is the cross sectional area of the ring, and a_o its spacing; and the increase in bending rigidity is represented by a second parameter

$$f = I_\varphi/[a_o h^3/12\ (1-\nu^2)\ ] \tag{9}$$

where I_φ is the effective moment of inertia on the ring and shell combination,

$$I_\varphi = I_{22} + A_2(e_2 - \bar{z}_2)^2 + [a_o h^3/12(1-\nu^2)] + [a_o h\ \bar{z}_2^{\,2}/(1-\nu^2)] \tag{10}$$

where I_{22} is the moment of inertia of the ring cross section about its centroid and the other geometrical quantities are shown in fig. 1.

Once the equivalent orthotropic shell has been defined, the orthotropic theory can be applied (see ref. 13). It should be noted that Bodner's approximation of unity for k (ref. 5) is verified, with an error of much less than one percent, also by calculations for conical shells. In Table 1, the critical pressures for typical ring-stiffened conical shells are again compared with those for corresponding isotropic shells.

Correlation with Equivalent Cylindrical Shells

For isotropic conical shells under hydrostatic pressure, Seide (ref. 11) showed that the critical pressures can be correlated to those of equivalent cylindrical shells, bringing out the taper ratio, $\psi = 1 - (R_1/R_2)$, as the significant parameter of conicity. The equivalent cylindrical shell is taken on the basis of Niordson's results (ref. 15) as one having a length equal to the slant length of the cone, 1, a radius equal to its average radius of curvature, ρ_{av}, and the same thickness h, where

$$
\begin{aligned}
l &= a\,(x_2 - 1) \\
\rho_{av} &= [a(1 + x_2)\,\tan \alpha/2] = (R_1 + R_2)/2 \cos \alpha \qquad (11) \\
\psi &= 1 - (1/x_2) = 1 - (R_1/R_2)
\end{aligned}
$$

The correlation yielded an approximate curve for the ratio of the critical pressure of conical shells to that of their equivalent cylindrical shells versus the taper ratio

$$(p/\overline{p}) = g(\psi) \qquad (12)$$

(fig. 2 of ref. 11). A very similar curve was obtained in reference 8 for conventional simple supports (which differ slightly from Seide's boundary conditions) verifying the significance of the taper ratio as the main geometrical parameter of the conicity in the case of external pressure loading.

It is therefore reasonable to expect that also for the case of orthotropic and ring-stiffened conical shells under external pressure the taper ratio will be the significant parameter of conicity. The orthotropy and ring-stiffening will probably affect cylindrical and conical shells in the same manner and hence the ratio of $(p/\overline{p})$ should be very nearly the same as for isotropic shells.

In order to investigate this hypothesis, the critical pressures for the equivalent cylindrical shells were computed by Bodner's method (ref. 5) for all the typical conical shells given in Table 1, which include orthotropic shells of fiberglass reinforced epoxy and plywood, and ring stiffened shells of steel. The ratios $(p/\overline{p})$ are plotted in fig. 2 and compared with the curve $g(\psi)$ taken from reference 8, since the present analysis is for the same simple supports assumed there. The comparison in fig. 2 verifies the hypothesis and yields a very convenient approximate method for the determination of the critical uniform external pressure of any orthotropic, or ring-stiffened, conical shell.

The procedure involves: (a) calculation of the dimensions of the equivalent cylindrical shell, using equations (11); (b) computation of the critical pressure $\bar{p}$ for this equivalent shell by Bodner's method (ref. 5), and (c) reading the correct g from fig. 2. Finally p_{cr} for the conical shell is obtained from $p = \bar{p}\, g$.

Buckling Under Axial Compression and Under Combined Axial Load and External or Internal Pressure

In reference 12, the solution of reference 8, is applied to a linear analysis of the asymmetrical buckling of thin isotropic conical shells under uniform axial compression, after the effect of axial constraint has been shown to be small also for this type of loading (ref. 16). Calculations for a typical isotropic conical shell yielded a slightly lower buckling load than by the corresponding linear axisymmetrical analysis (ref. 17). The analysis is now extended to orthotropic shells, and can then be applied directly to stiffened shells by consideration of an equivalent orthotropic shell.

The final set of linear equations of the stability analysis eqs. (7) have separate load terms which are not affected by the orthotropy and are hence identical to those for isotropic shells. Provided the same form of buckling displacement is possible, only this load term has to be changed, if instability under a different type of loading has to be investigated. For uniform axial compression the third term of equations (7) is therefore replaced, as for isotropic conical shells, by

$$K^4[(-1)^{m+n} x_2^{2\gamma-1} - 1]G_4(n,m)\,(P/E_x)\,(1/\pi ah \sin 2\alpha) \qquad (13)$$

where $G_4(n,m)$ is an algebraic expression given in reference 12.

It should be noted that though the same form of deflection functions is assumed as solutions in case of buckling under external pressure and under uniform axial compression, the calculations differ slightly, since the basic buckling mode has, instead of $n = 1$, a number of axial waves of the same order as t (see ref. 12).

As the analysis is linear, combinations of load terms may be added, subject to the above mentioned proviso of admissibility of displacement functions. Hence for combined axial compression and external or internal pressure the final simultaneous equations would be obtained directly by adding expression (13) to equation (7), changing the sign of p in the case of internal pressure or that of P in the case of a tensile axial load.

Insofar as linear theory can represent actual buckling shapes, it may be expected from similar analyses for isotropic conical and cylindrical shells, that asymmetrical modes will predominate for combinations of external pressure and axial compression, or tension, whereas in the presence of internal pressure symmetrical modes will appear (ref. 18).

MORE ACCURATE ANALYSIS FOR STIFFENED CONICAL SHELLS

Stress-Strain Relations

The instability of ring- or stringer- stiffened conical shells may be analysed more accurately by consideration of the separate distributed stiffness of the rings and stringers. The circumferential or longitudinal stiffeners are assumed each to be distributed evenly along one spacing (one half spacing each side), the middle surface of the shell being chosen as reference line. This approach is valid for closely spaced stiffeners which need not be necessarily evenly spaced and equal, and permits detection of differences in the critical load caused by their eccentricity.

The theory sets out with the formulation of stress-strain relations of the shell together with stiffeners. The stiffeners may have different elastic properties, and the strains are assumed to be identical at the contact surface of stiffeners and shell.

For the shell the stress and strain relations are

$$\sigma_x(z) = [E/(1-\nu^2)]\ [\epsilon_x + \nu\epsilon_\varphi - (z/a)(\kappa_x + \nu\kappa_\varphi)\ ] \tag{14}$$

$$\sigma_\varphi(z) = [E/(1-\nu^2)]\ [\epsilon_\varphi + \nu\epsilon_x - (z/a)\ (\kappa_\varphi + \nu\kappa_x)\ ]$$

while for the stiffeners they are

$$\begin{aligned} \sigma_x(z) &= E_1\ [\epsilon_x - (z/a)\ \kappa_x\ ] \\ \sigma_\varphi(z) &= E_2\ [\epsilon_\varphi - (z/a)\kappa_\varphi\ ] \end{aligned} \tag{15}$$

Hence the forces and moments acting on an element become

$$\begin{aligned} N_x &= [Eh/(1-\nu^2)]\ [\epsilon_x(1+\mu_1) + \nu\epsilon_\varphi - \chi_1\ \kappa_x\ ] \\ N_\varphi &= [Eh/(1-\nu^2)]\ [\epsilon_\varphi(1+\mu_2) + \nu\epsilon_x - \chi_2\ \kappa_\varphi\ ] \\ N_{x\varphi} &= N_{\varphi x} = [Eh/2\ (1+\nu)]\gamma_{x\varphi} \end{aligned} \tag{16}$$

and

$$\begin{aligned} M_x &= -\ (D/a)\ [\kappa_x(1+\eta_{o1}) + \nu\kappa_\varphi - \zeta_1\ \epsilon_x] \\ M_\varphi &= -\ (D/a)\ [\kappa_\varphi(1+\eta_{o2}) + \nu\kappa_x - \zeta_2\ \epsilon_\varphi] \\ M_{x\varphi} &= \quad (D/a)\ [(1-\nu) + \eta_{t1}\ ]\kappa_{x\varphi} \\ M_{\varphi x} &= -\ (D/a)\ [(1-\nu) + \eta_{t2}\ ]\kappa_{x\varphi} \end{aligned} \tag{17}$$

where μ_1 and μ_2 are the increases in effective cross sectional area of the shell due to stringers and rings respectively, χ_1 and χ_2 are the changes in extensional stiffness caused by the eccentricities of stringers and rings, η_{o1}, η_{o2}, η_{t1}, η_{t2} are the increases in bending and twisting stiffness of the shell due to stringers and rings, and ζ_1 and ζ_2 are the changes in bending stiffness caused by the eccentricities of stringers and rings.

With the aid of these relations the stability equations are obtained in terms of displacements. These equations are similar to those for isotropic shells, before uncoupling, though more complicated.

Buckling Under External Pressure

General instability under external pressure is analysed with the aid of the same solution employed for isotropic and orthotropic conical shells, equations (3). However, since the more accurate stiffened shell equations are not amenable to the direct solution, coupled with the Galerkin method for the third equation, employed for isotropic and orthotropic shells, a modified solution using successive correction factors and a variational approach, that is basically an extension of the Galerkin method, has been applied by M. Baruch in his unpublished doctoral dissertation under the guidance of the author. Since the successive correction factors converge rapidly, the method is not too laborious. The extended Galerkin approach also permits direct estimates and correction of the error involved in the partial compliance only with the boundary conditions for u and v. When the stiffeners have no eccentricity, or the eccentricity is neglected, the method reduces to the orthotropic analysis discussed above.

The Effect of Eccentricity of Stiffeners

By an analysis similar to that outlined above, Baruch and the author investigated the effect of eccentricity of stiffeners on the general instability of stiffened cylindrical shells under hydrostatic pressure. For typical shells with rings on the inside, the critical pressures were found to be 11.5 percent to 13.5 percent above those obtained with the identical rings on the outside. For typical conical shells similar magnitudes are obtained. For example, for two ring stiffened conical shells of the following properties

$$\alpha = 30^\circ \qquad a = 57.59 \text{ in.} \qquad (A_2/a_o h) = 0.1471 \qquad (e_2/h) = 1.653$$

$$\nu = 0.3 \qquad h = 0.1 \text{ in} \qquad [I_{22}/(a_o h^3/12)] = 0.7819 \qquad (\bar{z}_2/h) = 0.2119$$

one obtains

x_2	Taper Ratio $\psi=1-(1/x_2)$	(ρ_{av}/h)	$p/p_{\text{no eccentricity}}$ inside rings	outside rings	$\left(\frac{p_{\text{inside rings}}}{p_{\text{outside rings}}}\right)$
1.5	0.333	415	1.021	0.963	1.061
5.0	0.800	995	1.058	0.965	1.093

It may be concluded that from a buckling point of view the placing of stiffening rings (or frames) on the inside of the shell is advantageous. Further it should be noted that the effect of eccentricity of stiffening rings is of sufficient magnitude to require care in the interpretation of experimental results.

For stringers the effect of eccentricity is smaller and opposite: outside stringers yield higher instability pressures than inside ones, as has also been shown for cylindrical shells.

EXPERIMENTAL INVESTIGATION

The first phase of an experimental program on the instability of orthotropic and stiffened conical shells initiated at the Technion, is concerned with the general instability of ring stiffened shells under hydrostatic pressure. The test rig (fig.3) is similar to that used in previous investigations of isotropic shells (ref. 19), except that it is smaller and designed for higher pressures. The specimens (see fig.4a) are of mild steel, have rings on the outside, and are fabricated by careful machining (a tolerance of ± 0.001 in. was obtained on thickness of shell and rings). The specimens are clamped at the edges. As in reference 20, strain gages are installed around the circumference and opposite a stiffening ring near the estimated position of maximum buckling deflection. On the first test cone, an additional row of strain gages was installed opposite the centre of the bay between two stiffening rings, to detect early panel instability. No panel instability appeared before failure by general instability, and the fact that the strain values of the two rows did not differ appreciably during the whole test, seems to justify the assumption of effectiveness of complete bay length, a_o, implied in the theoretical analyses.

Three conical shells of similar geometry (R_1= 1.77 in., R_2= 5.67 in., and $\alpha = 20^o$) were tested. All failed by general instability. The strain gage readings indicated clearly the embryonic lobe formation discussed in references 19 and 20, and the cones buckled by a sudden formation of one large general instability wave in place of one of those embryonic lobes (see fig.4b). Since the buckling stress was not far from the yield stress of the material, the elastic buckling transformed immediately into plastic deformation. Attempts to raise the pressure again resulted in growth of the wave (and appearance of one further wave in two tests) with additional small plastic panel buckles.

Theoretical buckling pressures which would appear for perfect cones were determined from strain gage readings of the 3 tests by the extension of Southwell's method given in reference 21. The intercept and the slope methods, suggested there, yielded nearly similar perfect cone buckling pressures, which were about 8 percent higher than the observed ones, as follows:

Specimen no.	1	2	3
Theoretical buckling pressure (psi)	106.1	96.2	95.8
Observed failing pressure (psi)	99.2	94.0	84.6
Perfect cone buckling pressure (psi)	106	103	92

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The ratios of observed failing pressure and perfect cone buckling pressure to the equivalent cylinder buckling pressure ($\bar{p}$) are also plotted in figure 2. These preliminary results are in good agreement with the theory. It should be noted, however, that the test specimens were clamped while the theory is for simple supports. However, the effect of clamped edges should be rather small for ring stiffened shells due to their relatively large circumferential stiffness (for cylindrical shells it is usually neglected - see ref. 20).

FUTURE RESEARCH

Both the orthotropic theory and the more accurate theory for stiffened shells should be extended to torsion and combined loading and verified experimentally. Further experimental investigation of the buckling under external pressure of orthotropic and ring-stiffened conical and cylindrical shells is required. The tests should also aim at verifying the effect of eccentricity of stiffeners postulated by the theory. The effect of clamped edges should also be further clarified. Extensive tests of cylindrical and conical shells with closely spaced stiffeners under axial compression are needed to confirm the remarkable agreement with linear theory pointed out by Becker and Gerard (ref. 1) in the case of a recent test by Pugliese. Research along these lines is planned at the Technion.

APPENDIX A

STABILITY EQUATIONS FOR ORTHOTROPIC CONICAL SHELLS

For orthotropic conical shells the uncoupled Donnell type stability equations of reference 9 can be written in non-dimensional form (for zero surface forces) as

$$L_{10}(u) = \cot\alpha\left\{(L_5 + L_9)[x\,\nu_{\varphi x}\,w_{,x} - (\alpha_2/\alpha_1)w] - (1/\sin\alpha)(\alpha_2/\alpha_1)(\alpha_2/\alpha_3)L_7(w_{,\varphi})\right\} \tag{A1}$$

$$L_{10}(v) = (\alpha_2/\alpha_3)\cot\alpha\left\{-L_8[x\,\nu_{\varphi x}w_{,x} - (\alpha_2/\alpha_1)w] + (1/\sin\alpha)[L_5 + (\alpha_2/\alpha_1)L_6](w_{,\varphi})\right\} \tag{A2}$$

and

$$(1/x)L_1(w) - K^4\Big\{(\bar{\sigma}_x/E_x)w_{,xx} + (\bar{\sigma}_\varphi/E_x)[(w_{,\varphi\varphi}/x^2\sin^2\alpha) + (1/x)w_{,x}]$$
$$+ 2(\bar{\tau}_{x\varphi}/E_x)[(w_{,x\varphi}/x\sin\alpha) - (w_{,\varphi}/x^2\sin\alpha)]\Big\}$$
$$+ (1/x^3)(\alpha_2/\alpha_1)K^4\cot^2\alpha\, L_4^{-1}[(1/x)(x^3 w_{,xx})_{,xx}] = 0 \qquad (A3)$$

where the operators L_1 to L_{10} are defined in reference 9 or 13, and the inverse operator L_4^{-1} is defined by

$$L_4[L_4^{-1}(z)] = z \qquad (A4)$$

It should be noted that the radial stability equation, eq. (A3), is not the usual eighth order equation, which is a higher derivative of the radial equilibrium equation, but a Batdorf type modified equation which is more convenient for solution by the Galerkin method, since it ensures monotonic decrease of the approximate buckling loads with increase in number of terms of approximation.

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TABLE 1

CRITICAL PRESSURES OF TYPICAL ORTHOTROPIC AND RING-STIFFENED CONICAL SHELLS

(a) Orthotropic Shells (Cone angle 30°)

No	Material	Taper Ratio $1-(1/x_2)$	$\left(\frac{\rho_{av}}{h}\right)$	$\frac{E_x}{E_\varphi}$	$\nu_{\varphi x}$	$(p/E_x)\times 10^6$		
						Orthotropic Shell	Equivalent Cylinder	Isotropic Shell
1	Plywood	0.333	415	20.0	0.022	0.0409	0.0402	0.4050
2	Fibreglass Reinforced Epoxy(143)	0.333	415	2.59	0.090	0.2041	0.2003	0.4050
3		0.500	499	0.386	0.234	0.3048	0.2841	0.1538
4		0.600	582	2.59	0.090	0.0418	0.0373	0.5340
5		0.800	995	0.386	0.234	0.0307	0.0246	0.0159

(b) Ring Stiffened Shells (Material: steel; Cone angle: Nos. 6,7 - 30° No.8 - 20°)

No	Taper Ratio $1-(1/x_2)$	$\left(\frac{\rho_{av}}{h}\right)$	$\left(\frac{A_2}{a_o h}\right)$	$\left(\frac{12\, I_{22}}{a_o h^3}\right)$	$\frac{e_2}{h}$	$(p/E)\times 10^6$		
						Stiffened Shell	Equivalent Cylinder	Isotropic Shell
6	0.333	415	0.1471	0.7819	1.653	1.263	1.234	0.4050
7	0.800	995	0.1471	0.7819	1.653	0.0557	0.0450	0.0159
8	0.678	98.4	0.152	0.590	1.325	3.54	3.04	1.297

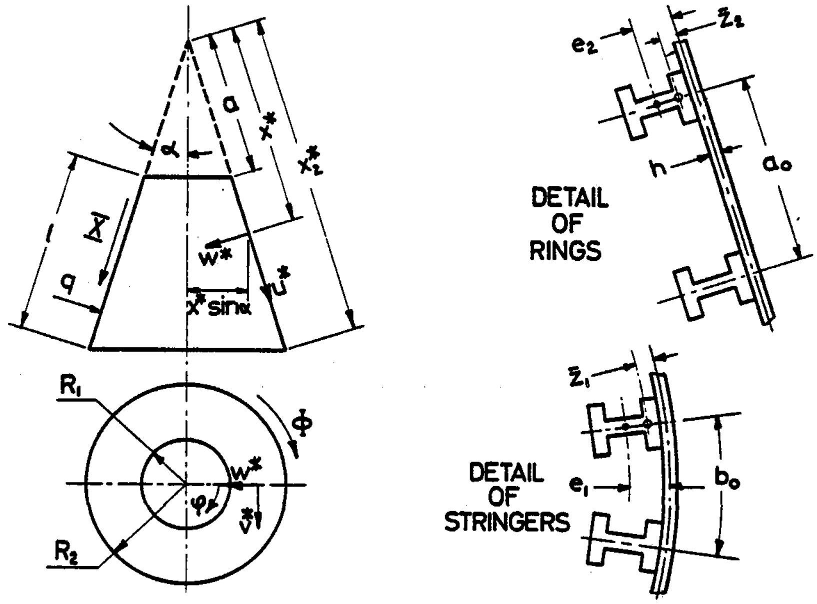

Figure 1.- Notation.

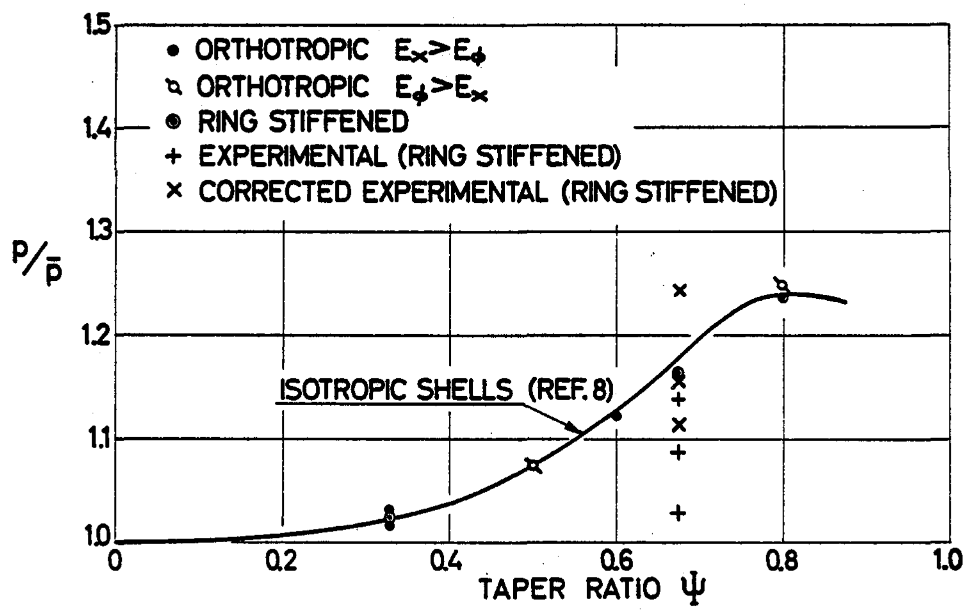

Figure 2.- Ratios of the buckling pressure of orthotropic and ring stiffened conical shells to that of equivalent cylindrical shells.

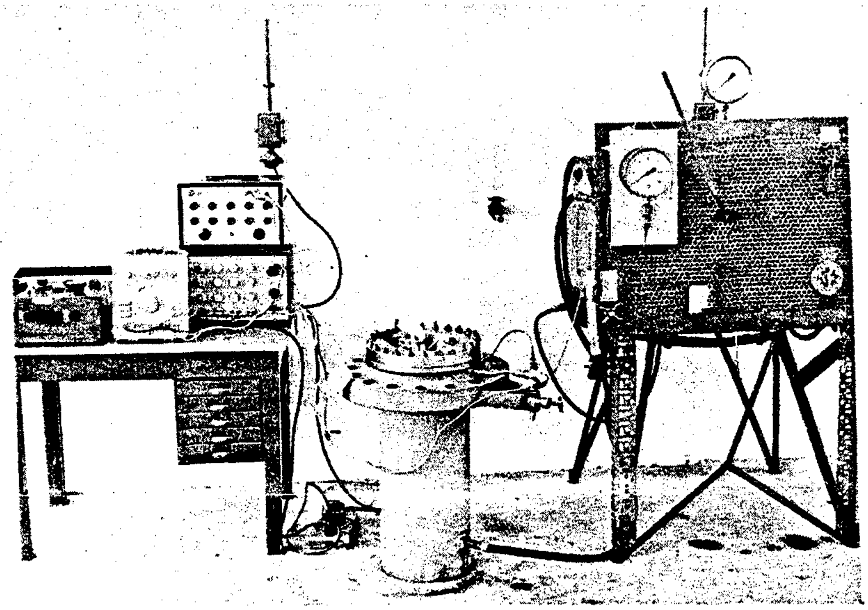

Figure 3.- Experimental set-up.

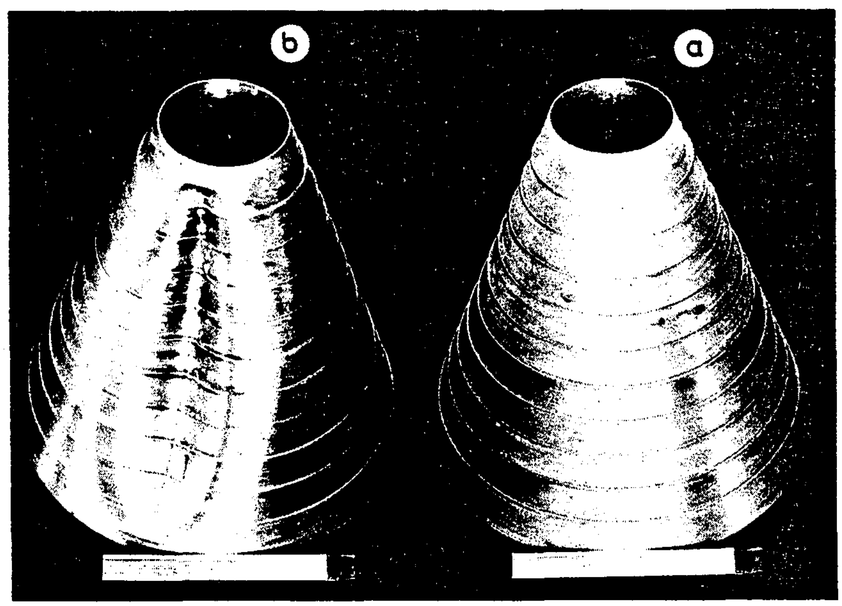

Figure 4.- Test specimen: (a) prior to test, (b) after failure by general instability.