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ON LAGRANGE'S THEORY OF THE THREE-BODY PROBLEM

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SUMMARY

Lagrange's famous studies of the three-body problem, presented in Tisserand's *Traité de Mécanique Céleste,* Vol. I, are redeveloped in modern mathematical expressions. This new portrayal of a very old problem of celestial mechanics gives rise to various considerations which can be very useful for the practical computation. In particular, the nine elements of the "reduced" three-body problem are defined as a symmetrically constructed system of three groups of three elements each, and their differential equations are shown. A by-product of this study is a representation of the coefficients of the equation of the fourth degree for Lagrange's quantity ρ in symmetrical form.

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INTRODUCTION

Lagrange's famous treatment of the three-body problem is still the point of departure of many studies on this difficult problem, which has never been solved in its totality. Although Lagrange's theory is interpretated excellently in its mathematical elegance and clarity in Tisserand's textbook of celestial mechanics,** it is intriguing and, for modern application, practical to bring forth even more the insight and beauty of this theory by using modern mathematical representation. The following considerations are intended mainly to serve the end of bringing this classic theory, in new trappings, into the focus of celestial mechanical study. Also, in the reconsideration of this old problem, certain previously ignored questions will be answered.

Lagrange presented the following postulate: The general three-body problem can be solved completely if it is possible to represent as functions of time all those factors which are independent of the special selection of the coordinate system. The *relative* three-body problem (i.e., the problem of the motion of two bodies in reference to a third or of all three in reference to the common center of gravity) requires twelve integrals, three of which determine the orientation of the system in space. Thus by Lagrange's theorem only nine integrals must be found, for the other three can then be determined easily by a simple quadrature based upon functions known.

According to Hesse, we call this problem of seeking the nine "geometric" integrals the "reduced" three-body problem.*** It is easy to see that this reduced problem can actually be satisfied by nine quantities which are invariable with respect to coordinate transformation. The geometric figure formed by the four vectors determining motion — the vectors of location and velocity of the two bodies with reference to the third — is determined by nine quantities which are independent of the coordinate system. These nine quantities are the values of these four vectors and the five angles by which the vectors' respective locations

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**Tisserand, F., "*Traité de Mécanique Céleste*," Paris: Gauthier-Villars et Fils, 1889, Vol. I.

***This is not to be confused with the *restricted* three-body problem.

are fixed. The construction of this figure from these nine elements is not significant in itself, since the components can be assembled in several different ways; but of the various possibilities there will always be only one that is compatible with the initial conditions given.

These facts, easily understandable geometrically, are reflected in the mathematical relationships between the elements of motion. The relative vectors of location of two bodies in reference to the third are $\mathbf{p}_1$, $\mathbf{p}_2$, and the relative velocity vectors $\dot{\mathbf{p}}_1$, $\dot{\mathbf{p}}_2$; and for convenience let

$$\mathbf{p}_1 = \mathbf{r}_1, \quad \dot{\mathbf{p}}_1 = \mathbf{r}_2, \quad \mathbf{p}_2 = \mathbf{r}_3, \quad \dot{\mathbf{p}}_2 = \mathbf{r}_4 .$$

The geometrical figure formed by these vectors (discounting the aforementioned additional possibilities) is determined by the scalar products possible between them:

$$p_{ab} = (\mathbf{r}_a \mathbf{r}_b) = x_a x_b + y_a y_b + z_a z_b , \tag{1}$$

which form the matrix

$$M = \begin{Vmatrix} p_{11} & \cdots & p_{14} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ p_{41} & \cdots & p_{44} \end{Vmatrix} . \tag{2}$$

Because $p_{ab} = p_{ba}$, this matrix will be formed from ten different elements among which the identity

$$|M| = \begin{vmatrix} p_{11} & \cdots & p_{14} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ p_{41} & \cdots & p_{44} \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_4 & y_4 & z_4 & 0 \end{vmatrix}^2 = 0 \tag{3}$$

holds; therefore only nine of the quantities (Equation 1) which are independent of coordinate transformation are arbitrary, whereas the tenth is a function of the others.

NEW FORM OF THE EQUATIONS OF LAGRANGIAN THEORY

The beauty of the Lagrange theory can be completely expressed only if we provide completely symmetrical formulas. This symmetry cannot be attained if a dominant position is assigned to one of the three bodies, as in the above discussion. Therefore we will introduce a reference system in which none of the vectors is favored. For this purpose, we will call the three masses m_1, m_3, m_5 (Figure 1a) or m_i, m_j, m_k (Figure 1b), where the indexes i, j, k traverse the values 1, 3, 5 or their cyclic permutations. The relative vectors between the masses m_j, m_k are then labeled

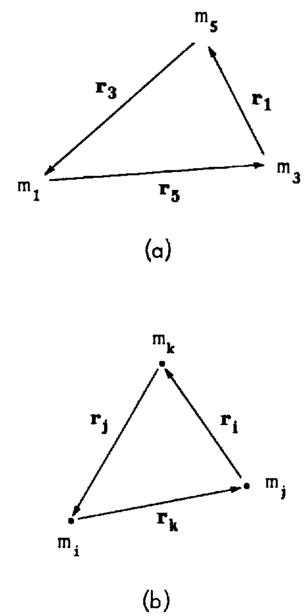

Figure 1—Relative positions of three bodies

$$\mathbf{r}_i = \overrightarrow{m_j m_k}$$

and form the closed train

$$\mathbf{r}_i + \mathbf{r}_j + \mathbf{r}_k = 0. \tag{4}$$

The same also applies to the relative velocity vectors; if we set

$$\dot{\mathbf{r}}_1 = \mathbf{r}_2, \quad \dot{\mathbf{r}}_3 = \mathbf{r}_4, \quad \dot{\mathbf{r}}_5 = \mathbf{r}_6,$$

or

$$\dot{\mathbf{r}}_i = \mathbf{r}_\alpha, \quad \dot{\mathbf{r}}_j = \mathbf{r}_\beta, \quad \dot{\mathbf{r}}_k = \mathbf{r}_\gamma,$$

where $\alpha = i+1$, $\beta = j+1$, $\gamma = k+1$ traverse the values 2, 4, 6 or their cyclic permutations, then it follows from the differentiation of Equation 4 that

$$\mathbf{r}_\alpha + \mathbf{r}_\beta + \mathbf{r}_\gamma = 0 . \tag{5}$$

From the six vectors $\mathbf{r}_1$, $\cdots$ r_6, 36 invariables $(\mathbf{r}_a \mathbf{r}_b) = p_{ab}$ can be formed, but only 21 of these are different since $p_{ab} = p_{ba}$. Twelve relationships exist among these quantities – Equation 3 and eleven others – by means of which the five quantities p_{15}, p_{25}, $\cdots$ p_{55} and the six quantities p_{16}, p_{66}, $\cdots$ p_{66} can be reduced to the ten invariables of the matrix (Equation 2) using the linear relations of Equations 4 and 5. For instance,

$$p_{15} = (\mathbf{r}_1 \mathbf{r}_5) = (\mathbf{r}_1, -\mathbf{r}_1 - \mathbf{r}_3) = -p_{11} - p_{13} , \tag{6}$$

$$p_{55} = (\mathbf{r}_5 \mathbf{r}_5) = (-\mathbf{r}_1 - \mathbf{r}_3)^2 = p_{11} + p_{33} + 2p_{13} . \tag{7}$$

The second of these equations is identical to the cosine theorem of plane trigonometry applied to the triangle of the three bodies in Figures 1a and 1b.

Equation 3 is not symmetrical since it contains only the invariables with the indexes 1 to 4; it can be made symmetrical, however, if we use Lagrange's quantity ρ.

Lagrange noted that the remarkable relationship

$$2\rho = p_{14} - p_{23} = p_{25} - p_{16} = p_{36} - p_{45} \tag{8}$$

exists among the invariables, for

$$p_{14} - p_{23} = -\left(p_{16} + p_{12}\right) + \left(p_{25} + p_{21}\right) = p_{25} - p_{16},$$

and

$$p_{14} - p_{23} = -\left(p_{34} + p_{54}\right) + \left(p_{43} + p_{63}\right) = p_{36} - p_{45},$$

On the other hand, if we set

$$\left.\begin{aligned} 2s_1 &= p_{36} + p_{45}, \\ 2s_3 &= p_{25} + p_{16}, \\ 2s_5 &= p_{14} + p_{23}, \end{aligned}\right\} \tag{9}$$

from Equations 7 and 8 we obtain

$$\begin{aligned} p_{14} &= s_5 + \rho, \quad & p_{25} &= s_3 + \rho, \quad & p_{36} &= s_1 + \rho \\ p_{23} &= s_5 - \rho, \quad & p_{16} &= s_3 - \rho, \quad & p_{45} &= s_1 - \rho. \end{aligned} \tag{10}$$

If Equation 10 is substituted into Equation 3 for p_{14} and p_{23},

$$M = \begin{vmatrix} p_{11} & p_{12} & p_{13} & s_5 + \rho \\ p_{21} & p_{22} & s_5 - \rho & p_{24} \\ p_{31} & s_5 - \rho & p_{33} & p_{34} \\ s_5 + \rho & p_{42} & p_{43} & p_{44} \end{vmatrix} = 0, \tag{11}$$

which is a fourth degree equation for ρ in which the cubic term is missing:

$$\rho^4 + A\rho^2 + B\rho + C = 0. \tag{12}$$

The coefficients of this equation are not constructed symmetrically however, if they are determined by the solution of the determinant (Equation 11). But since ρ is independent of the mass favored (Equation 8), it must be possible to give to the quantities A, B, and C a symmetric form based upon the three masses.

So far no effort has been made to show this *in extenso*. In order to do so, nine quantities which form a symmetrical system will be selected from the 21 invariables $p_{11}, p_{12}, \cdots p_{66}$, such as $p_{11}, p_{33}, p_{55}; p_{22}, p_{44}, p_{66}; p_{12}, p_{34}, p_{56}$, or the squares of the relative distances and velocities, and the scalar products of the three vectors of location with their respective velocity vectors. These nine quantities shall be called the *fundamental invariables* of the reduced three-body problem. If these quantities are known as functions of time, all others can be derived from them, some with linear relationships like Equations 6 and 7, some with the help of the Lagrange quantity ρ, which, because of Equation 12, will be a function of the fundamental invariables. Twelve of the 21 invariables can therefore be expressed with ρ and the nine quantities. In actuality, we find

$$\left.\begin{aligned}
2p_{13} &= p_{55} - p_{11} - p_{33} \,, \\
2p_{24} &= p_{66} - p_{22} - p_{44} \,, \\
2p_{35} &= p_{11} - p_{33} - p_{55} \,, \\
2p_{46} &= p_{22} - p_{44} - p_{66} \,, \\
2p_{51} &= p_{33} - p_{55} - p_{11} \,, \\
2p_{62} &= p_{44} - p_{66} - p_{22} \,.
\end{aligned}\right\} \qquad (13)$$

Further,

$$\left.\begin{aligned}
2s_1 &= p_{12} - p_{34} - p_{56} \,, \\
2s_3 &= p_{34} - p_{56} - p_{12} \,, \\
2s_5 &= p_{56} - p_{12} - p_{34} \,;
\end{aligned}\right\} \qquad (14)$$

and the remaining six invariables are obtained from Equation 10.

With the help of these relationships, it is also possible to express the coefficients of the biquadratic equation (Equation 12) with the nine fundamental quantities in symmetrical form. We shall present only the results of this somewhat complicated computation. If, for purposes of brevity, we introduce

$$\left.\begin{aligned}
\psi_{11} &= p_{11}p_{22} - p_{12}^2 \,, \\
\psi_{13} &= p_{11}p_{44} - p_{34}p_{12} \,, \\
\psi_{15} &= p_{11}p_{66} - p_{56}p_{12} \,, \\
\psi_{31} &= p_{33}p_{22} - p_{12}p_{34} \,, \\
\psi_{33} &= p_{33}p_{44} - p_{34}^2 \,, \\
\psi_{35} &= p_{33}p_{66} - p_{56}p_{34} \,, \\
\psi_{51} &= p_{55}p_{22} - p_{12}p_{56} \,, \\
\psi_{53} &= p_{55}p_{44} - p_{34}p_{56} \,, \\
\psi_{55} &= p_{55}p_{66} - p_{56}^2 \,, \\
X_{13} &= \psi_{13} + \psi_{31} \,, \\
X_{35} &= \psi_{35} + \psi_{53} \,, \\
X_{51} &= \psi_{51} + \psi_{15} \,,
\end{aligned}\right\} \tag{15}$$

we obtain

$$2A = \psi_{11} + \psi_{33} + \psi_{55} - \left(X_{13} + X_{35} + X_{51}\right) \,.$$

$$B = \begin{vmatrix} p_{11} & p_{33} & p_{55} \\ p_{22} & p_{44} & p_{66} \\ p_{12} & p_{34} & p_{56} \end{vmatrix} \tag{16}$$

and

$$4C = A^2 - \left(X_{13}X_{35} + X_{35}X_{51} + X_{51}X_{13}\right) + 2\left(\psi_{11}X_{35} + \psi_{33}X_{51} + \psi_{55}X_{13}\right) \,.$$

The unhandy necessity of solving Equation 12 can be avoided in all practical applications by using the always available integral of the constant impulse moment. Actually, the two known integrals of the reduced three-body problem can be expressed through the fundamental invariables and ρ. We have the energy integral

$$h = \frac{1}{2}\left(\frac{p_{22}}{m_1} + \frac{p_{44}}{m_3} + \frac{p_{66}}{m_5}\right) - \left(\frac{1}{m_1\sqrt{p_{11}}} + \frac{1}{m_3\sqrt{p_{33}}} + \frac{1}{m_5\sqrt{p_{55}}}\right) = \text{constant} ; \tag{17}$$

and, if

$$\psi_{ii} = p_{ii}p_{\alpha\alpha} - p_{i\alpha}^2 \qquad (i = 1, 3, 5; \ \alpha = 2, 4, 6)$$

are the diagonal terms of the matrix ψ_{ik} (Equation 15) and if

$$2\phi_{jk} = A + X_{jk} + 2\rho^2 ,$$

the impulse moment is

$$g^2 = \sum_i \frac{\psi_{ii}}{m_i^2} + 2\sum_{j,k} \frac{\phi_{jk}}{m_j m_k}$$

or

$$g^2 = G + \frac{m_1 + m_3 + m_5}{m_1 m_3 m_5}\left(A + 2\rho^2\right) = \text{constant} . \tag{18}$$

In Equation 18 g is the length of the impulse moment vector which stands vertically upon the invariable plane, and

$$G = \frac{\psi_{11}}{m_1^2} + \frac{\psi_{33}}{m_3^2} + \frac{\psi_{55}}{m_5^2} + \frac{X_{35}}{m_3 m_5} + \frac{X_{51}}{m_5 m_1} + \frac{X_{13}}{m_1 m_3} .$$

If the unit of mass is chosen so that $m_1 + m_3 + m_5 = 1$, it follows from Equation 18 that

$$2\rho^2 = m_1 m_3 m_5\left(g^2 - G\right) - A \tag{19}$$

or, if ρ^2 and ρ^4 are eliminated with the help of Equation 19 from Equation 12,

$$B\rho = \frac{A^2}{4} - C - \frac{1}{4} m_1^2 m_3^2 m_5^2 \left(g^2 - G\right)^2 . \tag{20}$$

Since g^2 is known from the initial conditions of each special problem, ρ can be obtained from Equation 19 or, if B is not equal to zero, from Equation 20.

The differential equations of the relative three-body problem can be written in vectorial form:

$$\ddot{\mathbf{r}}_i = -\mu_i \mathbf{r}_i + m_i \mathbf{R} , \qquad \left(i = 1,\ 3,\ 5;\quad \Sigma\, m_i = 1\right) \tag{21}$$

where

$$\left.\begin{aligned} \mu_i &= \frac{1}{r_i^3} , \\ \text{and} \qquad & \\ \mathbf{R} &= \mathbf{r}_1 \mu_1 + \mathbf{r}_3 \mu_3 + \mathbf{r}_5 \mu_5 \end{aligned}\right\} \tag{22}$$

These differential equations are not independent of one another because of the relationship shown in Equation 4. Thus it is sufficient to solve two — perhaps those for r_1, r_3 (two vectorial differential equations of the second order which form a twelfth order system). But this system is not symmetrical since one of the three masses, perhaps m_5, is given a favored position. Instead of this, however, the Lagrange theory makes it possible to relate the problem's solution to the integration of nine first order differential equations for the nine fundamental invariables, i.e., a completely symmetrical system with no unnecessary components. In actuality, if we differentiate the quantities

$$p_{ii} = \left(\mathbf{r}_i \mathbf{r}_i\right),\quad p_{i\alpha} = \left(\mathbf{r}_i \mathbf{r}_\alpha\right),\quad p_{\alpha\alpha} = \left(\mathbf{r}_\alpha \mathbf{r}_\alpha\right) \qquad (i = 1,\ 3,\ 5;\ \alpha = i + 1) \tag{23}$$

with reference to time and eliminate $\dot{\mathbf{r}}_\alpha = \ddot{\mathbf{r}}_i$ by means of Equation 21, then

$$\left.\begin{aligned} \frac{1}{2}\dot{p}_{ii} &= p_{i\alpha} , \\ \dot{p}_{i\alpha} &= p_{\alpha\alpha} - p_{ii}\mu_i + m_i p_i , \\ \frac{1}{2}\dot{p}_{\alpha\alpha} &= -p_{i\alpha}\mu_i + m_i p_\alpha , \end{aligned}\right\} \tag{24}$$

in which we abbreviate

$$\left.\begin{aligned} p_i &= \left(\mathbf{r}_i \mathbf{R}\right) = p_{i1}\mu_1 + p_{i3}\mu_3 + p_{i5}\mu_5 , \\ p_\alpha &= \left(\mathbf{r}_\alpha \mathbf{R}\right) = p_{\alpha 1}\mu_1 + p_{\alpha 3}\mu_3 + p_{\alpha 5}\mu_5 . \end{aligned}\qquad \begin{aligned} (i &= 1,\ 3,\ 5; \\ \alpha &= 2,\ 4,\ 6) . \end{aligned}\right\} \tag{25}$$

The formation of the quantities p_i and p_α from the fundamental invariables becomes even clearer if we write

$$\left.\begin{aligned} p_i &= p_{ij}(\mu_j - \mu_i) + p_{ik}(\mu_k - \mu_i) , \\ p_\alpha &= s_k(\mu_j - \mu_i) + s_j(\mu_k - \mu_i) + \rho(\mu_k - \mu_j) , \\ \text{where} \\ 2p_{ij} &= p_{kk} - p_{ii} - p_{jj} ; \quad 2s_k = p_{k\gamma} - p_{i\alpha} - p_{j\beta} . \end{aligned}\right\} \tag{26}$$

If Equations 24 are integrated (by using, for example, the numerical methods for the given initial conditions), the problem is solved in principle and the integrals (Equations 17 and 18) are available to check the results. The Lagrange's differential equation for ρ,

$$\begin{aligned} 2\dot{\rho} &= m_1 p_{35}(\mu_3 - \mu_5) + m_3 p_{51}(\mu_5 - \mu_1) + m_5 p_{13}(\mu_1 - \mu_3) \\ &= \sum m_i p_{jk}(\mu_j - \mu_k) , \end{aligned} \tag{27}$$

can be added to the system (Equation 24) as a tenth equation, and is remarkable in its simplicity and symmetry, thereby making possible another thorough check.

Under certain conditions — especially if one of the three bodies has a considerably larger mass than the other two and their orbits can be considered to be disturbed Kepler movements in conical sections — it is useful to introduce, in place of the fundamental invariables, derived invariables which are constructed to correspond with those used in the theory of the two-body problem.* If we set

$$\left.\begin{aligned} \mu_i &= \frac{1}{r_i^3} = p_{ii}^{-3/2} , \\ \sigma_i &= \frac{\dot{r}_i}{r_i} = \frac{p_{i\alpha}}{p_{ii}} , \\ \omega_i &= \frac{p_{\alpha\alpha}}{p_{ii}} , \\ \rho_i &= 2\mu_i - \omega_i , \\ \epsilon_i &= \omega_i - \mu_i , \\ \vartheta_i &= \omega_i - \sigma_i^2 , \end{aligned}\right\} \tag{28}$$

*Stumpff, K., "Calculation of Ephemerides from Initial Values," NASA Technical Note D-1415, in publication 1962.

then these quantities, as derived from Equation 24, satisfy the differential equations

$$
\left.
\begin{aligned}
\dot{\mu}_i &= -3\mu_i\sigma_i \quad \left(\text{or } \dot{r}_i = r_i\sigma_i\right), \\
\dot{\sigma}_i &= \epsilon_i - 2\sigma_i^2 + m_i \frac{p_i}{p_{ii}} , \\
\dot{\omega}_i &= -2\sigma_i\left(\mu_i + \omega_i\right) + 2m_i \frac{p_\alpha}{p_{ii}} , \\
\dot{\rho}_i &= -2\rho_i\sigma_i - 2m_i \frac{p_\alpha}{p_{ii}} , \\
\dot{\epsilon}_i &= -\sigma_i\left(\mu_i + 2\epsilon_i\right) + 2m_i \frac{p_\alpha}{p_{ii}} , \\
\vartheta_i &= -4\eta_i\sigma_i + 2\frac{m_i}{p_{ii}}\left(p_\alpha - p_i\sigma_i\right) .
\end{aligned}
\quad
\begin{aligned}
(i &= 1, 3, 5; \\
\alpha &= i + 1)
\end{aligned}
\right\} \quad (29)
$$

Since only three of the six quantities (Equation 28) are independent, only three of the equation systems (Equation 29) need to be considered, i.e., μ_i, σ_i, ϵ_i. If we also set

$$
\epsilon_i^* = \epsilon_i + m_1 \frac{p_i}{p_{ii}} , \qquad (30)
$$

we obtain the system:

$$
\begin{aligned}
\mu_i &= -3\mu_i\sigma_i , \quad \left(\text{or } \dot{r}_i = r_i\sigma_i\right) ; \\
\dot{\sigma}_i &= \epsilon_i^* - 2\sigma_i^2 ; \\
\dot{\epsilon}_i^* &= -\sigma_i\left(\mu_i + 2\epsilon_i^*\right) + \gamma_i ,
\end{aligned} \qquad (31)
$$

where

$$
\gamma_i = \frac{m_i}{p_{ii}}\left(2p_\alpha + \dot{p}_i\right) . \qquad (32)
$$

The γ_i are three functions of the invariables which, if m_i is one of the small masses, are called perturbations. To show this, we form the equations

$$\dot{r}_i = r_i \sigma_i ,$$

$$\ddot{r}_i = \dot{r}_i \sigma_i + r_i \dot{\sigma}_i = r_i (\epsilon_i^* - \sigma_i^2) ,$$

$$\dddot{r}_i = \dot{r}_i (\epsilon_i^* - \sigma_i^2) + r_i (\dot{\epsilon}_i^* - 2\sigma_i \dot{\sigma}_i)$$

$$= r_i \sigma_i \left[3 (\epsilon_i^* - \sigma_i^2) + \mu_i\right] + r_i \gamma_i$$

by differentiating the identity $r_i^2 = x_i^2 + y_i^2 + z_i^2$. After elimination of σ_i and $\epsilon_i^* - \sigma_i^2$ from these equations, there are three third order differential equations:

$$r_i \dddot{r}_i + 3\dot{r}_i \ddot{r}_i + \frac{\dot{r}_i}{r_i^2} = r_i^2 \gamma_i = m_i (2p_\alpha + \dot{p}_i) , \tag{33}$$

whose nine integrals also solve the reduced problem. Because $\Sigma p_\alpha = \Sigma \dot{p}_i = 0$, it follows from Equation 33 that

$$\sum \frac{1}{m_i} \left(r_i \dddot{r}_i + 3\dot{r}_i \ddot{r}_i + \frac{\dot{r}_i}{r_i^2} \right) = 0 ,$$

an equation whose integral

$$\sum \frac{1}{m_i} \left(r_i \ddot{r}_i + \dot{r}_i^2 - \frac{1}{r_i} \right) = \text{constant}$$

yields the energy law.

CONCLUSION

In the case of general three-body motion where the ratios of mass can have arbitrary values, we always have to rely on numerical integration methods. We have a choice of integrating either the nine first order differential equations (Equations 24 or 31) or the three third order differential equations (Equations 33). The preferable method is determined by experience and available facilities. The system (Equation 33) would probably be preferred for program-controlled electronic computers, since it requires only three, rather than nine, tables of differences.

For instance if the problem is to calculate disturbed planetary orbits, when m_1 denotes the mass of the sun, and m_3, m_5 two small planet masses (such as Jupiter, $m_3 < 10^{-3}$ and Saturn, $3m_5 < 10^{-4}$), and if the perturbations of the second order can be ignored, we use the six equations (Equations 24 or 31) or the two equations (Equation 33) with $i = 3, 5$. The disturbing functions γ_3, γ_5, which are compounded with small factors, are then calculated with the help of the system's undisturbed movement which is known from the initial conditions. This process cannot be used in equations where $i = 1$; the motion of the masses m_3 and m_5 relative to one another cannot (as can that of m_3 or m_5 relative to m_1) be considered in first approximation to be a Kepler ellipse. It is unnecessary, on the other hand, to solve the equation where $i = 1$, since its solution is implicitly a co-product of the integration process leading to the solution of the other equations.

It should also be mentioned that

$$\left.\begin{aligned} &\frac{d}{dt}\left(r_i^2 \rho_i\right) = -2m_i p_\alpha \\ &\text{and} \\ &\frac{d}{dt}\left(r_i^4 \vartheta_i\right) = 2m_i p_{ii} \left(p_\alpha - p_i \sigma_i\right) \end{aligned}\right\} \qquad (34)$$

can be derived from Equation 29. These equations are integrable for $m_i = 0$ and yield the known integrals of the two-body theory:

$$r_i^2 \rho_i = \frac{1}{a_i} = \text{constant} \qquad \text{(energy theorem)},$$

$$r_i^4 \vartheta_i = p_i = \text{constant} \qquad \text{(area theorem)}.$$

In this method of writing the invariables, the formulas (Equation 34) represent four of the differential equations of the elemental perturbations used in the theory of the special perturbations.

The task of carrying out the integration of the equations of perturbations according to a method which has been earlier described for the solution of the undisturbed problem will be treated in a later report. An attempt of this type has already been made for Hill's lunar problem,* a particularly simple variation of the restricted three-body problem. The expansion of this result to include the general problem of disturbed orbits will be of considerably more interest.

*Stumpff, K., "Remarks on Hill's Lunar Theory, I and II," NASA Technical Notes D-1450, D-1451, in publication 1962.

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