SOME RECENT RESULTS ON THE BUCKLING MECHANISM OF SPHERICAL CAPS*

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1. INTRODUCTION

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The curve of pressure versus deformation for a spherical cap subjected to external pressure is frequently assumed to be similar to that of Fig. 1. Here P is a loading parameter and D(P) is a measure of the deformation (e.g. maximum displacement). This curve implies that for $P > P_U$ and for $P < P_L$ there is only one equilibrium state. For each P in $P_L < P < P_U$ there are three equilibrium states and the cap must buckle at some P in this interval. We call P_L and P_U, respectively, the upper and lower buckling loads. Points on the branch OU correspond to unbuckled equilibrium states, those on the branch LN to buckled states and those on UL to unstable states.

The buckled and unbuckled equilibrium states are not "adjacent". Therefore when buckling occurs at some value of P the shell snaps from an unbuckled state to a buckled one. Of course, the buckling process is dynamic and a complete understanding of it requires a corresponding analysis. However, we believe that a static analysis can yield a description of the mechanism which initiates or "triggers" the dynamic process and can also yield the theoretical value, say $P = P_A$, at which this occurs.

Several ways of defining P_A are based on the curve of Figure 1. First von Kármán and Tsien [1], who proposed this figure, assumed that P_L should be the buckling load. Later Kaplan and Fung [2] applied the classical buckling criterion

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to conclude that P_U might be the appropriate value. All attempted calculations of P_U and P_L do not in general show agreement with the experimentally determined buckling loads, P_E.

Two other proposals for defining P_A introduce additional considerations. One assumes that "initial imperfections" are always present in the shell specimen and that they serve to change the corresponding value of P_U, say to P_U'. Then applying the classical criterion to such a shell yields different values for P_A. However, some rough calculations [3,4] with special forms of initial imperfection in the undeformed shape do not account for the discrepancy between P_U and P_E. Of course imperfection theory requires a more complete investigation.

The second proposal [5] assumes the existence of an intermediate load, P_M, in the interval $P_L < P_M < P_U$ such that for all P in $P_L \leq P < P_M$ the unbuckled states have less potential energy than the corresponding buckled states and conversely for P in $P_M < P \leq P_U$. It is further assumed that a shell will not snap to a state of higher energy and thus P_M is a lower bound on P_A. However, this theory does not at present define a precise value for P_A. Calculations for clamped caps [6] show that P_E is close to P_M for a specific range of caps and then they deviate sharply.

Recent analytical and theoretical results of the authors indicate that the curve of Figure 1 is usually not correct for clamped and other spherical shells. In fact a much more complicated behavior is found, allowing more than three equilibrium states for various loads. Consequently the previously described buckling criteria cannot be valid in general. In the remainder of this paper we shall present modifications of these criteria and some of our results which suggest them.

2. BUCKLING MECHANISMS

In analogy with the usual precise mathematical definition of stability we may say that an equilibrium state of a shell is stable at a given load if it depends continuously on the load. Thus if D(P) is an appropriate measure of the deformation at load P then no buckling is possible if D(P) is a continuous function of P. We therefore define "stability-loss" pressures as those values of P at which D(P) becomes discontinuous.

In all buckling problems known to us the discontinuities occur by virtue of D(P) becoming multivalued. Thus the stability-loss pressures are those values of P at which the equilibrium problem loses uniqueness. In analogy with our previous discussion we define P_L, the lower buckling load, as the least such pressure (i.e. there are unique solutions for all $P < P_L$ but nonunique ones for $P = P_L$). Similarly P_U is defined as the greatest stability-loss pressure (i.e. unique solutions exist for all $P > P_U$ but not for $P = P_U$). Both P_L and P_U need not exist in all problems (in fact we know of no case in which both have been proven to exist).

The sense of the inequalities is arbitrary in the above definitions. We could let $Q = -P$ in the formulation and then $Q_U = -P_L$ and $Q_L = -P_U$. Indeed this should be the case — for if the load on an elastically buckled shell is relaxed it will eventually unbuckle. We do not distinguish this unbuckling phenomenon from what is usually called buckling. In fact we shall later propose some experiments, based on this observation, to measure various critical pressures. (Such unbuckling experiments may eliminate some of the effects of initial imperfections.)

In addition to P_U and P_L we define as critical pressures all values of P at which the multiplicity of solutions of the equilibrium problem changes. The critical pressures thus defined need not include the load P_A. That is, the appearance of several equilibrium states at a given P does not insure that the shell will snap from one such state to another. However by extending Friedrichs' idea [5] we may define an order or preference of the equilibrium states for a fixed P as follows. Let $S_1(P), S_2(P), \ldots, S_n(P)$ denote the equilibrium states (note that n depends upon P) and let $E_i(P)$ denote the potential energy of the state $S_i(P)$. Then we order these states according to the magnitude of E. Thus $S_i(P)$ is "preferred" to $S_k(P)$ or has lower order if $E_i(P) < E_k(P)$. We now define the "intermediate" critical pressures as those P for which $E_i(P) = E_k(P)$ for some $i \neq k$. The load (or loads) P_A may be "close" to one of the intermediate critical pressures.

We assume that it is always possible for a shell to snap from any state to a preferred state. If the shell is in $S_k(P)$ and jumps to $S_i(P)$ which is preferred, we must still invoke some mechanism to initiate the jump. "Small" disturbances which are always present could trigger the snapping provided that $S_k(P)$ and $S_i(P)$ are reasonably close states, say nearly similar deformations, stresses, etc. However, if there is some intermediate state $S_j(P)$ preferred to $S_k(P)$ (i.e. $E_i < E_j < E_k$) and sufficiently close to $S_k(P)$ then a small disturbance would enable the shell to jump or rather start to jump from $S_k(P)$ to $S_j(P)$. Once this snapping is initiated the problem becomes dynamic and it is possible to end in the non-neighboring lower state $S_i(P)$. Thus, roughly speaking, we may say that the intermediate states such as $S_j(P)$ furnish some internal degrees of freedom to enable small disturbances to initiate a large snap through. Of course this mechanism allows asymmetric equilibrium states to play a role even though the buckling may be between symmetric states. Thus if these intermediate states actually exist the cap has essentially a "built-in" triggering mechanism.

The load at which buckling occurs in the above theory depends upon the "initial" state $S_k(P)$ and the "final" state $S_i(P)$. Thus if S_i and S_k are "close" no intermediate state may be required. However, if these states are no longer close then snap through becomes more difficult. Furthermore with changing geometric parameter intermediate states may appear or disappear between S_i and S_k either enhancing or retarding the ability to buckle.

To apply the above considerations we must investigate the existence of non-unique equilibrium states and the energies of these states. We will indicate below several results which have been obtained for a variety of spherical cap problems. Only axisymmetric deformations are considered in these studies.

3. BIFURCATION BUCKLING

The radially symmetric equilibrium equations for a shallow spherical cap have been formulated in terms of $\alpha(x)$, a slope of the deformed midsurface, and $\gamma(x)$, a stress function [6]. These equations are

$$(1) \qquad L\alpha = \rho(\alpha\gamma + Px^2) , \qquad L\gamma = \rho(x^2 - \alpha^2) ,$$

where P is the dimensionless loading parameter, ρ is a geometric parameter and $L\cdot = x \frac{d}{dx}(\frac{1}{x}\frac{d}{dx} x\cdot)$. Assuming regularity and symmetry at the center of the cap, $x = 0$, implies the boundary conditions:

$$(2) \qquad \alpha(0) = 0 , \qquad \gamma(0) = 0 .$$

At the edge of the cap, $x = 1$, we now consider two types of boundary conditions which are such that the uniformly compressed spherical cap (i.e. $\alpha(x) = x$, $\gamma(x) = -Px$) is a solution of the equilibrium problem:

$$(3) \qquad \begin{array}{lll} \text{A)} & \alpha(1) = 1 , & \gamma(1) = -P ; \\ \text{B)} & \alpha(1) = 1 , & \gamma'(1) - \nu\gamma(1) = -(1-\nu)P . \end{array}$$

These edge conditions are physically reasonable, the first in each case implying no rotation and the second implying no transverse shear force and that the meridional membrane stress in case A or the meridional displacement in case B is proportional to the external pressure.

The precise knowledge of the unbuckled state permits us, in these cases, to rigorously establish certain properties of the non-unique solutions. (The mathematical proofs and a detailed discussion of these results will be presented elsewhere.) We prove, using the bifurcation theory of Poincaré, [7] that for each ρ and all P in a sufficiently small neighborhood of each eigenvalue of an appropriate linearized problem there exists a solution of the full nonlinear problem. Furthermore the curve of D(P) versus P for each ρ can be shown to have the form of the solid portion of the curve in Figure 2. Some of the dotted portions of this figure are determined by numerical calculations. Assuming the potential energy expression to have a minimum we can establish the existence of an intermediate buckling load P_M (see Fig. 2) at which the unbuckled and "first" buckled states have equal energy. Upper and lower bounds on P_M have been determined as:

$$\frac{\omega_1^2}{\rho} \leq P_M < \underline{P}(\rho) \ .$$

Here $\underline{P}(\rho)$ is the lowest of the eigenvalues $P_k = \frac{\omega_k^2}{\rho} + \frac{2\rho}{\omega_k^2}$ of the linearized problem and ω_k is the k-th zero of the Bessel function $J_1(x)$. By using a minimization procedure rigorous upper bounds have been obtained which are considerably lower than $\underline{P}$ and which for a limited range of ρ, serve to closely bracket P_M. In addition we have extended our previously employed finite difference method [6] to the bifurcation problems and have determined accurate numerical approximations of P_M and P_L. These numerical results will be reported in detail elsewhere. We have, as yet, been unsuccessful in establishing the existence of any intermediate buckling loads at which the potential energies of two buckled solutions are equal. It seems likely that some of the bifurcation analysis can be extended with suitable modification to the unsymmetric bifurcation buckling of spherical caps.

4. RELAXATION BUCKLING

Shallow spherical shells clamped and fixed along the edge have been studied in a number of recent papers. The nonlinear boundary value problem is the same as in case B of the previous section if we set P = 0 in the γ boundary condition (3B).

We have extended the previous finite difference calculations [6] to obtain results for larger values of ρ. The new results obtained in this way agree in many respects with those of [3,8,9]. However they cover a larger range and we do not interpret them in the same manner.

Another set of calculations based on initial value problems, which are the most extensive yet known to us, clarifies many of the previous results. This method has also been used by Murray and Wright [10] in more limited calculations. Although these two calculations were initiated independently we have benefited from private communications with Murray and Wright before embarking on extensive numerical work. The details of the techniques employed will be published elsewhere and we shall summarize here only some preliminary results relating to multiplicity of solutions and energy loads.

In Figure 3 we show curves, in the $P - \sqrt{\rho}$ plane, which separate regions of different numbers of solutions. For example consider the line $\rho = \rho_1$. As P increases along this line it represents a particular clamped cap subjected to increasing uniform external load. From $P = 0$ to A_1 there we find a unique, "unbuckled", solution. At the load corresponding to point A_1 more equilibrium states are possible, hence P_{A_1} is apparently the lower buckling load. Between A_1 and B_1 only three equilibrium states were found and only one was found above B_1 (up to pressures of 3.6). If there are only unique solutions above B_1 then P_{B_1} is an upper buckling load and the cap must snap between P_{A_1} and P_{B_1}. Two of the solutions in this interval were found to have equal energy at point a_1. These solutions correspond to the unbuckled solution and what we term the "buckled" solution. The deformation of the buckled state is similar in shape to that of the unbuckled state but has a considerably larger amplitude. The third solution found here always has more energy than the buckled and unbuckled solutions (we call it an "unstable" state). Thus only one intermediate critical pressure can be defined for this value of ρ. This entire situation was observed in the approximate interval $2.8 < \sqrt{\rho} < 5$.

It was also observed that in this interval the experimentally determined buckling loads were quite close to the equal energy load. All of these results have also been obtained previously, by the finite difference calculations [6].

The situation changes at about $\rho = 25$. Consider the cap $\rho = \rho_2 > 25$ of Figure 3. Again we find a lower buckling load at P_{A_2}, three solutions above and an intermediate critical pressure at P_{a_2}. However the "buckled" and "unbuckled" states are much less similar than they were previously. Furthermore as P increases above P_{B_2} a new pair of solutions is found, for a total of five equilibrium states. One of these new solutions has energy between the "buckled" and "unbuckled" states at load P_{b_2} and a deformation closer to the unbuckled state. This state could function as the "trigger state" discussed in section 2. However the loads at which these solutions have been found are considerably above P_E. [Our calculations can only indicate existence and hence, of course, do not preclude the existence of other equilibrium states at lower pressures. In fact asymmetric states have not been included in our study but even other symmetric states could be present. See also the remarks in the next section.]

Having found the above solutions a closer study of the results of finite difference calculations revealed that they had actually "jumped" to one of the intermediate solutions at some $P > P_{B_2}$. For still higher pressures around P_{C_2} the solution jumped to the final buckled state.

The calculations are being continued and as many as nine non-unique solutions have recently been determined.

5. CONCLUDING REMARKS

To facilitate a comparison between theoretically and experimentally determined buckling loads extensive numerical results are required for a variety of other boundary conditions. The experiments of Kaplan and Fung and others were conducted for nominally clamped edges. However this condition cannot be precisely realized in practice. The crucial dependence of the solutions on the boundary conditions is illustrated by the bifurcation buckling problem discussed in section 3.

The results described in the preceding sections indicate the need for a series of carefully controlled experiments on accurately fabricated specimens subject to a variety of boundary conditions. If it is feasible, an experimental study of bifurcation buckling would be of great interest. It is particularly important to design experiments which will explore the mechanisms of buckling. For example, many of the effects of initial imperfections in the specimen could be eliminated by conducting an unbuckling test. In such a test the specimen is allowed to buckle elastically and the pressure is increased slightly above the buckling load. Then the pressure is lowered until the shell snaps back to an unbuckled equilibrium position. A comparison of the experimentally determined snap-back load with the appropriate critical pressures may give an indication of the buckling mechanism.

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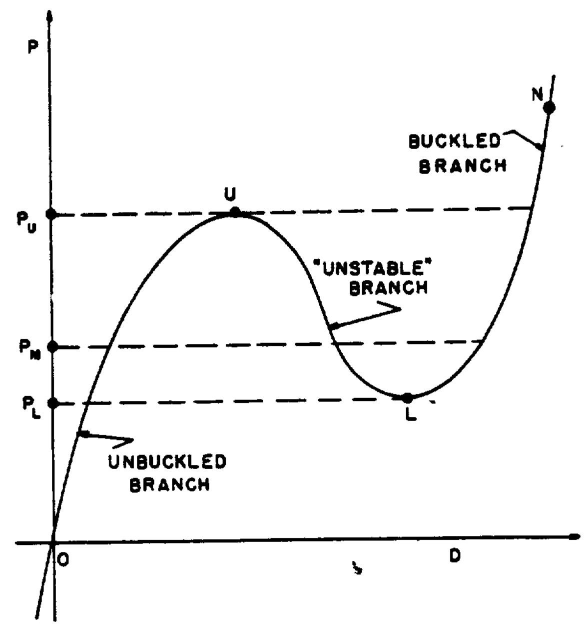

Figure 1.- Load deformation curve proposed by v. Kármán and Tsien.

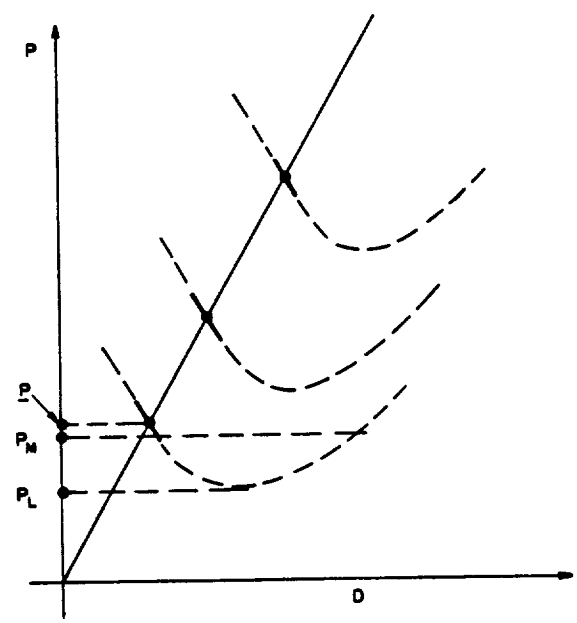

Figure 2.- Load deformation curve for bifurcation buckling (sketch).

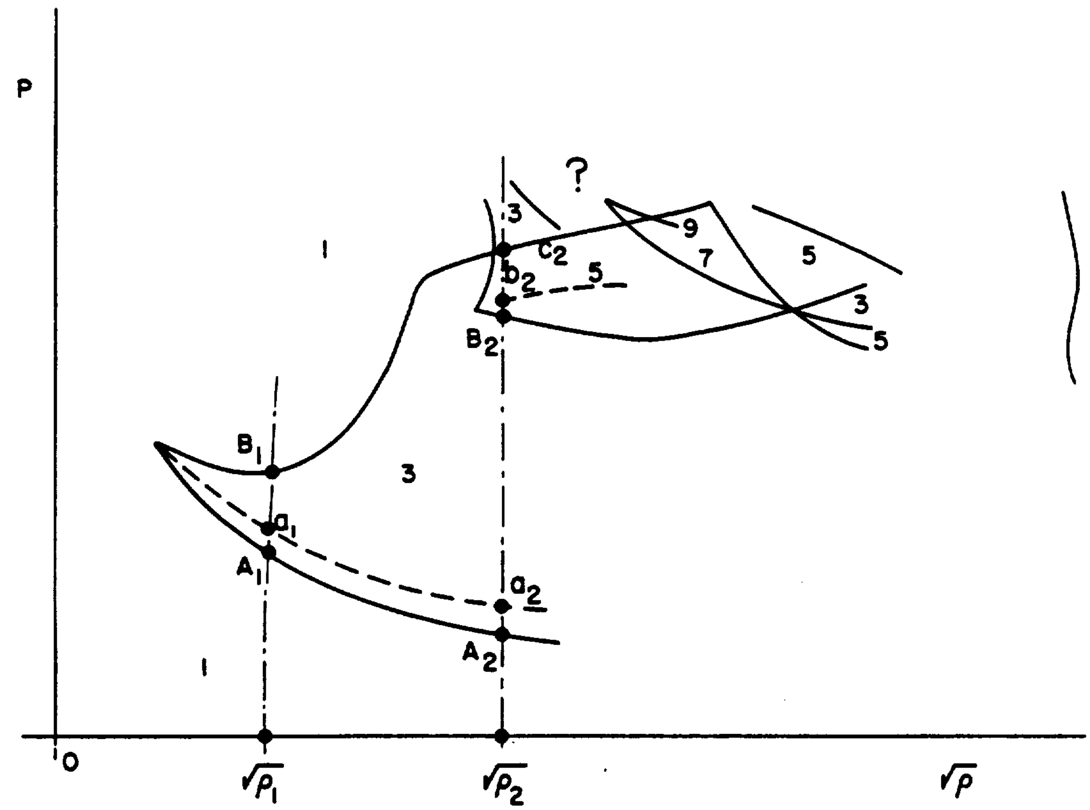

Figure 3.- Critical pressure curves for clamped caps (sketch).