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AN ANALYTIC REPRESENTATION OF MUSEN'S THEORY OF ARTIFICIAL SATELLITES IN TERMS OF THE ORBITAL TRUE LONGITUDE

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AN ANALYTICAL REPRESENTATION OF MUSEN'S THEORY OF ARTIFICIAL SATELLITES IN TERMS OF THE ORBITAL TRUE LONGITUDE

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SUMMARY

Analytic solutions to order k_2 of the differential equations of Musen's Theory of Artificial Satellites are presented. These solutions include long-period terms and terms with small divisors derived from the k_2^2 approximation as well as from the third and fourth harmonics of the earth's potential. The results obtained herein may be used for computing orbits of artificial satellites to the first order, for checking the machine programming of the numerical theory, and for comparing Musen's theory with other artificial satellite theories. In addition, these results may be incorporated into the numerical theory to broaden its applicability, particularly in the case of nearly circular orbits.

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AN ANALYTIC REPRESENTATION OF MUSEN'S THEORY OF ARTIFICIAL SATELLITES IN TERMS OF THE ORBITAL TRUE LONGITUDE

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INTRODUCTION

Musen's theory of artificial satellites (Reference 1) in terms of the orbital true longitude permits numerical calculation of the perturbations of a satellite's orbit due to the earth's potential field. The degree of accuracy is limited only by the knowledge of the earth's potential field and the capabilities of the computer. This theory is also a useful tool for improving our knowledge of the geodetic parameters of the earth's potential. However, any numerical theory has the disadvantage that the characteristics of the solution — such as its behaviour for nearly circular orbits or in the vicinity of the critical angle of 63.4 degrees — are not explicitly given. In addition, comparison with other satellite theories is virtually impossible.

For these reasons the development of an analytical solution of Musen's theory was undertaken. Since the theory has been adopted by NASA for orbit computation in its numerical form, an analytical solution is particularly useful — first as a check for the machine program, and secondly as a first approximation to the numerical series. For example, substituting the analytical expression of $(1 - c)$, the motion of the argument of perigee in the plane of the orbit, to order k_2^2, into the numerical program would eliminate a division by e_0 which would cause trouble in the case of a nearly circular orbit.

The basic arguments in this report are expressed in terms of the orbital true longitude v. They are the true anomaly of the auxiliary ellipse,

$$\xi = cv - \pi_0 = v - \left[(1 - c)\, v + \pi_0\right] ;$$

and the argument of latitude of the auxiliary satellite,

$$\eta = gv - \theta_0 = v - \left[(1 - g) v + \sigma_0 \right] .$$

Since the departure point is arbitrary, σ_0 is taken equal to θ_0. All other nomenclature has been kept the same as in Musen's paper (Reference 1):

a the semimajor axis,

e the eccentricity,

$h = \dfrac{1}{\sqrt{a(1 - e^2)}}$

i the inclination of the orbit plane to the Earth's equator,

$n = a^{-3/2}$, the mean motion,

$p = a(1 - e^2)$.

θ the right ascension of the ascending node,

σ the angular distance of the node from the departure point X,

φ the elliptic $cv - \pi_0$.

PERTURBATIONS IN THE ORBIT PLANE

The derivative of Hansen's W function in terms of the orbital true longitude as given by Musen is

$$\begin{aligned} \frac{dW}{dv} = \; & h_0^2 \left(\frac{1}{u^2} \frac{\partial \Omega}{\partial v} \right) \left\{ 2 \frac{u}{h_0^2} \cos(\xi - \varphi) - 1 - e_0 \cos \varphi \right. \\ & \left. + 2 \left(\frac{h}{h_0} \right)^2 \left[\cos(\xi - \varphi) - 1 \right] \right\} \frac{h}{h_0} \\ & - 2 \frac{\partial \Omega}{\partial u} \frac{h}{h_0} \sin(\xi - \varphi) \\ & + (1 - c) \left[\frac{\partial W}{\partial \varphi} - \left(1 + \frac{h_0}{h} \right) e_0 \sin \varphi \right] . \end{aligned}$$

For the first approximation of W, we let

$$u = \bar{u} = h_0^2 (1 + e_0 \cos \xi),$$

$$\frac{h}{h_0} = \frac{h_0}{h} = 1 ,$$

$$\frac{\partial W}{\partial \varphi} = 0 ,$$

$$\lambda_1 = \sin \frac{i_0}{2} ,$$

$$\lambda_4 = \cos \frac{i_0}{2} ,$$

$$\lambda_2 = \lambda_3 = 0 .$$

The disturbing function is taken to be the same as that given by Musen:

$$\Omega = k_2 u^3 (1 - 3\psi^2) + k_3 u^4 (3\psi - 5\psi^3) + k_4 u^5 (3 - 30\psi^2 + 35\psi^4) .$$

After substituting this into Equation 1 and expanding, we develop a trigonometric series for dW/dv. The quantity $(1 - c)$ is determined in such a way that no $\sin \varphi$ term appears in dW/dv. The first approximation to $(1 - c)$ is

$$(1 - c) = -\frac{3}{2} k_2 h_0^4 \left(1 - 3 \cos^2 i_0\right) .$$

From the second approximation of dW/dv we determine the value of this secular term to order k_2^2:

$$\begin{aligned} c = 1 &+ \frac{3}{2} k_2 h_0^4 \left(1 - 3 \cos^2 i_0\right) \\ &- \frac{15}{16} k_4 h_0^8 \left(4 + 3e_0^2\right)\left(3 - 30 \cos^2 i_0 + 35 \cos^4 i_0\right) \\ &+ \frac{3}{32} k_2^2 h_0^8 \left[\left(162 - 64 \sqrt{1 - e_0^2} + 25e_0^2\right)\right. \\ &\quad - \left(1060 - 480 \sqrt{1 - e_0^2} + 74e_0^2\right) \cos^2 i_0 \\ &\quad \left. + \left(1650 - 864 \sqrt{1 - e_0^2} + 9e_0^2\right) \cos^4 i_0\right] . \end{aligned} \quad (2)$$

The integration of dW/dv yields

$$W = C_0 + C_1 \cos \varphi + \sum A' \cos (i\xi + j\varphi + 2k\eta) + \sum B' \sin \left[i\xi + j\varphi + (2k + 1)\eta\right] . \quad (3)$$

The constants of integration C_0 and C_1 are determined from $dn_0 \delta z/dv$ in such a way that no constant and no $\cos \xi$ term should appear in that derivative. Finally $\overline{W}$ (the function needed to obtain $n_0 \delta z$, the perturbations in the mean anomaly; and $(1 + \nu)$, the perturbations in the radius vector) is itself obtained from W by setting $\varphi = \xi$:

$$\overline{W} = C_0 + C_1 \cos\xi + \sum A \cos\left[i\xi + 2k\eta\right] + \sum B \sin\left[i\xi + (2k+1)\eta\right] . \qquad (4)$$

The perturbations in the mean anomaly are developed from the derivative (Reference 1)

$$\frac{dn_0 \delta z}{dv} = \left(1 - e_0^2\right)^{\frac{3}{2}} \left(\frac{h_0^2}{\bar{u}}\right)^2 \frac{h_0^2 \frac{\overline{W}}{\bar{u}} + \nu^2}{1 + \frac{h_0^2 \overline{W}}{\bar{u}}} ; \qquad (5)$$

or, in a more convenient form,

$$\frac{dn_0 \delta z}{dv} = \left(1 - e_0^2\right)^{\frac{3}{2}} \left(\frac{\bar{r}}{p_0}\right)^3 \overline{W} + \left(1 - e_0^2\right)^{\frac{3}{2}} \left(\frac{\bar{r}}{p_0}\right)^2 \nu^2 - \left(\frac{\bar{r}}{p_0}\right) \overline{W} \frac{dn_0 \delta z}{dv} . \qquad (6)$$

where

$$\frac{h_0^2}{\bar{u}} = \frac{\bar{r}}{p_0} .$$

For the first approximation of $dn_0 \delta z / dv$ we let

$$\nu = 0, \qquad \frac{dn_0 \delta z}{dv} = 0$$

in the right-hand side of Equation 6, and we have

$$\left[\frac{dn_0 \delta z}{dv}\right]_1 = \left(1 - e_0^2\right)^{\frac{3}{2}} \left(\frac{\bar{r}}{p_0}\right)^3 \overline{W} . \qquad (7)$$

In the second approximation, Equation 7 is used in the last term of Equation 6 and ν^2 is obtained from $\bar{u} = u(1 + \nu)$.

$$\nu^2 = \left(\frac{\bar{r}}{p_0}\right)^2 \left(\frac{\delta u}{h_0^2}\right)^2 .$$

Thus, the second approximation to order k_2^2 becomes

$$\left[\frac{dn_0 \delta z}{dv}\right]_2' = \left(1 - e_0^2\right)^{\frac{3}{2}} \left(\frac{\bar{r}}{p_0}\right)^3 \overline{W} + \left(1 - e_0^2\right)^{\frac{3}{2}} \left(\frac{\bar{r}}{p_0}\right)^4 \left[\left(\frac{\delta u}{h_0^2}\right)^2 - \overline{W}^2\right] . \tag{8}$$

Considering the expression

$$W = \Xi + \Upsilon \cos \varphi + \Psi \sin \varphi ,$$

and the definitions of Ξ, Υ, and Ψ (Reference 1) we see that the next approximation after $h/h_0 = h_0/h = 1$ is

$$\delta \frac{h_0}{h} = -\frac{1}{3}\Xi \quad \text{and} \quad \delta \frac{h}{h_0} = +\frac{1}{3}\Xi , \tag{9}$$

whence

$$u = \bar{u} + \frac{1}{6}\Xi\bar{u} + \frac{1}{2} h_0^2 \overline{W} . \tag{10}$$

After determining a first approximation of all of the perturbations to order k_2, including the λ parameters which will be discussed below, we use the results in the original expression for the disturbing function and in Equation 1 to determine a second approximation of dW/dv. These terms were the k_2^2, k_3, and k_4 terms. In the process of integration some of the terms in dW/dv are divided by $(c - g)$ or a multiple of $(c - g)$. Individually, c and g are both of the order of 1, but $(c - g)$ is of the order of k_2. Hence, from the second approximation of dW/dv we obtain some first order terms in W.

The expression, to order k_2, that we obtained for Hansen's W function, is developed as follows:

$$W = \Xi + \Upsilon \cos \varphi + \Psi \sin \varphi , \tag{11}$$

where

$$\begin{aligned} \Xi = &+ 3k_2 h_0^4 \left(1 - 3\cos^2 i_0\right)\left[2 - \sqrt{1 - e_0^2}\right] \\ &+ \frac{3}{8} k_2 h_0^4 \left(1 - \cos^2 i_0\right) \left\{ - 12\cos 2\eta - 4e_0 \cos(\xi + 2\eta) - 12 e_0 \cos(\xi - 2\eta) \right. \\ &\left. + \left[e_0^2 - \frac{10 e_0^2 \cos^2 i_0}{1 - 5\cos^2 i_0}\right] \cos(2\xi - 2\eta) \right\} \end{aligned}$$

(equation continued next page)

$$+ \frac{3}{2} \frac{k_3}{k_2} h_0^2 \, e_0 \sin i_0 \sin (\xi - \eta)$$

$$- \frac{15}{4} \frac{k_4}{k_2} h_0^4 \, e_0^2 \left[1 - 3 \cos^2 i_0 - \frac{8 \cos^4 i_0}{1 - 5 \cos^2 i_0} \right] \cos (2\xi - 2\eta); \tag{12}$$

$$\Upsilon = + k_2 h_0^4 \left(1 - 3 \cos^2 i_0\right) \left[\frac{1}{2} e_0 + \frac{2e_0 + e_0^3}{1 + \sqrt{1 - e_0^2}} - \left(12 + 3e_0^2\right) \cos \xi - 6e_0 \cos 2\xi - e_0^2 \cos 3\xi \right]$$

$$+ \frac{1}{8} k_2 h_0^4 \left(1 - \cos^2 i_0\right) \Bigg\{ 24e_0 \cos 2\eta + \left(28 + 5e_0^2\right) \cos (\xi + 2\eta)$$

$$+ \left(12 - 3e_0^2\right) \cos (\xi - 2\eta) + 18e_0 \cos (2\xi + 2\eta)$$

$$+ \left[20e_0 + e_0^3 - \frac{\left(20e_0 + 10e_0^3\right) \cos^2 i_0}{1 - 5 \cos^2 i_0} \right] \cos (2\xi - 2\eta)$$

$$+ 3e_0^2 \cos (3\xi + 2\eta) + 3e_0^2 \cos (3\xi - 2\eta) \Bigg\}$$

$$+ \frac{1}{2} \frac{k_3}{k_2} h_0^2 \left(2 + e_0^2\right) \sin i_0 \sin (\xi - \eta)$$

$$- \frac{5}{4} \frac{k_4}{k_2} h_0^4 \left(2e_0 + e_0^3\right) \left[1 - 3 \cos^2 i_0 - \frac{8 \cos^4 i_0}{1 - 5 \cos^2 i_0} \right] \cos (2\xi - 2\eta); \tag{13}$$

and

$$\Psi = - \frac{1}{4} k_2 h_0^4 \left(1 - 3 \cos^2 i_0\right) \left[\left(12 + 9e_0^2\right) \sin \xi + 6e_0 \sin 2\xi + e_0^2 \sin 3\xi \right]$$

$$+ \frac{1}{8} k_2 h_0^4 \left(1 - \cos^2 i_0\right) \Bigg\{ 36e_0 \sin 2\eta + \left(28 + 11e_0^2\right) \sin (\xi + 2\eta)$$

$$+ \left(12 - 21e_0^2\right) \sin (\xi - 2\eta) + 18e_0 \sin (2\xi + 2\eta)$$

$$+ \left[20e_0 + e_0^3 - 20e_0 \cos^2 i_0 - \frac{\left(100e_0 - 20e_0^3\right) \cos^4 i_0}{1 - 5 \cos^2 i_0} + \frac{100e_0^3 \cos^6 i_0}{\left(1 - 5 \cos^2 i_0\right)^2} \right] \sin (2\xi - 2\eta)$$

$$+ \; 3e_0^2 \sin(3\xi + 2\eta) + 3e_0^2 \sin(3\xi - 2\eta) \Bigg\}$$

$$- \frac{k_3}{k_2} h_0^2 \sin i_0 \left[1 - \frac{2e_0^2 \cos^2 i_0}{1 - 5\cos^2 i_0} \right] \cos(\xi - \eta)$$

$$- \frac{5}{4} \frac{k_4}{k_2} h_0^4 \Bigg[2e_0 + e_0^3 - \left(6e_0 + 5e_0^3\right) \cos^2 i_0$$

$$\frac{\left(16e_0 + 12e_0^3\right)\cos^4 i_0}{1 - 5\cos^2 i_0} + \frac{16e_0^3 \cos^6 i_0}{\left(1 - 5\cos^2 i_0\right)^2} \Bigg] \sin(2\xi - 2\eta) \,. \tag{14}$$

Substituting Equation 12 into Equation 9 gives us the expressions for h_0/h and h/h_0 to order k_2:

$$\frac{h_0}{h} = 1 - k_2 h_0^4 \left(1 - 3\cos^2 i_0\right) \left(2 - \sqrt{1 - e_0^2}\right)$$

$$+ \frac{1}{8} k_2 h_0^4 \left(1 - \cos^2 i_0\right) \Bigg\{ 12 \cos 2\eta + 4e_0 \cos(\xi + 2\eta)$$

$$+ 12e_0 \cos(\xi - 2\eta) - \left[e_0^2 - \frac{10e_0^2 \cos^2 i_0}{1 - 5\cos^2 i_0} \right] \cos(2\xi - 2\eta) \Bigg\}$$

$$- \frac{1}{2} \frac{k_3}{k_2} h_0^2 e_0 \sin i_0 \sin(\xi - \eta)$$

$$+ \frac{5}{4} \frac{k_4}{k_2} h_0^4 e_0^2 \left[1 - 3\cos^2 i_0 - \frac{8\cos^4 i_0}{1 - 5\cos^2 i_0} \right] \cos(2\xi - 2\eta) \,, \tag{15}$$

and

$$\frac{h}{h_0} = 2 - \frac{h_0}{h} \,.$$

The perturbations of the radius vector can be expressed most easily as the perturbations of its reciprocal u from Equation 10. Remembering that $\overline{W} = W|_{\varphi=\xi}$ we have also

$$u = \bar{u} + \frac{1}{6} h_0^2 \left[\left(4 + e_0 \cos\xi\right) \Xi + 3\Upsilon \cos\xi + 3\Psi \sin\xi \right] \,. \tag{16}$$

From this we obtain

$$u = \bar{u} + \frac{1}{4} k_2 h_0^6 \left(1 - 3 \cos^2 i_0\right) \Bigg[10 - 3e_0^2 - 8\sqrt{1 - e_0^2}$$

$$+ \frac{4e_0\left(1 + e_0^2\right)}{1 + \sqrt{1 - e_0^2}} \cos \xi + e_0^2 \cos 2\xi \Bigg]$$

$$- \frac{1}{32} k_2 h_0^6 \left(1 - \cos^2 i_0\right) \Bigg\{ \left(16 + 24e_0^2\right) \cos 2\eta + 20e_0 \cos (\xi + 2\eta)$$

$$+ \left[8e_0 - 3e_0^3 + \frac{\left(40e_0 + 20e_0^3\right)\cos^2 i_0}{1 - 5\cos^2 i_0} - \frac{20e_0^3 \cos^4 i_0}{\left(1 - 5\cos^2 i_0\right)^2} \right] \cos (\xi - 2\eta)$$

$$+ 4e_0^2 \cos (2\xi + 2\eta) - 20e_0^2 \frac{1 - 9\cos^2 i_0}{1 - 5\cos^2 i_0} \cos (2\xi - 2\eta)$$

$$- e_0^3 \left[1 - \frac{20 \cos^2 i_0}{1 - 5\cos^2 i_0} - \frac{20 \cos^4 i_0}{\left(1 - 5 \cos^2 i_0\right)^2} \right] \cos (3\xi - 2\eta) \Bigg\}$$

$$- \frac{1}{4} \frac{k_3}{k_2} h_0^4 \sin i_0 \left[\left(2 + 5e_0^2\right) \sin \eta - 4e_0 \sin (\xi - \eta) + 3e_0^2 \sin (2\xi - \eta) \right]$$

$$- \frac{5}{16} \frac{k_4}{k_2} h_0^6 \left(1 - 3\cos^2 i_0 - \frac{8 \cos^4 i_0}{1 - 5\cos^2 i_0}\right) \Bigg[8e_0^2 \cos (2\xi - 2\eta)$$

$$+ \left(e_0^3 + \frac{2e_0^3 \cos^2 i_0}{1 - 5\cos^2 i_0}\right) \cos (3\xi - 2\eta)$$

$$+ \left(4e_0 + 3e_0^3 - \frac{2e_0^3 \cos^2 i_0}{1 - 5\cos^2 i_0}\right) \cos (\xi - 2\eta) \Bigg] . \tag{17}$$

The final perturbation in the orbit plane to be determined is the perturbation of the mean anomaly, $n_0 \delta z$. This is obtained by substituting the expressions for $\overline{W}$ and δu into Equation 8 and integrating. It is of interest to note that any terms with an argument of $(\xi - \eta)$ or any multiple of it will, when integrated, produce a term of a lower order than in the expression for $dn_0 \delta z/dv$. Hence, we develop $\overline{W}$ to order k_2^2 to determine all terms of order k_2 in $n_0 \delta z$. On the other hand it might be expected that from the long period terms of order k_2 in $dn_0 \delta z/dv$ we would obtain some terms of zero order in $n_0 \delta z$, but this was not the case. All the long period terms generated in $dn_0 \delta z/dv$ of order k_2 independently cancelled one another. Consequently, in our expression for $n_0 \delta z$ the short period terms come from $\overline{W}$ of order k_2, and the long period terms come from $\overline{W}$ of order k_2^2 and from the second term in Equation 8, also of order k_2^2. The equation for $n_0 \delta z$ (Equation 18) is given as Appendix A because of its length.

PERTURBATIONS OF THE POSITION OF THE ORBIT PLANE

The perturbations of the position of the orbit plane are developed by determining the λ parameters and the secular motions of the argument of perigee and of the ascending node. By definition, the osculating values of the argument of perigee and of the node are

$$\omega = (g - c)v + \pi_0 - \sigma_0 + N + K + \phi \tag{19}$$

and

$$\theta = (1 - h')v + \theta_0 - N + K . \tag{20}$$

The λ parameters are defined by

$$\begin{aligned} \lambda_1 &= \sin \frac{i}{2} \cos N , \\ \lambda_2 &= \sin \frac{i}{2} \sin N , \\ \lambda_3 &= \cos \frac{i}{2} \sin K , \\ \lambda_4 &= \cos \frac{i}{2} \cos K . \end{aligned} \tag{21}$$

They include all the periodic perturbations from the position of the mean node, from the mean argument of perigee, and from the inclination. The derivatives of the λ parameters as given by Musen are

$$\frac{d\lambda_1}{dv} = + \left(\frac{h' + g}{2} - 1 \right) \lambda_2 + \frac{h_0^2}{4} \left(\frac{h}{h_0} \right)^2 \frac{1}{u^2} \frac{\partial \Omega}{\partial \lambda_2} \cos i , \tag{22}$$

$$\frac{d\lambda_2}{dv} = - \left(\frac{h' + g}{2} - 1 \right) \lambda_1 - \frac{h_0^2}{4} \left(\frac{h}{h_0} \right)^2 \frac{1}{u^2} \frac{\partial \Omega}{\partial \lambda_1} \cos i , \tag{23}$$

$$\frac{d\lambda_3}{dv} = + \frac{h' - g}{2} \lambda_4 + \frac{h_0^2}{4} \left(\frac{h}{h_0} \right)^2 \frac{1}{u^2} \frac{\partial \Omega}{\partial \lambda_4} \cos i , \tag{24}$$

$$\frac{d\lambda_4}{dv} = -\frac{h' - g}{2}\lambda_3 - \frac{h_0^2}{4}\left(\frac{h}{h_0}\right)^2 \frac{1}{u^2}\frac{\partial\Omega}{\partial\lambda_3}\cos i\,. \tag{25}$$

By determining the derivatives of Ω with respect to the λ's to order k_2 and substituting them into the foregoing equations, we develop a series for each of the λ parameters. The secular terms h' and g are determined from Equations 23 and 24 in such a manner that no constant term should appear or the right-hand sides. And from the second approximations of the λ derivatives, we determine these secular terms to order k_2^2, that is,

$$\begin{aligned} g = {} & 1 + 3k_2 h_0^4 \cos^2 i_0 - \frac{15}{4} k_4 h_0^8 \left(2 + 3e_0^2\right)\left(3\cos^2 i_0 - 7\cos^4 i_0\right) \\ & + \frac{3}{8} k_2^2 h_0^8 \Big[\left(110 - 48\sqrt{1 - e_0^2} + 13e_0^2\right)\cos^2 i_0 \\ & \quad - \left(290 - 144\sqrt{1 - e_0^2} + 9e_0^2\right)\cos^4 i_0\Big]\,; \end{aligned} \tag{26}$$

$$\begin{aligned} (1 - h') = {} & -3k_2 h_0^4 \cos i_0 + \frac{15}{4} k_4 h_0^8 \left(2 + 3e_0^2\right)\left(3\cos i_0 - 7\cos^3 i_0\right) \\ & - \frac{3}{8} k_2^2 h_0^8 \Big[\left(88 - 40\sqrt{1 - e_0^2} + 9e_0^2\right)\cos i_0 \\ & \quad - \left(236 - 120\sqrt{1 - e_0^2} + 5e_0^2\right)\cos^3 i_0\Big]\,. \end{aligned} \tag{27}$$

From Equations 26 and 2 we obtain the secular motion of the argument of perigee:

$$\begin{aligned} g - c = {} & -\frac{3}{2} k_2 h_0^4 \left(1 - 5\cos^2 i_0\right) \\ & + \frac{15}{16} k_4 h_0^8 \left[\left(12 + 9e_0^2\right) - \left(144 + 126e_0^2\right)\cos^2 i_0 + \left(196 + 189e_0^2\right)\cos^4 i_0\right] \\ & - \frac{3}{32} k_2^2 h_0^8 \Big[\left(162 - 64\sqrt{1 - e_0^2} + 25e_0^2\right) \\ & \quad - \left(1500 - 672\sqrt{1 - e_0^2} + 126e_0^2\right)\cos^2 i_0 \\ & \quad + \left(2810 - 1440\sqrt{1 - e_0^2} + 45e_0^2\right)\cos^4 i_0\Big]\,. \end{aligned} \tag{28}$$

The analytic expressions for the λ parameters obtained are as follows:

$$\lambda_1 = \sin\frac{i_0}{2} + k_2 h_0^4 \sin i_0 \cos i_0 \cos\frac{i_0}{2}\left[\frac{3}{4}\cos 2\eta + \frac{1}{4} e_0 \cos(\xi + 2\eta)\right.$$

$$\left. + \frac{3}{4} e_0 \cos(\xi - 2\eta) - \frac{e_0^2\left(1 - 15\cos^2 i_0\right)}{16\left(1 - 5\cos^2 i_0\right)}\cos(2\xi - 2\eta)\right]$$

$$+ -\frac{1}{4}\frac{k_3}{k_2} h_0^2 e_0 \cos i_0 \cos\frac{i_0}{2}\sin(\xi - \eta)$$

$$+ \frac{5}{8}\frac{k_4}{k_2} h_0^4 e_0^2 \sin i_0 \cos i_0 \cos\frac{i_0}{2}\frac{\left(1 - 7\cos^2 i_0\right)}{\left(1 - 5\cos^2 i_0\right)}\cos(2\xi - 2\eta)\ ; \tag{29}$$

$$\lambda_2 = + k_2 h_0^4 \sin i_0 \cos i_0 \cos\frac{i_0}{2}\left\{\frac{3}{2} e_0 \sin\xi - \frac{3}{4}\sin 2\eta - \frac{1}{4} e_0 \sin(\xi + 2\eta)\right.$$

$$+ \frac{3}{4} e_0 \sin(\xi - 2\eta) + \left[-\frac{e_0^2\left(8 + 3\cos i_0 - 18\cos^2 i_0\right)}{8\left(1 - 5\cos^2 i_0\right)}\right.$$

$$\left.\left. + \frac{e_0^2\left(5 + \cos i_0 - 6\cos^2 i_0\right)\left(1 - 15\cos^2 i_0\right)}{16\left(1 - 5\cos^2 i_0\right)^2}\right]\sin(2\xi - 2\eta)\right\}$$

$$+ \frac{1}{4}\frac{k_3}{k_2} h_0^2 e_0 \cos i_0 \cos\frac{i_0}{2}\left(\frac{1 - 2\cos i_0 - 3\cos^2 i_0}{1 - 5\cos^2 i_0}\right)\cos(\xi - \eta)$$

$$+ \frac{5}{4}\frac{k_4}{k_2} h_0^4 e_0^2 \sin i_0 \cos i_0 \cos\frac{i_0}{2}\left[\frac{4 - 7\cos^2 i_0}{1 - 5\cos^2 i_0}\right.$$

$$\left. - \frac{\left(5 + \cos i_0 - 6\cos^2 i_0\right)\left(1 - 7\cos^2 i_0\right)}{2\left(1 - 5\cos^2 i_0\right)^2}\right]\sin(2\xi - 2\eta)\ . \tag{30}$$

$$\lambda_3 = - k_2 h_0^4 \sin i_0 \cos i_0 \sin\frac{i_0}{2}\left\{\frac{3}{2} e_0 \sin\xi - \frac{3}{4}\sin 2\eta - \frac{1}{4} e_0 \sin(\xi + 2\eta)\right.$$

$$+ \frac{3}{4} e_0 \sin(\xi - 2\eta) + \left[-\frac{e_0^2\left(8 - 3\cos i_0 - 18\cos^2 i_0\right)}{8\left(1 - 5\cos^2 i_0\right)}\right.$$

$$\left.\left. + \frac{e_0^2\left(5 - \cos i_0 - 6\cos^2 i_0\right)\left(1 - 15\cos^2 i_0\right)}{16\left(1 - 5\cos^2 i_0\right)^2}\right]\sin(2\xi - 2\eta)\right\}$$

(equation continued next page)

$$- \frac{1}{4}\frac{k_3}{k_2} h_0^2 e_0 \cos i_0 \sin\frac{i_0}{2}\left(\frac{1 + 2\cos i_0 - 3\cos^2 i_0}{1 - 5\cos^2 i_0}\right)\cos(\xi - \eta)$$

$$- \frac{5}{4}\frac{k_4}{k_2} h_0^4 \, e_0^2 \sin i_0 \cos i_0 \sin\frac{i_0}{2}\left[\frac{4 - 7\cos^2 i_0}{1 - 5\cos^2 i_0} - \frac{(5 - \cos i_0 - 6\cos^2 i_0)(1 - 7\cos^2 i_0)}{2(1 - 5\cos^2 i_0)^2}\right]\sin(2\xi - 2\eta) \quad ; \tag{31}$$

$$\lambda_4 = \cos\frac{i_0}{2} - k_2 h_0^4 \sin i_0 \cos i_0 \sin\frac{i_0}{2}\left[\frac{3}{4}\cos 2\eta + \frac{1}{4} e_0 \cos(\xi + 2\eta)\right.$$

$$\left. + \frac{3}{4} e_0 \cos(\xi - 2\eta) - \frac{1}{16} e_0^2 \frac{1 - 15\cos^2 i_0}{1 - 5\cos^2 i_0}\cos(2\xi - 2\eta)\right]$$

$$+ \frac{1}{4}\frac{k_3}{k_2} h_0^2 e_0 \cos i_0 \sin\frac{i_0}{2} \sin(\xi - \eta)$$

$$- \frac{5}{8}\frac{k_4}{k_2} h_0^4 e_0^2 \sin i_0 \cos i_0 \sin\frac{i_0}{2}\left(\frac{1 - 7\cos^2 i_0}{1 - 5\cos^2 i_0}\right)\cos(2\xi - 2\eta) \; . \tag{32}$$

CONCLUSION

An analytic development of Musen's true anomaly version of the Hansen theory to the first order has been presented. Expressions for the W function, u, h/h_0, $n_0 \delta z$, and the λ-parameters have been given, as well as analytic representations of the secular motions of the node and argument of perigee for this theory.

For nearly circular orbits the expression for $(1 - c)$ can be inserted directly into the numerical program of this theory, thus eliminating the only division by e_0 which could be a point of slow or no convergence.

We see that this theory, as well as other general perturbation theories (References 2, 3, and 4), cannot be used for orbits with inclinations in the neighborhood of 63.4 degrees: since that is the value for which $(1 - 5\cos^2 i_0)$ goes to zero, the series for the perturbations will not converge and will have no meaning.

The appearance of a coupling term $k_3 k_4/k_2^3$ in the perturbations of the mean anomaly was an unexpected result. This is due to the use of the pseudo time in Hansen's theory.

Finally we conclude that, whereas an analytic development is useful for understanding the makeup of this theory, for comparison with other general perturbation theories, and for checking out the coding of the numerical program, it is impracticable for general use even to the first order.

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Appendix A

Equation for the Perturbation of the Mean Anomaly

Because of its length Equation 18, the perturbation of the mean anomaly, was not included in the text and is given here.

To reduce the length of Equation 18, the following substitutions are used:

$$
\begin{aligned}
C_0' &= 4 - 2\sqrt{1 - e_0^2}\,, \\
C_1' &= 1 - \frac{2}{3}\sqrt{1 - e_0^2} + \frac{4}{3}\left(1 + \sqrt{1 - e_0^2}\right)^{-1}, \\
\beta &= \frac{1 - \sqrt{1 - e_0^2}}{e_0}, \\
A &= \left(1 + \beta^2\right)\left(10 - 5\beta^2 + \beta^4\right), \\
B &= \left(1 + \beta^2\right)^2\left(5 - 4\beta^2 + \beta^4\right), \\
D &= \left(1 + \beta^2\right)^3\left(7 - 7\beta^2 + 2\beta^4\right), \\
E &= \left(1 + \beta^2\right)^4\left(14 - 16\beta^2 + 5\beta^4\right), \\
F &= \left(1 + \beta^2\right)^5\left(12 - 15\beta^2 + 5\beta^4\right), \\
G &= \left(1 + \beta^2\right)\left(35 - 21\beta^2 + 7\beta^4 - \beta^6\right), \\
H &= \left(1 + \beta^2\right)^2\left(14 - 14\beta^2 + 6\beta^4 - \beta^6\right), \\
I &= \left(1 + \beta^2\right)^3\left(42 - 54\beta^2 + 27\beta^4 - 5\beta^6\right).
\end{aligned}
$$

With these substitutions Equation 18 is:

$$
\begin{aligned}
n_0 \delta z = {} & \frac{k_2 h_0^4\left(1 - 3\cos^2 i_0\right)}{\left(1 - e_0^2\right)} \Bigg\{ \left[e_0^2\left(-\frac{25}{8} + \frac{9}{4}C_0' - \frac{9}{8}C_1'\right) - \frac{17}{8}e_0^4 + \frac{3}{32}e_0^4\left(1 - C_1'\right)A + \frac{3}{64}e_0^6 B\right]\sin 2\xi \\
& + \left[e_0^3\left(-1 + \frac{3}{4}C_1'\right) + \frac{e_0^3}{8}\left(2 - C_0' + e_0^2\right)A - \frac{3}{32}e_0^5\left(1 - C_1'\right)B - \frac{e_0^7}{64}D\right]\sin 3\xi \\
& + \left[\frac{3}{16}e_0^4 + \frac{3}{64}e_0^4\left(1 - C_1'\right)A - \frac{9}{64}e_0^4\left(2 - C_0' + e_0^2\right)B + \frac{9}{256}e_0^6\left(1 - C_1'\right)D + \frac{e_0^8}{256}E\right]\sin 4\xi \\
& + \left[-\frac{e_0^5}{80}A - \frac{9}{160}e_0^5\left(1 - C_1'\right)B + \frac{9}{160}e_0^5\left(2 - C_0' + e_0^2\right)D - \frac{3}{320}e_0^7\left(1 - C_1'\right)E\right]\sin 5\xi \\
& + \left[\frac{e_0^6}{64}B + \frac{3}{128}e_0^6\left(1 - C_1'\right)D - \frac{e_0^6}{64}\left(2 - C_0' + e_0^2\right)E + \frac{3}{512}e_0^8\left(1 - C_1'\right)F\right]\sin 6\xi \Bigg\} \\
& + \frac{k_2 h_0^4\left(1 - \cos^2 i_0\right)}{\left(1 - e_0^2\right)} \Bigg\{ \left(\frac{1}{4} + \frac{7}{16}e_0^2 + e_0^4\right)\sin 2\eta + \left(\frac{3}{2}e_0 + \frac{15}{8}e_0^3\right)\sin\left(\xi - 2\eta\right)
\end{aligned}
$$

$$+\left[-\frac{e_0}{4}-\frac{e_0^3}{8}-\frac{e_0^5}{16}A\right]\sin(\xi+2\eta)+\left[\frac{3}{16}e_0^2-\frac{3}{16}e_0^4+\frac{3}{128}e_0^4A+\frac{9}{128}e_0^6B\right]\sin(2\xi+2\eta)$$

$$+\left[-\frac{27}{8}e_0^3-\frac{e_0^3}{16}\left(1-e_0^2\right)A\right]\sin(3\xi-2\eta)+\left[-\frac{e_0^3}{80}\left(1-e_0^2\right)A-\frac{9}{320}e_0^5B-\frac{9}{320}e_0^7D\right]\sin(3\xi+2\eta)$$

$$+\left[\frac{9}{8}e_0^4+\frac{3}{64}e_0^4A+\frac{3}{64}e_0^4\left(1-e_0^2\right)B\right]\sin(4\xi-2\eta)+\left[\frac{e_0^4}{64}\left(1-e_0^2\right)B+\frac{3}{256}e_0^6D+\frac{e_0^8}{128}E\right]\sin(4\xi+2\eta)$$

$$+\left[-\frac{e_0^5}{16}A-\frac{3}{64}e_0^5B-\frac{e_0^5}{64}\left(1-e_0^2\right)D\right]\sin(5\xi-2\eta)+\left[-\frac{3}{448}e_0^5\left(1-e_0^2\right)D-\frac{3}{896}e_0^7E\right]\sin(5\xi+2\eta)$$

$$+\left[\frac{9}{128}e_0^6B+\frac{9}{512}e_0^6D+\frac{e_0^6}{256}\left(1-e_0^2\right)E\right]\sin(6\xi-2\eta)+\left[\frac{e_0^6}{512}\left(1-e_0^2\right)E\right]\sin(6\xi+2\eta).\Bigg\}$$

$$+\frac{k_2h_0^4}{\left(1-e_0^2\right)\left(1-5\cos^2 i_0\right)}\Bigg\{\left[-\frac{15}{8}e_0^2+\frac{e_0^6}{64}A\right]\sin 2\eta+\left(-\frac{5}{2}e_0-\frac{17}{16}e_0^3+\frac{3}{16}e_0^5\right)\sin(\xi-2\eta)$$

$$+\left[\frac{5}{4}e_0^3+\frac{3}{16}e_0^5-\frac{e_0^5}{64}A-\frac{e_0^7}{64}B\right]\sin(\xi+2\eta)+\left[\left(-\frac{5}{64}e_0^4-\frac{3}{256}e_0^6\right)A+\frac{9}{512}e_0^6B+\frac{3}{512}e_0^8D\right]\sin(2\xi+2\eta)$$

$$+\left(\frac{47}{16}e_0^3+\frac{7}{16}e_0^5\right)\sin(3\xi-2\eta)+\left[\left(\frac{3}{32}e_0^5+\frac{9}{640}e_0^7\right)B-\frac{9}{1280}e_0^7D\right]\sin(3\xi+2\eta)$$

$$+\left[\frac{15}{32}e_0^4-\left(\frac{5}{32}e_0^4+\frac{3}{128}e_0^6\right)A\right]\sin(4\xi-2\eta)+\left[\left(-\frac{5}{128}e_0^6-\frac{3}{512}e_0^8\right)D+\frac{e_0^8}{512}E\right]\sin(4\xi+2\eta)$$

$$+\left[-\frac{e_0^5}{8}-\frac{e_0^5}{64}A+\left(\frac{5}{32}e_0^5+\frac{3}{128}e_0^7\right)B\right]\sin(5\xi-2\eta)+\left[\frac{5}{448}e_0^7E\right]\sin(5\xi+2\eta)$$

$$+\left[\frac{e_0^6}{128}A+\frac{9}{512}e_0^6B+\left(-\frac{15}{256}e_0^6-\frac{9}{1024}e_0^8\right)D\right]\sin(6\xi-2\eta)+\left[-\frac{15}{256}e_0^8F\right]\sin(6\xi+2\eta)\Bigg\}$$

$$+\frac{k_2h_0^4\cos^2 i_0}{\left(1-e_0^2\right)\left(1-5\cos^2 i_0\right)}\Bigg\{\left[\frac{105}{8}e_0^2-\frac{9}{16}e_0^4-\frac{13}{64}e_0^6A\right]\sin 2\eta+\left(\frac{35}{2}e_0+5e_0^3-\frac{9}{4}e_0^5\right)\sin(\xi-2\eta)$$

$$+\left[-\frac{35}{4}e_0^3-\frac{21}{8}e_0^5+\frac{e_0^5}{4}A+\frac{13}{64}e_0^7B\right]\sin(\xi+2\eta)+\left[\left(\frac{35}{64}e_0^4+\frac{21}{128}e_0^6\right)A-\frac{9}{32}e_0^6B-\frac{39}{512}e_0^8D\right]\sin(2\xi+2\eta)$$

$$+\left(-14e_0^3-\frac{25}{4}e_0^5\right)\sin(3\xi-2\eta)+\left[\left(-\frac{21}{32}e_0^5-\frac{63}{320}e_0^7\right)B+\frac{9}{80}e_0^7D\right]\sin(3\xi+2\eta)+\left[-\frac{111}{16}e_0^4+\left(\frac{35}{32}e_0^4+\frac{21}{64}e_0^6\right)A\right]\sin(4\xi-2\eta)$$

$$+\left[\left(\frac{35}{128}e_0^6+\frac{21}{256}e_0^8\right)D-\frac{e_0^8}{32}E\right]\sin(4\xi+2\eta)+\left[\frac{13}{8}e_0^5+\frac{e_0^5}{4}A+\left(-\frac{35}{32}e_0^5-\frac{21}{64}e_0^7\right)B\right]\sin(5\xi-2\eta)$$

$$+\left[-\frac{5}{64}e_0^7E\right]\sin(5\xi+2\eta)+\left[-\frac{13}{128}e_0^6A-\frac{9}{32}e_0^6B+\left(\frac{105}{256}e_0^6+\frac{63}{512}e_0^8\right)D\right]\sin(6\xi-2\eta)+\left[\frac{105}{256}e_0^8F\right]\sin(6\xi+2\eta)\Bigg\}$$

$$+\frac{k_2h_0^4\cos^4 i_0}{\left(1-e_0^2\right)\left(1-5\cos^2 i_0\right)}\Bigg\{\left[-\frac{45}{4}e_0^2+\frac{9}{16}e_0^4+\frac{3}{16}e_0^6A\right]\sin 2\eta+\left(-15e_0-\frac{63}{16}e_0^3+\frac{33}{16}e_0^5\right)\sin(\xi-2\eta)$$

$$+\left[\frac{15}{2}e_0^3+\frac{39}{16}e_0^5-\frac{15}{64}e_0^5A-\frac{3}{16}e_0^7B\right]\sin(\xi+2\eta)+\left[\left(-\frac{15}{32}e_0^4-\frac{39}{256}e_0^6\right)A+\frac{135}{512}e_0^6B+\frac{9}{128}e_0^8D\right]\sin(2\xi+2\eta)$$

$$+\left(\frac{177}{16}e_0^3+\frac{93}{16}e_0^5\right)\sin(3\xi-2\eta)+\left[\left(\frac{9}{16}e_0^5+\frac{117}{640}e_0^7\right)B-\frac{27}{256}e_0^7D\right]\sin(3\xi+2\eta)$$

$$+\left[\frac{207}{32}e_0^4+\left(-\frac{15}{16}e_0^4-\frac{39}{128}e_0^6\right)A\right]\sin(4\xi-2\eta)+\left[\left(-\frac{15}{64}e_0^6-\frac{39}{512}e_0^8\right)D+\frac{15}{512}e_0^8E\right]\sin(4\xi+2\eta)$$

$$+\left[-\frac{3}{2}e_0^5-\frac{15}{64}e_0^5A+\left(\frac{15}{16}e_0^5+\frac{39}{128}e_0^7\right)B\right]\sin(5\xi-2\eta)+\left[\frac{15}{224}e_0^7E\right]\sin(5\xi+2\eta)$$

$$+\left[\frac{3}{32}e_0^6A+\frac{135}{512}e_0^6B+\left(-\frac{45}{128}e_0^6-\frac{117}{1024}e_0^8\right)D\right]\sin(6\xi-2\eta)+\left[-\frac{45}{128}e_0^8F\right]\sin(6\xi+2\eta)\Bigg\}$$

$$+ \frac{k_3 h_0^2 \sin i_0}{k_2 (1 - e_0^2)} \Bigg\{ \left(1 + \frac{e_0^2}{2} - \frac{3}{2} e_0^4\right) \cos \eta + \left[- \frac{3}{4} e_0 - \frac{9}{16} e_0^3 + \frac{7}{64} e_0^5 A \right] \cos (\xi + \eta)$$

$$+ \frac{5}{2} e_0^2 (1 - e_0^2) \cos (2\xi - \eta) + \left[\frac{e_0^2}{2} \left(1 + \frac{9}{4} e_0^2\right) - \frac{e_0^4}{16} A - \frac{7}{64} e_0^6 B \right] \cos (2\xi + \eta)$$

$$+ \left[- \frac{39}{16} e_0^3 + \frac{e_0^3}{16} \left(1 + \frac{9}{4} e_0^2\right) A \right] \cos (3\xi - \eta) + \left[- \frac{e_0^3}{32} \left(1 + \frac{9}{4} e_0^2\right) A + \frac{9}{128} e_0^5 B + \frac{21}{512} e_0^7 D \right] \cos (3\xi + \eta)$$

$$+ \left[\frac{7}{8} e_0^4 + \frac{e_0^4}{16} A - \frac{e_0^4}{16} \left(1 + \frac{9}{4} e_0^2\right) B \right] \cos (4\xi - \eta) + \left[\frac{3}{80} e_0^4 \left(1 + \frac{9}{4} e_0^2\right) B - \frac{9}{320} e_0^6 D - \frac{7}{640} e_0^8 E \right] \cos (4\xi + \eta)$$

$$+ \left[- \frac{7}{128} e_0^5 A - \frac{9}{128} e_0^5 B + \frac{3}{128} e_0^5 \left(1 + \frac{9}{4} e_0^2\right) D \right] \cos (5\xi - \eta) + \left[- \frac{e_0^5}{64} \left(1 + \frac{9}{4} e_0^2\right) D + \frac{e_0^7}{128} E \right] \cos (5\xi + \eta)$$

$$+ \left[\frac{21}{320} e_0^6 B + \frac{9}{320} e_0^6 D - \frac{e_0^6}{160} \left(1 + \frac{9}{4} e_0^2\right) E \right] \cos (6\xi - \eta) + \left[\frac{e_0^6}{224} \left(1 + \frac{9}{4} e_0^2\right) E \right] \cos (6\xi + \eta) \Bigg\}$$

$$+ \frac{k_4 h_0^4 (1 - 8 \cos^2 i_0 + 7 \cos^4 i_0)}{k_2 (1 - e_0^2)(1 - 5 \cos^2 i_0)} \Bigg\{ \left[\frac{15}{8} e_0^2 - \frac{5}{32} e_0^6 A \right] \sin 2\eta + \left(\frac{5}{2} e_0 - \frac{5}{8} e_0^3 - \frac{15}{8} e_0^5 \right) \sin (\xi - 2\eta)$$

$$+ \left[- \frac{5}{4} e_0^3 \left(1 + \frac{3}{2} e_0^2\right) + \frac{5}{32} e_0^5 A + \frac{5}{32} e_0^7 B \right] \sin (\xi + 2\eta) + \left[\frac{5}{64} e_0^4 \left(1 + \frac{3}{2} e_0^2\right) A - \frac{45}{256} e_0^6 B - \frac{15}{256} e_0^8 D \right] \sin (2\xi + 2\eta)$$

$$+ \frac{35}{8} e_0^3 (1 - e_0^2) \sin (3\xi - 2\eta) + \left[- \frac{3}{32} e_0^5 \left(1 + \frac{3}{2} e_0^2\right) B + \frac{9}{128} e_0^7 D \right] \sin (3\xi + 2\eta)$$

$$+ \left[- \frac{75}{16} e_0^4 + \frac{5}{32} e_0^4 \left(1 + \frac{3}{2} e_0^2\right) A \right] \sin (4\xi - 2\eta) + \left[\frac{5}{128} e_0^6 \left(1 + \frac{3}{2} e_0^2\right) D - \frac{5}{256} e_0^8 E \right] \sin (4\xi + 2\eta)$$

$$+ \left[\frac{5}{4} e_0^5 + \frac{5}{32} e_0^5 A - \frac{5}{32} e_0^5 \left(1 + \frac{3}{2} e_0^2\right) B \right] \sin (5\xi - 2\eta) + \left[- \frac{5}{448} e_0^7 E \right] \sin (5\xi + 2\eta)$$

$$+ \left[- \frac{5}{64} e_0^6 A - \frac{45}{256} e_0^6 B + \frac{15}{256} e_0^6 \left(1 + \frac{3}{2} e_0^2\right) D \right] \sin (6\xi - 2\eta) \Bigg\}$$

$$+ \frac{k_2 h_0^4 (2 - 41 \cos^2 i_0 + 174 \cos^4 i_0 - 135 \cos^6 i_0)}{(1 - e_0^2)(1 - 5 \cos^2 i_0)^2} \Bigg\{ \left[\frac{3}{32} e_0^4 - \frac{1}{128} e_0^6 A \right] \sin 2\eta + \frac{e^3}{8} (1 - e_0^2) \sin (\xi - 2\eta)$$

$$+ \left[- \frac{1}{16} e_0^5 + \frac{1}{128} e_0^7 B \right] \sin (\xi + 2\eta) + \left[\frac{1}{256} e_0^6 A - \frac{3}{1024} e_0^8 D \right] \sin (2\xi + 2\eta) + \frac{e_0^3}{8} (1 - e_0^2) \sin (3\xi - 2\eta) - \frac{3}{640} e_0^7 B \sin (3\xi + 2\eta)$$

$$+ \left[- \frac{3}{32} e_0^4 + \frac{1}{128} e_0^6 A \right] \sin (4\xi - 2\eta) + \frac{1}{512} e_0^8 D \sin (4\xi + 2\eta) + \left[\frac{1}{16} e_0^5 - \frac{1}{128} e_0^7 B \right] \sin (5\xi - 2\eta) + \left[- \frac{1}{256} e_0^6 A + \frac{3}{1024} e_0^8 D \right] \sin (6\xi - 2\eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 (2 - 9 \cos^2 i_0)}{k_2 (1 - e_0^2)(1 - 5 \cos^2 i_0)} \Bigg\{ (- e_0^2 + e_0^4) \cos \eta + \left[\frac{3}{4} e_0^3 - \frac{1}{16} e_0^5 A \right] \cos (\xi + \eta) + (- e_0^2 + e_0^4) \cos (2\xi - \eta) + \left[- \frac{1}{2} e_0^4 + \frac{1}{16} e_0^6 B \right] \cos (2\xi + \eta)$$

$$+ \left[\frac{3}{4} e_0^3 - \frac{1}{16} e_0^5 A \right] \cos (3\xi - \eta) + \left[\frac{1}{32} e_0^5 A - \frac{3}{128} e_0^7 D \right] \cos (3\xi + \eta) + \left[- \frac{1}{2} e_0^4 + \frac{1}{16} e_0^6 B \right] \cos (4\xi - \eta) + \left[- \frac{3}{80} e_0^6 B + \frac{1}{160} e_0^8 E \right] \cos (4\xi + \eta)$$

$$+ \left[\frac{1}{32} e_0^5 A - \frac{3}{128} e_0^7 D \right] \cos (5\xi - \eta) + \frac{1}{64} e_0^7 D \cos (5\xi + \eta) + \left[- \frac{3}{80} e_0^6 B + \frac{1}{160} e_0^8 E \right] \cos (6\xi - \eta) - \frac{1}{224} e_0^8 E \cos (6\xi + \eta) \Bigg\}$$

$$+ \frac{k_4 h_0^4 (2 - 25 \cos^2 i_0 + 86 \cos^4 i_0 - 63 \cos^6 i_0)}{k_2 (1 - e_0^2)(1 - 5 \cos^2 i_0)^2} \Bigg\{ \left[- \frac{15}{16} e_0^4 + \frac{5}{64} e_0^6 A \right] \sin 2\eta - \frac{5}{4} e_0^3 (1 - e_0^2) \sin (\xi - 2\eta) + \left[\frac{5}{8} e_0^5 - \frac{5}{64} e_0^7 B \right] \sin (\xi + 2\eta)$$

$$+ \left[- \frac{5}{128} e_0^6 A + \frac{15}{512} e_0^8 D \right] \sin (2\xi + 2\eta) - \frac{5}{4} e_0^3 (1 - e_0^2) \sin (3\xi - 2\eta) + \frac{3}{64} e_0^7 B \sin (3\xi + 2\eta) + \left[\frac{15}{16} e_0^4 - \frac{5}{64} e_0^6 A \right] \sin (4\xi - 2\eta)$$

$$- \frac{5}{256} e_0^8 D \sin (4\xi + 2\eta) + \left[- \frac{5}{8} e_0^5 + \frac{5}{64} e_0^7 B \right] \sin (5\xi - 2\eta) + \left[\frac{5}{128} e_0^6 A - \frac{15}{512} e_0^8 D \right] \sin (6\xi - 2\eta) \Bigg\}$$

$$+ \frac{k_2 h_0^4}{(1 - e_0^2)\,(1 - 5\cos^2 i_0)} \Big[-\frac{15}{16} e_0^2 - \frac{23}{32} e_0^4 + \frac{7}{4} e_0^2 C_0' + \frac{5}{32} e_0^4 C_0' + \frac{15}{16} e_0^2 C_1' + \frac{15}{32} e_0^4 C_1'$$

$$+ \left(\frac{9}{128} e_0^4 + \frac{e_0^6}{32} - \frac{11}{384} e_0^4 C_0' - \frac{3}{128} e_0^4 C_1' \right) A + \left(-\frac{31}{2560} e_0^6 + \frac{1}{256} e_0^6 C_0' + \frac{3}{256} e_0^6 C_1' \right) B - \frac{e_0^8}{1024} D \Big] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4 \cos^2 i_0}{(1 - e_0^2)\,(1 - 5\cos^2 i_0)} \Big[-\frac{75}{8} e_0^2 + \frac{31}{4} e_0^4 - 7 e_0^2 C_0' - \frac{5}{8} e_0^4 C_0' - \frac{15}{4} e_0^2 C_1' - \frac{15}{8} e_0^4 C_1'$$

$$+ \left(-\frac{33}{64} e_0^4 - \frac{25}{128} e_0^6 + \frac{11}{96} e_0^4 C_0' + \frac{3}{32} e_0^4 C_1' \right) A + \left(\frac{47}{320} e_0^6 - \frac{1}{64} e_0^6 C_0' - \frac{3}{64} e_0^6 C_1' \right) B - \frac{e_0^8}{512} D \Big] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4 \cos^4 i_0}{(1 - e_0^2)\,(1 - 5\cos^2 i_0)} \Big[\frac{165}{16} e_0^2 - \frac{225}{32} e_0^4 + \frac{21}{4} e_0^2 C_0' + \frac{15}{32} e_0^4 C_0' + \frac{45}{16} e_0^2 C_1' + \frac{45}{32} e_0^4 C_1'$$

$$+ \left(\frac{57}{128} e_0^4 + \frac{21}{128} e_0^6 - \frac{11}{128} e_0^4 C_0' - \frac{9}{128} e_0^4 C_1' \right) A + \left(-\frac{69}{512} e_0^6 + \frac{3}{256} e_0^6 C_0' + \frac{9}{256} e_0^6 C_1' \right) B + \frac{3}{1024} e_0^8 D \Big] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4 (1 - 2\cos^2 i_0 + \cos^4 i_0)}{(1 - e_0^2)(1 - 5\cos^2 i_0)} \left[-\frac{75}{256} e_0^4 + \left(\frac{13}{256} e_0^4 + \frac{5}{1024} e_0^6 \right) A + \left(-\frac{3}{128} e_0^4 - \frac{3}{64} e_0^6 \right) B + \left(\frac{13}{1024} e_0^6 + \frac{9}{2048} e_0^8 \right) D - \frac{5}{3072} e_0^8 E \right] \sin (4\xi - 4\eta)$$

$$+ \frac{k_4 h_0^4 (1 - 8\cos^2 i_0 + 7\cos^4 i_0)}{k_2 (1 - e_0^2)(1 - 5\cos^2 i_0)} \left[-\frac{45}{4} e_0^2 + \frac{25}{8} e_0^4 + \frac{5}{16} e_0^6 - \frac{5}{64} e_0^4 (1 + e_0^2) A + \frac{e_0^6}{128} \left(\frac{9}{2} + e_0^2 \right) B - \frac{1}{512} e_0^8 D \right] \sin (2\xi - 2\eta)$$

$$+ \frac{k_4 h_0^4 (1 - 2\cos^2 i_0 + \cos^4 i_0)}{k_2 (1 - e_0^2)(1 - 5\cos^2 i_0)} \left[\frac{105}{64} e_0^4 + \frac{35}{256} e_0^6 - \frac{35}{256} e_0^4 \left(1 + \frac{e_0^2}{2} \right) A + \frac{3}{256} e_0^4 \left(7 + \frac{35}{2} e_0^2 + \frac{7}{4} e_0^4 \right) B - \frac{21}{512} e_0^6 \left(\frac{3}{2} + e_0^2 \right) D + \frac{13}{1024} e_0^8 E \right] \sin (4\xi - 4\eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0}{k_2 (1 - e_0^2)} \left[-\frac{23}{4} e_0 + \frac{11}{16} e_0^3 + \frac{5}{32} e_0^5 - \frac{e_0^5}{32} A + \frac{e_0^7}{256} B \right] \cos (\xi - \eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 (1 - \cos^2 i_0)}{k_2 (1 - e_0^2)(1 - 5\cos^2 i_0)} \left[\frac{85}{144} e_0^3 + \frac{5}{144} e_0^5 - \frac{5}{288} e_0^3 \left(1 + \frac{3}{4} e_0^2 \right) A + \frac{e_0^5}{128} \left(3 + \frac{e_0^2}{2} \right) B - \frac{7}{1536} e_0^7 D \right] \cos (3\xi - 3\eta)$$

$$+ \frac{k_2 h_0^4 (1 - 4\cos^2 i_0 + 3\cos^4 i_0)}{(1 - e_0^2)^2 (1 - 5\cos^2 i_0)} \Big[\frac{e_0^2}{2} \left(11 - \frac{23}{4} C_0' + \frac{3}{2} C_1' \right) + \frac{e_0^4}{192} (315 - 233 C_0' + 621 C_1') - \frac{e_0^6}{192} (141 - 35 C_0' - 105 C_1')$$

$$+ \frac{e_0^6}{1536} (15 - 13 C_0' - 15 C_1' + 8 e_0^2) G + \frac{e_0^8}{384} \left(1 + \frac{1}{2} C_0' + \frac{3}{2} C_1' \right) H \Big] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4}{(1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left[e_0^2 \left(-\frac{97}{16} + \frac{155}{48} C_0' - \frac{15}{16} C_1' \right) + e_0^4 \left(-\frac{31}{32} + \frac{91}{96} C_0' - \frac{105}{32} C_1' \right) + \frac{1}{24} e_0^6 C_0' \right] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4 \cos^2 i_0}{(1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left[e_0^2 \left(\frac{449}{8} - \frac{365}{12} \;\text{[illegible]}\; 75\, C_1' \right) + e_0^4 \left(\frac{229}{32} - \frac{649}{96} C_0' + \frac{915}{32} C_1' \right) - \frac{19}{24} e_0^6 C_0' \right] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4 \cos^4 i_0}{(1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left[e_0^2 \left(-\frac{2331}{16} + \frac{1275}{16} C_0' - \frac{405}{16} C_1' \right) - e_0^4 \left(\frac{513}{32} - \frac{471}{32} C_0' + \frac{2295}{32} C_1' \right) + \frac{21}{8} e_0^6 C_0' \right] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4 \cos^6 i_0}{(1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left[e_0^2 \left(\frac{765}{8} - \frac{105}{2} C_0' + \frac{135}{8} C_1' \right) + e_0^4 \left(\frac{315}{32} - \frac{285}{32} C_0' + \frac{1485}{32} C_1' \right) - \frac{15}{8} e_0^6 C_0' \right] \sin (2\xi - 2\eta)$$

$$+ \frac{k_3^2 (1 - \cos^2 i_0)}{k_2^3 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)} \left(-\frac{29}{48} e_0^2 - \frac{17}{48} e_0^4 + \frac{23}{24} e_0^6 \right) \sin (2\xi - 2\eta)$$

$$+ \frac{k_4 h_0^4 (1 - 11\cos^2 i_0 + 31\cos^4 i_0 - 21\cos^6 i_0)}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left[e_0^2 \left(\frac{25}{16} - \frac{65}{48} C_0' + \frac{15}{16} C_1' \right) - e_0^4 \left(\frac{25}{16} - \frac{85}{48} C_0' + \frac{15}{16} C_1' \right) - \frac{5}{12} e_0^6 C_0' \right] \sin (2\xi - 2\eta)$$

$$+ \frac{k_2 h_0^4 \left(1 - 2\cos^2 i_0 + \cos^4 i_0\right)}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)} \left[- \frac{223}{256} e_0^4 - \frac{169}{512} e_0^6 + \frac{e_0^6}{1536}\left(25 + \frac{7}{4} e_0^2\right) G - \frac{e_0^6}{768}\left(5 + \frac{21}{2} e_0^2\right) H + \frac{11}{8192} e_0^8 I\right] \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^2} \left[\frac{209}{128} e_0^4 + \frac{273}{1024} e_0^6 + \frac{5}{3072} e_0^8 - \left(\frac{25}{3072} e_0^6 + \frac{43}{24576} e_0^8\right) G + \frac{5}{1536} e_0^8 H\right] \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4 \cos^2 i_0}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^2} \left[- \frac{1417}{128} e_0^4 - \frac{2997}{1024} e_0^6 - \frac{241}{3072} e_0^8 + \left(\frac{25}{384} e_0^6 + \frac{671}{24576} e_0^8\right) G - \frac{5}{192} e_0^8 H\right] \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4 \cos^4 i_0}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^2} \left[\frac{2207}{128} e_0^4 + \frac{5175}{1024} e_0^6 + \frac{467}{3072} e_0^8 - \left(\frac{325}{3072} e_0^6 + \frac{1213}{24576} e_0^8\right) G + \frac{65}{1536} e_0^8 H\right] \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4 \cos^6 i_0}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^2} \left[- \frac{999}{128} e_0^4 - \frac{2451}{1024} e_0^6 - \frac{77}{1024} e_0^8 + \left(\frac{25}{512} e_0^6 + \frac{195}{8192} e_0^8\right) G - \frac{5}{256} e_0^8 H\right] \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(- \frac{265}{384} e_0^4 - \frac{553}{6144} e_0^6 - \frac{67}{6144} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4 \cos^2 i_0}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(\frac{5665}{768} e_0^4 + \frac{701}{384} e_0^6 + \frac{223}{768} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4 \cos^4 i_0}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(- \frac{18695}{768} e_0^4 - \frac{30263}{3072} e_0^6 - \frac{6737}{3072} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4 \cos^6 i_0}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(\frac{7505}{256} e_0^4 + \frac{1867}{128} e_0^6 + \frac{911}{256} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_2 h_0^4 \cos^8 i_0}{\left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(- \frac{2985}{256} e_0^4 - \frac{13251}{2048} e_0^6 - \frac{3369}{2048} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_4 h_0^4 \left(1 - 9\cos^2 i_0 + 15\cos^4 i_0 - 7\cos^6 i_0\right)}{k_2 \left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^2} \left[\frac{35}{64} e_0^4 - \frac{705}{512} e_0^6 - \frac{25}{1536} e_0^8 + \frac{5}{3072} e_0^6 \left(5 + \frac{43}{4} e_0^2\right) G - \frac{5}{1536} e_0^8 H\right] \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_4 h_0^4}{k_2 \left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(- \frac{875}{768} e_0^4 + \frac{1415}{1536} e_0^6 + \frac{335}{1536} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_4 h_0^4 \cos^2 i_0}{k_2 \left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(\frac{2345}{128} e_0^4 - \frac{875}{64} e_0^6 - \frac{595}{128} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_4 h_0^4 \cos^4 i_0}{k_2 \left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(- \frac{1435}{16} e_0^4 + \frac{15935}{256} e_0^6 + \frac{7025}{256} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_4 h_0^4 \cos^6 i_0}{k_2 \left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(\frac{49525}{384} e_0^4 - \frac{16735}{192} e_0^6 - \frac{16055}{384} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_4 h_0^4 \cos^8 i_0}{k_2 \left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left(- \frac{14455}{256} e_0^4 + \frac{19285}{512} e_0^6 + \frac{9625}{512} e_0^8\right) \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_4^2 h_0^4 \left(1 - 16\cos^2 i_0 + 78\cos^4 i_0 - 112\cos^6 i_0 + 49\cos^8 i_0\right)}{k_2^3 \left(1 - e_0^2\right)^2 \left(1 - 5\cos^2 i_0\right)^3} \left[\frac{125}{96} e_0^4 - \frac{325}{1536} e_0^6 - \frac{1675}{1536} e_0^8\right] \sin\left(4\xi - 4\eta\right)$$

$$+ \frac{k_3 h_0^2 \sin i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)} \left[e_0\left(\frac{9}{4} - \frac{13}{12} C_0' + \frac{3}{4} C_1'\right) + e_0^3\left(-\frac{9}{16} + \frac{17}{12} C_0' - \frac{3}{4} C_1'\right) + e_0^5\left(-\frac{289}{96} - \frac{1}{3} C_0'\right) + \frac{29}{32} e_0^7 + \frac{11}{768} e_0^7 G\right] \cos(\xi - \eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 \cos^2 i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)} \left[e_0\left(-\frac{19}{4} + \frac{13}{4} C_0' - \frac{9}{4} C_1'\right) + e_0^3\left(\frac{49}{16} - \frac{17}{4} C_0' + \frac{9}{4} C_1'\right) + e_0^5\left(\frac{289}{96} + C_0'\right) - \frac{29}{32} e_0^7 - \frac{11}{768} e_0^7 G\right] \cos(\xi - \eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(-\frac{5}{4} e_0 - \frac{125}{96} e_0^3 + \frac{181}{96} e_0^5 + \frac{2}{3} e_0^7\right) \cos(\xi - \eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 \cos^2 i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(\frac{35}{4} e_0 + \frac{221}{24} e_0^3 - \frac{211}{24} e_0^5 - \frac{55}{6} e_0^7\right) \cos(\xi - \eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 \cos^4 i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(-\frac{15}{2} e_0 - \frac{253}{32} e_0^3 + \frac{221}{32} e_0^5 + \frac{17}{2} e_0^7\right) \cos(\xi - \eta)$$

$$+ \frac{k_3 k_4 h_0^2 \sin i_0 (1 - 8\cos^2 i_0 + 7\cos^4 i_0)}{k_2^3 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(\frac{5}{4} e_0 + \frac{85}{48} e_0^3 + \frac{175}{48} e_0^5 - \frac{20}{3} e_0^7\right) \cos(\xi - \eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 (1 - \cos^2 i_0)}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)} \left[\frac{13}{24} e_0^3 - \frac{27}{32} e_0^5 + \frac{17}{288} e_0^7 + \frac{e_0^5}{1152}\left(5 + \frac{29}{2} e_0^2\right) G - \frac{e_0^7}{576} H\right] \cos(3\xi - 3\eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(-\frac{265}{288} e_0^3 + \frac{209}{288} e_0^5 + \frac{7}{36} e_0^7\right) \cos(3\xi - 3\eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 \cos^2 i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(\frac{511}{72} e_0^3 - \frac{323}{72} e_0^5 - \frac{47}{18} e_0^7\right) \cos(3\xi - 3\eta)$$

$$+ \frac{k_3 h_0^2 \sin i_0 \cos^4 i_0}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(-\frac{593}{96} e_0^3 + \frac{361}{96} e_0^5 + \frac{29}{12} e_0^7\right) \cos(3\xi - 3\eta)$$

$$+ \frac{k_3 k_4 h_0^2 \sin i_0 (1 - 8\cos^2 i_0 + 7\cos^4 i_0)}{k_2^3 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left(\frac{245}{144} e_0^3 + \frac{35}{144} e_0^5 - \frac{35}{18} e_0^7\right) \cos(3\xi - 3\eta)$$

$$+ \frac{k_2 h_0^4 \sin^4 i_0 (2 - 9\cos^2 i_0)(1 - 15\cos^2 i_0)}{(1 - e_0^2)^2 (1 - 5\cos^2 i_0)^3} \left[+\frac{1}{32} e_0^4 - \frac{27}{512} e_0^6 + \frac{13}{1536} e_0^8 + \frac{5}{12288} e_0^8 G\right] \sin(4\xi - 4\eta)$$

$$+ \frac{k_2 h_0^4 \sin^4 i_0 (2 - 9\cos^2 i_0)(1 - 15\cos^2 i_0)}{(1 - e_0^2)(1 - 5\cos^2 i_0)^4} \left[-\frac{5}{128} e_0^4 - \frac{5}{512} e_0^6 + \left(\frac{15}{64} e_0^4 + \frac{63}{512} e_0^6\right) \cos^2 i_0\right] \sin(4\xi - 4\eta)$$

$$+ \frac{k_2 h_0^4 \sin^4 i_0 (2 - 9\cos^2 i_0)^2 (1 - 15\cos^2 i_0)^2}{(1 - e_0^2)(1 - 5\cos^2 i_0)^5} \frac{e_0^6}{512} \sin(4\xi - 4\eta)$$

$$+ \frac{k_3 h_0^2 \sin^3 i_0 (2 - 9\cos^2 i_0)}{k_2 (1 - e_0^2)^2 (1 - 5\cos^2 i_0)^2} \left[-\frac{1}{3} e_0^3 + \frac{9}{16} e_0^5 - \frac{13}{144} e_0^7 - \frac{5}{1152} e_0^7 G\right] \left[3\cos(\xi - \eta) + \cos(3\xi - 3\eta)\right]$$

$$+ \frac{k_3 h_0^2 \sin^3 i_0 (2 - 9\cos^2 i_0)}{k_2 (1 - e_0^2)(1 - 5\cos^2 i_0)^3} \left[+\frac{7}{16} e_0^3 + \frac{3}{16} e_0^5 - \left(\frac{45}{16} e_0^3 + \frac{41}{16} e_0^5\right) \cos^2 i_0\right] \left[3\cos(\xi - \eta) + \cos(3\xi - 3\eta)\right]$$

$$- \frac{1}{24} \frac{k_3 h_0^2 e_0^5 \sin^3 i_0 (2 - 9\cos^2 i_0)^2 (1 - 15\cos^2 i_0)}{k_2 (1 - e_0^2)(1 - 5\cos^2 i_0)^4} \left[3\cos(\xi - \eta) + \cos(3\xi - 3\eta)\right]$$

$$+\frac{k_4 h_0^4 \sin^4 i_0 \left(2 - 9\cos^2 i_0\right)\left(1 - 7\cos^2 i_0\right)}{k_2\left(1 - e_0^2\right)^2\left(1 - 5\cos^2 i_0\right)^3}\left[-\frac{5}{16}e_0^4 + \frac{135}{256}e_0^6 - \frac{65}{768}e_0^8 - \frac{25}{6144}e_0^8 G\right]\sin(4\xi - 4\eta)$$

$$+\frac{k_4 h_0^4 \sin^4 i_0 \left(2 - 9\cos^2 i_0\right)\left(1 - 7\cos^2 i_0\right)}{k_2\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^4}\left[\frac{55}{128}e_0^4 + \frac{25}{128}e_0^6 - \left(\frac{375}{128}e_0^4 + \frac{345}{128}e_0^6\right)\cos^2 i_0\right]\sin(4\xi - 4\eta)$$

$$-\frac{5}{128}\frac{k_4 h_0^4 \sin^4 i_0 \left(2 - 9\cos^2 i_0\right)^2\left(1 - 15\cos^2 i_0\right)\left(1 - 7\cos^2 i_0\right)}{k_2\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^5}e_0^6\sin(4\xi - 4\eta)$$

$$-\frac{5}{8}\frac{k_3 k_4 h_0^2 \sin^3 i_0 \left(2 - 9\cos^2 i_0\right)\left(1 - 7\cos^2 i_0\right)}{k_2^3\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^3}\left(e_0^3 + 3e_0^5\right)\left[3\cos(\xi - \eta) + \cos(3\xi - 3\eta)\right]$$

$$+\frac{5}{12}\frac{k_3 k_4 h_0^2 \sin^3 i_0 \left(2 - 9\cos^2 i_0\right)^2\left(1 - 7\cos^2 i_0\right)e_0^5}{k_2^3\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^4}\left[3\cos(\xi - \eta) + \cos(3\xi - 3\eta)\right]$$

$$+\frac{1}{4}\frac{k_3^2 \sin^2 i_0 \left(2 - 9\cos^2 i_0\right)}{k_2^3\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^2}\left(e_0^2 + 4e_0^4\right)\sin(2\xi - 2\eta)$$

$$-\frac{1}{4}\frac{k_3^2 \sin^2 i_0 \left(2 - 9\cos^2 i_0\right)^2}{k_2^3\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^3}e_0^4\sin(2\xi - 2\eta)$$

$$-\frac{25}{128}\frac{k_4^2 h_0^4 \sin^4 i_0 \left(2 - 9\cos^2 i_0\right)\left(1 - 7\cos^2 i_0\right)^2}{k_2^3\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^4}\left(2e_0^4 + 5e_0^6\right)\sin(4\xi - 4\eta)$$

$$+\frac{25}{128}\frac{k_4^2 h_0^4 \sin^4 i_0 \left(2 - 9\cos^2 i_0\right)^2\left(1 - 7\cos^2 i_0\right)^2}{k_2^3\left(1 - e_0^2\right)\left(1 - 5\cos^2 i_0\right)^5}e_0^6\sin(4\xi - 4\eta) .$$