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TIME-DEPENDENT AIR FORCES ON WINGS WITH SUPERSONIC LEADING AND TRAILING EDGES AND SUBSONIC SIDE EDGES WITH APPLICATION TO A WING DEFORMING HARMONICALLY ACCORDING TO A GENERAL POLYNOMIAL EQUATION

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TIME-DEPENDENT AIR FORCES ON WINGS WITH SUPERSONIC LEADING AND TRAILING EDGES AND SUBSONIC SIDE EDGES WITH APPLICATION TO A WING DEFORMING HARMONICALLY ACCORDING TO A GENERAL POLYNOMIAL EQUATION

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SUMMARY

The integral form of the velocity potential and pressure distribution for a wing with supersonic leading edges and subsonic side edges in supersonic flow is derived herein for a wing undergoing any arbitrary time-dependent deformations. The expressions are simplified by assuming harmonic deformations and then expanding the integrand of the velocity potential to the third power of frequency. The special case is treated for which the side edge is parallel to the free stream and the oscillations are such that the amplitude of wing distortion can be represented by a polynomial of any desired degree in the span coordinate and third degree in the chordwise coordinate.

The equations are further reduced to the special cases of a rigid wing oscillating in pitch and translation and of a rigid wing in a sinusoidal gust, the results of which are presented in an appendix. Sample calculations are made for the total lift on a delta and rectangular wing and the results are presented in a table where a comparison is made with the exact values from linearized potential-flow theory.

INTRODUCTION

Time-dependent aerodynamic forces have been a subject of continuing theoretical development for many years. Most effort has been directed toward methods of predicting air forces due to simple harmonic motion since these methods can be applied directly to aircraft flutter problems. With suitable operations these harmonically varying forces, which were developed for application to flutter, can be used in the harmonic analysis of airplane response to continuous atmospheric turbulence. Thus, the accumulated knowledge of unsteady air forces due to harmonic motion of wings at various speeds may be applied to both flutter and response to turbulence.

The lift and moment for rigid wings of various planforms undergoing harmonic oscillations have been derived. (For example, see refs. 1 to 10.) The lift and moment on certain rigid restrained wings subjected to continuous sinusoidal gusts (or turbulence) have been presented in references 11 to 13. The results of references 1 to 13 have been compiled in reference 14 together with the unsteady air forces for additional planforms. A more complete list of references is given in the bibliography of reference 15.

The aerodynamic forces for application to nonrigid or deforming wings are available for special cases. For example, if the distorted shape of the wing can be represented by a quadratic equation in the chordwise and spanwise coordinates, references 16 to 19 may be used in the supersonic speed range. In reference 16, the velocity potential for a triangular wing with subsonic leading edges undergoing general second-degree forms of harmonic distortion in both the spanwise and chordwise coordinates is presented. The velocity potential therein is expanded to the third power of the oscillation frequency in order to obtain the forces and moments. Reference 17 is an extension of reference 16 wherein a higher degree of wing distortion is considered and the velocity potential is expanded to the fifth power of the frequency. In reference 18, the generalized forces for a harmonically oscillating rectangular wing are given. The downwash distribution is assumed to be a general polynomial in the spanwise and chordwise coordinates. In reference 19 a strip theory technique is used to obtain the generalized forces on a delta wing with supersonic leading edges. This procedure gives the exact pressure distribution for arbitrary chordwise variation of displacements and, at most, linear variation of displacements in the span direction.

In the present paper an integral expression is given for the pressure distribution on a wing with swept supersonic leading edges and arbitrarily swept subsonic side edges with an arbitrary time-dependent downwash distribution. The trailing edge is also arbitrary but must be supersonic at all points. The expression is simplified by considering the special case for a wing undergoing harmonic motion with side edge parallel to free stream. The deformed shape of the wing is represented by a polynomial of any desired degree in the span direction and third degree in the chord direction. The aerodynamic forces are obtained by expanding the equations to the third power of frequency. Reduction of the equations for application to a rigid wing oscillating in pitch and translation and to a rigid wing in a sinusoidal gust is presented in an appendix.

SYMBOLS

a	speed of sound
A_s	normalizing factors used in equation (46) to define displacement of wing
b	wing span
$\dot{h}$	sinking velocity of wing
I_s	quantity defined by equation (D3)
$J_n(x)$	Bessel function of first kind
k	reduced frequency
K_n	normalizing factor (see eq. (42))
$l(x,y)$	lift distribution due to downwash, $w = e^{i\omega t} \sum_{n=0}^{3} K_n \left(x - \frac{y}{\lambda}\right)^n \sum_{s=0}^{\infty} A_s y^s$
$L_k \Big\vert_{\eta_i}^{\eta_j}$	quantity defined by equation (47)
$\overline{L}_k = L_k(-\lambda) \Big\vert_{\eta_i}^{\eta_j}$	
M	Mach number
n,s,r	integers
P_s	quantity defined by equation (D10)
Δp	local pressure difference
$\overline{\Delta p}/q$	amplitude of pressure coefficient due to downwash, $w = K_n e^{i\omega t} x^n \delta(y-\eta)$

q	dynamic pressure, $\rho V^2/2$
Q_s	quantity defined by equation (D4)
r_0, r_1, r_2	quantities defined by equations (29)
r_3, r_4	quantities defined by equations (41)
R_s	quantities defined by equation (D9)
$R(y)$	quantity defined by equation (37)
t', t, t_2	time
t_1	transformed time (see eqs. (5))
$u = \beta\lvert y - \eta\rvert$	
$\bar{u} = \beta(y + \eta)$	
V	free-stream velocity
w	vertical velocity on surface of wing, positive up
w_0, w_1	amplitude of vertical velocity associated with the Dirac delta function
x', y', z'	rectangular coordinates fixed to wing
x_1, y_1, z_1	transformed coordinates (see eqs. (5))
x, y, z	rectangular coordinate system fixed to apex of wing
α	angle of attack
$\beta = \sqrt{M^2 - 1}$	
$\delta(x)$	Dirac delta function
η', η, η_1	position on y', y, and y_1 axes where downwash is applied, respectively
λ	slope of leading edge of wing

Λ β times slope of side edge of wing

ρ density

ϕ, χ, ψ velocity potentials

ω circular frequency

$\bar{\omega} = \frac{M^2\omega}{V\beta^2}$

ξ_1, ξ_2 coordinates

$\gamma, \zeta, \mu, \sigma$ dummy variable

ANALYSIS

Introductory Remarks

As a first step in the analysis, an integral expression is developed for the velocity potential associated with a downwash strip of Dirac delta form on a wing in supersonic flow with swept supersonic leading and trailing edges and subsonic side edges. By superimposing these strips over the wing planform, a general expression for the generalized forces is derived for any arbitrary time-dependent downwash distribution. These equations, although complicated, can be programed on the modern-day high-speed digital computers.

As a next step in the analysis, a simplification is made of the above-mentioned expressions by assuming simple harmonic motion. The velocity potential and pressure coefficients associated with the harmonically oscillating strip are then presented. These expressions are further simplified so as to pertain to the special case where the side edge is parallel to the free stream. The pressure coefficients are then expanded in powers of frequency and by superimposing the downwash strips over the wing planform the pressure distribution for various wing distortions is obtained.

The method used is that of Gardner (ref. 20) which reduces the nonsteady finite-wing problem to two "steady" finite-wing problems. Without deriving the method, its essential points are given herein.

Velocity Potential and Pressure Coefficients Associated

With a Downwash Strip of Dirac Delta Form

The linearized boundary-value problem.- The differential equation of the propagation of disturbances that must be satisfied by the velocity potential is (when referred to a moving coordinate system x',y',z')

$$\frac{1}{a^2}\left(\frac{\partial}{\partial t'} + V\frac{\partial}{\partial x'}\right)^2 \phi = \frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial z'^2} \tag{1}$$

where

$$\phi \equiv \phi(x',y',z',t')$$

The boundary conditions that must be satisfied by the velocity potential are

$$\phi(x',y',0,t') = 0 \tag{2}$$

ahead of the wing

$$\left(\frac{\partial\phi}{\partial z'}\right)_{z'\to 0} = V\frac{\partial z'}{\partial x'} + \frac{\partial z'}{\partial t'} = w(x',y',t') \tag{3}$$

on the wing.

The wing planform for which the velocity potentials and pressure distributions are to be obtained is shown in figure 1. The numbered regions on this figure will be discussed later in the paper. A convenient planform to analyze is shown in sketch 1 where by means of various transformations and superposition techniques the results can be applied to the planform shown in figure 1. The downwash associated with the planform in sketch 1 is assumed to be

$$w(x',y',t') = w_0(x',t')\ \delta(y'-\eta') \tag{4}$$

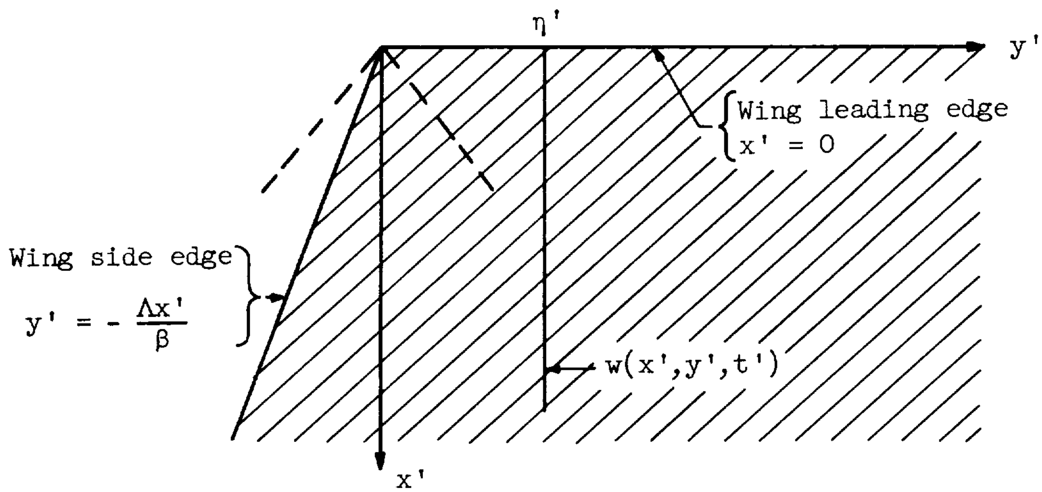
where $\delta(y')$ is the Dirac delta and is defined as

$$\int_{-\infty}^{\infty} \delta(y')\ dy' = 1$$

and

$$\int_{-\infty}^{\infty} F(y') \, \delta(y'-y) \, dy' = F(y)$$

and the slope of the side edge is expressed as Λ/β for convenience.



Sketch 1

Although this downwash distribution may appear to be physically unreasonable, it will be shown in appendix A that the results utilizing it reduce to known functions.

Transformation of the boundary-value problem.- With a transformation similar to that employed in reference 20,

$$\left.\begin{aligned}
x_1 &= \frac{\Lambda y' + x'/\beta}{\sqrt{1 - \Lambda^2}} \\
y_1 &= \frac{\Lambda x'/\beta + y'}{\sqrt{1 - \Lambda^2}} \\
t_1 &= \frac{Mx' - \beta^2 at'}{\beta} \\
z_1 &= z' \\
\eta_1 &= \eta'
\end{aligned}\right\} \tag{5}$$

equations (1) to (4) become

$$\phi_{x_1x_1} - \phi_{y_1y_1} - \phi_{z_1z_1} - \phi_{t_1t_1} = 0 \qquad (6)$$

$$\phi \equiv 0 \qquad \left(x_1 < \Lambda y_1\right) \qquad (7)$$

ahead of the wing

$$\left(\phi_{z_1}\right)_{z_1 \to 0} = w\left(x_1,y_1,t_1\right) \qquad \left(y_1 > 0\right) \qquad (8)$$

on the wing

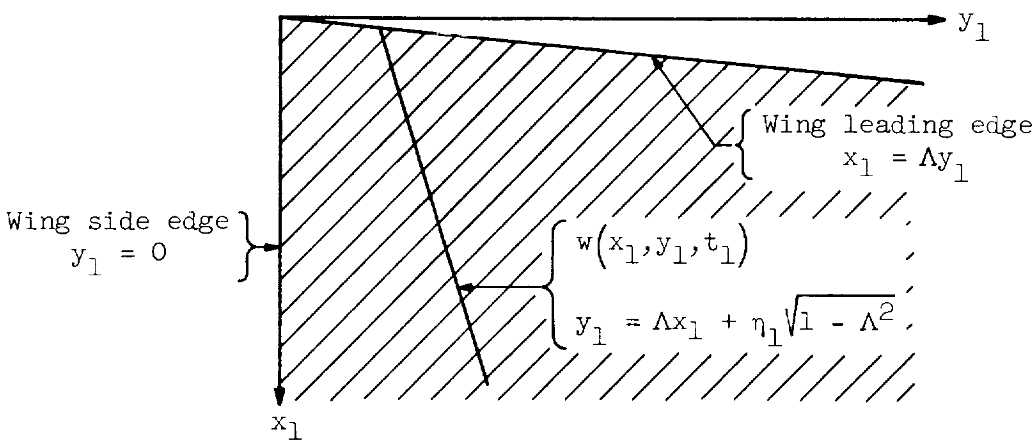
$$w\left(x_1,y_1,t_1\right) = w_0\left\{\frac{\beta\left(x_1 - \Lambda y_1\right)}{\sqrt{1 - \Lambda^2}}, \frac{1}{\beta a}\left[\frac{M\left(x_1 - \Lambda y_1\right)}{\sqrt{1 - \Lambda^2}} - t_1\right]\right\} \delta\left(\frac{y_1 - \Lambda x_1}{\sqrt{1 - \Lambda^2}} - \eta_1\right) \qquad (9a)$$

$$w\left(x_1,y_1,t_1\right) = w_1\left(x_1,y_1,t_1\right) \delta\left(\frac{y_1 - \Lambda x_1}{\sqrt{1 - \Lambda^2}} - \eta_1\right) \qquad (9b)$$

where

$$w_1\left(x_1,y_1,t_1\right) = w_0\left\{\frac{\beta\left(x_1 - \Lambda y_1\right)}{\sqrt{1 - \Lambda^2}}, \frac{1}{\beta a}\left[\frac{M\left(x_1 - \Lambda y_1\right)}{\sqrt{1 - \Lambda^2}} - t_1\right]\right\} \qquad (9c)$$

The wing in the transformed coordinate system is shown in sketch 2, where the shaded area of sketch 1 transforms into the shaded area of sketch 2, and the side edge and leading edge become $y_1 = 0$ and $x_1 = \Lambda y_1$, respectively.



Sketch 2

Gardner's method consists of introducing the variable ξ_1 and constructing a potential function $\psi(\xi_1,x_1,y_1,z_1,t_1)$ defined for all $\xi_1 \geqq 0$ such that ψ satisfies the following two differential equations:

$$\psi_{\xi_1\xi_1} - \psi_{y_1y_1} - \psi_{z_1z_1} = 0 \tag{10}$$

$$\psi_{x_1x_1} - \psi_{t_1t_1} - \psi_{\xi_1\xi_1} = 0 \tag{11}$$

and the conditions

$$\psi \equiv 0 \tag{12}$$

ahead of the wing and

$$\left(\psi_{z_1}\right)_{z_1=0} = \chi(\xi_1,x_1,y_1,t_1) \qquad (y_1 > 0) \tag{13}$$

and where

$$\chi(0,x_1,y_1,t_1) = w(x_1,y_1,t_1) \tag{14}$$

and $w(x_1,y_1,t_1)$ is defined by equation (9b). It can be seen that, by adding equations (10) and (11), the resulting equation has the same form as equation (6). Similarly, equations (12) and (7) have the same form and equations (13) and (14) reduce to equation (8) as ξ_1 approaches 0. Consequently, the velocity potential is found by setting $\xi_1 = 0$ such that

$$\phi(x_1,y_1,z_1,t_1) = \psi(0,x_1,y_1,z_1,t_1) \tag{15}$$

Differentiating equation (11) with respect to z_1 and defining a new potential function χ such that

$$\left(\frac{\partial\psi}{\partial z_1}\right)_{z_1\to 0} = \chi(\xi_1,x_1,y_1,t_1) \tag{16}$$

equation (11) becomes

$$\chi_{x_1x_1} - \chi_{t_1t_1} - \chi_{\xi_1\xi_1} = 0 \tag{17}$$

with the boundary conditions

$$\chi = 0 \tag{18}$$

ahead of the wing and

$$\chi = w(x_1,y_1,t_1) \tag{19}$$

for $\xi_1 = 0$. This will be considered the first boundary-value problem. The second boundary-value problem consists of equations (10), (12), and (13) and is restated here for convenience

$$\psi_{\xi_1\xi_1} - \psi_{y_1y_1} - \psi_{z_1z_1} = 0 \tag{20}$$

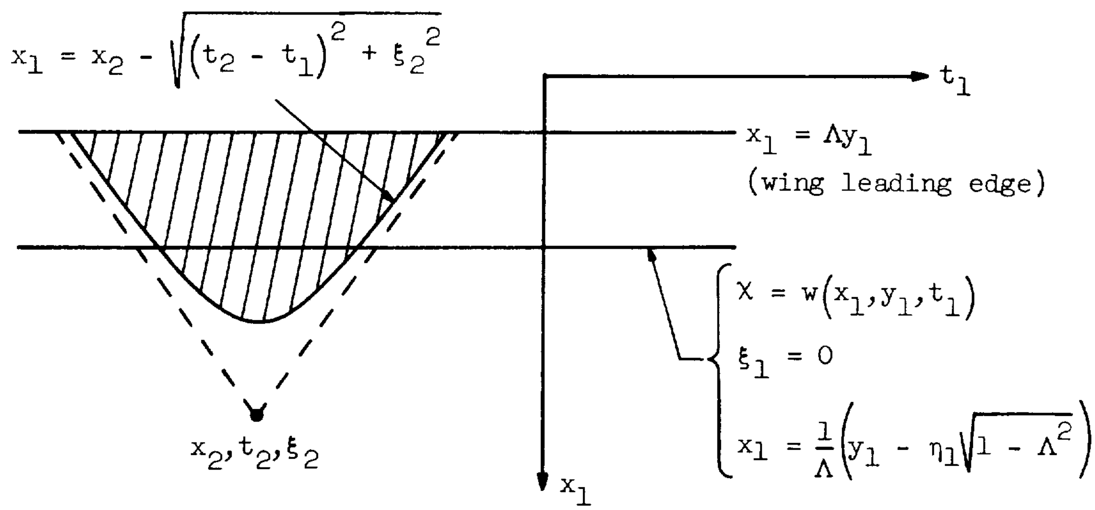
$$\psi \equiv 0 \tag{21}$$

ahead of wing and

$$\left(\psi_{z_1}\right)_{z_1=0} = \chi(\xi_1,x_1,y_1,t_1) \tag{22}$$

for $y > 0$. It can be seen that the solution of equations (17) to (19) becomes a boundary condition for equations (20) to (22).

Solution for X-function.- The boundary-value problem defined by equations (17), (18), and (19) is similar to the steady two-dimensional supersonic "wing" problem in x_1,t_1,ξ_1 space, where x_1 is in the downstream direction, t_1 in the span direction, and ξ_1 is normal to the wing as implied in sketch 3. The downwash, as can be seen from equations (9a) and (19), is concentrated along the line $x_1 = \frac{1}{\Lambda}\left(y_1 - \eta_1\sqrt{1 - \Lambda^2}\right)$ in the $\xi_1 = 0$ plane.



Sketch 3

The curve $x_1 = x_2 - \sqrt{(t_2 - t_1)^2 + \xi_2^2}$ represents the intersection of the plane $\xi_1 = 0$ with the characteristic forecone emanating from the point x_2, t_2, ξ_2.

The solution for χ can now be written in terms of simple sources

$$\chi = -\frac{1}{\pi}\frac{\partial}{\partial \xi_1}\iint_{S_0} \frac{w(x_2, y_1, t_2)\, dx_2\, dt_2}{\sqrt{(x_1 - x_2)^2 - (t_1 - t_2)^2 - \xi_1^2}} \tag{23}$$

where S_0 is the hatched region indicated in sketch 3. It might be noted that the expression for χ (eq. (23)) differs from the classic potential given for the two-dimensional steady-flow problem by the partial derivative $\partial/\partial \xi_1$. This is due to the fact that in the two-dimensional problem the vertical velocity, which is analogous to $\left(\frac{\partial \chi}{\partial \xi_1}\right)_{\xi_1=0}$, is specified on the wing, whereas, in this problem, the potential

$(X)_{\xi_1} = 0$ is specified on the wing. Substituting for $w(x_1,y_1,t_1)$ from equation (9b) into equation (23) and introducing the appropriate limits yields

$$X = -\frac{1}{\pi}\frac{\partial}{\partial\xi_1}\int_{t_1-\sqrt{(x_1-\Lambda y_1)^2-\xi_1^2}}^{t_1+\sqrt{(x_1-\Lambda y_1)^2-\xi_1^2}} dt_2 \int_{\Lambda y_1}^{x_1-\sqrt{(t_2-t_1)^2+\xi_1^2}} \frac{w_1(x_2,y_1,t_2)\,\delta\left(\frac{y_1 - \Lambda x_2}{\sqrt{1 - \Lambda^2}} - \eta_1\right)}{\sqrt{(x_1 - x_2)^2 - (t_1 - t_2)^2 - \xi_1^2}}\,dx_2 \tag{24}$$

By integrating equation (24) with respect to x_2 and then making the substitution $t_2 = t_1 + \sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_1^2}\cos\theta$, keeping in mind that the value of the integrand is concentrated along the line $x_1 = \frac{1}{\Lambda}\left(y_1 - \eta_1\sqrt{1 - \Lambda^2}\right)$ between $t_2 = t_1 \pm \sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_1^2}$ there is obtained

$$X = -\frac{\sqrt{1 - \Lambda^2}}{\pi}\frac{\partial}{\partial\xi_1}\int_0^{\pi} w_1\left[\frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}, y_1, t_1 + \sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_1^2}\cos\theta\right]d\theta$$

$$\left(\Lambda(x_1 - \xi_1) + \eta_1\sqrt{1 - \Lambda^2} > y_1 > \frac{\eta_1}{\sqrt{1 - \Lambda^2}}\right) \tag{25a}$$

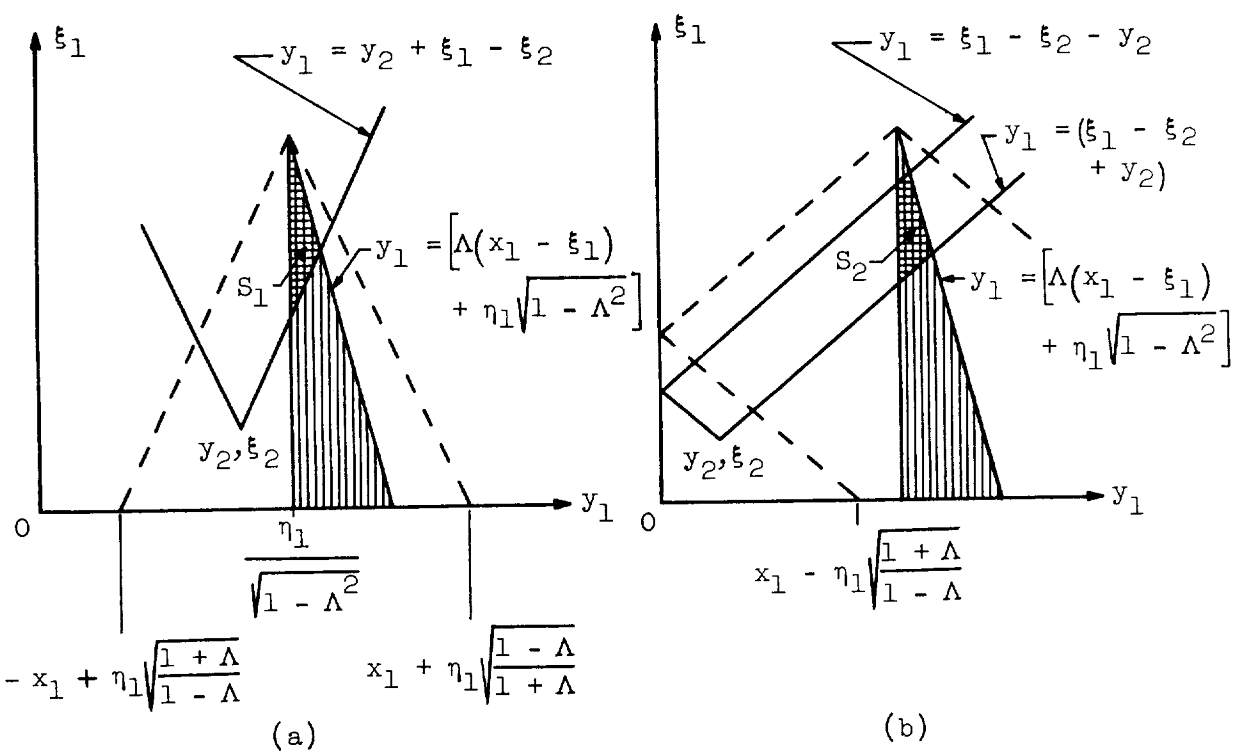
and $X = 0$ for all other regions. When w_1 is rewritten in terms of w_0 (eq. (9c)), equation (25a) becomes

$$X = -\frac{\sqrt{1 - \Lambda^2}}{\pi}\frac{\partial}{\partial\xi_1}\int_0^{\pi} w_0\left[\frac{\beta}{\Lambda}\left(y_1\sqrt{1 - \Lambda^2} - \eta_1\right), \frac{M}{\Lambda\beta a}\left(y_1\sqrt{1 - \Lambda^2} - \eta_1\right) - \frac{t_1}{\beta a} - \frac{1}{\beta a}\sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_1{}^2}\cos\theta\right]d\theta$$

$$\left(\Lambda\left(x_1 - \xi_1\right) + \eta_1\sqrt{1 - \Lambda^2} > y_1 > \frac{\eta_1}{\sqrt{1 - \Lambda^2}}\right) \qquad (25b)$$

and $X = 0$ for all other regions.

Solution for the velocity potential $\phi(x_1,y_1,0_1,t_1)$.- The boundary-value problem defined by equations (20), (21), and (22) is also similar to steady two-dimensional supersonic wing problem in the ξ_1,y_1,z_1 space, where ξ_1 is in the upstream direction, y_1 in the span direction, z_1 is normal to the ξ_1,y_1 plane as implied in sketch 4.



Sketch 4

The solution for ψ can now be written in terms of simple sources as

$$\psi\left(\xi_1,x_1,y_1,0,t_1\right) = \iint\limits_{S_1} \frac{\chi\left(\xi_2,x_1,y_2,t_1\right) d\xi_2 \, dy_2}{\sqrt{\left(\xi_1 - \xi_2\right)^2 - \left(y_1 - y_2{}^2\right)^2}} \tag{26}$$

and by means of equations (15) and (26)

$$\phi\left(x_1,y_1,0,t_1\right) = \lim_{\xi_1 \to 0} \iint\limits_{S_1} \frac{\chi\left(\xi_2,x_1,y_2,t_1\right) d\xi_2 \, dy_2}{\sqrt{\left(\xi_1 - \xi_2\right)^2 - \left(y_1 - y_2\right)^2}} \tag{27}$$

where S_1 is the crosshatched region indicated in sketch 4. The crosshatched and hatched regions in sketch 4 are the regions where the potential $\chi\left(x_1,y_1,\xi_1,t_1\right) \neq 0$ as dictated by the conditions imposed by equation (25). Integration over region S_1 will yield results for pure supersonic flow, whereas integration over region S_2 (sketch 4(b)) will yield results which contain the effects of the subsonic side and must be taken into account when $0 < y_1 < x_1 - \eta_1\sqrt{\frac{1 + \Lambda}{1 - \Lambda}}$.

By substituting the appropriate limits into equation (27),

$$\left.\begin{aligned}
\phi = \phi_1\left(x_1,y_1,0,t_1\right) &= -\frac{1}{\pi}\int_{\frac{\eta_1}{\sqrt{1-\Lambda^2}}}^{r_0} dy_2 \int_{y_2-y_1}^{r_1} \frac{\chi\left(\xi_2,x_1,y_2,t_1\right) d\xi_2}{\sqrt{\xi_2{}^2 - \left(y_1 - y_2\right)^2}} \\
&\qquad \left(x_1 + \eta_1\sqrt{\frac{1 - \Lambda}{1 + \Lambda}} > y_1 > x_1 - \eta_1\sqrt{\frac{1 + \Lambda}{1 - \Lambda}} > 0\right) \\
\phi = \phi_2\left(x_1,y_1,0,t_1\right) &= -\frac{1}{\pi}\int_{r_2}^{r_0} dy_2 \int_{y_2-y_1}^{r_1} \frac{\chi\left(\xi_2,x_1,y_2,t_1\right) d\xi_2}{\sqrt{\xi_2{}^2 - \left(y_1 - y_2\right)^2}} - \frac{1}{\pi}\int_{\frac{\eta_1}{\sqrt{1-\Lambda^2}}}^{r_2} dy_2 \int_{y_2-y_1}^{y_2+y_1} \frac{\chi\left(\xi_2,x_1,y_2,t_1\right) d\xi_2}{\sqrt{\xi_2{}^2 - \left(y_1 - y_2\right)^2}} \\
&\qquad \left(x_1 - \eta_1\sqrt{\frac{1 + \Lambda}{1 - \Lambda}} > y > 0\right)
\end{aligned}\right\} \tag{28}$$

where

$$\left.\begin{aligned} r_0 &= \frac{\Lambda(x_1 + y_1) + \eta_1\sqrt{1 - \Lambda^2}}{1 + \Lambda} \\ r_1 &= \frac{\eta_1\sqrt{1 - \Lambda^2} - y_2}{\Lambda} + x_1 \\ r_2 &= \frac{\Lambda(x_1 - y_1) + \eta_1\sqrt{1 - \Lambda^2}}{1 + \Lambda} \end{aligned}\right\} \tag{29}$$

If the potentials in equations (28) are transformed back to the true x',y',t' coordinates, the regions where these potentials exist are given below and are also shown in sketch 5

For $\eta' > 0$:

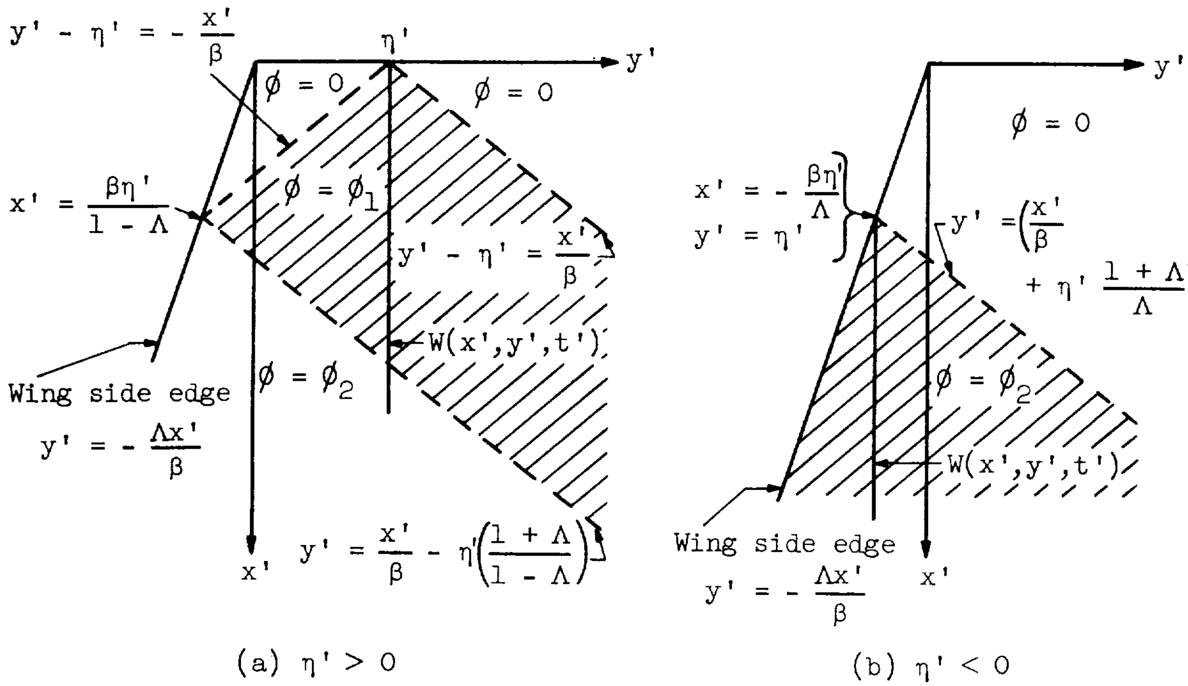
$$\left.\begin{aligned} \phi &= \phi_1(x',y',t') \qquad \left(\eta' - \frac{x'}{\beta} < y' < \eta' + \frac{x'}{\beta},\ 0 < x' < \frac{\beta\eta'}{1 - \Lambda}\right) \\ \phi &= \phi_1(x',y',t') \qquad \left[\frac{x'}{\beta} - \eta'\left(\frac{1 + \Lambda}{1 - \Lambda}\right) < y' < \eta' + \frac{x'}{\beta},\ x' > \frac{\beta\eta'}{1 - \Lambda}\right] \\ \phi &= \phi_2(x',y',t') \qquad \left[-\frac{\Lambda x'}{\beta} < y' < \frac{x'}{\beta} - \eta'\left(\frac{1 + \Lambda}{1 - \Lambda}\right),\ x' > \frac{\beta\eta'}{1 - \Lambda}\right] \end{aligned}\right\} \tag{30a}$$

and $\phi = 0$ for all other regions.

For $\eta' < 0$:

$$\phi = \phi_2(x',y',t') \qquad \left(-\frac{\Lambda x'}{\beta} < y' < \frac{x'}{\beta} + \frac{1 + \Lambda}{\Lambda}\eta',\ x' > -\frac{\beta\eta'}{\Lambda}\right) \tag{30b}$$

and $\phi = 0$ for all other regions.



Sketch 5

The pressure difference in terms of the physical x',y',t' coordinates is given by

$$\Delta p = 2\rho\left(\frac{\partial\phi}{\partial t'} + V\,\frac{\partial\phi}{\partial x'}\right) \tag{31}$$

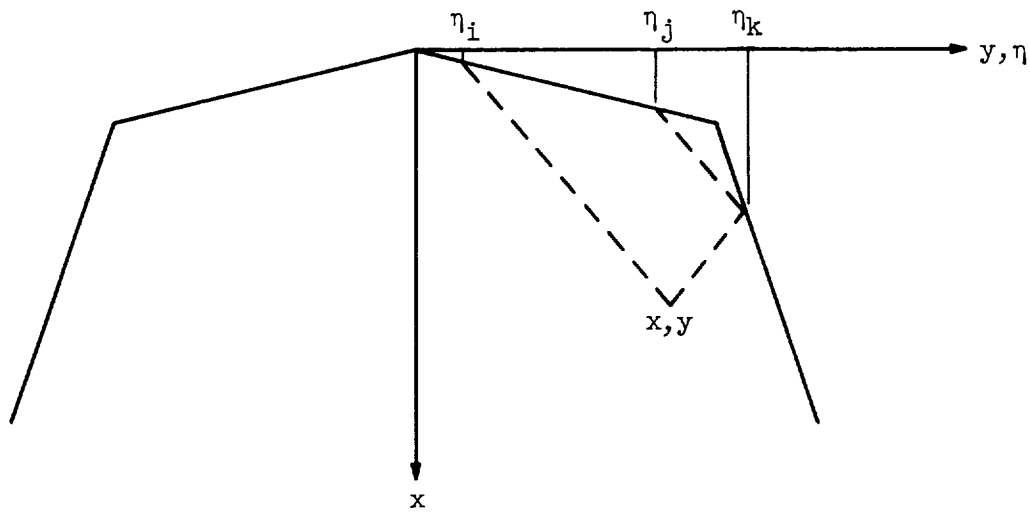
Subsequently, a coordinate system is employed where the origin is situated on the apex of a sweptback wing. The x-coordinate is parallel to the free-stream direction and the y-coordinate, in the span direction. The coordinate system will also have the additional characteristic that the Dirac delta strip will have its origin on the leading edge of the wing rather than on the y'-axis as indicated in sketch 5(a). If it is assumed that the potential functions given by equations (28) and the pressure coefficient as given by equation (31) have been transformed to this new x,y,t coordinate system, the lift distribution due to arbitrary downwash distribution can be written as

$$l(x,y) = \int_{\eta_i}^{\eta_j} f(\eta)\,\frac{\Delta p_1(x,y,t;\eta)}{q}\,d\eta + \int_{\eta_j}^{\eta_k} f(\eta)\,\frac{\Delta p_2(x,y,t;\eta)}{q}\,d\eta \tag{32}$$

where $\Delta p_1/q$ and $\Delta p_2/q$ are the pressure coefficients associated with the velocity potential ϕ_1 and ϕ_2, respectively, and $f(\eta)$ is a nondimensional spanwise downwash weighting function defined as

$$w(x,y,t) = w_0(x,t)\ f(y)$$

A method for obtaining the limits of η_s will be discussed in a subsequent section; however, an indication of their significance is shown in sketch 6 where η_i, η_j, and η_k are particular values of η_s



Sketch 6

It might be noted that, in order to obtain $\Delta p/q$ for use in equation (32), a triple integration is involved: one as indicated in equation (25b) and two more as indicated in equations (28). Therefore, in order to obtain the loading due to an arbitrary downwash distribution, four integrations are required. To obtain the generalized forces another integration is required, and if a transient phenomenon is present, a time-superposition integral is required. However, with the modern high-speed computers it does not seem unreasonable to undertake the integration of a quintuple or even a sextuple integral.

The remainder of this paper will deal with the evaluation of equations (25b) and (28) in order to obtain the lift distribution for a wing oscillating in simple harmonic motion with a polynomial downwash distribution in the chordwise and spanwise direction. The potential will first be derived for the oscillating strip, whose downwash can be represented by

$$\left.\begin{aligned} w(x',y',t') &= w_0(x',t')\,\delta(y' - \eta') \\ &= w_0(x')\,e^{i\omega t'}\,\delta(y' - \eta') \end{aligned}\right\} \tag{33}$$

The potential is then derived for this strip and a frequency expansion performed to obtain the corresponding pressure distribution. These expressions are further simplified for the special case where the side edge is parallel to the free stream and the downwash is assumed to vary as x'^n in the free-stream direction. Superposition techniques are then used to obtain the loading distribution for various wing distortions.

Velocity Potential and Pressure Coefficients

for Simple Harmonic Motion

Solution for the potentials χ and ϕ.- By assuming simple harmonic motion, the downwash distribution as given by equations (33) becomes

$$w\left(x_1,y_1,t_1\right) = w_0\left[\frac{\beta\left(x_1 - \Lambda y_1\right)}{\sqrt{1 - \Lambda^2}}\right] e^{\frac{i\omega}{\beta a}\left[\frac{M\left(x_1 - \Lambda y_1\right)}{\sqrt{1-\Lambda^2}} - t_1\right]}\,\delta\left(\frac{y_1 - \Lambda x_1}{\sqrt{1 - \Lambda^2}} - \eta_1\right) \tag{34}$$

when transformed to the x_1,y_1,t_1 coordinate system by means of equations (5). Comparing equations (34) and (9) gives

$$w_0\left(x_1,t_1\right) = w_0\left(x_1\right) e^{i\omega t_1} \tag{35}$$

Substituting equation (35) into equation (25b) gives

$$\chi = \frac{-\sqrt{1 - \Lambda^2}}{\pi\Lambda}\, w_0\left[\frac{\beta}{\Lambda}\left(y_1\sqrt{1 - \Lambda^2} - \eta_1\right)\right] e^{\frac{i\omega M}{\Lambda\beta a}\left(y_1\sqrt{1-\Lambda^2} - \eta_1\right) - \frac{i\omega t_1}{\beta a}}\,\frac{\partial}{\partial\xi_1}\int_0^{\pi} e^{-\frac{i\omega}{\beta a}\sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1-\Lambda^2}}{\Lambda}\right)^2 - \xi_1^2}\cos\theta}\,d\theta$$

$$\left(\Lambda\left(x_1 - \xi_1\right) + \eta_1\sqrt{1 - \Lambda^2} > y_1 > \frac{\eta_1}{\sqrt{1 - \Lambda^2}}\right) \tag{36a}$$

and $\chi = 0$ for all other regions or

$$\chi = -R(y_1)\frac{\partial}{\partial \xi_1} J_0\left[\frac{\omega}{\beta a}\sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_1{}^2}\right]$$

$$\left(\Lambda(x_1 - \xi_1) + \eta_1\sqrt{1 - \Lambda^2} > y_1 > \frac{\eta_1}{\sqrt{1 - \Lambda^2}}\right) \qquad (36b)$$

and $\chi = 0$ for all other regions where

$$R(y_1) = \frac{\sqrt{1 - \Lambda^2}}{\Lambda} w_0\left[\frac{\beta}{\Lambda}\left(y_1\sqrt{1 - \Lambda^2} - \eta_1\right)\right] e^{\frac{i\omega M}{\Lambda\beta a}\left(y_1\sqrt{1-\Lambda^2}-\eta_1\right) - \frac{i\omega t_1}{\beta a}} \qquad (37)$$

If the indicated differentiation is performed and it is kept in mind that there is a discontinuity in χ along the line $\xi_1 = \frac{\Lambda x_1 + \eta_1\sqrt{1 - \Lambda^2} - y_1}{\Lambda}$, equation (36b) becomes

$$\chi = R(y_1)\left[\delta\left(\xi_1 - \frac{\Lambda x_1 - y_1 + \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right) - \frac{d}{d\xi_1} J_0\left(\frac{\omega}{\beta a}\sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_1{}^2}\right)\right] \qquad (38)$$

The evaluation of potentials ϕ_1 and ϕ_2 (eqs. (28)) together with equation (38) is presented in appendix A and the final results for the velocity potential (eqs. (A29) and (A30)) in the true x',y',t' coordinate system are as follows:

$$\phi_1(x',y',t') = -\frac{1}{\pi} e^{i\omega t'} \int_{|y'-\eta'|}^{\frac{x'}{\beta}} \frac{w_0(x'-\beta\xi)\, e^{-i\omega M\xi/\beta a} \cos \tau}{\sqrt{\xi^2 - (y' - \eta')^2}}\, d\xi \qquad (39)$$

$$\phi_2(x',y',t') = -\frac{e^{i\omega t'}}{\pi}\int_{|y'-\eta'|}^{r_3}\frac{w_0(x'-\beta\xi)\ e^{-i\omega M\xi/\beta a}\cos\tau}{\sqrt{\xi^2-(y'-\eta')^2}}\,d\xi$$

$$+\frac{\omega}{\pi\beta a}\,e^{i\omega t'}\int_{r_3}^{\frac{x'}{\beta}} w_0(x'-\beta\xi)\ e^{-i\omega M\xi/\beta a}\int_0^{r_4}\frac{J_1\left(\tau\sqrt{1-\gamma^2}\right)}{\sqrt{1-\gamma^2}}\,d\gamma \tag{40}$$

where the regions where these potentials exist are given by equations (30a) and (30b) and are shown in sketch 5 and

$$\left.\begin{aligned}
\tau &= \frac{\omega}{\beta a}\sqrt{\xi^2-(y'-\eta')^2}\\
r_3 &= \frac{2}{\Lambda+1}\left(\frac{\Lambda x'}{\beta}+y'\right)-(y'-\eta')\\
r_4 &= \left\{\frac{4\left(\frac{\Lambda x'}{\beta}+y'\right)\left(\frac{\Lambda x'}{\beta}+\eta'-\Lambda\xi\right)}{\left(1-\Lambda^2\right)\left[\xi^2-(y'-\eta')^2\right]}\right\}^{1/2}
\end{aligned}\right\} \tag{41}$$

Frequency expansion of velocity potential and pressure coefficients.- An analytic evaluation of equations (39) and (40) does not seem possible at the present; therefore, a downwash distribution

$$w(x',y',t') = K_n x'^n\ e^{i\omega t'}\ \delta(y'-\eta') \tag{42}$$

is chosen, and a frequency expansion of the integrands of equations (39) and (40) is performed before integration. It might be noted that comparison of equations (42) and (33) indicates that

$$w_0(x') = K_n x'^n \tag{43}$$

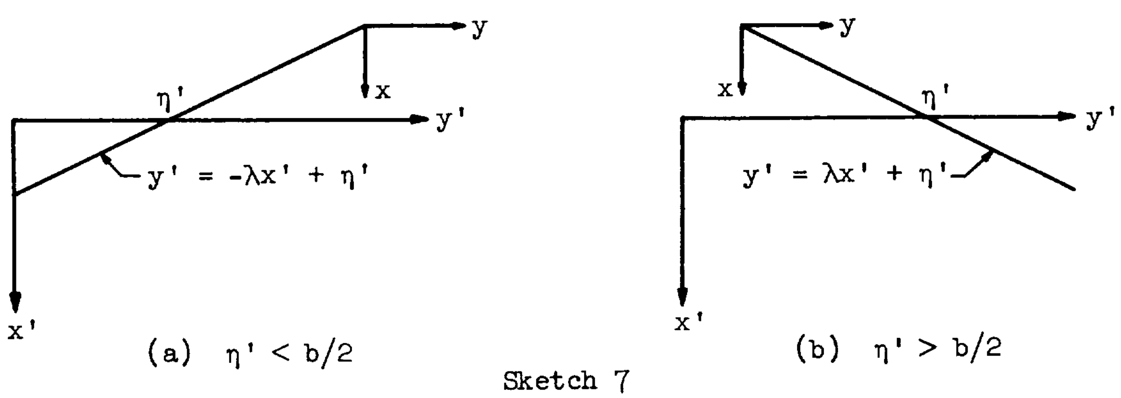
where K_n is a normalizing factor having the dimensions of velocity divided by the nth power of a length.

The results of the above-mentioned expansion are presented in appendix B. Equations (B5) to (B22) represent the pressure distribution for a wing with a supersonic leading edge and subsonic side edge $\left(y' = -\frac{\Lambda x'}{\beta}\right)$ in supersonic flow. The downwash distribution on this wing is given by equation (42). By superposition techniques, the pressure distribution over the entire wing for any harmonic deformations can be obtained; however, the amount of work and time involved becomes very lengthy. Therefore, the pressure coefficients are derived only for the special case where the side edge was parallel to the free stream. These coefficients are presented in appendix C as equations (C11) to (C21) for values of $n = 0$, 1, 2, and 3. The values of $n = 0$, 1, 2, and 3 represent a chordwise strip dy' of the wing at $y' = \eta'$ undergoing translation, pitching, parabolic bending, and cubic bending, respectively.

In most analyses it is desirable to have the origin on the center line of the wing. Therefore, in the section to follow a coordinate system is chosen so that the origin is at the apex of a sweptback wing. By assigning a given spanwise variation of deflection and integrating over the appropriate region of the wing, the pressure distribution can be obtained for any spanwise variation of deformations and up to a cubic in chordwise variation of deformations.

Loading Coefficients for Polynomial Downwash Distribution

Transformation to a coordinate system fixed on apex of wing.- A sketch of the wing together with the new coordinate system, fixed on the apex of the wing, is shown in sketch 7 where λ is the slope of the leading edge and b is the wing span. Two sketches are needed depending on whether η' is greater than or less than $b/2$.

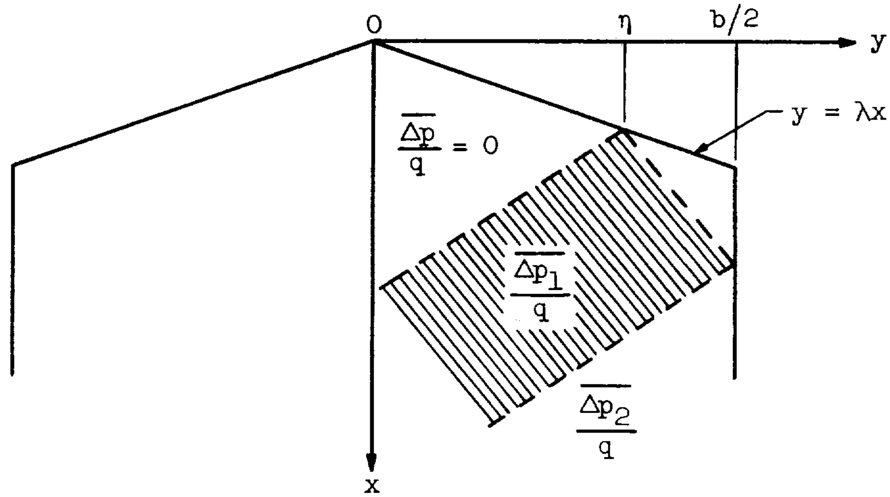


Sketch 7

Inspection of sketch 7 shows the appropriate transformation to be

$$\left.\begin{aligned} x' &= x - \frac{|\eta|}{\lambda} \\ y' &= y + \frac{b}{2} \\ \eta' &= \eta + \frac{b}{2} \end{aligned}\right\} \tag{44}$$

The resulting figure showing the wing, new coordinate system, and the regions where the appropriate pressure coefficients apply is shown in sketch 8:



Sketch 8

and the downwash equation becomes

$$w(x,y,t) = K_n e^{i\omega t}\left(x - \frac{|y|}{\lambda}\right)^n \delta(y-\eta) \tag{45}$$

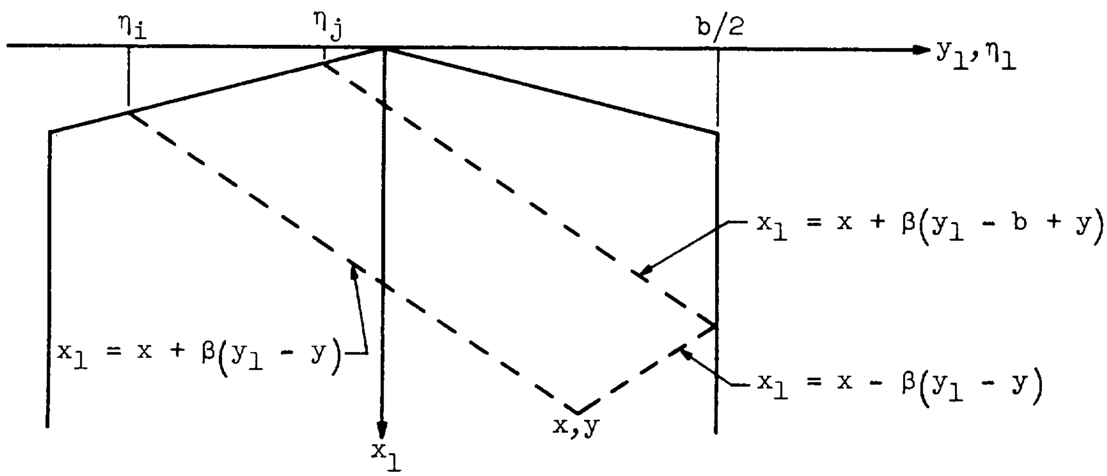
General loading coefficients.- If the wing is now divided into regions according to characteristic Mach wave reflections (see sketch 8) and the downwash is given by

$$w(x,y,t) = K_n e^{i\omega t}\left(x - \frac{|y|}{\lambda}\right)^n \sum_s A_s y^s \tag{46}$$

then the pressure at any point will consist of terms of the form

$$L_k\Big|_{\eta_i}^{\eta_j} = \int_{\eta_i}^{\eta_j} \sum_s A_s \eta^s \frac{\Delta p_k}{q}\left(x - \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta \tag{47}$$

where η_i and η_j are to be chosen according to the region in which the pressure coefficient is to be calculated and $\Delta p_k/q$ values are given in appendix C. As an example, consider the pressure at point x,y indicated in sketch 9



Sketch 9

from which it can be seen for this particular case

$$\eta_i = \lambda \frac{\beta y - x}{\lambda\beta + 1}$$

$$\eta_j = \lambda \frac{\left[\beta(b - y) - x\right]}{\lambda\beta + 1}$$

The pressure at point x,y can now be written as

$$l(x,y) = \sum_s A_s \left[\int_{\eta_i}^{\eta_j} \eta_s \frac{\overline{\Delta p_1}}{q}\left(x + \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta + \int_{\eta_j}^{0} \eta^s \frac{\overline{\Delta p_2}}{q}\left(x + \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta \right.$$

$$\left. + \int_0^{b/2} \eta^s \frac{\overline{\Delta p_2}}{q}\left(x - \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta \right] \tag{48}$$

The last term has the form of equation (47) and can be replaced by $L_2\Big|_0^{b/2}$.

The first term

$$\sum A_s \int_{\frac{\lambda(\beta y - x)}{\beta\lambda + 1}}^{\lambda\left[\frac{\beta(b-y)-x}{\beta\lambda+1}\right]} \eta^s \frac{\overline{\Delta p_1}}{q}\left(x + \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta$$

becomes

$$\sum A_s \int_{\frac{\beta y - x}{\beta\lambda - 1}}^{\lambda\left[\frac{\beta(b-y)-x}{\beta\lambda-1}\right]} \eta^s \frac{\overline{\Delta p_1}}{q}\left(x - \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta$$

when λ is replaced by $(-\lambda)$. This expression is now in the form of equation (47) and can be represented by $\overline{L}_1\Big|_{\eta_i(-\lambda)}^{\eta_j(-\lambda)}$ where the bar (-) is used to indicate that λ must be changed to $(-\lambda)$ before inserting the limits. This definition was adopted so as to utilize the definition of L_k given by equation (47) and thus eliminate the derivation of a new set of equations. Performing the same operation on the second term of equation (48) results in the following expression for the loading coefficients at x,y.

$$l(x,y) = \bar{L}_1\Bigg|_{\eta_i(-\lambda)}^{\eta_j(-\lambda)} + \bar{L}_2\Bigg|_{\eta_j(-\lambda)}^{0} + L_2\Bigg|_{0}^{b/2} \tag{49}$$

The L_1 and L_2 functions are derived in appendix D for $n = 0, 1, 2,$ and 3. The limits are left arbitrary so that the pressure coefficients at various positions on the wing (other than the position considered above) could be determined.

Loading coefficients for separate regions on the wing.- If the wing is now divided into the regions indicated in figure 1, the loading coefficients for each region $\left(l_I,\ l_{II},\ \text{etc.}\right)$ can be shown to be

$$l_I = L_1\Bigg|_{\eta_0}^{\eta_1} \tag{50}$$

$$l_{II} = L_1\Bigg|_{\eta_0}^{\eta_2} + L_2\Bigg|_{\eta_2}^{b/2} \tag{51}$$

$$l_{III} = L_1\Bigg|_{0}^{\eta_2} - \bar{L}_1\Bigg|_{0}^{\eta_0} + L_2\Bigg|_{\eta_2}^{b/2} = l_{II} + L_1\Bigg|_{0}^{\eta_0} - \bar{L}_1\Bigg|_{0}^{\eta_0} \tag{52}$$

$$l_{IV} = \bar{L}_1\Bigg|_{\eta_0}^{\eta_2} + \bar{L}_2\Bigg|_{\eta_2}^{0} + L_2\Bigg|_{0}^{b/2} \tag{53}$$

$$l_V = \bar{L}_1\Bigg|_{\eta_0}^{0} + L_1\Bigg|_{0}^{\eta_1} = l_I - l_{II} + l_{III} \tag{54}$$

$$l_{VI} = \bar{L}_1\Big|_{\eta_3}^{0} + L_1\Big|_{0}^{\eta_2} + \bar{L}_2\Big|_{-b/2}^{\eta_3} + L_2\Big|_{\eta_2}^{b/2} \tag{55}$$

$$l_{VII} = \bar{L}_1\Big|_{\eta_3}^{\eta_2} + \bar{L}_2\Big|_{-b/2}^{\eta_3} + \bar{L}_2\Big|_{\eta_2}^{0} + L_2\Big|_{0}^{b/2} = l_{VI} + l_{IV} - l_{III} \tag{56}$$

where

$$\left.\begin{aligned}
\eta_0 &= \frac{\lambda(\beta y - x)}{\beta\lambda - 1} \\
\eta_1 &= \frac{\lambda(\beta y + x)}{\beta\lambda + 1} \\
\eta_2 &= \frac{\lambda\left[\beta(b - y) - x\right]}{\beta\lambda - 1} \\
\eta_3 &= \frac{\lambda\left[x - \beta(y + b)\right]}{\beta\lambda + 1}
\end{aligned}\right\} \tag{57}$$

and the values of L_1 and L_2 are given in appendix D as equations (D11) to (D18) and, as pointed out previously, the bar (-) indicates that λ must be changed to $-\lambda$ before substituting in the limits.

RESULTS AND DISCUSSION

The integral expressions for the velocity potential and pressure coefficients associated with a wing with swept supersonic leading edges and arbitrarily swept subsonic side edges, deforming in any general time-dependent manner, are derived herein. The expressions are simplified by first assuming harmonic motion and then expanding to the third power of frequency. The special case is then treated for which (1) the side edge is parallel to free stream and (2) the oscillations are such that the distortion of the wing can be represented by a polynomial of any desired degree in the span coordinate and third degree in the chordwise coordinate.

Calculations are made to obtain the total lift coefficients for two wings - a 50° delta wing and a rectangular wing of aspect ratio 0.8 - both flying at a Mach number of 3.0. The wings are assumed to be subjected to continuous sinusoidal gusts and to harmonic sinking oscillations. No spanwise variation in downwash is considered. This analysis is presented in appendix E and the final results are tabulated in table I. As can be seen in table I, the results are in good agreement with those obtained by using reference 14 for reduced frequencies at least as high as those indicated in the table. It might be noted that the results for the wing in a sinusoidal gust are not as good as those for a wing undergoing harmonic sinking oscillations. It is believed that this difference is due to the fact that the sinusoidal gust wave is approximated by a cubic in the chord direction, whereas the downwash for harmonic sinking oscillations is exact.

CONCLUDING REMARKS

The integral expressions for the velocity potential and pressure coefficients associated with a wing with swept supersonic leading edges and arbitrarily swept subsonic side edges, deforming in any general time-dependent manner, are derived herein. The expressions are very complicated; however, with the modern high-speed computers it does not seem unreasonable to undertake such a task. As a possible check the equations are simplified by assuming simple harmonic motion and expanding to the third power of frequency. The special case is treated for which the side edge is parallel to the free stream and the oscillations are such that the distortions of the wing can be represented by a polynomial of any desired degree in the span coordinate and third degree in the chordwise coordinate.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., August 1, 1962.

APPENDIX A

REDUCTION OF THE VELOCITY POTENTIAL EQUATION

WITH APPLICATION TO A RECTANGULAR WING

Reduction of Velocity Potential Equations

The velocity potential as defined in the text by equations (28), (37), and (38) will be treated in six parts as indicated below. From equation (38), let L 5 3 0

$$X(\xi_1,x_1,y_1,t_1) = X_1(\xi_1,x_1,y_1,t_1) - X_2(\xi_1,x_1,y_1,t_1) \tag{A1}$$

where

$$\left.\begin{aligned}
X_1(\xi_1,x_1,y_1,t_1) &= R(y_1)\,\delta\left(\xi_1 - \frac{\Lambda x_1 - y_1 + \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right) \\
X_2(\xi_1,x_1,y_1,t_1) &= R(y_1)\frac{\partial}{\partial\xi_1}\,J_0\left[\frac{\omega}{\beta a}\sqrt{\left(x_1 - \frac{y_1 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_1^{\,2}}\right]
\end{aligned}\right\} \tag{A2}$$

By substituting equations (A1) and (A2) into equations (28) and defining the quantities

$$\left.\begin{aligned}
\theta_1(\xi_2,x_1,y_1,t_1;\ y_2) &= \frac{X_1(\xi_2,x_1,y_2,t_1)}{\sqrt{\xi_2^{\,2} - (y_1 - y_2)^2}} \\
\theta_2(\xi_2,x_1,y_1,t_1;\ y_2) &= \frac{X_2(\xi_2,x_1,y_2,t_1)}{\sqrt{\xi_2^{\,2} - (y_1 - y_2)^2}}
\end{aligned}\right\} \tag{A3}$$

the velocity potential of equations (28) can be written as

$$\phi_1(x_1,y_1,0,t_1) = \phi_{11} - \phi_{12} \tag{A4}$$

$$\phi_2\left(x_1,y_1,0,t_1\right) = \phi_{21} + \phi_{22} \tag{A5a}$$

$$\phi_{21} = \phi_{211} - \phi_{212} \tag{A5b}$$

$$\phi_{22} = \phi_{221} - \phi_{222} \tag{A5c}$$

where

$$\phi_{11} = -\frac{1}{\pi}\int_{\eta_1/\sqrt{1-\Lambda^2}}^{r_0} dy_2 \int_{y_2-y_1}^{r_1} \theta_1 \, d\xi_2 \tag{A6}$$

$$\phi_{12} = -\frac{1}{\pi}\int_{\eta_1/\sqrt{1-\Lambda^2}}^{r_0} dy_2 \int_{y_2-y_1}^{r_1} \theta_2 \, d\xi_2 \tag{A7}$$

$$\phi_{211} = -\frac{1}{\pi}\int_{r_2}^{r_0} dy_2 \int_{y_2-y_1}^{r_1} \theta_1 \, d\xi_2 \tag{A8}$$

$$\phi_{212} = -\frac{1}{\pi}\int_{r_2}^{r_0} dy_2 \int_{y_2-y_1}^{r_1} \theta_2 \, d\xi_2 \tag{A9}$$

$$\phi_{221} = -\frac{1}{\pi}\int_{\eta_1/\sqrt{1-\Lambda^2}}^{r_2} dy_2 \int_{y_2-y_1}^{y_2+y_1} \theta_1 \, d\xi_2 \tag{A10}$$

$$\phi_{222} = -\frac{1}{\pi}\int_{\eta_1/\sqrt{1-\Lambda^2}}^{r_2} dy_2 \int_{y_2-y_1}^{y_2+y_1} \theta_2 \, d\xi_2 \tag{A11}$$

and r_0, r_1, and r_2 are defined by equations (29).

Since the area of integration does not include the line $\xi_2 = \dfrac{\Lambda x_1 + \eta_1\sqrt{1 - \Lambda^2} - y_2}{\Lambda}$ (which is the argument of the Dirac delta),

$$\phi_{221} = 0 \tag{A12}$$

After the integration with respect to ξ_2, equations (A6) and (A8) may be represented by

$$\phi_k = -\frac{1}{\pi}\int_{r_i}^{r_j} \frac{R\left(y_2\right) dy_2}{\sqrt{\left(\dfrac{\Lambda x_1 - y_2 + \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \left(y_1 - y_2\right)^2}} \tag{A13}$$

By transforming equation (A13) to the true x',y' coordinates by means of equations (5) and making the substitution

$$y_2 = \frac{1}{\sqrt{1 - \Lambda^2}}\left(-\Lambda\xi + \eta' + \frac{\Lambda x'}{\beta}\right) \tag{A14}$$

equation (A13) becomes

$$\phi_k = \frac{1}{\pi} e^{i\omega t'} \int_{\overline{r_i}}^{\overline{r_j}} \frac{w_0(x'-\beta\xi)\, e^{-\frac{i\omega M}{\beta a}\xi}}{\sqrt{\xi^2 - (y' - \eta')^2}} \, d\xi \tag{A15}$$

With the appropriate substitutions for $\overline{r_i}$ and $\overline{r_j}$ it can be shown that equations (A6) and (A8) become, respectively,

$$\phi_{11} = -\frac{1}{\pi} e^{i\omega t'} \int_{-(y'-\eta')}^{x'/\beta} \frac{w_0(x'-\beta\xi)\, e^{-\frac{i\omega M}{\beta a}\xi}}{\sqrt{\xi^2 - (y' - \eta')^2}}\, d\xi \tag{A16}$$

$$\phi_{211} = -\frac{1}{\pi} e^{i\omega t'} \int_{-(y'-\eta')}^{\frac{2}{1+\Lambda}\left(\frac{\Lambda x'}{\beta} + y\right) - (y'-\eta')} \frac{w_0(x'-\beta\xi)\, e^{-\frac{i\omega M}{\beta a}\xi}}{\sqrt{\xi^2 - (y' - \eta')^2}}\, d\xi \tag{A17}$$

In order to reduce equations (A7), (A9), and (A11) the expression

$$\phi_m = -\frac{1}{\pi} \int_{r_i}^{r_j} dy_2 \int_{y_2-y_1}^{r_k} \theta_1\, d\xi_2 = -\frac{1}{\pi} \int_{r_i}^{r_j} R(y_2)\, dy_2 \int_{y_2-y_1}^{r_k} \frac{\frac{d}{d\xi_2} J_0\left(\frac{\omega}{\beta a}\sqrt{\left(x_1 - \frac{y_2 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - \xi_2^{\,2}}\right)}{\sqrt{\xi_2^{\,2} - (y_1 - y_2)^2}}\, d\xi_2 \tag{A18}$$

will be used to represent these three equations. By means of the substitution

$$\xi_2^{\,2} = \left[\left(x_1 - \frac{y_2 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - (y_1 - y_2)^2\right]\gamma^2 + (y_1 - y_2)^2 \tag{A19}$$

equation (A18) takes the form

$$\phi_m = -\frac{1}{\pi}\int_{r_i}^{r_j}\frac{R(y_2)\,dy_2}{\sqrt{\left(x_1 - \frac{y_2 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - (y_1 - y_2)^2}}\int_0^{\overline{r_k}}\frac{\frac{d}{d\gamma}J_0\left(\frac{\omega}{\beta a}\sqrt{\left[\left(x_1 - \frac{y_2 - \eta_1\sqrt{1 - \Lambda^2}}{\Lambda}\right)^2 - (y_1 - y_2)\right](1 - \gamma^2)}\right)}{\gamma}\,d\gamma \tag{A20}$$

which, after transforming to the true x',y',t' coordinates by means of equations (5) and making the substitution given by equation (A14), reduces to

$$\phi_m = \frac{1}{\pi}e^{i\omega t'}\int_{\overline{r_i}}^{\overline{r_j}}\frac{w_0(x'-\beta\zeta)\,e^{-\frac{i\omega M\zeta}{\beta a}}}{\sqrt{\zeta^2 - (y' - \eta')^2}}\,d\zeta\int_0^{\hat{r}_k}\frac{\frac{d}{d\gamma}J_0\left(\frac{\omega}{\beta a}\sqrt{\left[\zeta^2 - (y' - \eta')^2\right](1 - \gamma^2)}\right)}{\gamma}\,d\gamma$$

$$= \frac{\omega}{\pi\beta a}e^{i\omega t'}\int_{\overline{r_i}}^{\overline{r_j}}w_0(x'-\beta\zeta)\,e^{-\frac{i\omega M\zeta}{\beta a}}\,d\zeta\int_0^{\hat{r}_k}\frac{J_1\left(\frac{\omega}{\beta a}\sqrt{\left[\zeta^2 - (y' - \eta')^2\right](1 - \gamma^2)}\right)}{\gamma}\,d\gamma \tag{A21}$$

An expression that will be needed later is that for $\hat{r}_k = 1$ the second integral in equation (A21) becomes

$$\int_0^1\frac{J_1\left(\tau\sqrt{1 - \Lambda^2}\right)\,d\gamma}{\sqrt{1 - \gamma^2}} \tag{A22}$$

where

$$\tau = \frac{\omega}{\beta a}\sqrt{\zeta^2 - (y' - \eta')^2} \tag{A23}$$

By letting $\gamma = \sin\theta$, the integral (A22) becomes

$$\int_0^{\pi/2}J_1(\tau\sin\theta)\,d\theta \tag{A24a}$$

which is evaluated by Watson (ref. 21, p. 374) and is given as

$$\frac{1}{\tau}(1 - \cos \tau) \tag{A24b}$$

By substituting the appropriate values for $\overline{r_i}$, $\overline{r_j}$, and $\hat{r}_k$ into equation (A21), it can be shown that

$$\phi_{12} = -\frac{1}{\pi} e^{i\omega t'} \int_{-(y'-\eta')}^{x'/\beta} \frac{w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta}(1 - \cos \tau)\, d\zeta}{\sqrt{\zeta^2 - (y' - \eta')^2}} \tag{A25}$$

$$\phi_{212} = -\frac{1}{\pi} e^{i\omega t'} \int_{-(y'-\eta')}^{r_3} \frac{w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta}(1 - \cos \tau)}{\sqrt{\zeta^2 - (y' - \eta')^2}}\, d\zeta \tag{A26}$$

$$\phi_{222} = -\frac{\omega}{\pi\beta a} e^{i\omega t'} \int_{r_3}^{x'/\beta} w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta} d\zeta \int_0^{r_4} \frac{J_1\left(\tau\sqrt{1 - \gamma^2}\right)}{\sqrt{1 - \gamma^2}}\, d\gamma \tag{A27}$$

where

$$r_4{}^2 = \frac{4\left(\frac{\Lambda x'}{\beta} + y'\right)\left(\frac{\Lambda x'}{\beta} + \eta' - \Lambda\zeta\right)}{\left(1 - \Lambda^2\right)\left[\zeta^2 - (y' - \eta')^2\right]} \tag{A28}$$

Consequently, if equations (A12), (A16), (A17), (A25), (A26), and (A27) are substituted into equations (A4) and (A5a), the resulting form for the potential is

$$\phi_1(x',y',t') = -\frac{1}{\pi} e^{i\omega t'} \int_{|y'-\eta'|}^{x'/\beta} \frac{w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta} \cos\tau}{\sqrt{\zeta^2 - (y'-\eta')^2}}\, d\zeta \qquad \text{(A29)}$$

$$\phi_2(x',y',t') = -\frac{1}{\pi} e^{i\omega t'} \int_{|y'-\eta'|}^{r_3} \frac{w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta} \cos\tau}{\sqrt{\zeta^2 - (y'-\eta')^2}}\, d\zeta$$

$$+ \frac{\omega}{\pi\beta a} e^{i\omega t'} \int_{r_3}^{x'/\beta} w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta}\, d\zeta \int_0^{r_4} \frac{J_1\left(\tau\sqrt{1-\gamma^2}\right)}{\sqrt{1-\gamma^2}}\, d\gamma \qquad \text{(A30)}$$

Application to a Rectangular Wing

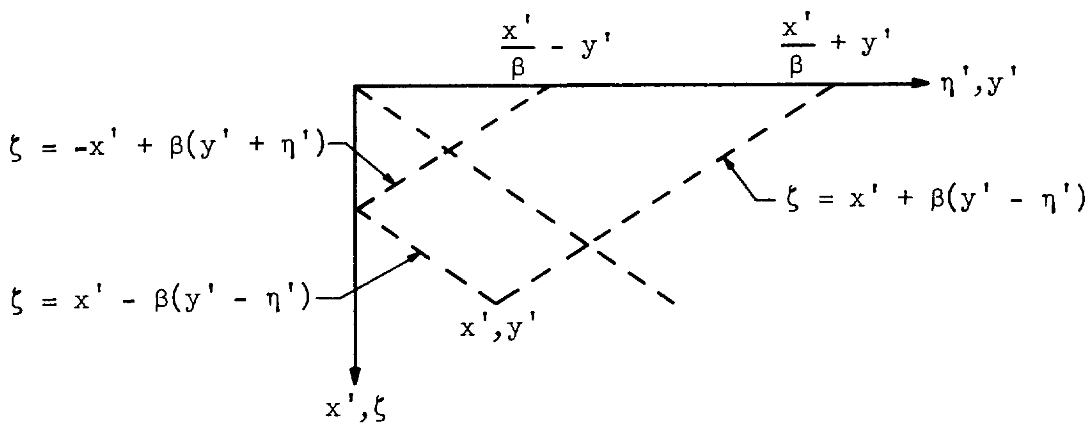
It might be noted that, as Λ approaches 0, equations (A29) and (A30) become

$$\left(\phi_1\right)_{\Lambda=0} = -\frac{1}{\pi} e^{i\omega t'} \int_{|y'-\eta'|}^{x'/\beta} \frac{w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta} \cos\tau}{\sqrt{\zeta^2 - (y'-\eta')^2}}\, d\zeta \qquad \text{(A31)}$$

$$(\phi_2)_{\Lambda=0} = -\frac{1}{\pi}\, e^{i\omega t'} \int_{|y'-\eta'|}^{y'+\eta'} \frac{w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta} \cos\tau}{\sqrt{\zeta^2 - (y'-\eta')^2}}\, d\zeta$$

$$+ \frac{\omega}{\pi\beta a}\, e^{i\omega t'} \int_{y'+\eta'}^{x'/\beta} w_0(x'-\beta\zeta)\, e^{-\frac{i\omega M}{\beta a}\zeta}\, d\zeta \int_0^{\left[\frac{4y'\eta'}{\zeta^2-(y'-\eta')^2}\right]^{1/2}} \frac{J_1\left(\tau\sqrt{1-\gamma^2}\right)}{\sqrt{1-\gamma^2}}\, d\gamma \qquad \text{(A32)}$$

It is now desired to find the potential for a rectangular wing for which $w(x',y',t') = w_0(x')\, e^{i\omega t'}$. Since equations (A31) and (A32) apply only for a downwash strip in the free-stream direction at $y' = \eta'$, these potentials must be integrated with respect to η' over the proper limits. From an examination of sketch 10



Sketch 10

the potential for the rectangular wing can be written as

$$\phi_{x'<\beta y'} = \int_{y'-\frac{x'}{\beta}}^{y'+\frac{x'}{\beta}} (\phi_1)_{\Lambda=0}\, d\eta' \qquad \text{(A33)}$$

$$\phi_{x'>\beta y'} = \int_0^{\frac{x'}{\beta}-y'} \left(\phi_2\right)_{\Lambda=0} d\eta' + \int_{\frac{x'}{\beta}-y'}^{y'+\frac{x'}{\beta}} \left(\phi_1\right)_{\Lambda=0} d\eta' \qquad (A34)$$

An evaluation of the integrals in equations (A33) and (A34) will yield the results given by equation (15) of reference 10.

APPENDIX B

FREQUENCY EXPANSION OF THE VELOCITY POTENTIAL AND PRESSURE COEFFICIENTS

Substitution of the downwash $w_0(x') = K_n x'^n$ (eq. (43)) into the expression for the velocity potential ϕ_1 (eq. (39)) and expanding the integrand in powers of frequency, and integrating yields the following results (where the primes have now been omitted from x, y, t, and η)

$$\phi_1(x,y,t) = -\frac{1}{\pi} e^{i\omega t} K_n \sum_{p=0} (-i\bar{\omega})^p h_p N_p \tag{B1}$$

where

$$\left.\begin{aligned} h_p &= \sum_{r=0}^{[p/2]} \frac{1}{M^{2r}(2r)!(p-2r)!} \sum_{s=0}^{n} \frac{(-1)^s n!\left(\beta\left|y-\eta\right|\right)^{s+p}}{s!(n-s)!} \sum_{m=0}^{r} \frac{(-1)^m r!}{(r-m)!m!} \\ N_p &= x^{n-s} \int_0^{\cosh^{-1}\frac{x}{\beta|y-\eta|}} \cosh^{s+p-2m}\phi \, d\phi \end{aligned}\right\} \tag{B2}$$

$$\bar{\omega} = \frac{M^2\omega}{V\beta^2} \tag{B3}$$

and $[p/2]$ is the integer part of $p/2$. The pressure coefficient associated with ϕ_1 is

$$\frac{\Delta p_1}{q} = e^{i\omega t}\,\frac{\overline{\Delta p_1}}{q} = \frac{4}{V^2}\left(\frac{\partial\phi_1}{\partial t} + V\,\frac{\partial\phi_1}{\partial x}\right) \tag{B4}$$

or

$$\frac{\overline{\Delta p_1}}{q} = \frac{4}{\pi V} K_n \sum_{p=0} (-i\bar{\omega})^p h_p \left[(-i\bar{\omega})\frac{\beta^2}{M^2} N_p - N_p'\right] \tag{B5}$$

and

$$\left.\begin{aligned} N_p' &= \frac{dN_p}{dx} = \frac{n - s}{x} N_p + \frac{x^{n-s}\left(\dfrac{x}{\beta|y - \eta|}\right)^{s+p-2m}}{\sqrt{x^2 - \beta^2(y - \eta)^2}} && (n \neq 0) \\ N_p' &= \frac{\left(\dfrac{x}{\beta|y - \eta|}\right)^{p-2m}}{\sqrt{x^2 - \beta^2(y - \eta)^2}} && (n = 0) \end{aligned}\right\} \tag{B6}$$

Similarly, equation (40) can be expanded to give

$$\frac{\overline{\Delta p_{21}}}{q} = \frac{4}{\pi V} K_n \sum_{p=0} (-i\bar{\omega})^p h_p \left[(-i\bar{\omega})\frac{\beta^2}{M^2} \overline{N_p} - \overline{N_p'}\right] \tag{B7}$$

$$\frac{\overline{\Delta p_{22}}}{q} = \frac{4(-i\bar{\omega})^2}{\pi V} K_n \sum_{m=0} (-i\bar{\omega})^m \sum_{q=0}^{m} \left[g'_{m-q}F_q + g_{m-q}F_q' - \frac{\beta^2}{M^2}(-i\bar{\omega})g_{m-q}F_q\right] \tag{B8}$$

where the prime indicates differentiation with respect to x, and

$$\overline{\Delta p_2} = \overline{\Delta p_{21}} + \overline{\Delta p_{22}} \tag{B9}$$

$$\overline{N_p} = x^{n-s} \int_0^{\cosh^{-1}\left[\frac{2\left(y + \frac{\Lambda x}{\beta}\right)}{(1+\Lambda)(y-\eta)} - 1\right]} \cosh^{s+p-2m}\phi \, d\phi \tag{B10}$$

$$g_{m-q} = - \frac{2\beta^{n+2}\left(x + \frac{\beta\eta}{\Lambda}\right)^{m-q}}{4M^2(-\Lambda)^{n+1}(m - q)!}\sqrt{\frac{4\left(y + \frac{\Lambda x}{\beta}\right)}{1 - \Lambda^2}} \tag{B11}$$

$$F_q = \sum_{k=0}^{2} b_{q-2k}\, d_k \sum_{s=0}^{n} c_s G_{q-2k,s,k} \tag{B12}$$

$$G_{q,s,k} = \int_{r_5}^{\eta} \mu^{q+s+\frac{1}{2}} f_k(\mu)\, d\mu \tag{B13}$$

$$r_5 = \frac{1 - \Lambda}{1 + \Lambda}\left[\eta(1 + \Lambda) - \Lambda\left(y - \frac{x}{\beta}\right)\right] \tag{B14}$$

$$b_q = \frac{\left(\frac{-\beta}{\Lambda}\right)^q}{q!} \tag{B15}$$

$$c_s = \frac{(-1)^s n!\,\eta^{n-s}}{s!(n - s)!} \tag{B16}$$

$$d_k = \left(\frac{i\beta}{2M}\right)^{2k} \tag{B17}$$

$$f_0(\mu) = 1 \tag{B18}$$

$$f_1(\mu) = - \frac{1}{2}\left(z^2 - \frac{\gamma\mu}{3}\right) \tag{B19}$$

$$f_2(\mu) = \frac{1}{2!3!}\left(z^4 - \frac{2}{3}\gamma\mu z^2 + \frac{\gamma^2\mu^2}{5}\right) \tag{B20}$$

$$\gamma = \frac{4\left(y + \frac{\Lambda x}{\beta}\right)}{1 - \Lambda^2} \tag{B21}$$

$$z^2 = \left(\frac{\eta + \frac{\Lambda x}{\beta} - \mu}{\Lambda}\right)^2 - (y - \eta)^2 \tag{B22}$$

With the limited number of $f_k(\mu)$ functions given (eqs. (B18) to (B20)) equation (B8) can only be expanded to the fourth power of frequency.

APPENDIX C

FREQUENCY EXPANSION OF PRESSURE COEFFICIENTS FOR SIDE EDGE PARALLEL TO FREE STREAM

Since equations (B5) and (B7) of appendix B are independent of the slope Λ attention need only be given to equation (B8). However, since singularities exist in equation (B8), $\left.\frac{\Delta p_{22}}{q}\right|_{\Lambda=0}$ was evaluated by using the second term in equation (A32) of appendix A. The resulting equations are (where the primes have been omitted from x, y, t, and η)

$$\left.\frac{\overline{\Delta p_{22}}}{q}\right|_{\Lambda=0} = K_n \frac{4\beta\sqrt{y\eta}}{\pi M^2 V}(-i\bar{\omega})^2 \sum_{q=0}(-i\bar{\omega})^q\left[(-i\bar{\omega})\frac{\beta^2}{M^2} H_{s,q,k} - H'_{s,q,k}\right] \tag{C1}$$

where the prime indicates differentiation with respect to x and

$$H_{s,q,k} = \sum_{k=0}^{2} a_{q-2k} \sum_{s=0}^{n} c_s \, d_k K_{s,q-2k,k} \tag{C2}$$

$$K_{s,q,k} = x^{n-s} \int_{\beta(y+\eta)}^{x} \sigma^{q+s} \vartheta_k(\sigma) \, d\sigma \tag{C3}$$

$$a_p = \frac{1}{p!} \tag{C4}$$

$$c_s = \frac{(-1)^s n!}{(n - s)!s!} \tag{C5}$$

$$d_k = \left(\frac{i\beta}{2M}\right)^{2k} \tag{C6}$$

$$\vartheta_0(\sigma) = 1 \tag{C7}$$

$$\vartheta_1(\sigma) = -\frac{1}{2}\left(z_1^{\,2} - \frac{4y\eta}{3}\right) \tag{C8}$$

$$\vartheta_2(\sigma) = \frac{1}{2!3!}\left(z_1^{\,4} - \frac{8y\eta}{3}\,z_1^{\,2} + \frac{16y^2\eta^2}{5}\right) \tag{C9}$$

$$z_1^{\,2} = \frac{1}{\beta^2}\left[\sigma^2 - \beta^2(y - \eta)^2\right] \tag{C10}$$

Evaluating the above expressions for values of $n = 0, 1, 2, 3$ and retaining only terms to the third power of frequency, the following results are obtained. In the following equations $u = \beta|y - \eta|$ and $\bar{u} = \beta(y + \eta)$.

For $n = 0$:

$$\frac{\overline{\Delta p_1}}{q} = -\frac{4}{\pi}\frac{K_0}{V}\left(\frac{1}{\sqrt{x^2 - u^2}} - i\bar{\omega}\left(-\frac{\beta^2}{M^2}\cosh^{-1}\frac{x}{u} + \frac{x}{\sqrt{x^2 - u^2}}\right)\right.$$

$$+ \frac{\bar{\omega}^2}{2M^2\sqrt{x^2 - u^2}}\left[x^2\left(M^2 - 3\right) - u^2\left(2M^2 - 3\right)\right] - \frac{i\bar{\omega}^3}{4}\left\{\frac{\beta^4}{M^4}u^2\cosh^{-1}\frac{x}{u}\right.$$

$$\left.\left.+ \frac{x}{3M^4\sqrt{x^2 - u^2}}\left[x^2\left(M^4 - 6M^2 - 3\right) - 3u^2\left(M^4 - 2M^2 - 1\right)\right]\right\}\right) \tag{C11}$$

$$\frac{\overline{\Delta p_{2i}}}{q} = -\frac{4}{\pi}\frac{K_0}{V}\left\{\frac{i\bar{\omega}\beta^2}{M^2}\cosh^{-1}\frac{\bar{u}}{u} + \frac{\bar{\omega}^2\beta^2}{M^2}\,2\beta\sqrt{y\eta}\right.$$

$$\left.- \frac{i\bar{\omega}^3}{4M^4}\,\beta^2\left[u^2\beta^2\cosh^{-1}\frac{\bar{u}}{u} + \left(M^2 + 1\right)\bar{u}2\beta\sqrt{y\eta}\right]\right\} \tag{C12}$$

$$\frac{\overline{\Delta p_{22}}}{q} = -\frac{4}{\pi}\frac{K_0}{V}\left[-\frac{\bar{\omega}^2}{M^2}\beta\sqrt{y\eta} + \frac{i\bar{\omega}^3}{M^4}\beta\sqrt{y\eta}\left(x + \beta^2\bar{u}\right)\right] \tag{C13}$$

For $n = 1$:

$$\frac{\overline{\Delta p_1}}{q} = -\frac{4K_1}{\pi V}\left(\cosh^{-1}\frac{x}{u} - \frac{i\bar{\omega}}{M^2}\left[-\beta^2 x\cosh^{-1}\frac{x}{u} + (2M^2 - 1)\sqrt{x^2 - u^2}\right]\right.$$

$$+ \frac{\bar{\omega}^2}{4M^2}\left[-3\beta^2 u^2\cosh^{-1}\frac{x}{u} + \left(M^2 - 3\right)x\sqrt{x^2 - u^2}\right]$$

$$\left. - \frac{i\bar{\omega}^3}{4M^4}\left\{\beta^4 xu^2\cosh^{-1}\frac{x}{u} + \frac{\sqrt{x^2 - u^2}}{9}\left[x^2\left(M^4 - 6M^2 - 3\right) - 2u^2\left(8M^4 - 12M^2 + 3\right)\right]\right\}\right) \tag{C14}$$

$$\frac{\overline{\Delta p_{21}}}{q} = -\frac{4K_1}{\pi V}\left(\cosh^{-1}\frac{\bar{u}}{u} - \frac{i\bar{\omega}}{M^2}\left[-\beta^2 x\cosh^{-1}\frac{\bar{u}}{u} + \left(2M^2 - 1\right)2\beta\sqrt{y\eta}\right]\right.$$

$$+ \frac{\bar{\omega}^2}{4M^2}\left\{-3\beta^2 u^2\cosh^{-1}\frac{\bar{u}}{u} + 2\beta\sqrt{y\eta}\left[4\beta^2 x - \left(3M^2 - 1\right)\bar{u}\right]\right\} - \frac{i\bar{\omega}^3}{M^4}\left\{\frac{\beta^4 xu^2}{4}\cosh^{-1}\frac{\bar{u}}{u}\right.$$

$$\left.\left. + \frac{\beta\sqrt{y\eta}}{18}\left[9\left(M^4 - 1\right)x\bar{u} - 2\left(4M^4 + 3M^2 - 3\right)\bar{u}^2 - \left(16M^4 - 24M^2 + 6\right)u^2\right]\right\}\right) \tag{C15}$$

$$\frac{\overline{\Delta p_{22}}}{q} = -\frac{4K_1}{\pi V}\left\{-\frac{\bar{\omega}^2}{M^2}\beta\sqrt{y\eta}(x - \bar{u}) + \frac{i\bar{\omega}^3}{4M^4}2\beta\sqrt{y\eta}(x - \bar{u})\left[x + (2M^2 - 1)\bar{u}\right]\right\} \tag{C16}$$

For $n = 2$:

$$\frac{\overline{\Delta p_1}}{q} = -\frac{4K_2}{\pi V}\Bigg(2\left(x \cosh^{-1}\frac{x}{u} - \sqrt{x^2 - u^2}\right) - \frac{i\bar{\omega}}{2M^2}\Bigg\{-\left[2x^2\beta^2 + \left(3M^2 - 1\right)u^2\right]\cosh^{-1}\frac{x}{u}$$

$$+ \left(5M^2 - 3\right)x\sqrt{x^2 - u^2}\Bigg\} + \frac{\bar{\omega}^2}{2M^2}\Bigg\{-3\beta^2 xu^2 \cosh^{-1}\frac{x}{u} + \frac{\sqrt{x^2 - u^2}}{3}\left[\left(M^2 - 3\right)x^2 + 2\left(4M^2 - 3\right)u^2\right]\Bigg\}$$

$$- i\bar{\omega}^3\Bigg\{\frac{\beta^2 u^2}{16M^4}\left[4\beta^2 x^2 + \left(5M^2 - 1\right)u^2\right]\cosh^{-1}\frac{x}{u} + \frac{x\sqrt{x^2 - u^2}}{144M^4}\left[2\left(M^4 - 6M^2 - 3\right)x^2 - \left(83M^4 - 138M^2 + 39\right)u^2\right]\Bigg\}\Bigg) \quad \text{(C17)}$$

$$\frac{\overline{\Delta p_{21}}}{q} = -\frac{4K_2}{\pi V}\Bigg(2x \cosh^{-1}\frac{\bar{u}}{u} - 4\beta\sqrt{y\eta} - \frac{i\bar{\omega}}{2M^2}\Bigg\{-\left[2\beta^2 x^2 + \left(3M^2 - 1\right)u^2\right]\cosh^{-1}\frac{\bar{u}}{u} + 2\beta\sqrt{y\eta}\left[4\left(2M^2 - 1\right)x - \left(3M^2 - 1\right)\bar{u}\right]\Bigg\}$$

$$+ \frac{\bar{\omega}^2}{2M^2}\Bigg\{-3\beta^2 xu^2 \cosh^{-1}\frac{\bar{u}}{u} + 4\beta\sqrt{y\eta}\left[x^2\beta^2 - \frac{\left(3M^2 - 1\right)}{2}x\bar{u} + \frac{2M^2}{3}\bar{u}^2 + \frac{\left(4M^2 - 3\right)}{3}u^2\right]\Bigg\} - \frac{i\bar{\omega}^3}{16M^4}\Bigg\{\beta^2 u^2\Bigg[4\beta^2 x^2$$

$$+ \left(5M^2 - 1\right)u^2\Bigg]\cosh^{-1}\frac{\bar{u}}{u} + 16\beta\sqrt{y\eta}\Bigg[\frac{\left(M^4 - 1\right)}{2}x^2\bar{u} - \frac{2}{9}\left(4M^4 + 3M^2 - 3\right)x\bar{u}^2 + \frac{1}{12}\left(5M^4 + 6M^2 - 3\right)\bar{u}^3$$

$$- \frac{2}{9}\left(8M^4 - 12M^2 + 3\right)xu^2 + \frac{1}{8}\left(5M^4 - 6M^2 + 1\right)\bar{u}u^2\Bigg]\Bigg\}\Bigg) \quad \text{(C18)}$$

$$\frac{\overline{\Delta p_{22}}}{q} = -\frac{4K_2}{\pi V}\left\{\frac{\bar{\omega}^2}{M^2}\beta\sqrt{\gamma\eta}(x - \bar{u})^2 + \frac{i\bar{\omega}^3}{3M^4}\beta\sqrt{\gamma\eta}(x - \bar{u})^2\left[x + \left(3M^2 - 1\right)\bar{u}\right]\right\} \tag{C19}$$

For $n = 3$:

$$\begin{aligned}\frac{\overline{\Delta p_1}}{q} = -\frac{4K_3}{\pi V}\Bigg(&3\left[\left(x^2 + \frac{u^2}{2}\right)\cosh^{-1}\frac{x}{u} - \frac{3}{2}x\sqrt{x^2 - u^2}\right] - \frac{i\bar{\omega}}{2M^2}\Bigg\{-\left[2\beta^2x^3 + \left(3M^2 - 1\right)3xu^2\right]\cosh^{-1}\frac{x}{u}\\
&+ \frac{1}{3}\sqrt{x^2 - u^2}\left[\left(17M^2 - 11\right)x^2 + 4\left(4M^2 - 1\right)u^2\right]\Bigg\} + \frac{\bar{\omega}^2}{16M^2}\Bigg\{-u^2\left[36\beta^2x^2 + 3\left(5M^2 - 3\right)u^2\right]\cosh^{-1}\frac{x}{u}\\
&+ x\sqrt{x^2 - u^2}\left[2\left(M^2 - 3\right)x^2 + \left(49M^2 - 39\right)u^2\right]\Bigg\} - \frac{i\bar{\omega}^3}{240M^4}\Bigg\{15\beta^2xu^2\left[4\beta^2x^2 + 3\left(5M^2 - 1\right)u^2\right]\cosh^{-1}\frac{x}{u}\\
&+ \sqrt{x^2 - u^2}\left[2\left(M^4 - 6M^2 - 3\right)x^4 - \left(159M^4 - 274M^2 + 83\right)x^2u^2 - 16\left(8M^4 - 8M^2 + 1\right)u^4\right]\Bigg\}\Bigg)\end{aligned} \tag{C20}$$

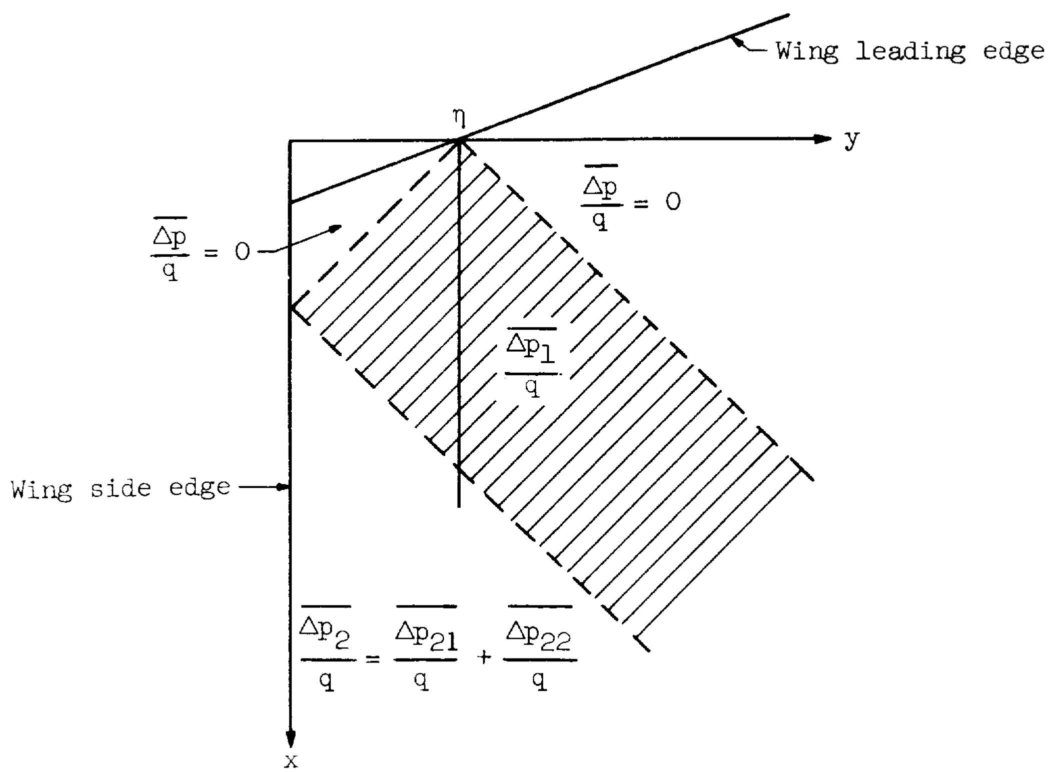
$$\begin{aligned}\frac{\overline{\Delta p_{21}}}{q} = -\frac{4K_3}{\pi V}\Bigg(&\left(3x^2 + \frac{3u^2}{2}\right)\cosh^{-1}\frac{\bar{u}}{u} + 2\beta\sqrt{\gamma\eta}\left(-6x + \frac{3}{2}\bar{u}\right) - \frac{i\bar{\omega}}{M^2}\Bigg\{-x\left[\beta^2x^2 + 3\left(3M^2 - 1\right)\frac{u^2}{2}\right]\cosh^{-1}\frac{\bar{u}}{u} + 2\beta\sqrt{\gamma\eta}\Bigg[3\left(2M^2 - 1\right)x^2\\
&- \frac{3}{2}\left(3M^2 - 1\right)x\bar{u} + \frac{\left(4M^2 - 1\right)}{3}\bar{u}^2 + \frac{2\left(4M^2 - 1\right)}{3}u^2\Bigg]\Bigg\} + \frac{\bar{\omega}^2}{M^2}\Bigg\{-\frac{u^2}{16}\left[36\beta^2x^2 + 3\left(5M^2 - 3\right)u^2\right]\cosh^{-1}\frac{\bar{u}}{u}\\
&+ 2\beta\sqrt{\gamma\eta}\left[\beta^2x^3 - \frac{3\left(3M^2 - 1\right)}{4}x^2\bar{u} + 2M^2x\bar{u}^2 + \left(4M^2 - 3\right)xu^2 - \frac{5M^2 + 1}{8}\bar{u}^3 - \frac{3\left(5M^2 - 3\right)}{16}\bar{u}u^2\right]\Bigg\}\\
&- \frac{i\bar{\omega}^3}{M^4}\Bigg\{\frac{\beta^2xu^2}{16}\left[4\beta^2x^2 + 3\left(5M^2 - 1\right)u^2\right]\cosh^{-1}\frac{\bar{u}}{u} + 2\beta\sqrt{\gamma\eta}\Bigg[\frac{M^4 - 1}{4}x^3\bar{u} - \frac{4M^4 + 3M^2 - 3}{6}x^2\bar{u}^2\\
&+ \frac{5M^4 + 6M^2 - 3}{8}x\bar{u}^3 - \frac{2M^4 + 3M^2 - 1}{10}\bar{u}^4 - \frac{8M^4 - 12M^2 + 3}{6}x^2u^2 + \frac{3\left(5M^4 - 6M^2 + 1\right)}{16}x\bar{u}u^2\\
&- \frac{8M^4 - 8M^2 + 1}{30}\bar{u}^2u^2 - \frac{8M^4 - 8M^2 + 1}{15}u^4\Bigg]\Bigg\}\Bigg)\end{aligned} \tag{C21}$$

$$\frac{\overline{\Delta p_{22}}}{q} = -\frac{4K_3}{\pi V}\left\{-\frac{\bar{\omega}^2}{M^2}\beta\sqrt{\gamma\eta}(x - \bar{u})^3 + \frac{i\bar{\omega}^3}{M^4}\beta\sqrt{\gamma\eta}(x - \bar{u})^3\left[x + \left(4M^2 - 1\right)\bar{u}\right]\right\} \tag{C22}$$

Equations (C11) to (C22) now represent the pressure distribution on a wing (in supersonic flow) with a supersonic leading edge and a side edge parallel to free stream for which the downwash distribution, as mentioned previously (with primes omitted) is

$$w(x,y,t) = K_n e^{i\omega t} x^n \delta(y-\eta) \quad (n = 0, 1, 2, \text{ and } 3)$$

The values of $n = 0, 1, 2$, and 3 represent a chordwise strip of the wing at $y = \eta$ undergoing translation, pitching, parabolic bending, and cubic bending, respectively. The wing and the regions where $\overline{\Delta p}/q$ apply are presented in sketch 11.



Sketch 11

APPENDIX D

EVALUATION OF EQUATION (47)

Equation (47) as given in the text is restated here for convenience. The two forms are

$$L_1\Big|_{\eta_i}^{\eta_j} = L_1 = \sum_s A_s \int_{\eta_i}^{\eta_j} \eta^s \frac{\overline{\Delta p_1}}{q}\left(x - \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta \tag{D1}$$

$$L_2\Big|_{\eta_i}^{\eta_j} = L_2 = \sum_s A_s \int_{\eta_i}^{\eta_j} \eta^s \frac{\overline{\Delta p_2}}{q}\left(x - \frac{\eta}{\lambda},\ y + \frac{b}{2},\ \eta + \frac{b}{2}\right) d\eta \tag{D2}$$

Examination of $\overline{\Delta p_1}/q$ in equations (C11), (C14), (C17), and (C20) indicate the presence of two particular integrals I_s and Q_s which are associated with L_1 and will be defined as

$$I_s = \int_{\eta_i}^{\eta_j} \frac{\eta^s \, d\eta}{\sqrt{\left(x - \frac{\eta}{\lambda}\right)^2 - \beta^2(y - \eta)^2}} \tag{D3}$$

$$Q_s = \int_{\eta_i}^{\eta_j} (y - \eta)^s \cosh^{-1} \frac{x - \frac{\eta}{\lambda}}{\beta|y - \eta|} \, d\eta \tag{D4}$$

where

$$I_0 = \frac{-|\lambda|}{\sqrt{\beta^2\lambda^2 - 1}} \sin^{-1} \frac{\left(\beta^2\lambda^2 y - \lambda x\right) - \left(\beta^2\lambda^2 - 1\right)\eta}{\left|\beta\lambda(\lambda x - y)\right|}\Bigg|_{\eta=\eta_i}^{\eta_j} \tag{D5}$$

$$I_1 = \frac{-|\lambda|}{\beta^2\lambda^2 - 1} \sqrt{(\lambda x - \eta)^2 - \beta^2\lambda^2(y - \eta)^2}\Bigg|_{\eta=\eta_i}^{\eta_j} + \frac{\beta^2\lambda^2 y - \lambda x}{\beta^2\lambda^2 - 1} I_0 \tag{D6}$$

For $s \geqq 2$

$$I_s = \frac{1}{s\left(\beta^2\lambda^2 - 1\right)}\left[-|\lambda|\eta^{s-1}\sqrt{(\lambda x - \eta)^2 - \beta^2\lambda^2(y - \eta)^2}\Bigg|_{\eta=\eta_i}^{\eta_j}\right.$$

$$\left. + (2s - 1)\left(\beta^2\lambda^2 y - \lambda x\right)I_{s-1} - \lambda^2(s - 1)\left(\beta^2\lambda^2 - x^2\right)I_{s-2}\right] \tag{D7}$$

and for $s \geqq 2$

$$Q_s = -\frac{(y - \eta)^{s+1}}{s + 1} \cosh^{-1} \frac{x - \frac{\eta}{\lambda}}{\beta|y - \eta|}\Bigg|_{\eta=\eta_i}^{\eta_j} + \frac{\lambda x - y}{\lambda(s + 1)} \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s - r)!} I_r \tag{D8}$$

Examination of the expressions for $\overline{\Delta p_{21}}/q$ and $\overline{\Delta p_{22}}/q$ given in appendix C indicate the presence of two particular integrals R_s and P_s associated with L_2 and will be defined as

$$R_s = \int_{\eta_i}^{\eta_j} \left(\frac{b}{2} - \eta\right)^s \sqrt{\frac{b}{2} - \eta}\, d\eta = \frac{-\left(\frac{b}{2} - \eta\right)^{s+3/2}}{s + \frac{3}{2}}\Bigg|_{\eta=\eta_i}^{\eta_j} \tag{D9}$$

$$P_s = \int_{\eta_i}^{\eta_j} (y - \eta)^s \cosh^{-1} \frac{b - y - \eta}{|y - \eta|}\, d\eta$$

$$= -\frac{(y - \eta)^{s+1}}{s + 1} \cosh^{-1} \frac{b - y - \eta}{|y - \eta|}\Bigg|_{\eta=\eta_i}^{\eta_j} + \frac{\sqrt{\frac{b}{2} - y}}{s + 1} \sum_{r=0}^{s} \frac{s!\left(y - \frac{b}{2}\right)^{s-r}}{r!(s - r)!} R_{r-1} \tag{D10}$$

By substituting equations (C11) to (C21) into the appropriate expression for L_1 or L_2 and using the quantities defined by equations (D3), (D4), (D9), and (D10), the following equations for L_1 and L_2 are obtained.

For $w = K_0 e^{i\omega t} \sum A_s y^s$:

$$\begin{aligned} L_1 = -\frac{4K_0}{\pi V} \sum A_s \Bigg(& I_s - i\bar{\omega}\left[x I_s - \frac{1}{\lambda} I_{s+1} - \frac{\beta^2}{M^2} \sum_{r=0}^{s} \frac{s!\, y^{s-r}(-1)^r}{r!(s - r)!} Q_r\right] + \frac{\bar{\omega}^2}{2M^2}\Bigg\{\left[\left(M^2 - 3\right)x^2 - \beta^2 y^2\left(2M^2 - 3\right)\right] I_s \\ & - \left[\frac{2x}{\lambda}\left(M^2 - 3\right) - 2\beta^2 y\left(2M^2 - 3\right)\right] I_{s+1} + \left[\frac{M^2 - 3}{\lambda^2} - \beta^2\left(2M^2 - 3\right)\right] I_{s+2}\Bigg\} - \frac{i\bar{\omega}^3}{4}\Bigg\{\frac{\beta^6}{M^4} \sum_{r=0}^{s} \frac{s!\, y^{s-r}(-1)^r}{r!(s - r)!} Q_{r+2} \\ & + \frac{M^4 - 6M^2 - 3}{3M^4}\left(x^3 I_s - \frac{3x^2}{\lambda} I_{s+1} + \frac{3x}{\lambda^2} I_{s+2} - \frac{1}{\lambda^3} I_{s+3}\right) - \frac{\beta^2\left(M^4 - 2M^2 - 1\right)}{M^4}\Bigg[x y^2 I_s - \left(2yx + \frac{y^2}{\lambda}\right) I_{s+1} \\ & + \left(x + \frac{2y}{\lambda}\right) I_{s+2} - \frac{1}{\lambda} I_{s+3}\Bigg]\Bigg\}\Bigg) \end{aligned} \tag{D11}$$

$$\begin{aligned} L_2 = -\frac{4K_0}{\pi V} \sum A_s \Bigg(& + \frac{i\bar{\omega}\beta^2}{M^2} \sum_{r=0}^{s} \frac{s!\, y^{s-r}(-1)^r}{r!(s - r)} P_r + \frac{\bar{\omega}^2 \beta\left(2M^2 - 3\right)}{M^2} \sqrt{\frac{b}{2} - y} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s - r)!} R_r \\ & - i\bar{\omega}^3\Bigg\{\frac{\beta^6}{4M^4} \sum_{r=0}^{s} \frac{s!\, y^{s-r}(-1)^r}{r!(s - r)!} P_{r+2} + \frac{\beta}{2M^4}\sqrt{\frac{b}{2} - y} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s - r)!}\Bigg[-2\left(x - \frac{b}{2\lambda}\right) R_r \\ & - \frac{2}{\lambda} R_{r+1} + \beta^5\left(\frac{b}{2} - y\right) R_r + \beta^5 R_{r+1}\Bigg]\Bigg\}\Bigg) \end{aligned} \tag{D12}$$

For $w = K_1 e^{i\omega t}\left(x - \frac{|y|}{\lambda}\right) \sum A_s y^s$:

$$
\begin{aligned}
L_1 = -\frac{4K_1}{\pi V} \sum A_s \Bigg(& \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{r!(s-r)!} Q_r - \frac{i\bar{\omega}}{M^2} \Bigg\{ -\beta^2 \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{r!(s-r)!} \left[\left(x - \frac{y}{\lambda}\right) Q_r + \frac{1}{\lambda} Q_{r+1}\right] + \left(2M^2 - 1\right)\Bigg[\left(x^2 - \beta^2 y^2\right) I_s \\
& + \left(\frac{-2x}{\lambda} + 2\beta^2 y\right) I_{s+1} + \left(\frac{1}{\lambda^2} - \beta^2\right) I_{s+2}\Bigg]\Bigg\} + \frac{\bar{\omega}^2}{4M^2}\Bigg(-3\beta^4 \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{r!(s-r)!} Q_{r+2} + \left(M^2 - 3\right)\Bigg\{x\left(x^2 - \beta^2 y^2\right) I_s \\
& + \left[x\left(\frac{-2x}{\lambda} + 2\beta^2 y\right) - \frac{1}{\lambda}\left(x^2 - \beta^2 y^2\right)\right] I_{s+1} + \left[x\left(\frac{1}{\lambda^2} - \beta^2\right) - \frac{1}{\lambda}\left(-\frac{2x}{\lambda} + 2\beta^2 y\right)\right] I_{s+2} - \frac{1}{\lambda}\left(\frac{1}{\lambda^2} - \beta^2\right) I_{s+3}\Bigg\}\Bigg) \\
& - \frac{i\bar{\omega}^3}{4M^4}\Bigg\{\beta^6 \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{r!(s-r)!}\left[\left(x - \frac{y}{\lambda}\right) Q_{r+2} + \frac{1}{\lambda} Q_{r+3}\right] + \frac{M^4 - 6M^2 - 3}{9}\left(x^4 I_s - \frac{4x^3}{\lambda} I_{s+1} + \frac{6x^2}{\lambda^2} I_{s+2} - \frac{4x}{\lambda^3} I_{s+3} + \frac{1}{\lambda^4} I_{s+4}\right) \\
& - \frac{\beta^2}{9}\left(17M^4 - 30M^2 + 3\right)\left[x^2 y^2 I_s - \left(2yx^2 + \frac{2xy^2}{\lambda}\right) I_{s+1} + \left(x^2 + \frac{4xy}{\lambda} + \frac{y^2}{\lambda^2}\right) I_{s+2} - \left(\frac{2x}{\lambda} + \frac{2y}{\lambda^2}\right) I_{s+3} + \frac{1}{\lambda^2} I_{s+4}\right] \\
& + \frac{2\beta^4\left(8M^4 - 12M^2 + 3\right)}{9}\left(y^4 I_s - 4y^3 I_{s+1} + 6y^2 I_{s+2} - 4y I_{s+3} + I_{s+4}\right)\Bigg\}\Bigg)
\end{aligned} \tag{D13}
$$

$$
\begin{aligned}
L_2 = -\frac{4K_1}{\pi V} \sum A_s \Bigg[& \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{(s-r)!r!} P_r - i\bar{\omega}\Bigg\{-\frac{\beta^2}{M^2} \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{(s-r)!r!}\left[\left(x - \frac{y}{\lambda}\right) P_r + \frac{1}{\lambda} P_{r+1}\right] + 2\beta\sqrt{\frac{b}{2} - y}\left(\frac{2M^2 - 1}{M^2}\right) \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!} R_r\Bigg\} \\
& + \bar{\omega}^2\Bigg\{-\frac{3\beta^4}{4M^2} \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{r!(s-r)!} P_{r+2} + \frac{\beta\sqrt{\frac{b}{2} - y}}{2M^2} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!}\Bigg[2\left(2M^2 - 3\right)\left(x - \frac{b}{2\lambda}\right) R_r + \frac{2\left(2M^2 - 3\right)}{\lambda} R_{r+1} - 3\beta^3\left(\frac{b}{2} - y\right) R_r \\
& - 3\beta^3 R_{r+1}\Bigg]\Bigg\} - i\bar{\omega}^3\Bigg(\frac{\beta^6}{4M^4} \sum_{r=0}^{s} \frac{s!y^{s-r}(-1)^r}{r!(s-r)!}\left[\left(x - \frac{y}{\lambda}\right) P_{r+2} + \frac{1}{\lambda} P_{r+3}\right] + \frac{\beta\sqrt{\frac{b}{2} - y}}{18M^4} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!}\Bigg\{-9\Bigg[\left(x - \frac{b}{2\lambda}\right)^2 R_r + \frac{2}{\lambda}\left(x - \frac{b}{2\lambda}\right) R_{r+1} \\
& + \frac{1}{\lambda^2} R_{r+2}\Bigg] + 9\beta^5\left[\left(x - \frac{b}{2\lambda}\right)\left(\frac{b}{2} - y\right) R_r + \left(x - \frac{y}{\lambda}\right) R_{r+1} + \frac{1}{\lambda} R_{r+2}\right] - \left(8M^4 - 12M^2 + 3\right)\beta^2\left[\left(\frac{b}{2} - y\right)^2 R_r + 2\left(\frac{b}{2} - y\right) R_{r+1} + R_{r+2}\right] \\
& - \left(16M^4 - 24M^2 + 6\right)\beta^2\left[\left(\frac{b}{2} - y\right)^2 R_r - 2\left(\frac{b}{2} - y\right) R_{r+1} + R_{r+2}\right]\Bigg\}\Bigg)\Bigg]
\end{aligned} \tag{D14}
$$

For $w = K_2 e^{i\omega t}\left(x - \frac{|y|}{\lambda}\right)^2 \sum A_s y^s$:

$$
\begin{aligned}
L_1 = -\frac{4K_2}{\pi V} \sum A_s \Bigg[& 2 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\left[\left(x - \frac{y}{\lambda}\right)Q_r + \frac{1}{\lambda} Q_{r+1}\right] - 2\left(x^2 - \beta^2 y^2\right)I_s + 4\left(\frac{x}{\lambda} - \beta^2 y\right)I_{s+1} - 2\left(\frac{1}{\lambda^2} - \beta^2\right)I_{s+2} \\
& - \frac{i\bar{\omega}}{2M^2}\Bigg(-\sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\Bigg\{2\beta^2\left[\left(x - \frac{y}{\lambda}\right)^2 Q_r + \frac{2}{\lambda}\left(x - \frac{y}{\lambda}\right)Q_{r+1} + \frac{1}{\lambda^2} Q_{r+2}\right] + \left(3M^2 - 1\right)\beta^2 Q_{r+2}\Bigg\} + \left(5M^2 - 3\right)\Bigg[\left(x^3 - \beta^2 y^2 x\right)I_s \\
& - \left(\frac{3x^2}{\lambda} - \frac{\beta^2 y^2}{\lambda} - 2\beta^2 yx\right)I_{s+1} + \left(\frac{3x}{\lambda^2} - \beta^2 x - \frac{2\beta^2 y}{\lambda}\right)I_{s+2} - \left(\frac{1}{\lambda^3} - \frac{\beta^2}{\lambda}\right)I_{s+3}\Bigg]\Bigg) - \frac{\bar{\omega}^2}{2M^2}\Bigg\{3\beta^4 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\left[\left(x - \frac{y}{\lambda}\right)Q_{r+2} + \frac{1}{\lambda} Q_{r+3}\right] \\
& - \frac{M^2 - 3}{3}\left(x^4 I_s - \frac{4x^3}{\lambda} I_{s+1} + \frac{6x^2}{\lambda^2} I_{s+2} - \frac{4x}{\lambda^3} I_{s+3} + \frac{1}{\lambda^4} I_{s+4}\right) - \frac{\beta^2\left(7M^2 - 3\right)}{3}\Bigg[x^2 y^2 I_s - 2\left(x^2 y + \frac{x}{\lambda} y^2\right)I_{s+1} \\
& + \left(x^2 + \frac{4yx}{\lambda} + \frac{y^2}{\lambda^2}\right)I_{s+2} - 2\left(\frac{x}{\lambda} + \frac{y}{\lambda^2}\right)I_{s+3} + \frac{1}{\lambda^2} I_{s+4}\Bigg] + \frac{\beta^4\left(8M^2 - 6\right)}{3}\left(y^4 I_s - 4y^3 I_{s+1} + 6y^2 I_{s+2} - 4y I_{s+3} + I_{s+4}\right)\Bigg\} \\
& - \frac{i\bar{\omega}^3}{16M^4}\Bigg(\beta^4 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\Bigg\{4\beta^2\left[\left(x - \frac{y}{\lambda}\right)^2 Q_{r+2} + \frac{2}{\lambda}\left(x - \frac{y}{\lambda}\right)Q_{r+3} + \frac{1}{\lambda^2} Q_{r+4}\right] + \left(5M^2 - 1\right)\beta^2 Q_{r+4}\Bigg\} + \frac{2\left(M^4 - 6M^2 - 3\right)}{9}\Bigg(x^5 I_s \\
& - \frac{5x^4}{\lambda} I_{s+1} + \frac{10x^3}{\lambda^2} I_{s+2} - \frac{10x^2}{\lambda^3} I_{s+3} + \frac{5x}{\lambda^4} I_{s+4} - \frac{1}{\lambda^5} I_{s+5}\Bigg) - \frac{\beta^2}{9}\left(85M^4 - 150M^2 + 33\right)\Bigg[y^2 x^3 I_s - \left(\frac{3x^2 y^2}{\lambda} + 2yx^3\right)I_{s+1} \\
& + \left(\frac{3xy^2}{\lambda^2} + \frac{6x^2 y}{\lambda} + x^3\right)I_{s+2} - \left(\frac{y^2}{\lambda^3} + \frac{6xy}{\lambda^2} + \frac{3x^2}{\lambda}\right)I_{s+3} + \left(\frac{2y}{\lambda^3} + \frac{3x}{\lambda^2}\right)I_{s+4} - \frac{1}{\lambda^3} I_{s+5}\Bigg] + \frac{\beta^4}{9}\left(83M^4 - 138M^2 + 39\right)\Bigg[xy^4 I_s \\
& - \left(4y^3 x + \frac{y^4}{\lambda}\right)I_{s+1} + \left(6xy^2 + \frac{4y^3}{\lambda}\right)I_{s+2} - \left(4xy + \frac{6y^2}{\lambda}\right)I_{s+3} + \left(x + \frac{4y}{\lambda}\right)I_{s+4} - \frac{1}{\lambda} I_{s+5}\Bigg]\Bigg)\Bigg]
\end{aligned}
\tag{D15}
$$

$$
\begin{aligned}
L_2 = &- \frac{4K_2}{\pi V} \sum A_s \Bigg[2 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg[\Big(x - \frac{y}{\lambda} \Big) P_r + \frac{1}{\lambda} P_{r+1} \bigg] - 4\beta \sqrt{\frac{b}{2} - y} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!} R_r - \frac{i\bar{\omega}}{M^2} \Bigg(- \frac{\beta^2}{2} \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg[2 \Big(x - \frac{y}{\lambda} \Big)^2 P_r \\
&+ \frac{4}{\lambda} \Big(x - \frac{y}{\lambda} \Big) P_{r+1} + \frac{2}{\lambda^2} P_{r+2} + \big(3M^2 - 1 \big) P_{r+2} \bigg] + \beta \sqrt{\frac{b}{2} - y} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!} \bigg\{ 4 \big(2M^2 - 1 \big) \bigg[\Big(x - \frac{b}{2\lambda} \Big) R_r + \frac{1}{\lambda} R_{r+1} \bigg] \\
&- \big(3M^2 - 1 \big) \beta \bigg[\Big(\frac{b}{2} - y \Big) R_r + R_{r+1} \bigg] \bigg\} \Bigg) + \frac{\bar{\omega}^2}{M^2} \Bigg(- \frac{3\beta^4}{2} \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg[\Big(x - \frac{y}{\lambda} \Big) P_{r+2} + \frac{1}{\lambda} P_{r+3} \bigg] + 2\beta \sqrt{\frac{b}{2} - y} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!} \bigg\{ \beta^2 \bigg[\Big(x - \frac{b}{2\lambda} \Big)^2 R_r \\
&+ \frac{2}{\lambda} \Big(x - \frac{b}{2\lambda} \Big) R_{r+1} + \frac{1}{\lambda^2} R_{r+2} \bigg] - \frac{\beta\big(3M^2 - 1\big)}{2} \bigg[\Big(x - \frac{b}{2\lambda} \Big) \Big(\frac{b}{2} - y \Big) R_r + \Big(x - \frac{y}{\lambda} \Big) R_{r+1} + \frac{1}{\lambda} R_{r+2} \bigg] + \frac{2M^2\beta^2}{3} \bigg[\Big(\frac{b}{2} - y \Big)^2 R_r + 2 \Big(\frac{b}{2} - y \Big) R_{r+1} + R_{r+2} \bigg] \\
&+ \frac{\big(4M^2 - 3\big)\beta^2}{3} \bigg[\Big(\frac{b}{2} - y \Big)^2 R_r - 2 \Big(\frac{b}{2} - y \Big) R_{r+1} + R_{r+2} \bigg] - \frac{1}{2} \bigg[x - \frac{b}{2\lambda} - \beta \Big(\frac{b}{2} - y \Big) \bigg]^2 R_r + \bigg[x - \frac{b}{2\lambda} - \beta \Big(\frac{b}{2} - y \Big) \bigg] \Big(\beta - \frac{1}{\lambda} \Big) R_{r+1} - \frac{1}{2} \Big(\beta - \frac{1}{\lambda} \Big)^2 R_{r+2} \bigg\} \Bigg) \\
&- \frac{i\bar{\omega}^3}{M^4} \Bigg(\frac{\beta^6}{16} \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg[4 \Big(x - \frac{y}{\lambda} \Big)^2 P_{r+2} + \frac{8}{\lambda} \Big(x - \frac{y}{\lambda} \Big) P_{r+3} + \frac{4}{\lambda^2} P_{r+4} + \big(5M^2 - 1 \big) P_{r+4} \bigg] + \beta^2 \sqrt{\frac{b}{2} - y} \sum_{r=0}^{s} \frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!} \bigg\{ R_r \bigg[- \frac{1}{3\beta} \Big(x - \frac{b}{2\lambda} \Big)^3 \\
&+ \frac{\beta^4}{2} \Big(x - \frac{b}{2\lambda} \Big)^2 \Big(\frac{b}{2} - y \Big) - \frac{\beta}{3} \Big(x - \frac{b}{2\lambda} \Big) \Big(\frac{b}{2} - y \Big)^2 \big(8M^4 - 12M^2 + 3 \big) + \frac{5\beta^4}{24} \big(5M^2 - 1 \big) \Big(\frac{b}{2} - y \Big)^3 \bigg] + R_{r+1} \bigg[\Big(\frac{\beta^4}{2} - \frac{1}{\beta\lambda} \Big) \Big(x - \frac{b}{2\lambda} \Big)^2 + \beta \Big(\frac{\beta^3}{\lambda} \\
&+ 2 \, \frac{8M^4 - 12M^2 + 3}{9} \Big) \Big(x - \frac{b}{2\lambda} \Big) \Big(\frac{b}{2} - y \Big) + \beta^2 \Big(- \frac{8M^4 - 12M^2 + 3}{3\beta\lambda} + \beta^2 \, \frac{5M^2 - 1}{8} \Big) \Big(\frac{b}{2} - y \Big)^2 \bigg] + R_{r+2} \bigg[\beta \Big(- \frac{1}{\beta^2\lambda^2} + \frac{\beta^4}{\beta\lambda} - \frac{8M^4 - 12M^2 + 3}{3} \Big) \Big(x - \frac{b}{2\lambda} \Big) \\
&+ \beta^2 \Big(\frac{\beta^4}{2\beta^2\lambda^2} + 2 \, \frac{8M^4 - 12M^2 + 3}{9\beta\lambda} + \beta^2 \, \frac{5M^2 - 1}{8} \Big) \Big(\frac{b}{2} - y \Big) \bigg] + R_{r+3} \beta^2 \Big(- \frac{1}{3\beta^3\lambda^3} + \frac{\beta^4}{2\beta^2\lambda^2} - \frac{8M^4 - 12M^2 + 3}{3\beta\lambda} + 5\beta^2 \, \frac{5M^2 - 1}{24} \Big) \bigg\} \Bigg) \Bigg]
\end{aligned}
\qquad \text{(D16)}
$$

For $w = K_3 e^{i\omega t}\left(x - \frac{|y|}{\lambda}\right)^3 \sum A_s y^s$:

$$
\begin{aligned}
L_1 = &- \frac{4K_3}{\pi V} \sum A_s \Bigg[3 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg[\Big(x - \frac{y}{\lambda}\Big)^2 Q_r + \frac{2}{\lambda}\Big(x - \frac{y}{\lambda}\Big) Q_{r+1} + \Big(\frac{1}{\lambda^2} + \frac{\beta^2}{2}\Big) Q_{r+2} \bigg] - \frac{9}{2}\Big(x^3 I_s - \frac{3x^2}{\lambda} I_{s+1} + \frac{3x}{\lambda^2} I_{s+2} - \frac{1}{\lambda^3} I_{s+3}\Big) + \frac{9\beta^2}{2}\bigg[xy^2 I_s \\
&- \Big(2xy + \frac{y^2}{\lambda}\Big) I_{s+1} + \Big(x + \frac{2y}{\lambda}\Big) I_{s+2} - \frac{1}{\lambda} I_{s+3} \bigg] - \frac{i\bar{\omega}}{2M^2}\Bigg(-\beta^2 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg\{ 2\bigg[\Big(x - \frac{y}{\lambda}\Big)^3 Q_r + \frac{3}{\lambda}\Big(x - \frac{y}{\lambda}\Big)^2 Q_{r+1} + \frac{3}{\lambda^2}\Big(x - \frac{y}{\lambda}\Big) Q_{r+2} + \frac{1}{\lambda^3} Q_{r+3} \bigg] \\
&+ 3(3M^2 - 1)\bigg[\Big(x - \frac{y}{\lambda}\Big) Q_{r+2} + \frac{1}{\lambda} Q_{r+3} \bigg] \bigg\} + \frac{17M^2 - 11}{3}\Big(x^4 I_s - \frac{4x^3}{\lambda} I_{s+1} + \frac{6x^2}{\lambda^2} I_{s+2} - \frac{4x}{\lambda^3} I_{s+3} + \frac{1}{\lambda^4} I_{s+4} \Big) - \frac{\beta^2}{3}(M^2 - 7)\bigg[x^2 y^2 I_s \\
&- 2\Big(x^2 y + \frac{xy^2}{\lambda}\Big) I_{s+1} + \Big(x^2 + \frac{4xy}{\lambda} + \frac{y^2}{\lambda^2}\Big) I_{s+2} - 2\Big(\frac{x}{\lambda} + \frac{y}{\lambda^2}\Big) I_{s+3} + \frac{1}{\lambda^2} I_{s+4} \bigg] - \frac{4\beta^4(4M^2 - 1)}{3}\Big(y^4 I_s - 4y^3 I_{s+1} + 6y^2 I_{s+2} - 4y I_{s+3} + I_{s+4} \Big) \Bigg) \\
&+ \frac{\bar{\omega}^2}{16M^2}\Bigg(-3\beta^4 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg\{ 12\bigg[\Big(x - \frac{y}{\lambda}\Big)^2 Q_{r+2} + \frac{2}{\lambda}\Big(x - \frac{y}{\lambda}\Big) Q_{r+3} + \frac{1}{\lambda^2} Q_{r+4} \bigg] + (5M^2 - 3) Q_{r+4} \bigg\} + 2(M^2 - 3)\Big(x^5 I_s - \frac{5x^4}{\lambda} I_{s+1} + \frac{10x^3}{\lambda^2} I_{s+2} \\
&- \frac{10x^2}{\lambda^3} I_{s+3} + \frac{5x}{\lambda^4} I_{s+4} - \frac{1}{\lambda^5} I_{s+5} \Big) + \beta^2(47M^2 - 33)\bigg[x^3 y^2 I_s - \Big(2x^3 y + \frac{3x^2 y^2}{\lambda}\Big) I_{s+1} + \Big(x^3 + \frac{6x^2 y}{\lambda} + \frac{3xy^2}{\lambda^2}\Big) I_{s+2} - \Big(\frac{3x^2}{\lambda} + \frac{6xy}{\lambda^2} + \frac{y^2}{\lambda^3}\Big) I_{s+3} \\
&+ \Big(\frac{3x}{\lambda^2} + \frac{2y}{\lambda^3}\Big) I_{s+4} - \frac{1}{\lambda^3} I_{s+5} \bigg] - \beta^4(49M^2 - 39)\bigg[xy^4 I_s - \Big(4xy^3 + \frac{y^4}{\lambda}\Big) I_{s+1} + \Big(6y^2 x + \frac{4y^3}{\lambda}\Big) I_{s+2} - \Big(4yx + \frac{6y^2}{\lambda}\Big) I_{s+3} + \Big(x + \frac{4y}{\lambda}\Big) I_{s+4} - \frac{1}{\lambda} I_{s+5} \bigg] \Bigg) \\
&- \frac{i\bar{\omega}^3}{16M^4}\Bigg(\beta^6 \sum_{r=0}^{s} \frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!} \bigg\{ 4\bigg[\Big(x - \frac{y}{\lambda}\Big)^3 Q_{r+2} + \frac{3}{\lambda}\Big(x - \frac{y}{\lambda}\Big)^2 Q_{r+3} + \frac{3}{\lambda^2}\Big(x - \frac{y}{\lambda}\Big) Q_{r+4} + \frac{1}{\lambda^3} Q_{r+5} \bigg] + 3(5M^2 - 1)\bigg[\Big(x - \frac{y}{\lambda}\Big) Q_{r+4} + \frac{1}{\lambda} Q_{r+5} \bigg] \bigg\} \\
&+ \frac{2(M^4 - 6M^2 - 3)}{15}\Big(x^6 I_s - \frac{6x^5}{\lambda} I_{s+1} + \frac{15x^4}{\lambda^2} I_{s+2} - \frac{20x^3}{\lambda^3} I_{s+3} + \frac{15x^2}{\lambda^4} I_{s+4} - \frac{6x}{\lambda^5} I_{s+5} + \frac{1}{\lambda^2} I_{s+6} \Big) - \frac{\beta^2}{15}(161M^4 - 286M^2 + 77)\bigg[x^4 y^2 I_s \\
&- \Big(2yx^4 + \frac{4x^3 y^2}{\lambda}\Big) I_{s+1} + \Big(x^4 + \frac{8x^3 y}{\lambda} + \frac{6x^2 y^2}{\lambda^2}\Big) I_{s+2} - 4\Big(\frac{x^3}{\lambda} + \frac{3x^2 y}{\lambda^2} + \frac{xy^2}{\lambda^3}\Big) I_{s+3} + \Big(\frac{6x^2}{\lambda^2} + \frac{8xy}{\lambda^3} + \frac{y^2}{\lambda^4}\Big) I_{s+4} - \Big(\frac{4x}{\lambda^3} + \frac{2y}{\lambda^4}\Big) I_{s+5} + \frac{1}{\lambda^4} I_{s+6} \bigg] \\
&+ \frac{\beta^4}{15}(31M^4 - 146M^2 + 67)\bigg[x^2 y^4 I_s - \Big(4x^2 y^3 + \frac{2xy^4}{\lambda}\Big) I_{s+1} + \Big(6x^2 y^2 + \frac{8xy^3}{\lambda} + \frac{y^4}{\lambda^2}\Big) I_{s+2} - \Big(4yx^2 + \frac{12xy^2}{\lambda} + \frac{4y^3}{\lambda^2}\Big) I_{s+3} + \Big(x^2 + \frac{8xy}{\lambda} + \frac{6y^2}{\lambda^2}\Big) I_{s+4} \\
&- \Big(\frac{2x}{\lambda} + \frac{4y}{\lambda^2}\Big) I_{s+5} + \frac{1}{\lambda^2} I_{s+6} \bigg] + \frac{16\beta^6}{15}(8M^4 - 8M^2 + 1)\Big(y^6 I_s - 6y^5 I_{s+1} + 15y^4 I_{s+2} - 20y^3 I_{s+3} + 15y^2 I_{s+4} - 6y I_{s+5} + I_{s+6} \Big) \Bigg) \Bigg]
\end{aligned}
\tag{D17}
$$

$$
\begin{aligned}
L_2 = &-\frac{4K_3}{\pi V}\sum A_s\Bigg\{3\sum_{r=0}^{s}\frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\Big[\left(x-\frac{y}{\lambda}\right)^2P_r+\frac{2}{\lambda}\left(x-\frac{y}{\lambda}\right)P_{r+1}+\frac{1}{\lambda^2}P_{r+2}+\frac{\beta^2}{2}P_{r+2}\Big]+2\beta\sqrt{\frac{b}{2}-y}\sum_{r=0}^{s}\frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!}\Big[-6\left(x-\frac{b}{2\lambda}\right)R_r-\frac{6}{\lambda}R_{r+1}\\
&+\frac{3\beta}{2}\left(\frac{b}{2}-y\right)R_r+\frac{3\beta}{2}R_{r+1}\Big]-\frac{i\bar{\omega}}{M^2}\Bigg(-\beta^2\sum_{r=0}^{s}\frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\Big[\left(x-\frac{y}{\lambda}\right)^3P_r+\frac{3}{\lambda}\left(x-\frac{y}{\lambda}\right)^2P_{r+1}+\frac{3}{\lambda^2}\left(x-\frac{y}{\lambda}\right)P_{r+2}+\frac{1}{\lambda^3}P_{r+3}+\frac{3}{2}(3M^2-1)\left(x-\frac{y}{\lambda}\right)P_{r+2}\\
&+\frac{3}{2}(3M^2-1)\frac{1}{\lambda}P_{r+3}\Big]+2\beta\sqrt{\frac{b}{2}-y}\sum_{r=0}^{s}\frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!}\Big\{3(2M^2-1)\Big[\left(x-\frac{b}{2\lambda}\right)^2R_r+\frac{2}{\lambda}\left(x-\frac{b}{2\lambda}\right)R_{r+1}+\frac{1}{\lambda^2}R_{r+2}\Big]-\frac{3\beta(3M^2-1)}{2}\Big[\left(x-\frac{b}{2\lambda}\right)\left(\frac{b}{2}-y\right)R_r\\
&+\left(x-\frac{y}{\lambda}\right)R_{r+1}+\frac{1}{\lambda}R_{r+2}\Big]+\frac{\beta^2(4M^2-1)}{3}\Big[\left(\frac{b}{2}-y\right)^2R_r+2\left(\frac{b}{2}-y\right)R_{r+1}+R_{r+2}\Big]+\frac{2(4M^2-1)}{3}\beta^2\Big[\left(\frac{b}{2}-y\right)^2R_r-2\left(\frac{b}{2}-y\right)R_{r+1}+R_{r+2}\Big]\Big\}\Bigg)\\
&+\frac{\bar{\omega}^2}{M^2}\Bigg(-\frac{3\beta^4}{16}\sum_{r=0}^{s}\frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\Big[12\left(x-\frac{y}{\lambda}\right)^2P_{r+2}+\frac{24}{\lambda}\left(x-\frac{y}{\lambda}\right)P_{r+3}+\frac{12}{\lambda^2}P_{r+4}+(5M^2-3)P_{r+4}\Big]+2\beta\sqrt{\frac{b}{2}-y}\sum_{r=0}^{s}\frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!}\Big\{R_r\Big[\left(x-\frac{b}{2\lambda}\right)^3\left(\frac{2M^2-3}{2}\right)\\
&-\frac{9\beta^3}{4}\left(x-\frac{b}{2\lambda}\right)^2\left(\frac{b}{2}-y\right)+\frac{3\beta^2}{2}(4M^2-3)\left(x-\frac{b}{2\lambda}\right)\left(\frac{b}{2}-y\right)^2-\frac{5\beta^3}{16}(5M^2-3)\left(\frac{b}{2}-y\right)^3\Big]+R_{r+1}\Big\{\beta\left(x-\frac{b}{2\lambda}\right)^2\Big[\frac{3(2M^2-3)}{2\beta\lambda}-\frac{9\beta^2}{4}\Big]-\beta^2\left(x-\frac{b}{2\lambda}\right)\left(\frac{b}{2}-y\right)\left(\frac{9\beta^2}{2\beta\lambda}+4M^2-3\right)\\
&+\beta^3\left(\frac{b}{2}-y\right)^2\Big[\frac{3(4M^2-3)}{2\beta\lambda}-\frac{3(5M^2-3)}{16}\Big]\Big\}+R_{r+2}\Big\{\beta^2\left(x-\frac{b}{2\lambda}\right)\Big[\frac{3(2M^2-3)}{2\beta^2\lambda^2}-\frac{9\beta^2}{2\beta\lambda}+\frac{3(4M^2-3)}{2}\Big]-\beta^3\left(\frac{b}{2}-y\right)\Big[\frac{9\beta^2}{4\beta^2\lambda^2}+\frac{4M^2-3}{\beta\lambda}+\frac{3(5M^2-3)}{16}\Big]\Big\}\\
&+R_{r+3}\beta^3\Big[\frac{2M^2-3}{2\beta^3\lambda^3}-\frac{9\beta^2}{4\beta^2\lambda^2}+\frac{3(4M^2-3)}{2\beta\lambda}-\frac{5(5M^2-3)}{16}\Big]\Big\}\Bigg)-\frac{i\bar{\omega}^3}{M^4}\Bigg[\frac{\beta^6}{16}\sum_{r=0}^{s}\frac{s!\,y^{s-r}(-1)^r}{r!(s-r)!}\Big[4\left(x-\frac{y}{\lambda}\right)^3P_{r+2}+\frac{12}{\lambda}\left(x-\frac{y}{\lambda}\right)^2P_{r+3}+\left(\frac{12}{\lambda^2}+15M^2-3\right)\left(x-\frac{y}{\lambda}\right)P_{r+4}\\
&+\frac{1}{\lambda}\left(\frac{4}{\lambda^2}+15M^2-3\right)P_{r+5}\Big]+2\beta\sqrt{\frac{b}{2}-y}\sum_{r=0}^{s}\frac{s!\left(\frac{b}{2}\right)^{s-r}(-1)^r}{r!(s-r)!}\Bigg(R_r\Big[-\frac{1}{8}\left(x-\frac{b}{2\lambda}\right)^4+\frac{\beta^5}{4}\left(x-\frac{b}{2\lambda}\right)^3\left(\frac{b}{2}-y\right)-\frac{\beta^2}{4}(8M^4-12M^2+3)\left(x-\frac{b}{2\lambda}\right)^2\left(\frac{b}{2}-y\right)^2\\
&+\frac{5\beta^5}{16}(5M^2-1)\left(x-\frac{b}{2\lambda}\right)\left(\frac{b}{2}-y\right)^3-\frac{\beta^4(8M^4-8M^2+1)}{8}\left(\frac{b}{2}-y\right)^4\Big]+R_{r+1}\Big\{\frac{\beta}{4}\left(\beta^4-\frac{2}{\beta\lambda}\right)\left(x-\frac{b}{2\lambda}\right)^3+\frac{\beta^2}{12}\Big[\frac{9\beta^4}{\beta\lambda}+2(8M^4-12M^2+3)\Big]\left(x-\frac{b}{2\lambda}\right)^2\left(\frac{b}{2}-y\right)\\
&-\beta^3\Big[\frac{1}{2\beta\lambda}(8M^4-12M^2+3)-\frac{3}{16}(5M^4-6M^2+1)\Big]\left(x-\frac{b}{2\lambda}\right)\left(\frac{b}{2}-y\right)^2+\beta^4\Big[\frac{5}{16\beta\lambda}(5M^4-6M^2+1)+\frac{8M^4-8M^2+1}{6}\Big]\left(\frac{b}{2}-y\right)^3\Big\}+R_{r+2}\Big\{\frac{\beta^2}{4}\left(x-\frac{b}{2\lambda}\right)^2\Big[-\frac{3}{\beta^2\lambda^2}\\
&+\frac{3\beta^4}{\beta\lambda}-(8M^4-12M^2+3)\Big]+\left(x-\frac{b}{2\lambda}\right)\left(\frac{b}{2}-y\right)\beta^3\Big[\frac{3\beta^4}{4\beta^2\lambda^2}+\frac{8M^4-12M^2+3}{3\beta\lambda}+\frac{3}{16}(5M^4-6M^2+1)\Big]+\beta^4\left(\frac{b}{2}-y\right)^2\Big[-\frac{8M^4-12M^2+3}{4\beta^2\lambda^2}+\frac{3(5M^4-6M^2+1)}{16\beta\lambda}\\
&-\frac{29(8M^4-8M^2+1)}{60}\Big]\Big\}+R_{r+3}\Big\{\left(x-\frac{b}{2\lambda}\right)\beta^3\Big[-\frac{1}{2\beta^3\lambda^3}+\frac{3\beta^4}{4\beta^2\lambda^2}-\frac{8M^4-12M^2+3}{2\beta\lambda}+\frac{5(5M^4-6M^2+1)}{16}\Big]+\beta^4\left(\frac{b}{2}-y\right)\Big[\frac{\beta^4}{4\beta^3\lambda^3}+\frac{8M^4-12M^2+3}{6\beta^2\lambda^2}\\
&+\frac{3(5M^4-6M^2+1)}{16\beta\lambda}+\frac{8M^4-8M^2+1}{6}\Big]\Big\}+R_{r+4}\beta^4\Big[-\frac{1}{8\lambda^4\beta^4}+\frac{\beta^4}{4\lambda^3\beta^3}-\frac{8M^4-12M^2+3}{4\beta^2\lambda^2}+\frac{5(5M^4-6M^2+1)}{16\beta\lambda}-\frac{8M^4-8M^2+1}{8}\Big]\Bigg)\Bigg]\Bigg\} \qquad \text{(D18)}
\end{aligned}
$$

APPENDIX E

EXPANSION OF DOWNWASH FUNCTION

Rigid Wing in Continuous Sinusoidal Gust

In order to utilize equations (D11) to (D18) for a wing in a sinusoidal gust, it is assumed that the downwash $w = w_0 e^{i\omega\left(t - \frac{x}{V}\right)}$, when expanded to the third power of x, will adequately define the gust. Although, in general, this is not true, a very good approximation to the gust function can be made if the reduced frequency k is restricted to permit only a third of a wavelength of the gust to be on any chordwise strip at any instant of time. This restriction on k is not unduly severe since for most analyses the useful frequency range is well within the limits stipulated.

On the basis of these assumptions, the downwash can be rewritten as

$$w = w_0 e^{i\omega\left(t - \frac{x}{V}\right)} = w_0 e^{i\omega t} e^{-i\lambda xp} \approx w_0 e^{i\omega t}\left(1 - i\lambda px - \frac{\lambda^2 p^2 x^2}{2} + i\,\frac{\lambda^3 p^3 x^3}{6}\right) \tag{E1}$$

where $\lambda p = \frac{\beta^2 \bar{\omega}}{M^2}$. By means of the following identities,

$$x \equiv \left(x - \frac{y}{\lambda}\right) + \frac{y}{\lambda}$$

$$x^2 \equiv \left(x - \frac{y}{\lambda}\right)^2 + \frac{2y}{\lambda}\left(x - \frac{y}{\lambda}\right) + \left(\frac{y}{\lambda}\right)^2$$

$$x^3 \equiv \left(x - \frac{y}{\lambda}\right)^3 + \frac{3y}{\lambda}\left(x - \frac{y}{\lambda}\right)^2 + \frac{3y^2}{\lambda^2}\left(x - \frac{y}{\lambda}\right) + \frac{y^3}{\lambda^3}$$

equation (E1) can be rewritten as

$$w = w_0 e^{i\omega t}\left[\left(1 - ipy - \frac{p^2y^2}{2} + i\,\frac{p^3y^3}{6}\right) + \left(x - \frac{y}{\lambda}\right)\left(-ip\lambda - p^2\lambda y + \frac{ip^3\lambda y^2}{2}\right)\right.$$
$$\left. + \left(x - \frac{y}{\lambda}\right)^2\left(-\frac{p^2\lambda^2}{2} + \frac{ip^3\lambda^2 y}{2}\right) + \frac{ip^3\lambda^3}{6}\left(x - \frac{y}{\lambda}\right)^3\right] \tag{E2}$$

Examination of equation (E2) and equations (D11) to (D18) shows that for a wing in a sinusoidal gust the values of A_s can be defined as follows:

for equations (D11) and (D12),

$$A_0 = 1$$

$$A_1 = -ip$$

$$A_2 = \frac{-p^2}{2}$$

$$A_3 = \frac{ip^3}{6}$$

for equations (D13) and (D14),

$$A_0 = -i\lambda p$$

$$A_1 = -p^2\lambda$$

$$A_3 = \frac{ip^3\lambda}{2}$$

for equations (D15) and (D16),

$$A_0 = \frac{-p^2\lambda^2}{2}$$

$$A_1 = \frac{ip^3\lambda^2}{2}$$

and for equations (D17) and (D18),

$$A_0 = \frac{ip^3\lambda^3}{6}$$

Here the unit of length associated with K_n was chosen as one so that $\frac{K_n}{V} = \frac{w_0}{V} = \alpha = 1$. On the basis of these values of A_s, the problem was set up on the IBM 650 data processing machine to retain only the third power of frequency. For example, in equations (D11) and (D12) all the terms in the bracket were retained for $s = 0$, whereas only the first three terms for $s = 1$, the first two terms for $s = 2$, and the first term for $s = 3$ were retained.

Oscillating Rigid Wing

This abbreviated program is not restricted to repeated sinusoidal gusts. For example, for a wing undergoing sinking and pitching oscillations, the downwash can be written as

$$\begin{aligned} w &= e^{i\omega t}\left[V\alpha + \dot{h} + i\omega\left(x - x_0\right)\alpha\right] \\ &= Ve^{i\omega t}\left[\alpha + \frac{\dot{h}}{V} - \frac{i\omega x_0\alpha}{V} + \frac{i\omega\alpha}{V\lambda}\, y + \frac{i\omega\alpha}{V}\left(x - \frac{y}{\lambda}\right)\right] \end{aligned} \tag{E3}$$

where, for equations (D11) and (D12),

$$A_0 = \alpha + \frac{\dot{h}}{V} - \frac{i\omega\alpha x_0}{V}$$

$$A_1 = \frac{i\omega\alpha}{V\lambda}$$

and for equations (D13) and (D14),

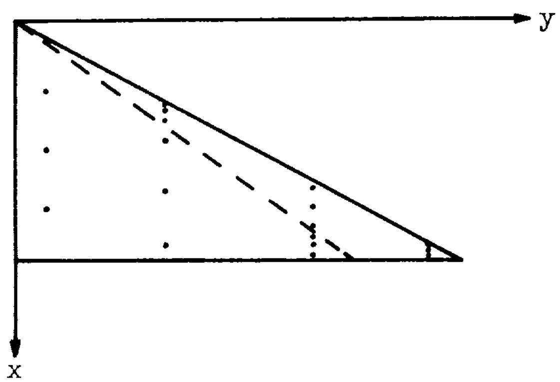
$$A_0 = \frac{i\omega\alpha}{V}$$

All other values of A_s are zero. Again the unit of length associated with K_n was chosen as 1; thus, $K_n/V = 1$.

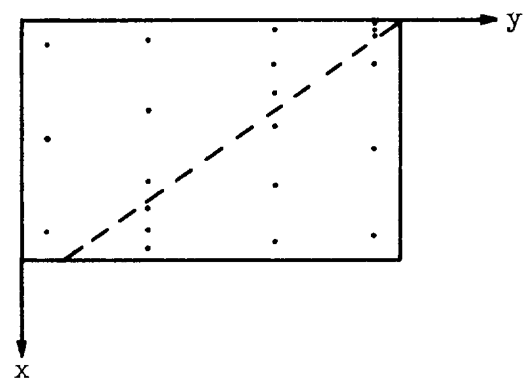
Application to a Delta and Rectangular Wing

Limited forms of equations (50), (51), and (54) have been programed on the IBM 650 data processing machine. The amount of information programed was dictated by the form of the downwash function for a wing in a continuous sinusoidal gust field as indicated by equation (E2). In order to check out the program, calculations were made for two wings, a 50° delta wing and an almost rectangular wing of aspect ratio 0.8, both flying at a Mach number of 3.0. The tangent of the leading-edge sweep of the almost rectangular wing was 10^{-6} instead of zero, since for $\lambda = 0$ singularities arise for which no provisions were made in the program. The position of the points for which the pressure coefficients were calculated was determined by means of a Gaussian distribution formula.

Four semispan stations were used and each region along a chordwise strip was divided into three stations. This procedure allowed for calculating the pressure coefficients at 18 points on the half-span delta wing and 21 points on the rectangular wing as indicated in sketches 12 and 13.



Sketch 12



Sketch 13

This procedure for positioning the various points permitted the use of a fifth-degree (three point) Gaussian integrating formula for each chordwise region and a seventh-degree (four point) formula spanwise to obtain the total lift on each wing. In table I the results are presented for a 50° delta wing flying at a Mach number of 3.0 in a sinusoidal gust field. As can be seen, the total lift coefficients are in good agreement with those obtained by using equation (67) of reference 14 for values of reduced frequencies at least as high as those indicated in the table.

As a further check, the total lift coefficients for a 50° delta and an almost rectangular wing of aspect ratio 0.8, both at a Mach number of 3 and undergoing harmonic sinking motion ($\alpha = 0$ in eq. (E3)), are also presented in table I. The results obtained for these cases were in very good agreement with those obtained by using equations (66) and (71) of reference 14 in the same frequency range.

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TABLE I.- TOTAL LIFT COEFFICIENTS FOR A DELTA AND RECTANGULAR WING SUBJECTED TO SINUSOIDAL GUSTS AND TO SINUSOIDAL SINKING OSCILLATIONS

(a) Sinusoidal gusts

k	C_L for a delta wing at M = 3.0 from -	
	Present theory	Equation (67) of reference 14
0	1.4148	1.4142
.08889	1.4028 - 0.1684i	1.4009 - 0.1773i
.17778	1.3671 - .3293i	1.3634 - .3508i
.26667	1.3076 - .4922i	1.3008 - .5157i
.35556	1.2248 - .6091i	1.2157 - .6680i

(b) Harmonic sinking oscillations

k	C_L for a delta wing at M = 3.0 from -	
	Present theory	Equation (66) of reference 14
0	1.4148	1.4142
.0444	1.4145 - 0.0056i	1.4140 - 0.0052i
.1333	1.4128 - .0167i	1.4126 - .0138i
.2222	1.4093 - .0274i	1.4093 - .0255i
.3333	1.4025 - .0396i	1.4035 - .0370i

k	C_L for a rectangular wing of aspect ratio 0.8 at M = 3.0 from -	
	Present theory	Equation (71) of reference 14
0	1.1015	1.1017
.0889	1.1011 + 0.0078i	1.1015 + 0.0075i
.1778	1.1004 + .0156i	1.1010 + .0152i
.2667	1.0988 + .0240i	1.1001 + .0233i
.3556	1.0986 + .0314i	1.0992 + .0322i

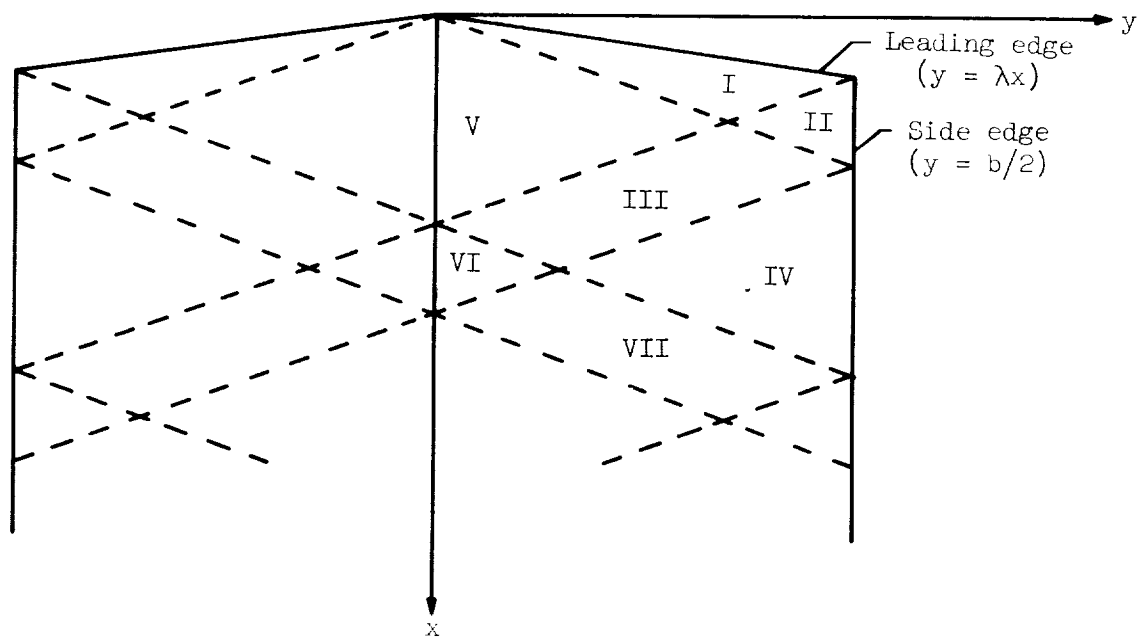

Figure 1.- Sketch illustrating the various regions for which pressure distributions have been derived.