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N63-12063

NASA TN D-1668

NASA TN D-1668

TECHNICAL NOTE

D-1668

MATHEMATICAL ANALYSIS FOR THE ORIENTATION AND CONTROL OF THE ORBITING ASTRONOMICAL OBSERVATORY SATELLITE

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON January 1963

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SUMMARY

A mathematical model is developed by which the following satellite orientation and control problems may be resolved: (1) determining attitude for maximum area of solar cells in sunlight; (2) generating slewing commands for a change in attitude; (3) computing star tracker gimbal angles for maintaining proper orientation; and (4) determining when guide stars are occulted by the earth, sun, and moon.

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INTRODUCTION

Sometime in 1964 the NASA expects to launch the first Orbiting Astronomical Observatory (OAO). The OAO, consisting of the spacecraft and the observatory (experimenter's package and equipment), is expected to have a lifetime of one year. The spacecraft, orbiting at an altitude of approximately 500 miles, will carry highly refined equipment to conduct astronomical experiments free from the disturbing influence of the earth's atmosphere.

The spacecraft will be capable of directing the experimenter's equipment anywhere in space and of maintaining this direction with a high degree of accuracy. This will be accomplished by a complex stabilization and control system whose major elements are: (1) sun sensors, rate gyros, and a jet system to initially orient and stabilize the spacecraft; (2) coarse inertia wheels to reorient the spacecraft, and fine inertia wheels to maintain the desired orientation; (3) a set of six star trackers to determine the orientation of the OAO; and (4) a magnetic unloading system to keep the momentum in the fine inertia wheels from building up excessively.

To place the OAO in the proper orientation in space, the precise slewing angles needed to accomplish this feat must be known, as well as the gimbal angles of the star trackers when they are locked on their respective guide stars. In addition to these problems various physical constraints such as star occultation by various bodies including the earth, sun, and moon must be considered. There are also many constraints due to the spacecraft itself, which include such things as obtaining maximum power by proper positioning of the solar paddles, keeping the optical axis away from the sun's damaging rays, and restricting the gimbal angles within their limits. This report describes the development of a mathematical model to aid in the solution of these problems.

COORDINATE SYSTEMS

The relative positions of the various bodies in space will be defined by giving their directions with respect to a fixed right-handed rectangular coordinate system with axes u, v, w and origin at the earth's center of gravity. The u-v plane will be the equatorial plane at some epoch T, the positive u axis pointed toward the vernal equinox and the positive w axis toward the true north pole at time T.

The position of a star will be denoted by its right ascension α and declination δ. In most applications the direction cosines a, b, c of the star will be needed; these are given by

$$a = \cos\delta\cos\alpha ,$$

$$b = \cos\delta\sin\alpha ,$$

$$c = \sin\delta .$$

Since the earth revolves about the sun, the position of the stars is not exactly fixed in this coordinate system; however, the maximum stellar parallax due to this motion is about 0.8 second of arc for the nearest star.

The positions of the sun, moon, and OAO will be given at any time t (measured from the epoch T) by their respective orbital elements, which are:

n = mean angular motion in the plane of the orbit

e = the eccentricity of the orbit

Ω = the right ascension of the ascending node; $0 \leq \Omega < 2\pi$

i = the inclination of orbital plane to the u-v plane; $0 \leq i \leq \pi$

ω = argument of perigee; $0 \leq \omega < 2\pi$

M_0 = the mean anomaly at time T

Since no orbit is truly Keplerian, first and higher derivatives of the various elements may be necessary according to the accuracy required. It may also be necessary to update these elements from time to time.

A secondary coordinate system with axes x, y, z is defined with respect to the OAO. The origin of the system is at the geometric center of the OAO with the optical axis coincident with the x axis. The positive slewing motions of yaw, pitch, and roll are defined to be clockwise rotations, as seen from the origin, about the positive z, y, and x axes respectively.

The attitude of the OAO in space is described by specifying the right ascension α and declination δ of the x axis and the angle β, which is the angle the y axis makes with the u-v plane measured in the y-z plane (positive direction toward the positive z axis).

Because of the great distances involved, the angular coordinates of the stars and the sun will be considered to be the same whether the origin of the coordinate system is at the center of the earth or the spacecraft. In the case of the sun, placing the center at the OAO instead of the earth produces a maximum error of less than 10 seconds of arc because of parallax (for a 500 mile circular orbit). For the stars this error is completely negligible. However, for the moon the error can become as great as 1 degree.

For each star tracker on the OAO we define a coordinate system with axes x_i, y_i, z_i such that the outer gimbal angle σ is in the $x_i - y_i$ plane (positive angle measured from positive x_i towards positive y_i) and the inner gimbal angle μ is the angle from the $x_i - y_i$ plane (positive towards negative z_i). Thus the direction cosines of a star (with gimbal angles σ and μ) in this coordinate system are given by

$$a = \cos\mu\cos\sigma ,$$

$$b = \cos\mu\sin\sigma ,$$

$$c = -\sin\mu .$$

TRANSFORMATIONS

The mathematical approach here is based on the algebra of rotations (matrix algebra); but, since we are concerned only with directions, the rectangular coordinates of a vector will be identical to its direction cosines. Thus, from the preceding definitions, if the attitude of the OAO is given by α, δ, β and if (u_0, v_0, w_0) is any vector in the u, v, w system, then the coordinates x_0, y_0, z_0 of this vector in the OAO system are given by

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = T_\beta T_\delta T_\alpha \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} ,$$

where

$$T_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_\delta = \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{pmatrix},$$

$$T_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} .$$

In addition, if the spacecraft is yawed, pitched, or rolled by ψ, θ, or ϕ respectively, the coordinates x_1, y_1, z_1 of the vector in this new coordinate system are given by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = T_\psi \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \quad T_\theta \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \quad \text{or} \quad T_\phi \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix},$$

where

$$T_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_\theta = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$

$$T_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}.$$

In order that all angles γ may be defined uniquely, we use the function

$$\gamma = \tan^{-1}(a/b) \tag{1}$$

with the understanding that the sign of a is the same as that of $\sin\gamma$ and the sign of b is the same as that of $\cos\gamma$. Thus the signs of a and b in Equation 1 determine the proper quadrant, and the inverse tangent of a divided by b determines the proper value. For all other inverse functions the principle value shall be taken. The range of all angles defined above is as follows:

Right ascension of x axis α	$0 \leq \alpha < 2\pi$
Declination of x axis δ	$-\pi/2 \leq \delta \leq \pi/2$
Roll of y axis β	$-\pi \leq \beta < \pi$
Yaw of OAO ψ	$-\pi \leq \psi < \pi$
Pitch of OAO θ	$-\pi \leq \theta < \pi$
Roll of OAO ϕ	$-\pi \leq \phi < \pi$

DETERMINATION OF FINAL ROLL

If α_2 and δ_2 are the right ascension and declination of a new target star S_2, then we wish to determine the roll β_2 which will orient the solar paddles such that they receive maximum sunlight when the optical axis points towards the star S_2. This will be the case when the angle γ between the sunline and a normal line of the paddle plane is a minimum. Let u_s, v_s, w_s be the direction cosines of the sunline in the u, v, w coordinate system. The direction cosines x_s, y_s, z_s of the sunline in the x, y, z system (optical axis pointing toward S_2) are then found as follows:

$$\begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = T_{\delta_2} T_{\alpha_2} \begin{pmatrix} u_s \\ v_s \\ w_s \end{pmatrix}.$$

The direction cosines of the sunline in the x, y, z system after a roll of β are given by

$$x_s ,$$
$$y_s \cos\beta + z_s \sin\beta ,$$
$$z_s \cos\beta - y_s \sin\beta .$$

If x_p, y_p, z_p are the direction cosines of a directed normal from one side of the paddle plane, the cosine of the angle γ between this normal and the sunline is expressed by

$$\begin{aligned} f(\beta) &= \cos\gamma \\ &= x_p x_s + y_p(y_s \cos\beta + z_s \sin\beta) + z_p(z_s \cos\beta - y_s \sin\beta). \end{aligned} \tag{2}$$

The requirement that γ be a minimum implies that $\cos\gamma$ be a maximum; that is,

$$\begin{aligned} \frac{df}{d\beta} &= -\sin\gamma \frac{d\gamma}{d\beta} \\ &= y_p(z_s \cos\beta - y_s \sin\beta) - z_p(z_s \sin\beta + y_s \cos\beta) \\ &= (y_p z_s - z_p y_s) \cos\beta - (y_p y_s + z_p z_s) \sin\beta \\ &= 0 , \end{aligned}$$

or

$$\tan\beta = \frac{y_p z_s - z_p y_s}{y_p y_s + z_p z_s} . \tag{3}$$

There are two possible values of β that will satisfy Equation 3: β_0 and β_1. However, the value that maximizes Equation 2 is the desired value of β; denote this value by β_0.

The above analysis has considered only one side of the paddle plane, but the other side is handled in exactly the same way with the normal whose direction cosines are $-x_p$, $-y_p$, $-z_p$. This requires that the negative of Equation 2 be a maximum, which again leads to Equation 3. The proper solution this time will be the other value of β to satisfy Equation 3, namely, β_1. Thus, to determine the best value of β, we evaluate $f(\beta_0)$ and $-f(\beta_1)$, and pick the value of β that gives the maximum of the two.

GENERATION OF SLEWING COMMANDS

If the OAO has an initial attitude of α_1, δ_1, β_1 and it is desired to reorient in order to obtain an attitude of α_2, δ_2, β_2, the slewing commands needed to accomplish this reorientation must be determined. Since the OAO may be rotated about any one of three axes, there are twelve possible slewing sequences. These slewing sequences are listed as follows:

yaw - pitch - roll	yaw - pitch - yaw
roll - pitch - yaw	roll - pitch - roll
yaw - roll - pitch	yaw - roll - yaw
pitch - roll - yaw	pitch - roll - pitch
pitch - yaw - roll	pitch - yaw - pitch
roll - yaw - pitch	roll - yaw - roll

The analysis for determining the amount of slewing required is similar regardless of the slewing sequence; therefore we shall refer to a general slewing sequence of i-j-k. The matrices of these i-j-k rotations will be denoted by T_i, T_j, and T_k respectively.

If v is any vector with coordinates given in the OAO system with attitude α_1, δ_1, β_1, the matrix to find the coordinates of v in the OAO system with attitude α_2, δ_2, β_2 can be obtained as a product of six matrices:

$$T_{\beta_2} T_{\delta_2} T_{\alpha_2} T_{\alpha_1}^{-1} T_{\delta_1}^{-1} T_{\beta_1}^{-1} ,$$

where T^{-1} indicates the inverse of T. Likewise, if a slewing sequence of i-j-k is given when the OAO has an attitude of α_1, δ_1, β_1, which causes the spacecraft to have a final attitude of α_2, δ_2, β_2, the matrix of the transformation from the initial attitude to the final is found from the matrix product $T_k T_j T_i$. Therefore T_i, T_j, and T_k must satisfy the matrix equation:

$$T_k T_j T_i = T_{\beta_2} T_{\delta_2} T_{\alpha_2} T_{\alpha_1}^{-1} T_{\delta_1}^{-1} T_{\beta_1}^{-1} \cdot \tag{4}$$

The right-hand side of Equation 4 is a 3×3 matrix that can be determined from α_1, δ_1, β_1, α_2, δ_2, β_2. Therefore this matrix is independent of the slewing sequence. This right-hand matrix shall be denoted as C with elements C_{ij}. Thus for each slewing sequence the left-hand side of Equation 4 can be compared with the C matrix to determine the amount of slewing. Several examples are given below:

SEQUENCE: YAW-PITCH-ROLL

$$T_\phi T_\theta T_\psi = \begin{pmatrix} \cos\theta\cos\psi , & \cos\theta\sin\psi , & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\theta\sin\psi, & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi, & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi, & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi, & \cos\phi\cos\theta \end{pmatrix} ;$$

and comparing with the C matrix implies that

$$\sin\theta = -C_{13} , \quad \sin\psi = C_{12}/\cos\theta , \quad \sin\phi = C_{23}/\cos\theta ,$$

$$\cos\theta = \pm\sqrt{C_{11}^2 + C_{12}^2} = \pm\sqrt{C_{23}^2 + C_{33}^2} , \quad \cos\psi = C_{11}/\cos\theta , \quad \cos\phi = C_{33}/\cos\theta \cdot$$

SEQUENCE: ROLL-YAW-PITCH

$$T_\theta T_\psi T_\phi = \begin{pmatrix} \cos\theta\cos\psi, & \cos\theta\sin\psi\cos\phi + \sin\theta\sin\phi, & \cos\theta\sin\psi\sin\phi - \sin\theta\cos\phi \\ -\sin\psi, & \cos\psi\cos\phi, & \cos\psi\sin\phi \\ \sin\theta\cos\psi, & \sin\theta\sin\psi\cos\phi - \cos\theta\sin\phi, & \sin\theta\sin\psi\sin\phi + \cos\theta\cos\phi \end{pmatrix};$$

and comparing this with the C matrix implies that

$$\sin\psi = -C_{21}, \quad \sin\theta = C_{31}/\cos\psi, \quad \sin\phi = C_{23}/\cos\psi,$$
$$\cos\psi = \pm\sqrt{C_{11}^2 + C_{31}^2} = \pm\sqrt{C_{22}^2 + C_{23}^2}, \quad \cos\theta = C_{11}/\cos\psi, \quad \cos\phi = C_{22}/\cos\psi.$$

SEQUENCE: YAW-PITCH-YAW

$$T_{\psi_2} T_\theta T_{\psi_1} = \begin{pmatrix} \cos\psi_2\cos\theta\cos\psi_1 - \sin\psi_1\sin\psi_2, & \cos\psi_2\cos\theta\sin\psi_1 + \sin\psi_2\cos\psi_1, & -\cos\psi_2\sin\theta \\ -(\sin\psi_2\cos\theta\cos\psi_1 + \sin\psi_1\cos\psi_2), & -\sin\psi_2\cos\theta\sin\psi_1 + \cos\psi_2\cos\psi_1, & \sin\psi_2\sin\theta \\ \sin\theta\cos\psi_1, & \sin\theta\sin\psi_1, & \cos\theta \end{pmatrix};$$

and comparison to the C matrix gives

$$\sin\theta = \pm\sqrt{C_{31}^2 + C_{32}^2} = \pm\sqrt{C_{13}^2 + C_{23}^2}, \quad \sin\psi_1 = C_{32}/\sin\theta, \quad \sin\psi_2 = C_{23}/\sin\theta,$$
$$\cos\theta = C_{33}, \quad \cos\psi_1 = C_{31}/\sin\theta, \quad \cos\psi_2 = -C_{13}/\sin\theta.$$

Thus for each slewing sequence we obtain two solutions corresponding to the plus and minus sign of the radical. Therefore there are actually twenty-four possible slewing commands.

We could continue the above process for all twelve combinations of the matrix $T_k\ T_j\ T_i$, compare each with the C matrix, and find that there is a definite pattern as to which elements of the C matrix to choose for a given slewing sequence. Let the numbers 1, 2, and 3 be used to represent roll, pitch, and yaw respectively (2-1-3 would indicate a pitch-roll-yaw sequence); then, for all sequences i-j-k where the same type of slew is not used more than once, we have the following relations:

$$\sigma_{123} = \sigma_{231} = \sigma_{312} = -\sigma_{132} = -\sigma_{213} = -\sigma_{321} = 1,$$

$$\sin j = \sigma_{ijk} C_{ki},$$

$$\cos j = \pm\sqrt{C_{ii}^2 + C_{ji}^2} = \pm\sqrt{C_{kj}^2 + C_{kk}^2},$$

$$\sin i = -\sigma_{ijk} C_{kj}/\cos j, \quad \sin k = -\sigma_{ijk} C_{ji}/\cos j,$$
$$\cos i = C_{kk}/\cos j, \quad \cos k = C_{ii}/\cos j.$$

In the degenerate case where $C_{kj} = C_{kk} = C_{ii} = C_{ji} = 0$, the value of j is either plus or minus 90 degrees. In this case i and k must satisfy the following:

$$\sin(k \pm \sigma_{ijk}\, i) = \sigma_{ijk} C_{ij} ,$$

$$\cos(k \pm \sigma_{ijk}\, i) = \mp \sigma_{ijk} C_{ik} .$$

The upper signs are taken when j is +90 degrees, and the lower sign when j is −90 degrees.

Example 1 — Assume the matrix C is given. Determine the slewing angles for a pitch-roll-yaw sequence.

By definition i is 2, j becomes 1, and k is 3. Here σ is − 1; therefore,

$$\sin\phi = -C_{32} ,$$

$$\cos\phi = \pm\sqrt{C_{22}^2 + C_{12}^2} ,$$

$$\sin\theta = C_{31}/\cos\phi , \quad \sin\psi = C_{12}/\cos\phi ,$$

$$\cos\theta = C_{33}/\cos\phi , \quad \cos\psi = C_{22}/\cos\phi .$$

If the slewing sequence includes the same type of slew twice (of the form i_1-j-i_2), and k is the slew not used, then the slewing angles are defined by

$$\sin j = \pm\sqrt{C_{ij}^2 + C_{ik}^2} = \pm\sqrt{C_{ji}^2 + C_{ki}^2} ,$$

$$\cos j = C_{ii} ,$$

$$\sin i_1 = C_{ij}/\sin j , \quad \sin i_2 = C_{ji}/\sin j ,$$

$$\cos i_1 = \sigma_{ji} C_{ik}/\sin j , \quad \cos i_2 = -\sigma_{ji} C_{ki}/\sin j ,$$

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = -\sigma_{13} = -\sigma_{21} = -\sigma_{32} = 1 .$$

When $C_{ij} = C_{ik} = C_{ji} = C_{ki} = 0$, the angle j is either 0 or 180 degrees, depending on whether C_{ii} is plus or minus. In this case the angles i_1 and i_2 must satisfy the relations

$$\sin(i_2 \pm i_1) = \sigma_{ji} C_{kj} ,$$

$$\cos(i_2 \pm i_1) = \pm C_{kk},$$

where the plus sign is taken if C_{ii} is positive and the minus sign if C_{ii} is negative.

Example 2 – Determine the slewing angles for a yaw-pitch-yaw sequence.

In this case i becomes 3, j is 2, k equals 1, and σ is +1:

$$\sin\theta = \pm\sqrt{C_{32}^2 + C_{31}^2} = \pm\sqrt{C_{23}^2 + C_{13}^2} ,$$

$$\cos\theta = C_{33} ,$$

$$\sin\psi_1 = C_{32}/\sin\theta , \quad \sin\psi_2 = C_{23}/\sin\theta ,$$

$$\cos\psi_1 = C_{31}/\sin\theta , \quad \cos\psi_2 = -C_{13}/\sin\theta .$$

DETERMINATION OF GIMBAL ANGLES

The determination of the star tracker gimbal angles will depend on the physical mounting of each star tracker with respect to the OAO's coordinate system. Let T_i ($i = 1, \cdots, 6$) be the transformation that determines the coordinates of a vector in the OAO system from the coordinates of the vector in the i^{th} star tracker system. Thus, if σ_i and μ_i are the outer and inner gimbal angles respectively of the i^{th} star tracker, the direction cosines in this star tracker system are given by $\cos\mu_i \cos\sigma_i$, $\cos\mu_i \sin\sigma_i$, $-\sin\mu_i$ and the direction cosines in the OAO system are obtained from the following relation:

$$T_i \begin{pmatrix} \cos\mu_i \cos\sigma_i \\ \cos\mu_i \sin\sigma_i \\ -\sin\mu_i \end{pmatrix} .$$

Thus, if the OAO has an attitude defined by α_2, δ_2, β_2 and the i^{th} star tracker is locked on a star with right ascension α_i and declination δ_i, the direction cosines x_i, y_i, z_i of the star with respect to this star tracker system can be determined from

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = T_i^{-1} T_{\beta_2} T_{\delta_2} T_{\alpha_2} \begin{pmatrix} \cos\delta_i \cos\alpha_i \\ \cos\delta_i \sin\alpha_i \\ \sin\delta_i \end{pmatrix} .$$

The gimbal angles σ_i and μ_i are then found from the following expressions:

$$\sigma_i = \tan^{-1}\left(y_i/x_i\right) ,$$

$$\mu_i = \tan^{-1}\left(-z_i\Big/\sqrt{x_i^2 + y_i^2}\right) .$$

If the OAO is slewed with a j-k-l slewing sequence and gimbal angles σ_{i_2}, μ_{i_2} after the slewing are desired as a function of the gimbal angles σ_{i_1}, μ_{i_1} before the slewing, the following formulas may be applied:

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = T_i^{-1}\, T_l T_k T_j T_i \begin{pmatrix} \cos\mu_{i_1} \cos\sigma_{i_1} \\ \cos\mu_{i_1} \sin\sigma_{i_1} \\ -\sin\mu_{i_1} \end{pmatrix} ,$$

$$\sigma_{i_2} = \tan^{-1}\left(y_i/x_i\right) ,$$

$$\mu_{i_2} = \tan^{-1}\left(-z_i\Big/\sqrt{x_i^2 + y_i^2}\right) .$$

SATELLITE CONSTRAINTS

Of the twenty-four possible slewing sequences there may be several that are not applicable because of restrictions imposed by the spacecraft itself. These restrictions may be investigated by the same techniques employed in the earlier sections.

One such restriction is that the sun shade may not protect the experimenter's equipment from the sun's rays if the optical x-axis is within 45 degrees of the sun. The first OAO also will have an experiment at the opposite end of the optical axis; thus in this case the minus x-axis must also be kept 45 degrees from the sun.

Although the sun shades are designed to shut if either experimental axis comes within the prohibited area of the sun, it is desirable to avoid a slewing sequence that would require such action. The avoidance of such slews will eliminate damage to the experimental equipment even if the sun shade fails to work properly.

Another satellite constraint is that at least two star trackers must not exceed their gimbal limits during the entire slewing sequence. If this is not possible, new star assignments must be made at intermediate slews.

To determine whether a target star with right ascension α and declination δ lies within the prohibited area of sun, we merely determine the angle from the star to the sun. The cosine of this angle γ is given by

$$\cos\gamma = x_s \cos\delta \cos\alpha + y_s \cos\delta \sin\alpha + z_s \sin\delta \quad ,$$

where x_s, y_s, z_s are the direction cosines of the sun. Thus, if γ is less than 45 degrees, this star may not be viewed. In the case of the double-ended OAO, γ must lie between 45 and 135 degrees before viewing is permissible.

To check whether a slewing sequence will cause the optical axis to come within 45 degrees of the sun, the angle γ between the sun and optical axis can be written as a function of the slewing angle. From this functional relation determine the domain of the slewing angle that makes $\gamma = 45$ degrees or less. If the desired slew lies within this domain, the slew is prohibited. If α_1, δ_1, β_1 define the attitude of the OAO before the slewing sequence begins and if u_s, v_s, w_s are the direction cosines of the sun in the inertial system, the direction cosines of the sun in the OAO system x_s, y_s, z_s are given by

$$\begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = T_{\beta_1} T_{\delta_1} T_{\alpha_1} \begin{pmatrix} u_s \\ v_s \\ w_s \end{pmatrix} .$$

After a slew of λ, where λ may be either a yaw, pitch, or roll, the direction cosines of the sun x_s', y_s', z_s' after the slew are defined as

$$\begin{pmatrix} x_s' \\ y_s' \\ z_s' \end{pmatrix} = T_\lambda \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} .$$

The cosine of the angle between the sun and optical axis as a function of λ is then

$$x_s \cos\lambda + b \sin\lambda \quad ,$$

where $b = y_s$ if λ indicates a yaw and $b = -z_s$ if λ is a pitch. A roll slew need not be considered, since a roll does not affect the angle between the sun and optical axis.

If λ_0 is the desired slew, the slew will be allowable if

$$x_s \cos\lambda + b \sin\lambda \leq \cos 45^\circ \tag{5}$$

for all λ between 0 and λ_0. Equation 5 may be written as

$$\frac{x_s}{\sqrt{x_s^2 + b^2}} \cos\lambda + \frac{b}{\sqrt{x_s^2 + b^2}} \sin\lambda \leq \frac{\cos 45^\circ}{\sqrt{x_s^2 + b^2}} ,$$

or

$$\cos\epsilon \cos\lambda + \sin\epsilon \sin\lambda = \cos(\epsilon - \lambda) \leq \frac{\cos 45^\circ}{\sqrt{x_s^2 + b^2}} ,$$

where

$$\epsilon = \tan^{-1}\left(b/x_s\right) , \quad -\pi \leq \epsilon \leq \pi .$$

If

$$\left| \frac{\cos 45^\circ}{\sqrt{x_s^2 + b^2}} \right| \geq 1 ,$$

the slew is always permissible. If

$$\left| \frac{\cos 45^\circ}{\sqrt{x_s^2 + b^2}} \right| < 1 ,$$

the slew is permissible only if none of the following angles lie between 0 and λ_0:

$$\epsilon - \eta ,$$

$$\epsilon + \eta ,$$

$$\epsilon - \eta + 2\pi ,$$

$$\epsilon + \eta - 2\pi ,$$

where

$$\eta = \cos^{-1}\left(\frac{\cos 45^\circ}{\sqrt{x_s^2 + b^2}}\right) .$$

The next slew in the sequence may be checked in exactly the same way after replacing x_s, y_s, z_s by x_s', y_s', z_s'.

The gimbal angles after each slew of a sequence may be obtained in the same fashion that the final gimbal angles are determined. If σ_{i_0} and μ_{i_0} are the outer and inner gimbal angles of the i^{th} star tracker and a slewing sequence of l-m-n is to be performed, the gimbal angles after the j^{th} slew are given by

$$\left.\begin{aligned} \sigma_{ij} &= \tan^{-1}\left(y_{ij}/x_{ij}\right) \\ \mu_{ij} &= \tan^{-1}\left(-z_{ij}\Big/\sqrt{x_{ij}^2 + y_{ij}^2}\right) \end{aligned}\right\} \quad j = 1, 2, 3 \;,$$

where

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = T_l T_i \begin{pmatrix} \cos\mu_{i_0}\cos\sigma_{i_0} \\ \cos\mu_{i_0}\sin\sigma_{i_0} \\ -\sin\mu_{i_0} \end{pmatrix},$$

$$\begin{pmatrix} x_{i_1} \\ y_{i_1} \\ z_{i_1} \end{pmatrix} = T_i^{-1} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \qquad \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = T_m \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix},$$

$$\begin{pmatrix} x_{i_2} \\ y_{i_2} \\ z_{i_2} \end{pmatrix} = T_i^{-1} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \qquad \begin{pmatrix} x_{i_3} \\ y_{i_3} \\ z_{i_3} \end{pmatrix} = T_i^{-1} T_n \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}.$$

OCCULTATION

The knowledge of stellar occultation is necessary for several important reasons: First, the attitude of the spacecraft cannot be maintained if less than two star trackers are tracking at any time; second, the occultation of the target star during an experiment would result in wasted time.

The three bodies that cause stellar occultation are the earth, sun, and moon. Occultation due to the sun and moon will be less frequent than that due to the earth. Because of this, the time of occultation due to the sun and moon can be determined in a similar manner. In both cases the origin of the coordinate system is assumed to be centered at the OAO. This introduces an error due to parallax of about 10 seconds of arc for the sun and about 1 degree for the moon.

Let i and Ω be the inclination and right ascension respectively of either the moon's or sun's orbit, and let α_i and δ_i be the right ascension and declination of the i^{th} star. The direction cosines a, b, c of this star in a coordinate system whose x-y plane is the plane of the orbit are given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\delta_i \;\; \cos\alpha_i \\ \cos\delta_i \;\; \sin\alpha_i \\ \sin\delta_i \end{pmatrix} .$$

The cosine of the angle γ between the star and the sun or moon is then

$$\cos\gamma = a\cos(\omega+\mu) + b\sin(\omega+\mu) ,$$

where ω is the argument of perigee and μ is the true anomaly. The general requirement is to determine when the angle γ will be less than some fixed angle λ (45 degrees for the sun, one-half the angle subtended by the moon plus errors for the moon). When γ just equals λ, the corresponding time t is the time of immersion or emersion for that star. This requirement of equality may be expressed as

$$a\cos(\omega+\mu) + b\sin(\omega+\mu) = \cos\lambda , \tag{6}$$

or

$$\cos\epsilon\cos(\omega+\mu) + \sin\epsilon\sin(\omega+\mu) = \frac{\cos\lambda}{\sqrt{a^2+b^2}} , \tag{7}$$

where

$$\epsilon = \tan^{-1}(b/a) , \qquad 0 \le \epsilon < 2\pi .$$

Equation 7 may also be written as

$$\cos(\epsilon-\omega-\mu) = \frac{\cos\lambda}{\sqrt{a^2+b^2}}$$

so that

$$\mu = \epsilon - \omega - \eta ,$$

where

$$\eta = \pm\left[\cos^{-1}\left(\frac{\cos\lambda}{\sqrt{a^2+b^2}}\right) + 2k\pi\right] , \qquad k = 0, 1, 2, \cdots .$$

Thus each value of η determines the value of μ at an immersion or emersion of the i^{th} star. This value of μ will correspond to an immersion if

$$\sin(\epsilon - \omega - \mu) > 0 \quad ,$$

and to an emersion if

$$\sin(\epsilon - \omega - \mu) < 0 \quad .$$

Once μ is known, the corresponding value of time can be obtained by the following relations:

$$E = \tan^{-1}\left(\frac{\sqrt{1 - e^2}\,\sin\mu}{e + \cos\mu}\right) + 2\pi\left[\frac{\mu}{2\pi}\right]^{*}$$

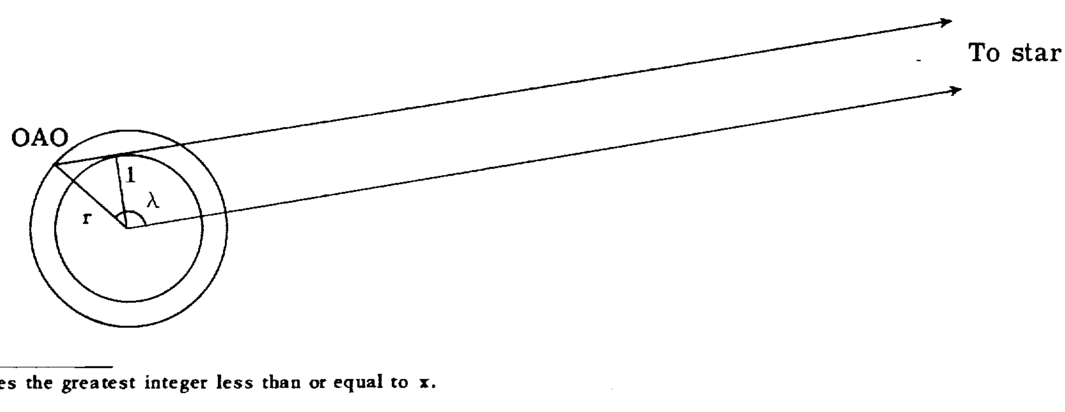
$$M = E - e\sin E$$

$$t = \frac{M - M_0}{n} \tag{8}$$

where e is the eccentricity of the orbit, n the mean motion of the body in the plane of the orbit, M_0 the mean anomaly at epoch, and t the time from epoch.

Although occultation due to the earth is more frequent and troublesome than that due to the sun and moon, it can be handled in a similar manner. The angle λ (immersion or emersion occurs when the angle between OAO and the star equals λ) in this case is not a constant but a function of the OAO's range. Thus immersion and emersion can be determined analytically only when the spacecraft is in a circular orbit. For small eccentricity, however, the circular solution using the mean range should give sufficient accuracy.

Because of the great distances of the stars we may assume that the line from the origin to a star is parallel to the line from the OAO to the star. Then, with the additional assumption of a spherical earth, the cosine of λ (see sketch) may be determined by simple trigonometry.



* [x] indicates the greatest integer less than or equal to x.

Thus

$$\cos \lambda = \frac{-\sqrt{r^2 - 1}}{r} ,$$

where the unit of distance is the earth's radius. Hence, if r is constant, occultation by the earth may be handled in the same manner as that of the sun and moon, that is, by Equations 6 through 8, where the elements are those of the OAO's and where the x-y plane of the coordinate system is the OAO's orbital plane. The test for immersion or emersion in this case is the reverse of that given for sun-moon occultation.

CONCLUDING REMARKS

The analysis and mathematical models contained in this report are intended to be quite general. In many cases the formulas may be simplified if accuracy requirements warrant it. In other cases a different coordinate system will simplify some models; for example, the ecliptic system would reduce any model involving the sun's coordinates. Thus the formulas contained herein are not dependent on any particular coordinate system, and simplifications may be made by simply omitting terms in various expressions.

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