967

ASYMMETRIC BUCKLING OF CLAMPED SHALLOW SPHERICAL SHELLS*

By Hubertus J. Weinitschke

Hughes Aircraft Company, Ground Systems Group

SUMMARY

The problem of buckling of clamped shallow spherical shells has recently been considered in several theoretical investigations. Buckling loads under uniform external pressure were obtained in these investigations which show a surprisingly good agreement with each other, but show a marked disagreement with available experimental values. In all previous studies it has been assumed that the shell deformations are rotationally symmetric. In this paper, the buckling problem is re-examined by introducing asymmetric modes of deformation. The approach is to superimpose small asymmetric deflections on finite axisymmetric deflections, and to show that the symmetric states of deformation are unstable over certain ranges of load and geometry parameter. Numerical results are obtained by means of a digital computer and are compared with previous theoretical and experimental results.

INTRODUCTION

Problems of elastic stability of thin shells that require a large deflection analysis have been of considerable interest lately. In some well-known examples, approximate theoretical results are in reasonably good agreement with available experimental information; but the author does not know of a single problem that has yielded to a mathematically satisfactory and accurate solution, which at the same time is in good agreement with experiments. This unsatisfactory situation holds true even for the problem of buckling under uniform pressure of a shallow spherical shell, which is in a sense the simplest of shell stability problems and which is the subject of this paper. A reason for this situation may be found in the fact that the majority of theoretical large deflection analyses of stability problems have been carried out by means of Rayleigh-Ritz type approximations, whose accuracy is in

*This work was carried out at the Massachusetts Institute of Technology, supported by the Office of Naval Research. The author is indebted to Prof. E. Reissner whose help and encouragement was of great value to him. The calculations were carried out at M. I. T. Computation Center.

general difficult to assess in nonlinear problems. Frequently, shells exhibit deflection patterns that are difficult to represent in terms of such approximations. (For instance, using a one- or two-term Ritz approximate solution of the axisymmetric buckling of a spherical cap, one finds increasingly incorrect buckling loads with increasing shell heights.) It seems that at the present state of knowledge, recourse to numerical methods of well-defined accuracy should be considered as a powerful (if temporary) alternate approach in theoretical studies of shell instability. In view of the complexity of large deflection equations, such an approach may involve extensive calculations on a high speed digital computer; however, a more detailed understanding of the buckling mechanism which one may gain from such calculations may render this approach worth-while. The problem treated in the present paper is believed to give some support to this point of view.

The buckling under uniform pressure of a shallow spherical shell, clamped along its boundary, has generally been considered to be of the snapping type. On the basis of a theory of finite axisymmetric deflections, one obtains a nonlinear load deflection curve which shows a local maximum of the pressure, except for extremely shallow shells which do not buckle. This maximum pressure p_c determines the critical load at which snapping occurs, provided that the classical buckling criterion is assumed to be valid. What is of particular interest is the stability curve that shows how p_c varies with the shell geometry parameter μ defined below. Several recent investigations have been concerned with the calculation of this stability curve (refs. 1 to 4). The results of these studies, which were obtained by entirely different techniques, show good agreement with each other. They were all based on a system of nonlinear differential equations for finite axisymmetric deformations. However, experimental results are generally in serious disagreement with the results of the axisymmetrical buckling theory. For some time, it has been believed that these discrepancies might be due to imperfections in geometrical shape. Recent results of Budiansky (see ref. 1) for certain types of imperfections of reasonable magnitudes tend to discourage such speculations.

The present analysis is based on the assumption that asymmetrical deflection modes are significant in the process of buckling. The buckling of spherical caps appears then to be a bifurcation rather than a snapping phenomenon: axisymmetric deformation takes place until a critical value is reached, at which point, bifurcation of solutions of the basic equations occurs. One branch of solutions corresponds to axisymmetric states of equilibrium, other branches correspond to asymmetric states, which in the vicinity of the bifurcation point differ from the axisymmetric states by infinitesimal amounts. The axisymmetric states are therefore unstable for pressures above the critical value, hereafter referred to as the asymmetrical buckling load. The main result of this paper is a new stability curve, based on asymmetric deflections of the form $w = w(r)\cos n\theta$. Two different techniques are employed; one amounts to calculation of the second variation of an appropriate potential energy functional, the other reduces the stability problem to a nonlinear eigenvalue problem.

BASIC EOUATIONS

The basic equations for finite bending of shallow shells have been derived by Marguerre (ref. 5) under the traditional assumptions of thin shell theory, that is, neglection of transverse shear deformability and of tangential displacement components u, v in the nonlinear terms. These equations, when written in polar coordinates and specified to a spherical cap, can be put in the form

$$\begin{aligned} A^{-1}\nabla^4F + 2H\nabla^2 w + \tfrac{1}{2}K[w,w] &= 0 \\ D\nabla^4 w - 2H\nabla^2 F - K[F,w] &= pb^4 \end{aligned} \qquad (1)$$

where b is the base radius of the shell, H is the shell " rise", p is the external pressure, A and D are stretching and bending stiffness factors respectively and ∇^2 is the Laplace operator in the polar coordinates r and θ. Stress resultants and couples are related to the stress function F and the axial displacement w by the formulas

$$N_r = r^{-2}F_{,\theta\theta} + r^{-1}F_{,r} \qquad N_\theta = F_{,rr} \qquad N_{r\theta} = r^{-2}F_{,\theta} - r^{-1}F_{,r\theta}$$

$$Q_r = -D(\nabla^2 w)_{,r} \qquad Q_\theta = -Dr^{-1}(\nabla^2 w)_{,\theta}$$

$$M_r = -D(w_{,rr} + \nu Tw) \qquad M_\theta = -D(Tw + \nu w_{,rr})$$

where $Tw = r^{-1}w_{,r} + r^{-2}w_{,\theta\theta}$, $D_\theta = D(1-\nu)$ and ν is Poisson's ratio. A comma followed by subscripts indicates differentiation with respect to the subscripted variable(s). The nonlinear terms in Eqs. (1) are expressed in terms of the differential operator K as follows

$$K[F,w] = \rho^{-1} F_{,\rho\rho}(w_{,\rho} + \rho^{-1}w_{,\theta\theta}) + w_{,\rho\rho}(F_{,\rho} + \rho^{-1}F_{,\theta\theta}) + 2\rho^{-2}\cdot$$
$$\left[\rho^{-1}(F_{,\rho\theta}w_{,\theta} + F_{,\theta}w_{,\rho\theta}) - F_{,\rho\theta}w_{,\rho\theta} - \rho^{-2}F_{,\theta}w_{\theta}\right]$$

where $\rho = rb^{-1}$.

The boundary conditions corresponding to a clamped edge are

$$u = v = w = w_{,r} = 0 \quad \text{at} \quad r = b(\rho = 1). \qquad (3)$$

In order to express the first two conditions in terms of F and w, consider the stress-strain relations

$$A_1F \equiv N_r - \nu N_\theta = A(u,_r + rw,_r R^{-1} + \frac{1}{2}w,_r^2)$$

$$A_2F = N_\theta - \nu N_r = Ar^{-1}(u + v,_\theta + \frac{1}{2}r^{-1}w_\theta^2)$$

$$A_3F \equiv 2(1+\nu)N_{r\theta} = A[v,_r - r^{-1}(v - u,_\theta) + R^{-1}w,_\theta + r^{-1}w,_r w,_\theta]$$

Expressing N_r, N_θ, and $N_{r\theta}$ in terms of F as indicated by the differential operators A_1, A_2, A_3, it can be verified that conditions (3) are equivalent to

$$A_2F = (rA_2F),_r - A_1F - (A_3F),_\theta = w = w,_r = 0 \quad \text{at } r = b. \tag{4}$$

The boundary value problem (1) and (4) constitutes the basis for the present analysis.

A convenient dimensionless form of Eqs. (1) is obtained by introducing the functions $g(\rho, \theta) = D^{-1}F$ and $h(\rho, \theta) = mt^{-1}w$, and the relevant geometry and load parameters $\mu^2 = 2mHt^{-1}$ and $\gamma = pb^4m(4Dt)^{-1}$ (t = shell thickness, $m^2 = 12(1 - \nu^2)$). With this, we have the following equations for g and h

$$\nabla^4 g + \mu^2 \nabla^2 h + \frac{1}{2}K[h, h] = 0, \qquad \nabla^4 h - \mu^2 g - K[g, h] = 4\gamma \tag{5}$$

For convenience of describing the buckling process, a dimensionless deformed volume is introduced by $V = \iint h(\rho, \theta)\rho d\rho\, d\theta$, and a load parameter by $P = \gamma\mu^{-4}$.

RESULTS OF PREVIOUS WORK

In all previous theoretical studies, the assumption was made that deformations are rotationally symmetrical (except in ref. 6). In that case, Eqs. (5) can be simplified considerably. With $p = g'(\rho)$, $q = h'(\rho)$, where the primes denote differentiation with respect to ρ, it can be shown that Eqs. (5) reduce to the following ordinary differential equations:

$$(p' + \rho^{-1}p)' = \mu^2 q - \frac{1}{2}\rho^{-1}q^2, \qquad (q' + \rho^{-1}q)' = \mu^2 p + 2\gamma\rho + \rho^{-1}pq. \tag{6}$$

The corresponding boundary conditions are

$$p(0) = q(0) = 0, \qquad p(1) = q'(1) - \nu q(1) = 0. \tag{7}$$

A great deal of effort has been devoted to solving Eqs. (6) and (7) and calculating the resulting stability curve $P_c(\mu)$. If the curve $P = P(V)$ is plotted for a fixed shell geometry, the variation of P with increasing V is roughly as follows: P increases until it reaches a local maximum P_c, then decreases to a local minimum P_ℓ (unstable states of equilibrium), and then increases again (stable post-buckling states of equilibrium).

In some recent papers, buckling criteria have been discussed that are based on a finite-jump buckling mechanism according to which the minimum load P_ℓ is to be considered as the actual collapse load of the shell. The shortcomings of these criteria, which have no logical basis, have been pointed out, e.g., see the comprehensive review article on shell instability by Fung and Sechler (ref. 7). It has further been shown (ref. 8) that for simply supported spherical caps under uniform pressure, P_ℓ is negative for certain shell geometries. Similar difficulties are encountered in the application of so-called energy buckling criteria to the spherical cap problem (see ref. 7). In the following, the term " buckling load" is to be understood in the classical sense.

The results of the earlier studies of Eqs. (6) and (7) which were obtained by Ritz-type approximations, perturbation techniques, and power series methods are generally in disagreement with each other (see refs. 1 and 2 for a more complete discussion and additional references). The power series approach proved promising; however, convergence difficulties for larger values of μ^2 made it impossible to obtain solutions for values of γ up to the critical load $\gamma_c = \mu^4 p_c$.

The results of the more recent investigations are based on iterative numerical solution of the above differential equations. A number of entirely different techniques have been used successfully to overcome the difficulties in the numerical solution, which are related to the increasing waviness of the normal deflection w with increasing shell parameter μ^2. A modified power series approach, expanding the solutions of Eqs. (6) in both powers of ρ and powers of $1 - \rho$ was employed by Weinitschke (ref. 2). In the work of Budiansky (ref. 1), the problem was formulated in terms of two nonlinear integral equations which were solved by means of matrix approximations. A different integral equation formulation was employed by Thurston (ref. 3), and a finite difference solution of Eqs. (6) and (7) was given by Archer (ref. 4). The stability curves based on the results of refs. 1 and 4 are in almost perfect agreement with each other (see curve S in Figure 1). However, except for the range $\mu \leq 5.5$, the experimental values shown in Figure 1 are in serious disagreement with the axisymmetric buckling theory.

It is interesting to note that the range $\mu \leq 5.5$ corresponds to the simple deflection mode: $w(\rho)$ has its maximum at the apex ($\rho = 0$) and is monotonically decreasing towards the edge. For larger shell rises, $\mu \geq 5.5$, the shell deforms according to axisymmetric theory into a deflection pattern of a higher degree of waviness before it snaps through at large values of P. However, in view of the results discussed below, the more wavy axisymmetric states become unstable in the asymmetric buckling theory.

THE BIFURCATION PROBLEM

The results discussed in the previous section lead to the conclusion that the low experimental values of P_c cannot be explained on the basis of axisymmetric deformations. The strong increase of the ratio $|N_\theta/N_r|$ for increasing P and for $\mu \geq 5.5$ observed in ref. 2 indicates the possibility of wrinkling in the circumferential direction of the cap. Furthermore, there is strong experimental evidence (see ref. 1) for asymmetrical buckling. Recently, buckling loads have been calculated by Gjelsvik and Bodner (ref. 6) using a Rayleigh-Ritz procedure, where $w(r, \theta)$ is assumed to be nonsymmetrical involving one free parameter. The resulting numerical values of Pc are larger than the values of axisymmetric theory (curve S in Figure 1); therefore, it is not evident from their work whether asymmetric deflection theory is able to explain the low experimental values.

A new approach to the buckling problem will now be outlined which takes into account asymmetrical deflection modes of the type $W(r)\cos n\theta$. It is assumed that axisymmetric deformation takes place until a critical value P^* is reached at which point bifurcation of solutions of the basic equations (6) occurs. The branch of axisymmetric solutions becomes unstable for pressure parameters $P \geq P^*$, and the shell deforms in an unsymmetrical mode. In other words, the smallest load for which bifurcation occurs is considered as the limit of stability, in accordance with general principles of elastic stability theory. However, the possibility must be admitted that the loss of stability represented by points of bifurcation on the axisymmetric load deflection curve $P = P(V)$ may be rather localized. Further investigation of the branches of asymmetrical states of equilibrium is necessary in order to determine the complete behavior of the shell.

In order to obtain the stability curve on the basis of this approach, small asymmetrical deflections $w(r, \theta)$ must be superimposed on finite axisymmetrical deflections. Since the latter are known, the equations for $w(r, \theta)$ and for the stresses are linear. Two ways of formulating the present approach analytically are described below which are at the same time suitable for obtaining numerical results.

VARIATIONAL METHOD

The potential energy of a shallow shell subject to uniform pressure can be written as follows

$$E_1[\epsilon_{ik}, w] = \tfrac{1}{2}A(1-\nu^2)^{-1}\iint[(\epsilon_{xx}+\epsilon_{yy})^2 - 2(1-\nu)(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}^2)]\,dxdy + \tfrac{1}{2}D\iint[(\nabla^2 w)^2 - 2(1-\nu)(w,_{xx}w,_{yy} - w,_{xy}^2)]\,dxdy + \iint pw\,dxdy \quad (8)$$

where the quantities ϵ_{ik} denote the midsurface strains of the shell. Eqs. (1) and (4) are now equivalent to the variational problem: minimize E_1 with respect to all (smooth) functions u, v, w satisfying $u = v = w = w' = 0$ along the edge $r = b$. The above integrals are to be taken over the projection of the shell midsurface on the x, y-plane. Using the stress strain relations as constraints, one can transform this variational problem into an equivalent one where u and v are eliminated. This leads to a new functional $E_2[w, F]$, where E_2 is to be minimized with respect to all functions w satisfying $w = w' = 0$. Here the function F is given in terms of w by the first of Eqs. (1) subject to the boundary conditions (4).

The details of this transformation and of the subsequent solution of the stability problem are toolengthy to be reproduced here and will be given in a future paper. Briefly, the procedure is as follows. Let $F_0(r)$, $W_0(r)$ refer to a given axisymmetric solution and define $\bar{f}$, $\bar{w}$ by $F = F_0 + \bar{f}$, $w = W_0 + \bar{w}$. In order to show that the solutions F_0, W_0 become unstable for values of the pressure exceeding a certain limit, one has to calculate the second variation $\delta^2 E_2[w, F]$ of the functional E_2 and show that it can be made negative for suitably chosen functions $w(r, \theta)$. The application of this method is thus reduced to numerical evaluation of certain double integrals and to solving a linear biharmonic equation to find the $\bar{f}$ corresponding to a given $\bar{w}$.

DIFFERENTIAL EQUATIONS METHOD

Assume the dimensionless stress g and displacement h in the form

$$g = g_0(\rho) + \epsilon g_1(\rho, \theta) + O(\epsilon^2), \qquad h = h_0(\rho) + \epsilon h_1(\rho, \theta) + O(\epsilon^2) \tag{9}$$

where g_0, h_0 denote the axisymmetric solution, ϵ is a small parameter and $O(\epsilon^2)$ stands for terms of order ϵ^2. Substituting (9) into (5), noting that g_0, h_0 satisfy Eqs. (6), and collecting terms of order ϵ, one obtains the desired equations for small asymmetric deflections. These deflections, represented by g_1, h_1 are

$$\begin{aligned} \nabla^4 g_1 + \mu^2 \nabla^2 h_1 + K[h_0, h_1] &= 0 \\ \nabla^4 h_1 - \mu^2 \nabla^2 g_1 - K[g_0, h_1] - K[h_0, g_1] &= 0 \end{aligned} \tag{10}$$

A similar process leads to appropriate boundary conditions for the functions g_1, h_1. These conditions together with Eqs. (10) constitute the basic 8th order eigenvalue problem to be solved. Although Eqs. (10) are linear in g_1, h_1, the problem is nonlinear insofar as the eigenvalue γ enters via the functions g_0, h_0, which depend on γ through the nonlinear equations (6).

In view of the periodicity in θ, we set $g_1 = G(\rho)\cos n\theta$, $h_1 = H(\rho)\cos n\theta$.

This leads to the following equations for G and H

$$LLG + \mu^2 LH + M[h_o, H] = 0$$
$$LLH - \mu^2 LG - M[g_o, H] - M[h_o, G] = 0 \qquad (11)$$

where

$$L = (\ldots)'' + \rho^{-1}(\ldots)' - n^2 \rho^{-2}(\ldots)$$
$$\rho M[X, Y] = X''(Y' - n^2\rho^{-1}Y) + X'Y''; \quad X = X(\rho), \; Y = Y(\rho)$$

L
3
1
0
9

The boundary conditions for G, H can be written in the form

$$\left.\begin{aligned} &H = H' = G'' - \nu(G' - n^2 G) = 0 \\ &G''' + G'' - (1 + 2n^2 + \nu n^2)G' + (3 + \nu)n^2 G = 0 \end{aligned}\right\} \text{at } \rho = 1 \qquad (12)$$

In addition, G, H, G', H' must be regular at $\rho = 0$.

As mentioned above, the solution of the axisymmetric problem has been obtained in ref. 2 in terms of power series in ρ and $1 - \rho$; therefore, it is indicated to calculate the solutions G, H also in this form, that is,

$$G(\rho) = \rho^{\lambda} \sum_{k=0}^{\infty} G_k \rho^{2k}, \qquad H(\rho) = \rho^{\lambda} \sum_{k=0}^{\infty} H_k \rho^{2k} \qquad (13)$$

and similar expansions with respect to $1 - \rho$. Substitution of (13) into (11) leads to the fourth degree indicial equation for the roots λ and to recurrence relations for the coefficients G_k, H_k from which four regular linear independent solutions can be constructed. Satisfaction of the boundary conditions (12) by an appropriate linear combination of these solutions leads to the condition of vanishing of a certain determinant $I(\mu, n; P)$, where $P = \gamma\mu^{-4}$, which determines a smallest eigenvalue $P^*(n)$ for each n. For a fixed parameter μ, the asymmetric buckling load parameter P_c is therefore determined by the smallest of the $P^*(n)$, that is, $P_c(\mu) = \underset{n}{\mathrm{Min}}\, P^*(n)$.

RESULTS AND DISCUSSION

In the application of the procedure outlined in the preceding section it is important to obtain sufficiently accurate solutions of Eqs. (11) because of the loss of some accuracy in the evaluation of the determinant $I(\mu, n, P)$. The roots of $I = 0$ were found by plotting I versus P keeping μ and n fixed. The use of a digital computer (IBM-7090) was essential in calculating I for sufficiently large ranges of the three parameters μ, n, and P.

The results of these calculations are shown in Fig. 1. The critical loads corresponding to a fixed circumferential wave number n are plotted versus μ, from which the scalloped asymmetric stability curve $P_c(\mu)$ (labeled A in Fig. 1) is obtained as the lower envelope. It is seen that axisymmetric snap-buckling prevails only over the narrow range $3.4 \leq \mu \leq 4$. For $\mu > 4$, there is bifurcation-buckling caused by asymmetric deflection modes showing an increasing number of circumferential waves with increasing μ. Although no attempt has been made to extend the stability curve beyond $\mu = 10$, it is significant that the calculated buckling loads are in reasonably good agreement with the observed data, thus essentially closing the large gap between theory and experiment that has hitherto existed.

In conclusion, it must be admitted that although a bifurcation phenomenon seems to determine the onset of buckling, the process of buckling at large may well appear as a snap-through phenomenon, that is, the asymmetric modes may become unstable with increasing deformation so that the final (post-buckled) state is again axisymmetrical. A theoretical confirmation would involve finite asymmetric post-buckling deflections and would be of great value for a more detailed understanding of the buckling process.

REFERENCES

1. Budiansky, B.: Buckling of Clamped Shallow Spherical Shells. IUTAM Symposium on the Theory of Thin Elastic Shells (Delft), 1959, pp. 64-94.

2. Weinitschke, H. J. : On the Stability Problem of Shallow Spherical Shells, Jour. Math. Phys., vol. 38, 1960, pp. 209-231.

3. Thurston, G. A.: A Numerical Solution of the Non-linear Equations for Axisymmetric Bending of Shallow Spherical Shells. Jour. Appl. Mech., vol. 28, 1961, pp. 557-568.

4. Archer, R. R. : On the Numerical Solution of the Nonlinear Equations for Shells of Revolution. Jour. Math. Phys., to appear.

5. Marguerre, K.: Zur Theorie der gekrümmten Platte grosser Formänderung. Proc. 5th. Int. Congress Appl. Mech. (Cambridge), 1938, pp. 93-101.

6. Gjelsvik, A., and Bodner, S. R.: The Non-symmetrical Snap-Buckling of the Clamped Spherical Cap. Tech. Report No. 30, Brown Univ. (Providence), 1962.

7. Fung, Y. C., and Sechler, E. E.: Instability of Thin Elastic Shells. Proc. 1st. Sympos. Naval Struct. Mechanics (Stanford), 1960, pp. 115-168.

8. Weinitschke, H. J.: On the Nonlinear Theory of Shallow Spherical Shells. Jour. Soc. Ind. Appl. Math., vol. 6, 1958, pp. 209-232.

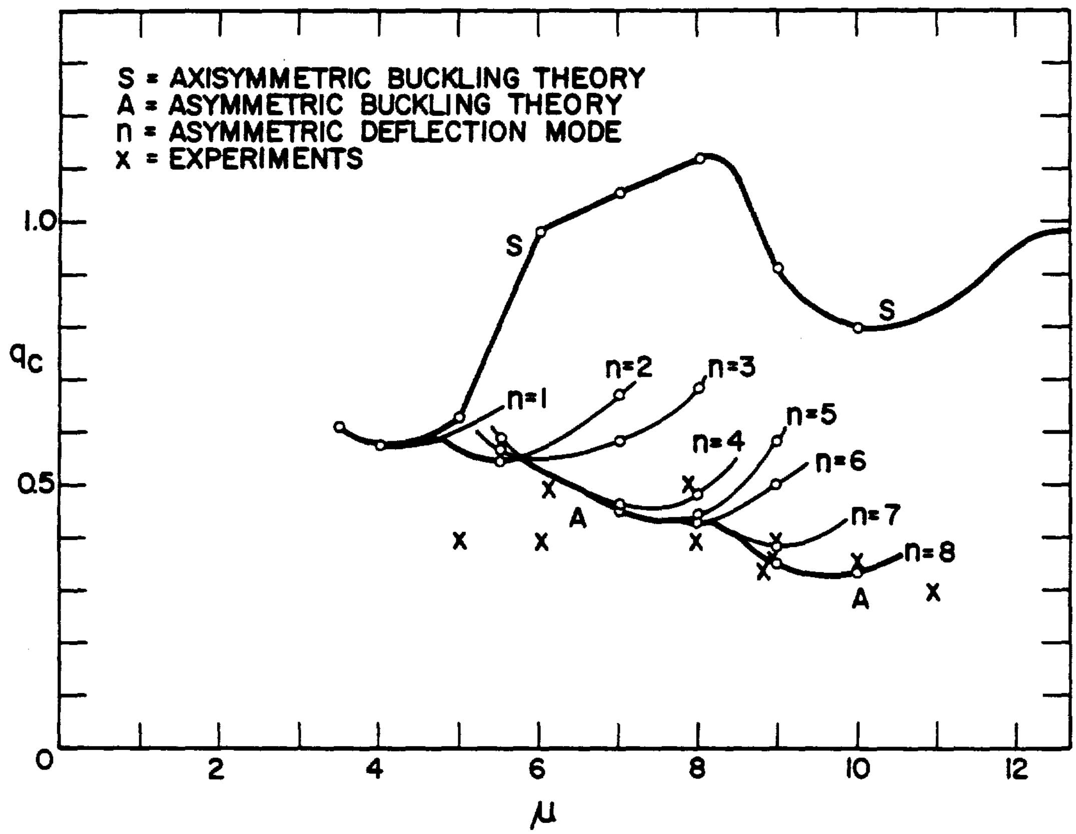

Figure 1.- Theoretical buckling pressures for clamped shallow spherical shells and experimental data.