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THE IN-PLANE VIBRATIONS OF A FLAT SPINNING DISK

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SUMMARY

An analysis of the influence of spinning on the natural frequencies of vibration of a thin elastic disk is first made and it is shown that this influence is negligible. Next, the natural frequencies for a stationary disk corresponding to the two extreme cases of clamping to a rigid hub and no clamping whatsoever are calculated and presented in graphical and tabular form. Approximate expressions for the two natural frequencies of a clamped disk which approach zero as the ratio of hub diameter to disk diameter approaches zero are given. Finally, there is presented an orthogonality relation between any two natural modes of vibration.

INTRODUCTION

The first writer to consider the in-plane vibrations of a spinning disk was Grammel in 1935, who treated the special but important case of rotationally symmetric torsional and radial vibrations. (See footnote, p. 37 of ref. 1.) On the basis of his analysis, Grammel reached the conclusion that, within the accuracy of the basic elastic equations, the effect of rotation on the natural frequencies of vibration was negligible. This conclusion was later (1939) adopted a priori by Biezeno and Grammel in an analysis of the more general case of torsional and radial vibrations with diametral nodes. (See ref. 1, pp. 59-64.)

In 1952, Yamada (ref. 2) attempted to improve Grammel's original analysis of the rotationally symmetric vibrations of a spinning disk by including in the equations of motion certain inertia forces which, it appeared to him, Grammel had neglected from the start. Yamada found that the additional terms he included could have appreciable influence on the lowest natural frequency of the purely torsional oscillation mode if the disk were clamped to a rigid hub of sufficiently small diameter and concluded that the lowest torsional frequency approaches zero as the velocity of rotation approaches the lowest natural frequency of the disk at rest - an erroneous conclusion contradicting that of Grammel.

In an earlier version of the present report, the author attempted to extend Yamada's analysis to the more general case of vibrations with nodal diameters and included the same inertia forces as had Yamada. Thus, among other results, Yamada's erroneous conclusion concerning the vanishing of the lowest natural torsional frequency was reasserted.

That this conclusion is erroneous was first brought to the author's attention by Messrs. D. G. Seymour and B. Wood of Bristol Siddeley Engines Ltd. They based their arguments on physical grounds and did not discuss the source of the discrepancy between Yamada's and the author's results and those of Grammel.

The purpose of this report is to resolve this discrepancy and to tabulate the upper and lower bounds on the natural frequencies of vibration of a thin elastic disk attached to a central hub. Also presented are explicit expressions for the dynamic stresses and displacement, approximate expressions for the two lowest natural frequencies for the case of clamping to a rigid hub, and an orthogonality relation. Some of the frequencies tabulated herein may be found in a paper by Singh and Nandeeswaraiya (ref. 3).

The work presented in this report has been motivated primarily by the need for a better understanding of some of the structural problems associated with the contemplated use of large rotating disks of thin aluminized plastic for optical and radar reflectors and solar sails in space.

SYMBOLS

a	hub radius of disk (fig. 1)
a_{jk}	matrix element
B_1,B_2	arbitrary constants associated with ϕ
B_3,B_4	arbitrary constants associated with ψ
b	outer radius of disk (fig. 1)
E	Young's modulus
$E_{\alpha\beta}$	covariant components of strain tensor
e	dilatational strain invariant, E^{α}_{α}
$\bar{e}_r,\bar{e}_\theta,\bar{k}$	unit vectors in r, θ, and z directions, respectively (fig. 1)

$F = \frac{E}{1 - \mu^2}$

G	shear modulus, $\frac{E}{2(1 - \mu)}$
J_m	Bessel function of the first kind, order m
m	number of nodal diameters
r	radial coordinate
$\overline{S}$	displacement vector, function of r, θ, and sometimes t
t	time
U,V	displacements in r and θ directions, respectively, functions of r, θ, and sometimes t
U^α, U_α	contravariant and covariant components, respectively, of $\overline{S}$
x,y,z	rotating set of axes (fig. 1)
x_o,y_o,z_o	inertial set of axes (fig. 1)
Y_m	Bessel function of the second kind, order m
∇	two-dimensional del operator
∇^2	two-dimensional Laplacian operator
ϵ	order of magnitude of dynamic strains in disk
ϵ^o	order of magnitude of static strains in disk
θ	angular coordinate (fig. 1)
μ	Poisson's ratio

$\xi = \frac{r}{b}$

ρ	mass per unit volume of disk material
σ_r	radial stress (fig. 1)

σ_θ circumferential stress (fig. 1)

τ shearing stress (fig. 1)

$\tau^{\alpha\beta}$ contravariant components of plane stress tensor

Φ vibratory dilatational displacement potential, function of r, θ, and t

ϕ vibratory dilatational displacement potential, function of r only

Ψ vibratory distortional displacement potential, function of r, θ, and t

ψ vibratory distortional displacement potential, function of r only

Ω frequency of rotation of disk

ω natural frequency of vibration

$\omega_1 = \sqrt{\frac{\rho}{F}}\, b\omega$

$\omega_2 = \sqrt{\frac{\rho}{G}}\, b\omega$

Superscript:

o static quantity

Dots over symbols indicate differentiation with respect to time. Primes indicate differentiation with respect to r.

EQUATIONS OF MOTION FOR A THIN SPINNING ELASTIC DISK

In this section the equations of motion for a thin spinning elastic disk are presented and it is shown how these equations reduce to those considered by Biezeno and Grammel in reference 1. Figure 1 indicates the coordinate system and stress notation used in this report.

In a nonvibrating spinning disk, the centrifugal loading induces a static stress field and the disk, if not perfectly rigid, undergoes a static deformation. If the disk material is elastic in this statically loaded state, the disk is capable of executing sinusoidal oscillations of infinitesimal amplitude about its static equilibrium configuration. By replacing all accelerations by inertial forces in accordance with d'Alembert's principle, the dynamic equations of equilibrium are reduced to the static equations of a body subject to a known initial stress. If these equations are written with respect to a set of coordinates in the undeformed body, the classical equations of the infinitesimal theory must be supplemented by terms of the form

(Initial stresses) × (Angles of rotation of element fibers)

Yamada's erroneous conclusion results from neglecting such initial stress terms in the equations of motion.

For a thin elastic disk spinning at a constant rotational frequency Ω these equations, referred to a set of polar coordinates r and θ in the undeformed disk, may be shown to take the form

$$\frac{d\left(r\sigma_r{}^o\right)}{dr} - \sigma_\theta{}^o + \rho r^2\Omega^2 = 0 \tag{1}$$

$$\frac{\partial(r\sigma_r)}{\partial r} + \frac{\partial\tau}{\partial\theta} - \sigma_\theta + \frac{\partial}{\partial r}\left(r\,\frac{\partial U}{\partial r}\,\sigma_r{}^o\right) + \frac{\partial}{\partial\theta}\left[\left(\frac{1}{r}\,\frac{\partial U}{\partial\theta} - \frac{V}{r}\right)\sigma_\theta{}^o\right]$$

$$- \left(\frac{U}{r} + \frac{1}{r}\,\frac{\partial V}{\partial\theta}\right)\sigma_\theta{}^o - \rho r\left(\ddot{U} - 2\Omega\dot{V} - \Omega^2 U\right) = 0 \tag{2}$$

$$\frac{\partial(r\tau)}{\partial r} + \frac{\partial\sigma_\theta}{\partial\theta} + \tau + \frac{\partial}{\partial r}\left(r\,\frac{\partial V}{\partial r}\,\sigma_r{}^o\right) + \frac{\partial}{\partial\theta}\left[\left(\frac{U}{r} + \frac{1}{r}\,\frac{\partial V}{\partial\theta}\right)\sigma_\theta{}^o\right]$$

$$+ \left(\frac{1}{r}\,\frac{\partial U}{\partial\theta} - \frac{V}{r}\right)\sigma_\theta{}^o - \rho r\left(\ddot{V} + 2\Omega\dot{U} - \Omega^2 V\right) = 0 \tag{3}$$

To obtain a determinant set of equations, equations (1) to (3) must be supplemented by stress-displacement relations and boundary conditions. These are supplied in later sections.

A quantitative argument may now be used to simplify these equations considerably. From equation (1) it can be seen that the initial stresses, which depend only on r, are of order $\rho b^2\Omega^2$, where b is the outer radius of the disk. Therefore let $\epsilon^o \equiv \frac{\rho b^2\Omega^2}{E}$ denote the order of magnitude of the static strains in the disk and let ϵ denote the order of magnitude of the dynamic strains in the disk. It then follows that the dynamic displacements U and V can be expressed as the sum of a rigid-body displacement plus a distortional term of order ϵ. That is,

$$U(r,\theta,t) = A(t)\cos\left[\theta - \alpha(t)\right] + \epsilon b U^*(r,\theta,t) \tag{4}$$

$$V(r,\theta,t) = -A(t)\sin\left[\theta - \alpha(t)\right] + B(t)r + \epsilon b V^*(r,\theta,t) \tag{5}$$

where A is the amplitude of the rigid-body displacement, B is the amplitude of the rigid-body rotation, α is the angle between the x_o inertial axis and the x-axis in the undeformed body, and $U^*,V^* = O(1)$. The amplitude B appears only in the case of no diametral nodes and A appears only in the case of one diametral node. Thus all terms in equations (2) and (3) of the form

(Initial stress) × (Rotation angle)

can be neglected as being of order ϵ^o compared with unity except those of the form

(Initial stress) × (Rigid-body rotation)

Under these conditions, equations (2) and (3) reduce to

$$\frac{\partial(r\sigma_r)}{\partial r} + \frac{\partial\tau}{\partial\theta} - \sigma_\theta - \rho r\left(\ddot{U} - 2\Omega\dot{V} - \Omega^2 U\right) = 0 \tag{6}$$

$$\frac{\partial(r\tau)}{\partial r} + \frac{\partial\sigma_\theta}{\partial\theta} + \tau + B\left[\frac{\partial(r\sigma_r{}^o)}{\partial r} - \sigma_\theta{}^o\right] - \rho r\left(\ddot{V} + 2\Omega\dot{U} - \Omega^2 V\right) = 0 \tag{7}$$

By using equation (1), equation (7) may be written as

$$\frac{\partial(r\tau)}{\partial r} + \frac{\partial\sigma_\theta}{\partial\theta} + \tau - \rho r\left[\ddot{V} + 2\Omega\dot{U} - \Omega^2(V - Br)\right] = 0 \tag{8}$$

In this report only the two extreme cases of clamping to a rigid hub and no clamping whatsoever (equivalent to a disk with a concentric hole) are considered. For these two cases certain restrictions must be placed on A and α. For clamping to a rigid hub, the condition that the displacements be zero at the hub means that $A = O(\epsilon)$. For the case of no clamping, any rigid-body motion of the disk must consist of movement along a straight line in inertial space at constant velocity in order to conserve linear momentum. Hence, in this case, $\alpha = -\Omega t$.

Consider now equations (6) and (8) for the case of vibrations with (a) no diametral nodes, (b) one diametral node, and (c) two or more diametral nodes:

(a) For no diametral nodes, equations (6) and (8) read

$$\frac{\partial(r\sigma_r)}{\partial r} - \sigma_\theta - \rho r(\ddot{U} - 2\Omega\dot{V}) + \epsilon\rho r b\Omega^2 U^* = 0 \tag{9}$$

$$\frac{\partial(r\tau)}{\partial r} + \tau - \rho r(\ddot{V} + 2\Omega\dot{U}) + \epsilon\rho r b\Omega^2 V^* = 0 \tag{10}$$

But $\epsilon\rho r b\Omega^2 = O(\epsilon\epsilon^o)$ and $\sigma_\theta, \sigma_r, \tau = O(\epsilon)$. Hence the last term in equation (9) and equation (10) is of order ϵ^o compared with unity, which is negligible. Therefore, these equations may be written in the simplified form

$$\frac{\partial(r\sigma_r)}{\partial r} - \sigma_\theta - \rho r(\ddot{U} - 2\Omega\dot{V}) = 0 \tag{11}$$

$$\frac{\partial(r\tau)}{\partial r} + \tau - \rho r(\ddot{V} + 2\Omega\dot{U}) = 0 \tag{12}$$

(b) For one diametral node, the B term does not appear, and equations (6) and (8) read

$$\frac{\partial(r\sigma_r)}{\partial r} + \frac{\partial\tau}{\partial\theta} - \sigma_\theta - \rho r\left(\ddot{U} - 2\Omega\dot{V} - \Omega^2 U\right) = 0 \tag{13}$$

$$\frac{\partial(r\tau)}{\partial r} + \frac{\partial\sigma_\theta}{\partial\theta} + \tau - \rho r\left(\ddot{V} + 2\Omega\dot{U} - \Omega^2 V\right) = 0 \tag{14}$$

If the disk is clamped to a rigid hub, then, for these one-diametral-node cases, $U, V = O(b\epsilon)$. Thus, since terms of order ϵ^o are being neglected compared with unity, the last terms in equations (13) and (14) may each be omitted. For the case of no clamping, in equation (13),

$$\ddot{U} - 2\Omega\dot{V} - \Omega^2 U = A\left(-\Omega^2 + 2\Omega^2 - \Omega^2\right)\cos(\theta + \Omega t) + \epsilon b\left(\ddot{U}^* - 2\Omega\dot{V}^* - \Omega^2 U^*\right) \tag{15}$$

and in equation (14),

$$\ddot{V} + 2\Omega\dot{U} - \Omega^2 V = A\left(\Omega^2 - 2\Omega^2 + \Omega^2\right)\sin(\theta + \Omega t) + \epsilon b\left(\ddot{V}^* + 2\Omega\dot{U}^* - \Omega^2 V^*\right) \tag{16}$$

Thus, in both cases the equations of motion are simplified to

$$\frac{\partial(r\sigma_r)}{\partial r} + \frac{\partial\tau}{\partial\theta} - \sigma_\theta - \rho r(\ddot{U} - 2\Omega\dot{V}) = 0 \tag{17}$$

$$\frac{\partial(r\tau)}{\partial r} + \frac{\partial\sigma_\theta}{\partial\theta} + \tau - \rho r(\ddot{V} + 2\Omega\dot{U}) = 0 \tag{18}$$

(c) For two or more diametral nodes, $U,V = O(b\epsilon)$ and equations (2) and (3) are readily seen to reduce to equations (17) and (18). But now by appealing to Biezeno and Grammel's arguments (pp. 41 and 59 of ref. 2), the Coriolis terms in equations (17) and (18) may also be neglected. Thus the equations of motion reduce to those for a stationary disk:

$$\frac{\partial(r\sigma_r)}{\partial r} + \frac{\partial\tau}{\partial\theta} - \sigma_\theta - \rho r\ddot{U} = 0 \tag{19}$$

$$\frac{\partial(r\tau)}{\partial r} + \frac{\partial\sigma_\theta}{\partial\theta} + \tau - \rho r\ddot{V} = 0 \tag{20}$$

These are the equations set down by Biezeno and Grammel. However, their derivation of these equations is somewhat misleading, especially in view of the statement on page 60 of reference 1 that any initial stresses due to rotation may be neglected. Certainly this statement is not true if the equations of motion are referred to an undeformed set of coordinates. Yet nowhere in the derivation of reference 1 is it implied that a deformed coordinate system has been used.

VECTOR FORM OF THE EQUATIONS OF MOTION

The dynamic stresses in the disk are assumed to be given in terms of U and V by the following linear stress-displacement relations of plane stress theory.

$$\tau = G\left(\frac{\partial V}{\partial r} - \frac{V}{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\right) \tag{21}$$

$$\sigma_r = F\left[\frac{\partial U}{\partial r} + \mu\left(\frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\right)\right] \tag{22}$$

$$\sigma_\theta = F\left(\frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r} + \mu\frac{\partial U}{\partial r}\right) \tag{23}$$

where

$$F = \frac{E}{1 - \mu^2} \tag{24}$$

$$G = \frac{E}{2(1 + \mu)} \tag{25}$$

E is Young's modulus, and μ is Poisson's ratio. If the displacement of a point of the disk during the time-dependent motion is expressed in the vector form

$$\bar{S}(r,\theta,t) = U(r,\theta,t)\bar{e}_r(\theta) + V(r,\theta,t)\bar{e}_\theta(\theta) \tag{26}$$

then with the aid of equations (21) to (23) and equation (26), equations (19) and (20) of the preceding section may be cast into the vector form

$$F\nabla(\nabla \cdot \bar{S}) - G\nabla \times (\nabla \times \bar{S}) = \rho\ddot{\bar{S}} \tag{27}$$

BOUNDARY CONDITIONS

The outer edge of the disk is to be stress-free; hence, at $r = b$, $\sigma_r = \tau = 0$. Assume for a moment that at the inner boundary the disk is clamped to an elastic hub and the disk is oscillating in one of its natural modes. Now if the flexibility of the hub could be continuously decreased - that is, be made more nearly rigid - it would be found that the frequency of the disk in this mode would continuously increase (the mode shape, of course, would continuously change as well) in accordance with Rayleigh's principle, since the constraints on the displacements were continually increasing. Thus, any frequencies encountered in practice will be bounded below and above, respectively, by the frequencies corresponding to the two extreme cases of a perfectly flexible hub (no clamping) and a perfectly rigid hub (rigid clamping). This report considers these two extreme cases only, together with the limiting case of a solid disk. It should be noted that two of the modes for the case of no clamping represent a rigid-body translation and a rigid-body rotation, and hence have zero natural frequency.

SOLUTION OF THE EQUATIONS OF MOTION

The solution of equation (27) is most easily obtained if $\bar{S}$ is written as follows:

$$\bar{S}(r,\theta,t) = \nabla\Phi(r,\theta,t) + \nabla \times \bar{k}\Psi(r,\theta,t) \tag{28}$$

where $\nabla\Phi$ represents the dilatational (or irrotational) component of $\bar{S}$ and $\nabla \times \bar{k}\Psi$ represents the distortional (or equivoluminal) component of $\bar{S}$. Substituting equation (28) into equation (27) gives

$$\nabla\left(F\nabla^2\Phi - \rho\ddot{\Phi}\right) + \nabla \times \bar{k}\left(G\nabla^2\Psi - \rho\ddot{\Psi}\right) = 0 \tag{29}$$

where ∇^2 is the two-dimensional Laplacian operator in polar coordinates:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{30}$$

For the present problem it will be sufficient to consider the solution of the equations

$$F\nabla^2\Phi - \rho\ddot{\Phi} = 0 \tag{31}$$

$$G\nabla^2\Psi - \rho\ddot{\Psi} = 0 \tag{32}$$

which clearly provide solutions which satisfy equation (29). To eliminate the angle and time dependence from these equations, set

$$\Phi(r,\theta,t) = \phi(r)\cos(m\theta + \omega t) \tag{33}$$

$$\Psi(r,\theta,t) = \psi(r)\sin(m\theta + \omega t) \tag{34}$$

Where m is an integer which determines the number of nodal diameters for a given motion. Equations (31) and (32) then read

$$\phi'' + \frac{\phi'}{r} - m^2 \frac{\phi}{r^2} + \frac{\rho\omega^2\phi}{F} = 0 \tag{35}$$

$$\psi'' + \frac{\psi'}{r} - m^2 \frac{\psi}{r^2} + \frac{\rho\omega^2\psi}{G} = 0 \tag{36}$$

where primes denote differentiation with respect to r.

Expressions in terms of ϕ and ψ are derived for the displacement (from eqs. (26) and (28)) and for the stresses (from eqs. (21) to (23)):

$$U = \left(\phi' + m \frac{\psi}{r}\right)\cos(m\theta + \omega t) \tag{37}$$

$$V = -\left(\psi' + m \frac{\phi}{r}\right)\sin(m\theta + \omega t) \tag{38}$$

$$\tau = -G\left(\psi'' - \frac{\psi'}{r} + m^2 \frac{\psi}{r^2} + 2m \frac{\phi'}{r} - 2m \frac{\phi}{r^2}\right)\sin(m\theta + \omega t) \tag{39}$$

$$\sigma_r = F\left[\phi'' + m \frac{\psi'}{r} - m \frac{\psi}{r^2} + \mu\left(\frac{\phi'}{r} - m^2 \frac{\phi}{r^2} - m \frac{\psi'}{r} + m \frac{\psi}{r^2}\right)\right] \cos(m\theta + \omega t) \tag{40}$$

$$\sigma_\theta = F\left[\frac{\phi'}{r}\ m^2 \frac{\phi}{r^2} - m \frac{\psi'}{r} + m \frac{\psi}{r^2} + \mu\left(\phi'' + m \frac{\psi'}{r} - m \frac{\psi}{r^2}\right)\right]\cos(m\theta + \omega t) \tag{41}$$

In view of equations (35) and (36), the expressions for the stresses may be written in the alternate form

$$\tau = \left[\rho\omega^2\psi + \frac{2G}{r}\left(\psi' - m^2 \frac{\psi}{r} - m\phi + m \frac{\phi}{r}\right)\right] \sin(m\theta + \omega t) \tag{42}$$

$$\sigma_r = -\left[\rho\omega^2\phi + \frac{2G}{r}\left(\phi' - m^2\frac{\phi}{r} - m\psi' + m\frac{\psi}{r}\right)\right]\cos(m\theta + \omega t) \tag{43}$$

$$\sigma_\theta = \left[-\mu\rho\omega^2\phi + \frac{2G}{r}\left(\phi' - m^2\frac{\phi}{r} - m\psi' + m\frac{\psi}{r}\right)\right]\cos(m\theta + \omega t) \tag{44}$$

Equations (35) and (36) are Bessel equations having the solutions

$$\phi = B_1 J_m\left(\sqrt{\frac{\rho}{F}}\,\omega r\right) + B_2 Y_m\left(\sqrt{\frac{\rho}{F}}\,\omega r\right) \tag{45}$$

$$\psi = B_3 J_m\left(\sqrt{\frac{\rho}{G}}\,\omega r\right) + B_4 Y_m\left(\sqrt{\frac{\rho}{G}}\,\omega r\right) \tag{46}$$

Inserting these solutions into equations (37) to (41) and defining, for simplicity, the nondimensional radial distance

$$\xi = \frac{r}{b} \tag{47}$$

yields

$$\frac{U}{b} = \frac{1}{\xi}\left[\sum_{k=1}^{4} a_{1k}(\xi)B_k\right]\cos(m\theta + \omega t) \tag{48}$$

$$\frac{V}{b} = -\frac{1}{\xi}\left[\sum_{k=1}^{4} a_{2k}(\xi)B_k\right]\sin(m\theta + \omega t) \tag{49}$$

$$\frac{\tau}{G} = \frac{1}{\xi^2}\left[\sum_{k=1}^{4} a_{3k}(\xi)B_k\right]\sin(m\theta + \omega t) \tag{50}$$

$$\frac{\sigma_r}{F} = \frac{1}{\xi^2}\left[\sum_{k=1}^{4} a_{4k}(\xi)B_k\right]\cos(m\theta + \omega t) \tag{51}$$

$$\frac{\sigma_\theta}{F} = -\frac{1}{\xi^2}\left[\sum_{k=1}^{4} a_{5k}(\xi)B_k\right]\cos(m\theta + \omega t) \tag{52}$$

The nondimensional frequencies

$$\omega_1 = \sqrt{\frac{\rho}{F}}\, b\omega \qquad\qquad \omega_2 = \sqrt{\frac{\rho}{G}}\, b\omega \tag{53}$$

are used in defining the following symbols which appear in equations (48) to (52):

$$\left.\begin{aligned}
a_{11}(\xi) &= mJ_m(\omega_1\xi) - (\omega_1\xi)J_{m+1}(\omega_1\xi) \\
a_{12}(\xi) &= mY_m(\omega_1\xi) - (\omega_1\xi)Y_{m+1}(\omega_1\xi) \\
a_{13}(\xi) &= mJ_m(\omega_2\xi) \\
a_{14}(\xi) &= mY_m(\omega_2\xi) \\
a_{21}(\xi) &= mJ_m(\omega_1\xi) \\
a_{22}(\xi) &= mY_m(\omega_1\xi) \\
a_{23}(\xi) &= mJ_m(\omega_2\xi) - (\omega_2\xi)J_{m+1}(\omega_2\xi) \\
a_{24}(\xi) &= mY_m(\omega_2\xi) - (\omega_2\xi)Y_{m+1}(\omega_2\xi) \\
a_{31}(\xi) &= 2M\left[(m - 1)J_m(\omega_1\xi) - (\omega_1\xi)J_{m+1}(\omega_1\xi)\right] \\
a_{32}(\xi) &= 2m\left[(m - 1)Y_m(\omega_1\xi) - (\omega_1\xi)Y_{m+1}(\omega_1\xi)\right] \\
a_{33}(\xi) &= \left[2m(m - 1) - (\omega_2\xi)^2\right]J_m(\omega_2\xi) + 2(\omega_2\xi)J_{m+1}(\omega_2\xi) \\
a_{34}(\xi) &= \left[2m(m - 1) - (\omega_2\xi)^2\right]Y_m(\omega_2\xi) + 2(\omega_2\xi)Y_{m+1}(\omega_2\xi) \\
a_{41}(\xi) &= \left[m(m - 1)(1 - \mu) - (\omega_1\xi)^2\right]J_m(\omega_1\xi) + (1 - \mu)(\omega_1\xi)J_{m+1}(\omega_1\xi) \\
a_{42}(\xi) &= \left[m(m - 1)(1 - \mu) - (\omega_1\xi)^2\right]Y_m(\omega_1\xi) + (1 - \mu)(\omega_1\xi)Y_{m+1}(\omega_1\xi) \\
a_{43}(\xi) &= m(1 - \mu)\left[(m - 1)J_m(\omega_2\xi) - (\omega_2\xi)J_{m+1}(\omega_2\xi)\right] \\
a_{44}(\xi) &= m(1 - \mu)\left[(m - 1)Y_m(\omega_2\xi) - (\omega_2\xi)Y_{m+1}(\omega_2\xi)\right] \\
a_{51}(\xi) &= \left[m(m - 1)(1 - \mu) + \mu(\omega_1\xi)^2\right]J_m(\omega_1\xi) + (1 - \mu)(\omega_1\xi)J_{m+1}(\omega_1\xi) \\
a_{52}(\xi) &= \left[m(m - 1)(1 - \mu) + \mu(\omega_1\xi)^2\right]Y_m(\omega_1\xi) + (1 - \mu)(\omega_1\xi)Y_{m+1}(\omega_1\xi) \\
a_{53}(\xi) &= m(1 - \mu)\left[(m - 1)J_m(\omega_2\xi) - (\omega_2\xi)J_{m+1}(\omega_2\xi)\right] \\
a_{54}(\xi) &= m(1 - \mu)\left[(m - 1)Y_m(\omega_2\xi) - (\omega_2\xi)Y_{m+1}(\omega_2\xi)\right]
\end{aligned}\right\} \tag{54}$$

The arbitrary constants appearing in equations (48) to (52) are determined by the boundary conditions specified in the preceding section and lead to the following matrix equations:

For no clamping,

$$\begin{bmatrix} a_{31}(a/b) & a_{32}(a/b) & a_{33}(a/b) & a_{34}(a/b) \\ a_{41}(a/b) & a_{42}(a/b) & a_{43}(a/b) & a_{44}(a/b) \\ a_{31}(1) & a_{32}(1) & a_{33}(1) & a_{34}(1) \\ a_{41}(1) & a_{42}(1) & a_{43}(1) & a_{44}(1) \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} = 0 \qquad (55)$$

and for clamping to a rigid hub,

$$\begin{bmatrix} a_{11}(a/b) & a_{12}(a/b) & a_{13}(a/b) & a_{14}(a/b) \\ a_{21}(a/b) & a_{22}(a/b) & a_{23}(a/b) & a_{24}(a/b) \\ a_{31}(1) & a_{32}(1) & a_{33}(1) & a_{34}(1) \\ a_{41}(1) & a_{42}(1) & a_{43}(1) & a_{44}(1) \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} = 0 \qquad (56)$$

For the special case of an unconstrained solid disk, the condition of regularity at the origin requires that equations (55) and (56) both reduce to the form

$$\begin{bmatrix} a_{31}(1) & a_{33}(1) \\ a_{41}(1) & a_{43}(1) \end{bmatrix} \begin{Bmatrix} B_1 \\ B_3 \end{Bmatrix} = 0 \qquad (57)$$

For $m = 0$ (radially symmetric modes), equations (55), (56), and (57) can be simplified as follows:

For no clamping,

$$\begin{bmatrix} (\omega_2 a/b)J_0(\omega_2 a/b) - 2J_1(\omega_2 a/b) & (\omega_2 a/b)Y_0(\omega_2 a/b) - 2Y_1(\omega_2 a/b) \\ \omega_2 J_0(\omega_2) - 2J_1(\omega_2) & \omega_2 Y_0(\omega_2) - 2Y_1(\omega_2) \end{bmatrix} \begin{Bmatrix} B_3 \\ B_4 \end{Bmatrix} = 0 \qquad (58)$$

and

$$\begin{bmatrix} (\omega_1 a/b)J_0(\omega_1 a/b) - (1 - \mu)J_1(\omega_1 a/b) & (\omega_1 a/b)Y_0(\omega_1 a/b) - (1 - \mu)Y_1(\omega_1 a/b) \\ \omega_1 J_0(\omega_1) - (1 - \mu)J_1(\omega_1) & \omega_1 Y_0(\omega_1) - (1 - \mu)Y_1(\omega_1) \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = 0 \qquad (59)$$

For clamping to a rigid hub,

$$\begin{bmatrix} J_1(\omega_2 a/b) & Y_1(\omega_2 a/b) \\ \omega_2 J_0(\omega_2) - 2J_1(\omega_2) & \omega_2 Y_0(\omega_2) - 2Y_1(\omega_2) \end{bmatrix} \begin{Bmatrix} B_3 \\ B_4 \end{Bmatrix} = 0 \qquad (60)$$

and

$$\begin{bmatrix} J_1(\omega_1 a/b) & Y_1(\omega_1 a/b) \\ \omega_1 J_0(\omega_1) - (1 - \mu)J_1(\omega_1) & \omega_1 Y_0(\omega_1) - (1 - \mu)Y_1(\omega_1) \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = 0 \qquad (61)$$

For the unrestrained solid disk,

$$\left[\omega_1 J_0(\omega_1) - (1 - \mu)J_1(\omega_1)\right]\{B_1\} = 0 \qquad (62)$$

and

$$\left[\omega_2 J_0(\omega_2) - 2J_1(\omega_2)\right]\{B_3\} = 0 \qquad (63)$$

The fact that equations (55) and (56) each break up into two independent 2×2 matrix equations for $m = 0$ means that distortional and dilatational waves for these modes are uncoupled.

In order for nontrivial solutions to exist for the constants B_k appearing in these various matrix equations, the determinant of the corresponding a_{jk} matrices must be zero. The frequencies which lead to the vanishing of these various determinants are the natural frequencies of vibration.

VIBRATIONAL MODES CONSIDERED

The natural frequencies of the following modes only have been calculated:

(a) For $m = 0$, the lowest three distortional ($U = 0$) and dilatational ($V = 0$) natural frequencies

(b) For $m = 1$, the lowest three natural frequencies for $\mu = 0$, 0.25, and 0.5.

These correspond to the first three natural frequencies associated with each of equations (58) to (63) and to the first three associated with each of equations (55) to (57) for $m = 1$. These frequencies, which were computed with the aid of an IBM 7090 electronic data processing system at the Langley Research Center, are listed in tables I to IV and presented graphically in figures 2 to 10. The radially symmetric torsional modes, as equations (58) and (60) show, are independent of Poisson's ratio μ. In some of the other modes the dependence on Poisson's ratio may be virtually negligible (fig. 8) or it may be considerable (fig. 10). The rather abrupt jump in the frequency curves near $a/b = 0$ in figures 8, 9, and 10 for the rigid-clamping case is due to the presence of logarithmic terms in the frequency determinant.

APPROXIMATE EXPRESSIONS FOR THE LOWEST NATURAL FREQUENCY IN THE $m = 0$ AND $m = 1$ MODES WHEN $(a/b)^2 \ll 1$

The $m = 0$ Mode

If terms of order $(a/b)^2$ and ${\omega_2}^2$ in equation (60) are retained, the vanishing of the determinant of $\left\{a_{jk}\right\}$ yields the approximate expression

$$\omega_2 = \sqrt{\frac{\rho}{G}}\, b\omega \approx 2\sqrt{2}\ a/b \qquad (a/b) \ll 1) \quad (64)$$

for the lowest natural frequency of vibration.

In figure 11 the exact value of ω_2 as determined by a numerical computation of the roots of the determinant of the matrix of coefficients in equation (60) is compared with the approximate value of ω_2 as given by equation (64). As may be seen, equation (64) approximates the curve of ω_2 against a/b by a straight line tangent to the exact curve at the origin.

Equation (64) may be given a simple physical interpretation with the aid of figure 12, which is a sketch of the first $m = 0$ mode shape for $a/b = 0.10$.

It seems reasonable to surmise from this sketch that as a/b approaches zero, the disk oscillates primarily as a rigid body, the chief distortions occurring within an annular region whose width is roughly of the same order of magnitude as the hub radius a.

The simplified model which this figure suggests is therefore an oscillating rigid disk restrained in rotation by a torsional spring at its center. The polar moment of inertia of such a disk of unit thickness is $\rho \frac{\pi b^4}{2}$. If the torsional stiffness of the spring is estimated to be of the order of magnitude of G times the area of a circle of radius $2a$ times the unit thickness $\left(\text{that is, } 4\pi a^2 G\right)$, the frequency of vibration of such an oscillating disk is then

$$\omega = \sqrt{\frac{\text{Torsional stiffness}}{\text{Moment of inertia}}} = 2\sqrt{\frac{2G}{\rho}}\frac{a}{b^2}$$

which agrees with the approximate expression derived previously.

The $m = 1$ Mode

For $m = 1$, the series expansions of the elements of the 4×4 matrix $\left\{a_{jk}\right\}$ of equation (56) are as follows, with the first two terms explicitly written out:

$$a_{11}\left(\omega_1 a/b\right) = \frac{\omega_1 a/b}{2} - \frac{3\left(\omega_1 a/b\right)^3}{16} + \dots$$

$$a_{12}\left(\omega_1 a/b\right) = \frac{1}{\pi}\left[\frac{2}{\omega_1 a/b} + \left(\omega_1 a/b\right)\log\left(\omega_1 a/b\right) + \dots\right]$$

$$a_{13}\left(\omega_2 a/b\right) = \frac{\omega_2 a/b}{2} - \frac{\left(\omega_2 a/b\right)^3}{16} + \dots$$

$$a_{14}(\omega_1 a/b) = -\frac{1}{\pi}\left[\frac{2}{\omega_2 a/b} - (\omega_2 a/b)\log(\omega_2 a/b) + \ldots\right]$$

$$a_{21}(\omega_1 a/b) = \frac{\omega_1 a/b}{2} - \frac{(\omega_1 a/b)^3}{16} + \ldots$$

$$a_{22}(\omega_1 a/b) = -\frac{1}{\pi}\left[\frac{2}{\omega_1 a/b} - (\omega_1 a/b)\log(\omega_1 a/b) + \ldots\right]$$

$$a_{23}(\omega_2 a/b) = \frac{\omega_2 a/b}{2} - \frac{3(\omega_2 a/b)^3}{16} + \ldots$$

$$a_{24}(\omega_2 a/b) = \frac{1}{\pi}\left[\frac{2}{\omega_2 a/b} + (\omega_2 a/b)\log(\omega_2 a/b) + \ldots\right]$$

$$a_{31}(\omega_1) = -\frac{\omega_1^3}{4} + \frac{\omega_1^5}{48} + \ldots$$

$$a_{32}(\omega_1) = \frac{1}{\pi}\left(\frac{8}{\omega_1} + 2\omega_1 + \ldots\right)$$

$$a_{33}(\omega_2) = -\frac{\omega_2^3}{4} + \frac{\omega_2^5}{24} + \ldots$$

$$a_{34}(\omega_2) = -\frac{1}{\pi}\left(\frac{8}{\omega_2} + \omega_2^3 \log \omega_2 + \ldots\right)$$

$$a_{41}(\omega_1) = -\frac{3 + \mu}{8}\omega_1^3 + \frac{5 + \mu}{96}\omega_1^5 + \ldots$$

$$a_{42}(\omega_1) = -\frac{1}{\pi}\left[\frac{4(1 - \mu)}{\omega_1} - (1 + \mu)\omega_1 + \ldots\right]$$

$$a_{43}(\omega_2) = -\frac{1 - \mu}{8}\omega_2^3 + \frac{1 - \mu}{96}\omega_2^5 + \ldots$$

$$a_{44}(\omega_2) = \frac{1 - \mu}{\pi}\left(\frac{4}{\omega_2} + \omega_2 + \ldots\right)$$

If the determinant of $\{a_{jk}\}$ is calculated by means of these series expansions and the lowest order terms in $(a/b)^2$ and ω_2 are retained, the following approximate expression is obtained for the lowest natural frequency:

$$\omega_2 \approx 2\sqrt{\frac{2}{(3 - \mu)\log(b/a)}} \qquad (a/b \ll 1) \qquad (65)$$

Figure 13 is a graphical comparison between the exact value of ω_2 and the approximate value given by equation (65). It will be observed that although the rather sudden increase in ω_2 near $a/b = 0$ is approximated well by equation (65), the approximation is valid for a much smaller range of a/b than the corresponding approximation (eq. (64)) for $m = 0$. Moreover, no simple physical interpretation for equation (65) is apparent.

CONCLUDING REMARKS

The equations of motion governing the linear vibrations of a thin spinning elastic disk have been reexamined, and Grammel's conclusion that the rotation of the disk has no essential effect on its natural frequencies of vibration has been reaffirmed. Upper and lower limits on the natural frequencies of vibration of an elastic disk clamped to an elastic hub have been calculated for the case of rotationally symmetric vibrations ($m = 0$) and vibrations with one diametral node ($m = 1$). These upper and lower limits correspond, respectively, to the natural frequencies of an elastic disk clamped to a rigid hub and one clamped to a perfectly flexible hub (which is equivalent to no clamping). These frequencies have been presented in both tabular and graphical form, and approximate formulas have been developed for the natural frequencies of disks clamped to rigid hubs of small diameter. Finally, an orthogonality relation has been stated which permits, for example, an approximate determination of the elastic response of the disk to any arbitrary motion of the hub.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., July 6, 1962.

APPENDIX

ORTHOGONALITY

The condition of orthogonality of any two modes of vibration of an elastic disk may be readily established with the aid of tensors. For a full discussion of the tensor concept in elasticity, the reader is referred to reference 4.

For harmonic motion $\ddot{\overline{S}} = -\omega^2\overline{S}$ and the dynamic equations of equilibrium may be written in the tensor form

$$\tau^{\alpha\beta}\Big|_{\beta} + \rho\omega^2 U^{\alpha} = 0 \qquad (\alpha = 1,\ 2) \qquad \text{(A1)}$$

where $\tau^{\alpha\beta}$ are the contravariant components of the plane stress tensor, U^{α} are the contravariant components of the relative displacement vector $\overline{S}$, and the vertical bar denotes covariant differentiation based upon the metric tensor $g_{\alpha\beta}$ of the coordinate system chosen to locate points in the midplane of the disk.

Let equation (A1) hold for some given vibrational mode and let a tilde denote quantities associated with any other distinct vibrational mode. Thus

$$\tilde{\tau}^{\alpha\beta}\Big|_{\beta} + \rho\tilde{\omega}^2\tilde{U}^{\alpha} = 0 \qquad (\tilde{\omega} \neq \omega) \qquad \text{(A2)}$$

Multiplying equation (A1) by $\tilde{U}_{\alpha}$ and equation (A2) by U_{α} and integrating both the resulting expressions over A, the area of the disk, results in

$$\iint \left(\tau^{\alpha\beta}\Big|_{\beta}\tilde{U}_{\alpha} + \rho\omega^2 U^{\alpha}\tilde{U}_{\alpha} \right) dA = 0 \qquad \text{(A3)}$$

$$\iint \left(\tilde{\tau}^{\alpha\beta}\Big|_{\beta}U_{\alpha} + \rho\tilde{\omega}^2\tilde{U}^{\alpha}U_{\alpha} \right) dA = 0 \qquad \text{(A4)}$$

Next equation (A4) is subtracted from equation (A3) and the following form of Green's theorem is used:

$$\iint \tau^{\alpha\beta}\Big|_\beta \tilde{U}_\alpha \, dA = \oint_{r=a} \tau^{\alpha\beta}\tilde{U}_\alpha m_\beta \, ds + \oint_{r=b} \tau^{\alpha\beta}\tilde{U}_\alpha n_\beta \, ds - \iint \tau^{\alpha\beta}\tilde{U}_\alpha\Big|_\beta \, dA \tag{A5}$$

where ds is an element of length taken along an edge of the disk and where m_β and n_β are the covariant components of the unit vectors normal to the circles $r = a$ and $r = b$, respectively. The resulting equation is

$$\iint \left[\tilde{\tau}^{\alpha\beta}U_\alpha\Big|_\beta - \tau^{\alpha\beta}\tilde{U}_\alpha\Big|_\beta + \rho\left(\omega^2 - \tilde{\omega}^2\right)U^\alpha\tilde{U}_\alpha\right] dA$$

$$+ \oint_{r=a}\left(\tau^{\alpha\beta}\tilde{U}_\alpha - \tilde{\tau}^{\alpha\beta}U_\alpha\right)m_\beta \, ds + \oint_{r=b}\left(\tau^{\alpha\beta}\tilde{U}_\alpha - \tilde{\tau}^{\alpha\beta}U_\alpha\right)n_\beta \, ds = 0 \tag{A6}$$

In view of the boundary conditions for the clamped and unrestrained disks, the two line integrals appearing in equation (A6) vanish. Furthermore, since the stress tensor is symmetric,

$$\tilde{\tau}^{\alpha\beta}U_\alpha\Big|_\beta - \tau^{\alpha\beta}\tilde{U}_\alpha\Big|_\beta = \tilde{\tau}^{\alpha\beta}E_{\alpha\beta} - \tau^{\alpha\beta}\tilde{E}_{\alpha\beta} \tag{A7}$$

where

$$E_{\alpha\beta} = \frac{1}{2}\left(U_\alpha\Big|_\beta + U_\beta\Big|_\alpha\right) \tag{A8}$$

is the infinitesimal elastic strain tensor. But the tensor form of the stress-strain relation of plane stress theory is as follows:

$$\tau^{\alpha\beta} = 2G\left(E^{\alpha\beta} + \frac{\mu}{1 - \mu}\, g^{\alpha\beta}e\right) \tag{A9}$$

where

$$e \equiv g^{\alpha\beta}E_{\alpha\beta} = E^\alpha_\alpha \tag{A10}$$

Hence

$$\tilde{\tau}^{\alpha\beta}E_{\alpha\beta} - \tau^{\alpha\beta}\tilde{E}_{\alpha\beta} = 2G\left[\tilde{E}^{\alpha\beta}E_{\alpha\beta} - E^{\alpha\beta}\tilde{E}_{\alpha\beta} + \frac{\mu}{1 - \mu}(\tilde{e}e - e\tilde{e})\right]$$

$$= 0 \tag{A11}$$

Thus, since $\tilde{\omega} \neq \omega$, the condition of orthogonality is

$$\iint U^{\alpha}\tilde{U}_{\alpha}\, dA = \iint \bar{S} \cdot \tilde{\bar{S}}\, dA$$

$$= 0 \tag{A12}$$

By use of equations (26), (48), and (49), equation (A12) is further reduced to

$$\int_{a/b}^{1} \left\{\left[\sum_{j=1}^{4} a_{1j}(\xi)B_j\right]\left[\sum_{k=1}^{4} \tilde{a}_{1k}(\xi)\tilde{B}_k\right] + \left[\sum_{j=1}^{4} a_{2j}(\xi)B_j\right]\left[\sum_{k=1}^{4} \tilde{a}_{2k}(\xi)\tilde{B}_k\right]\right\} \frac{d\xi}{\xi} = 0$$

where now a_{jk} and $\tilde{a}_{jk}$ are to be computed for the same value of m.

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TABLE I

UPPER AND LOWER LIMITS ON THE THREE LOWEST RADIALLY SYMMETRIC

NATURAL TORSIONAL (U = 0) FREQUENCIES

a/b	$\sqrt{\frac{\rho}{G}}\, b\omega$ for -					
	No clamping			Clamping to a rigid hub		
	First mode (Lowest mode is rigid-body rotation)	Second mode	Third mode	First mode	Second mode	Third mode
0	0	5.1356	8.4172	0	5.1356	8.4172
.10	0	5.1423	8.4574	*.2866	5.3312	8.8521
.20	0	5.2218	8.8039	.5956	5.8247	9.7985
.30	0	5.4702	9.6003	.9524	6.5763	11.1348
.40	0	5.9659	10.8944	1.3920	7.6429	12.9679
.50	0	6.8138	12.8555	1.9732	9.1775	15.5629
.60	0	8.2272	15.9036	2.8098	11.5088	19.4743
.70	0	10.7199	21.0708	4.1650	15.4183	26.0083
.80	0	15.8553	31.4903	6.8250	23.2597	39.0900
.90	0	31.4821	62.8650	14.7175	46.8124	78.3537
1.00	0	∞	∞	∞	∞	∞

*For $\frac{a}{b} < 0.10$ use the approximate formula $\sqrt{\frac{\rho}{G}}\, b\omega \approx 2\sqrt{2}\, a/b$.

TABLE II

UPPER AND LOWER LIMITS ON THE THREE RADIALLY SYMMETRIC

NATURAL DILATATIONAL (V = 0) FREQUENCIES

a/b	$\sqrt{\frac{\rho}{G}}\, b\omega$ for -					
	No clamping			Clamping to rigid hub		
	First mode	Second mode	Third mode	First mode	Second mode	Third mode
$\mu = 0$						
0	2.6038	7.5398	12.0722	2.6038	7.5398	12.0722
.10	2.5505	7.2650	11.5954	2.6577	7.8237	12.6921
.20	2.4114	7.0157	11.9262	2.8191	8.5316	14.0348
.30	2.2374	7.2654	13.1639	3.1007	9.6013	15.9268
.40	2.0673	8.0032	15.1084	3.5377	11.1137	18.5204
.50	1.9158	9.2842	17.9696	4.2038	13.2857	22.1907
.60	1.7848	11.3718	22.3461	5.2541	16.5831	27.7224
.70	1.6721	14.9791	29.7037	7.0558	22.1111	36.9626
.80	1.5745	22.3131	44.4781	10.7176	33.1989	55.4627
.90	1.4893	44.4726	88.8795	21.7915	66.5053	110.9895
1.00	1.4142	∞	∞	∞	∞	∞
$\mu = 0.25$						
0	3.2940	8.7851	13.9881	3.2940	8.7851	13.9881
.10	3.1888	8.2487	13.0928	3.3575	9.1150	14.7054
.20	2.9297	7.8231	13.5394	3.5462	9.9350	16.2570
.30	2.6415	8.1521	15.0546	3.8726	11.1720	18.4423
.40	2.3901	9.0667	17.3501	4.3770	12.9195	21.4374
.50	2.1844	10.5972	20.6852	5.1441	15.4280	25.6756
.60	2.0172	13.0466	25.7599	6.3530	19.2357	32.0632
.70	1.8794	17.2413	34.2711	8.4283	25.6189	42.7329
.80	1.7640	25.7325	51.3426	12.6503	38.4219	64.0949
.90	1.6659	51.3381	102.6220	25.4303	76.8807	128.2116
1.00	1.5811	∞	∞	∞	∞	∞
$\mu = 0.50$						
0	4.3317	10.8549	17.1909	4.3317	10.8549	17.1909
.10	4.0913	9.6969	15.4642	4.4120	11.2613	18.0710
.20	3.5630	9.0875	16.2616	4.6484	12.2689	19.9728
.30	3.0789	9.6358	18.2508	5.0540	13.7860	22.6500
.40	2.7142	10.8695	21.1295	5.6768	15.9277	26.3186
.50	2.4424	12.8203	25.2544	6.6202	19.0009	31.5096
.60	2.2342	15.8726	31.4963	8.1032	23.6648	39.3328
.70	2.0698	21.0480	41.9393	10.6458	31.4828	52.4005
.80	1.9364	31.4760	62.8617	15.8162	47.1632	78.5636
.90	1.8257	62.8584	125.6770	31.4667	94.2654	157.0902
1.00	1.7321	∞	∞	∞	∞	∞

TABLE III

UPPER AND LOWER LIMITS ON THE THREE LOWEST NATURAL FREQUENCIES

FOR ONE DIAMETRAL NODE (m = 1)

$$\left[\frac{a}{b} = 0(0.10)1.00\right]$$

a/b	$\sqrt{\frac{\rho}{G}}\,b\omega$ for - No clamping, First mode	Second mode	Third mode	Clamping to a rigid hub, First mode	Second mode	Third mode
			$\mu = 0$			
0	0 (Lowest mode is a rigid-body translation)	2.4744	5.0032	0	2.4744	5.0032
.10	0	2.4827	5.1385	1.2210	2.7481	5.9868
.20	0	2.5007	5.4711	1.5097	2.9309	6.4488
.30	0	2.5131	5.9185	1.7914	3.1904	7.0659
.40	0	2.5033	6.4519	2.1138	3.5963	8.0001
.50	0	2.4623	7.2144	2.5486	4.2404	9.4287
.60	0	2.3925	8.5150	3.2304	5.2797	11.6797
.70	0	2.3031	10.9091	4.4414	7.0771	15.5286
.80	0	2.2038	15.9662	6.9808	10.7357	23.3237
.90	0	2.1014	31.5313	14.7811	21.8035	46.8407
1.00	0	*2.0000	∞	∞	∞	∞
			$\mu = 0.25$			
0	0	2.6978	5.7746	0	2.6978	5.7746
.10	0	2.7131	5.9426	1.2588	3.1696	6.5308
.20	0	2.7491	6.2819	1.5466	3.4711	6.5981
.30	0	2.7839	6.5219	1.8292	3.8471	7.2909
.40	0	2.7918	6.7773	2.1579	4.3667	8.1464
.50	0	2.7565	7.3729	2.6022	5.1357	9.5241
.60	0	2.6809	8.5920	3.2899	6.3442	11.7415
.70	0	2.5792	10.9468	4.4986	8.4207	15.5673
.80	0	2.4659	15.9843	7.0262	12.6456	23.3459
.90	0	2.3499	31.5385	14.8068	25.4286	46.8504
1.00	0	*2.2361	∞	∞	∞	∞
			$\mu = 0.50$			
0	0	2.8572	6.5527	0	2.8572	6.5527
.10	0	2.8797	6.7790	1.2994	3.5736	7.3933
.20	0	2.9357	7.0977	1.5840	4.0544	7.4850
.30	0	2.9987	7.0153	1.8639	4.6393	7.8126
.40	0	3.0339	7.0426	2.1935	5.3946	8.5222
.50	0	3.0139	7.5099	2.6416	6.4327	9.7812
.60	0	2.9386	8.6612	3.3322	7.9814	11.9087
.70	0	2.8281	10.9816	4.5395	10.5695	15.6695
.80	0	2.7027	16.0012	7.0592	15.7724	23.4021
.90	0	2.5745	31.5453	14.8257	31.4473	46.8740
1.00	0	*2.4495	∞	∞	∞	∞

*For $\frac{a}{b} \rightarrow 1$, $\sqrt{\frac{\rho}{G}}\,\omega b \rightarrow 2\sqrt{1 + \mu}$, the value predicted for longitudinal vibrations of a thin elastic ring.

TABLE IV

LOWEST NATURAL FREQUENCIES FOR ONE DIAMETRAL NODE ($m = 1$)

FOR CLAMPING TO A RIGID HUB WHEN $a/b \ll 1$

$$\left[\frac{a}{b} = 0(0.01)0.10\right]$$

a/b	$\sqrt{\frac{\rho}{G}}\,b\omega$ for -		
	$\mu = 0$	$\mu = 0.25$	$\mu = 0.50$
0	0	0	0
.01	*.8161	*.8482	*.8842
.02	.8956	.9297	.9677
.03	.9539	.9891	1.0283
.04	1.0025	1.0386	1.0784
.05	1.0455	1.0822	1.1224
.06	1.0847	1.1218	1.1624
.07	1.1213	1.1587	1.1994
.08	1.1560	1.1936	1.2343
.09	1.1891	1.2268	1.2675
.10	1.2210	1.2588	1.2994

*For $\frac{a}{b} < 0.010$, use the approximate formula

$$\sqrt{\frac{\rho}{G}}\,b\omega \approx 2\sqrt{\frac{2}{(3 + \mu)\log(b/a)}}$$

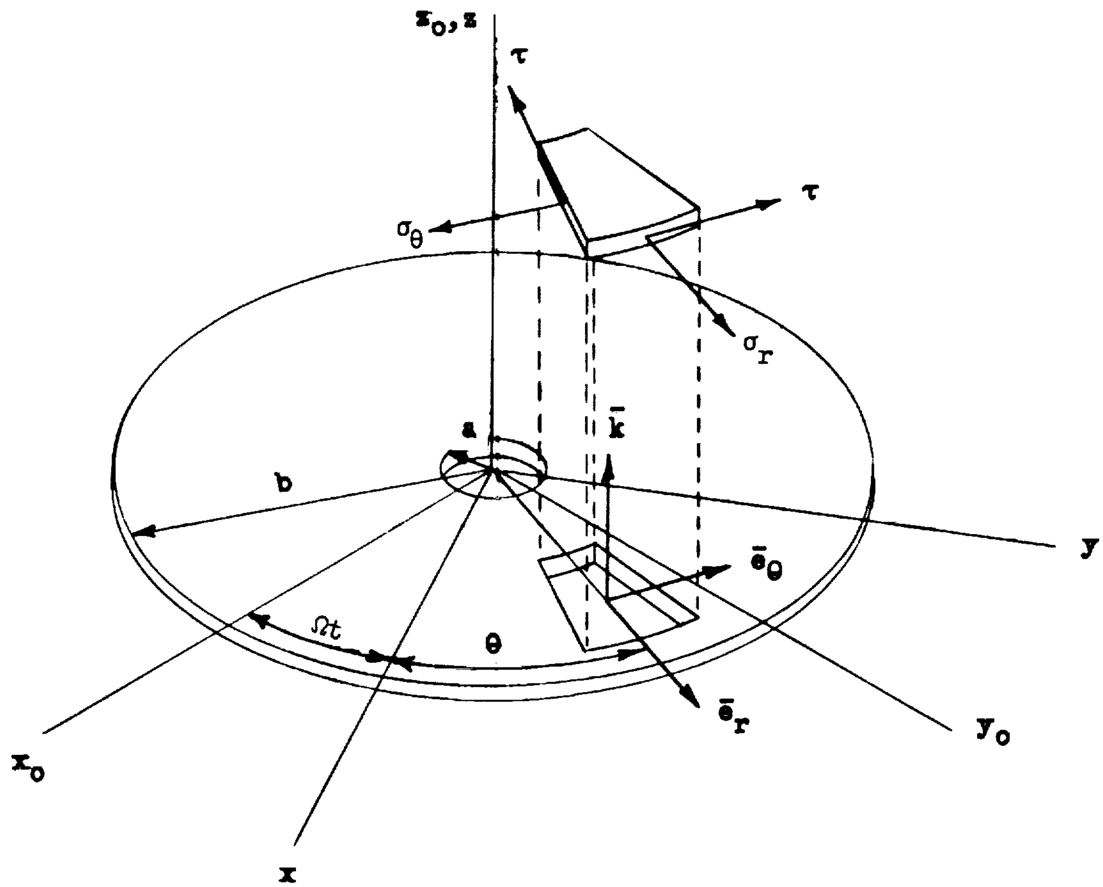

Figure 1.- Coordinate system and stress notation for the flat spinning disk.

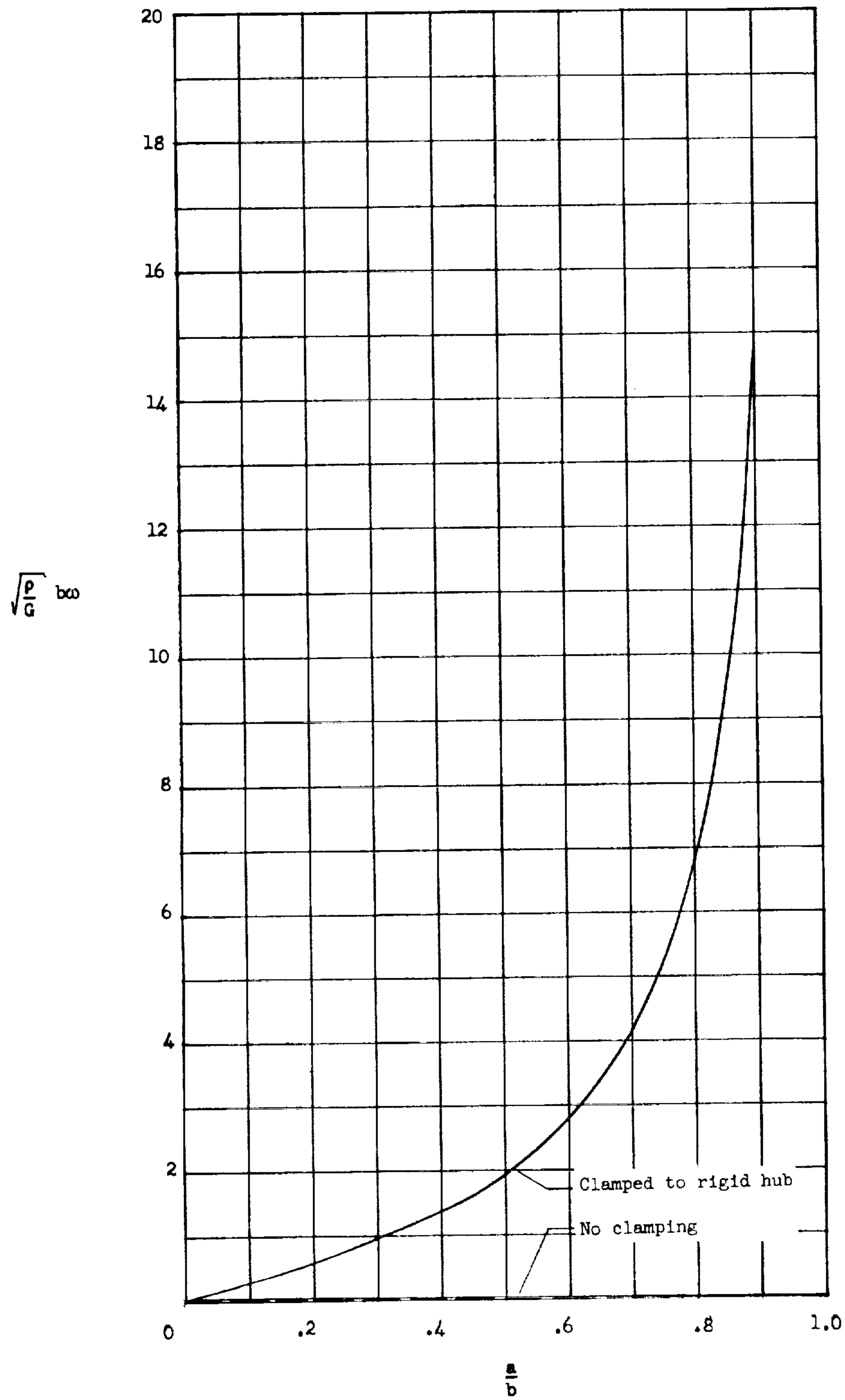

Figure 2.- Lowest natural torsional frequency (m = 0).

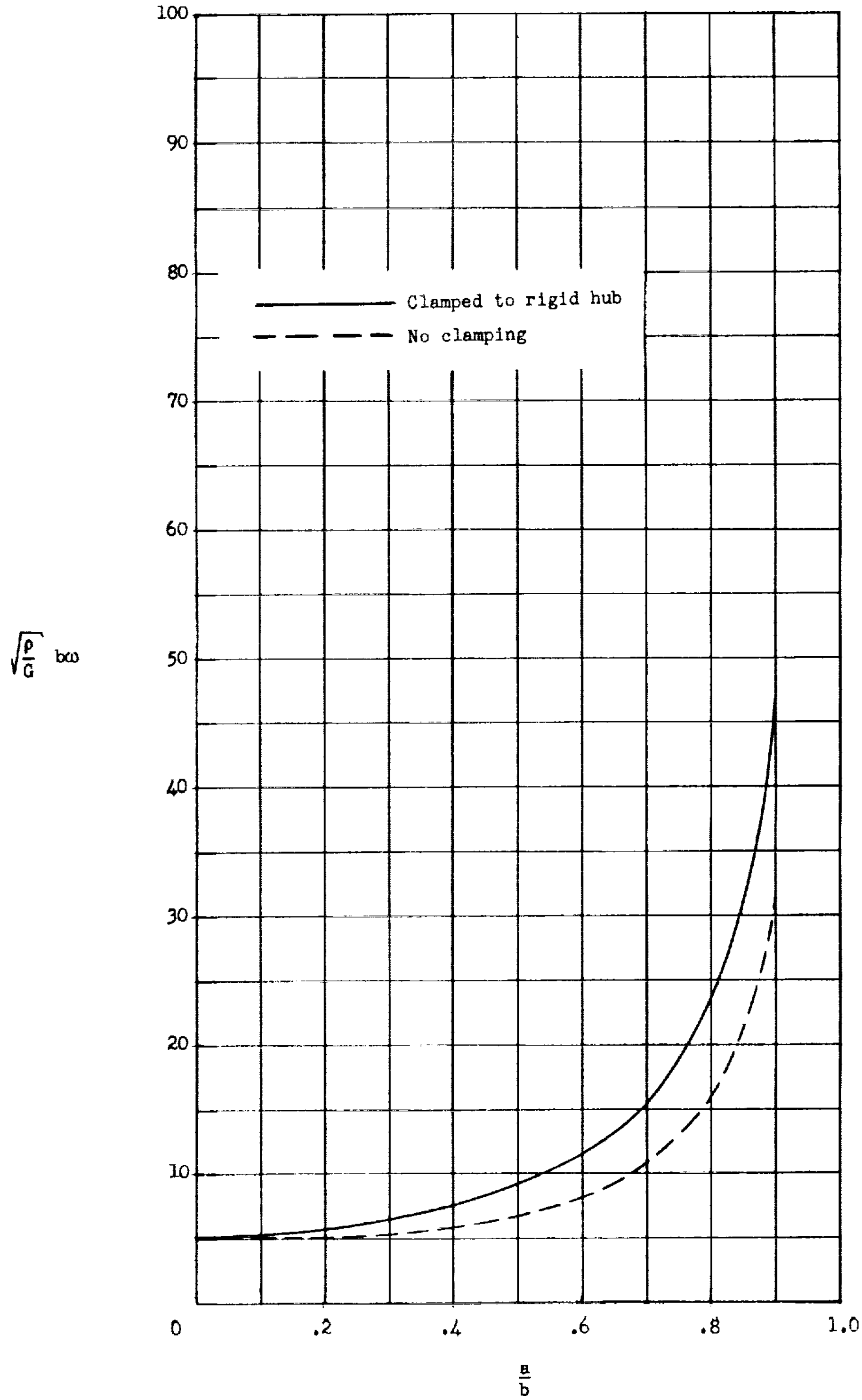

Figure 3.- Second lowest natural torsional frequency (m = 0).

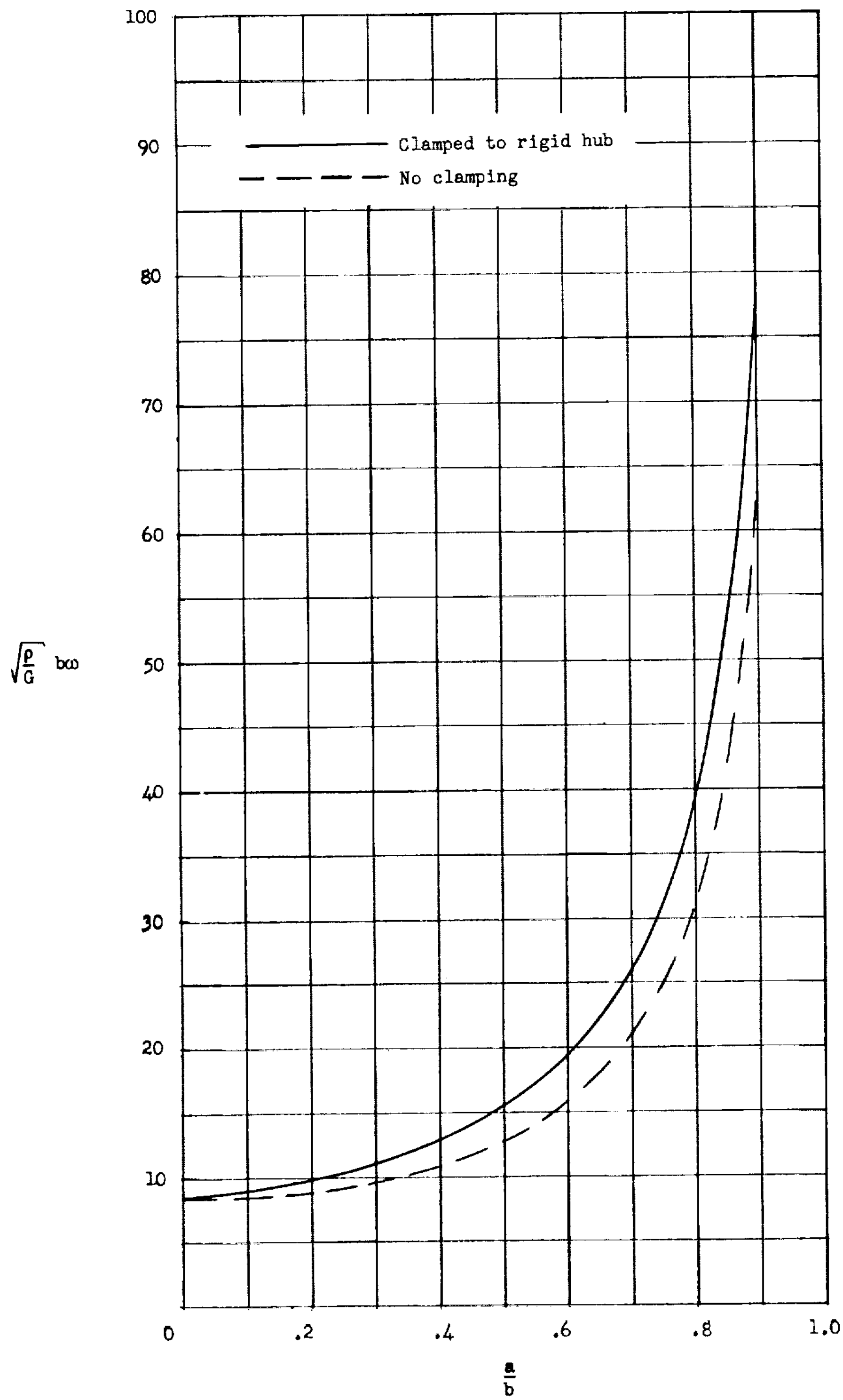

Figure 4.- Third lowest natural torsional frequency (m = 0).

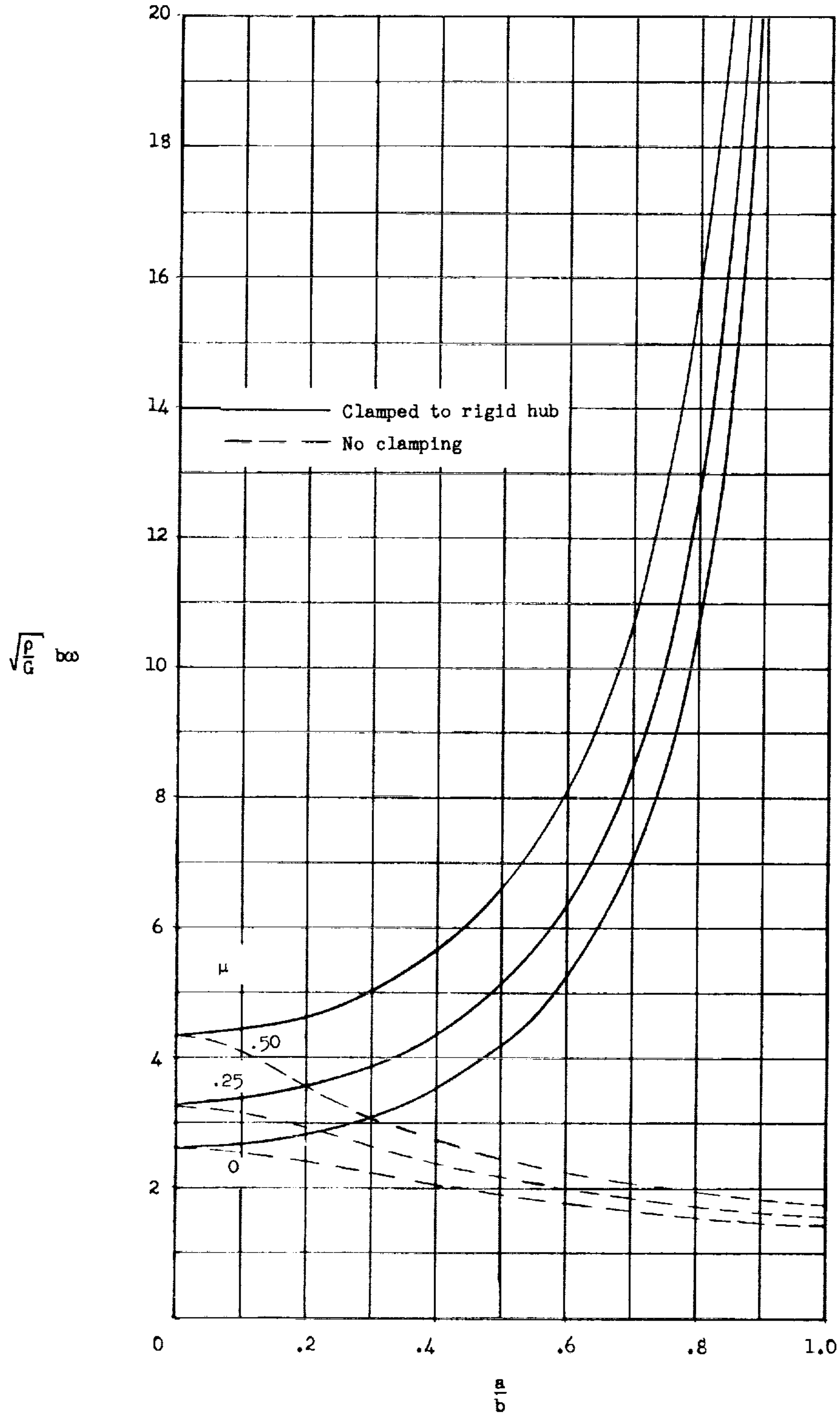

Figure 5.- Lowest natural dilatational frequency (m = 0).

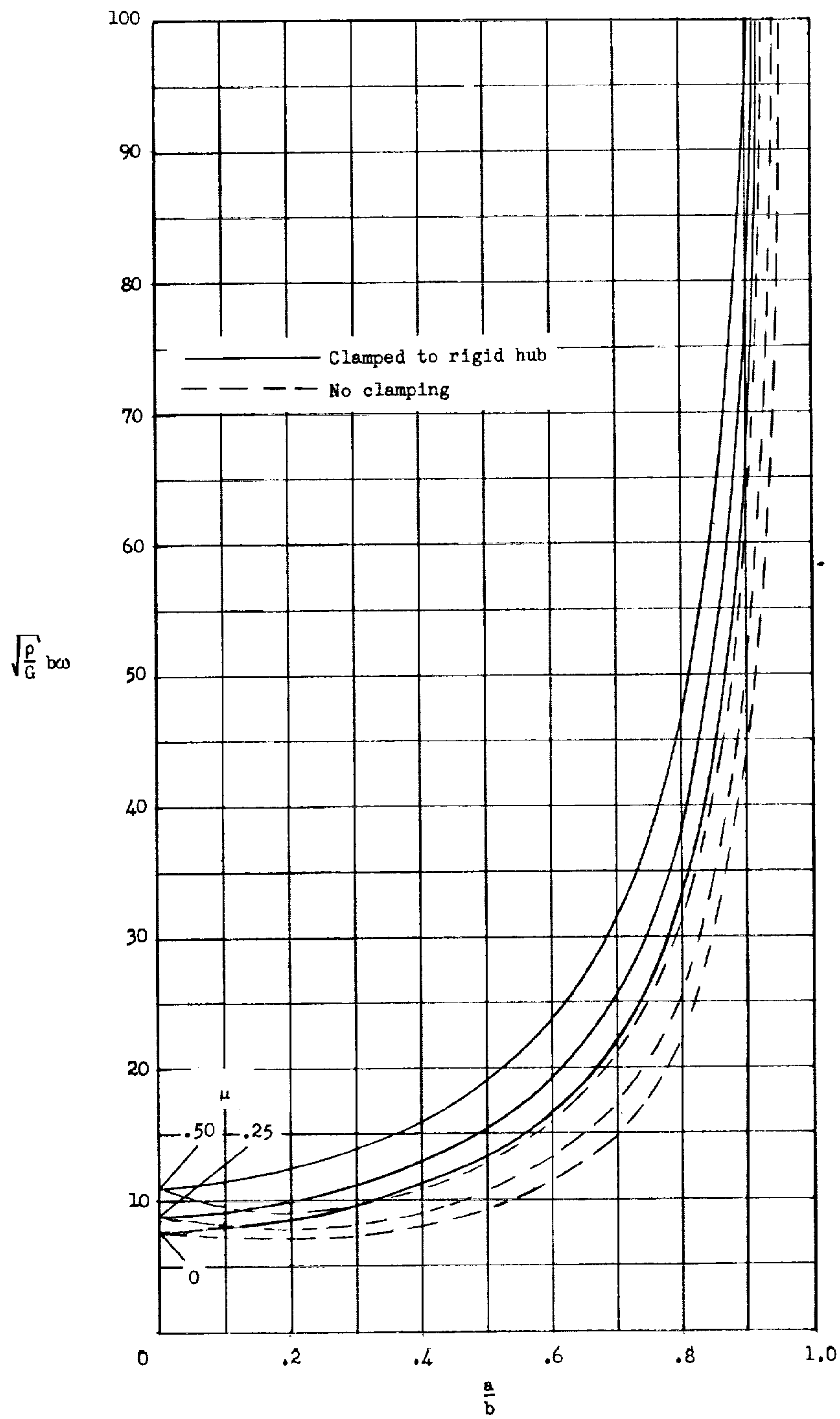

Figure 6.- Second lowest natural dilatational frequency (m = 0).

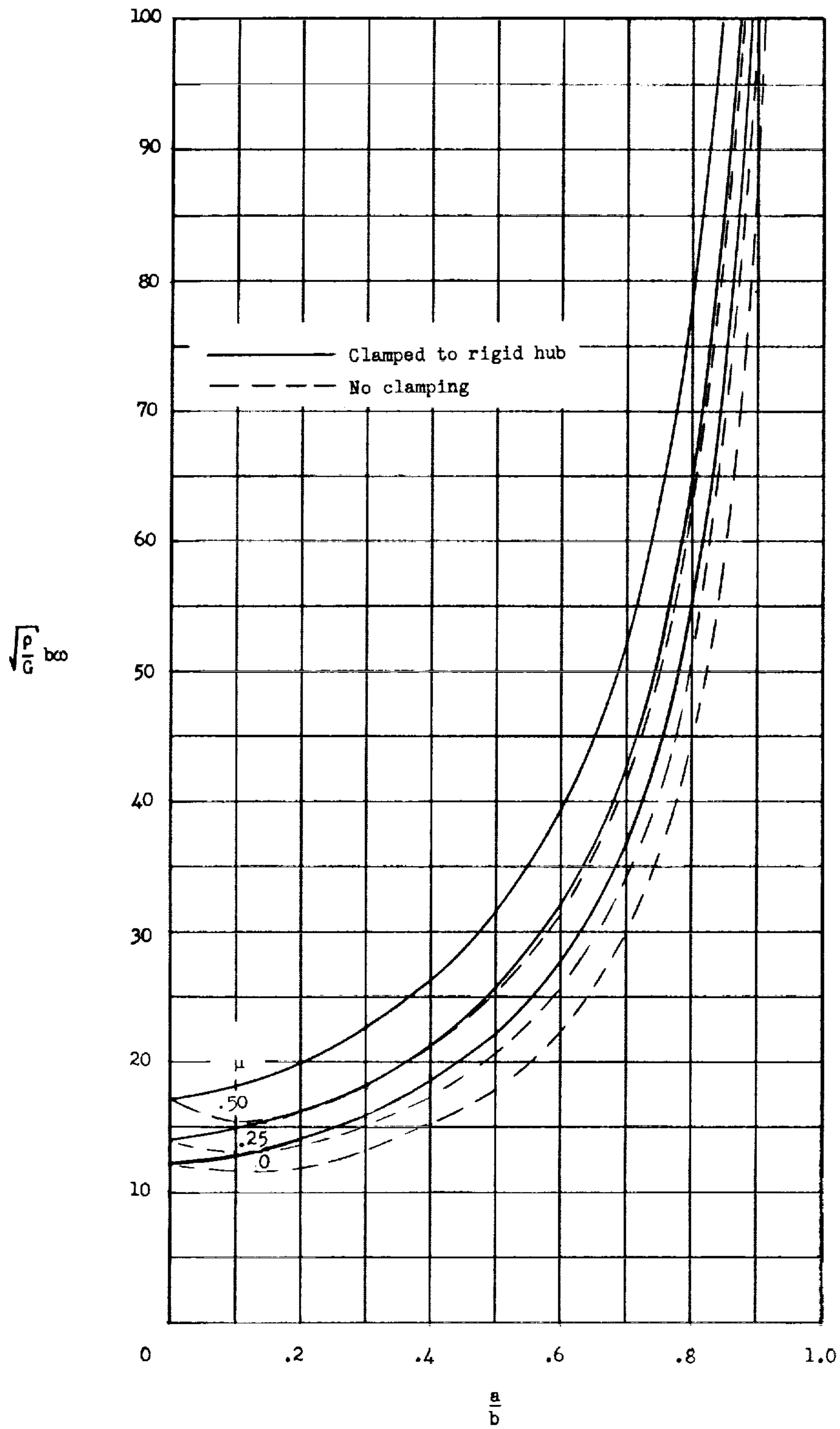

Figure 7.- Third lowest natural dilatational frequency (m = 0).

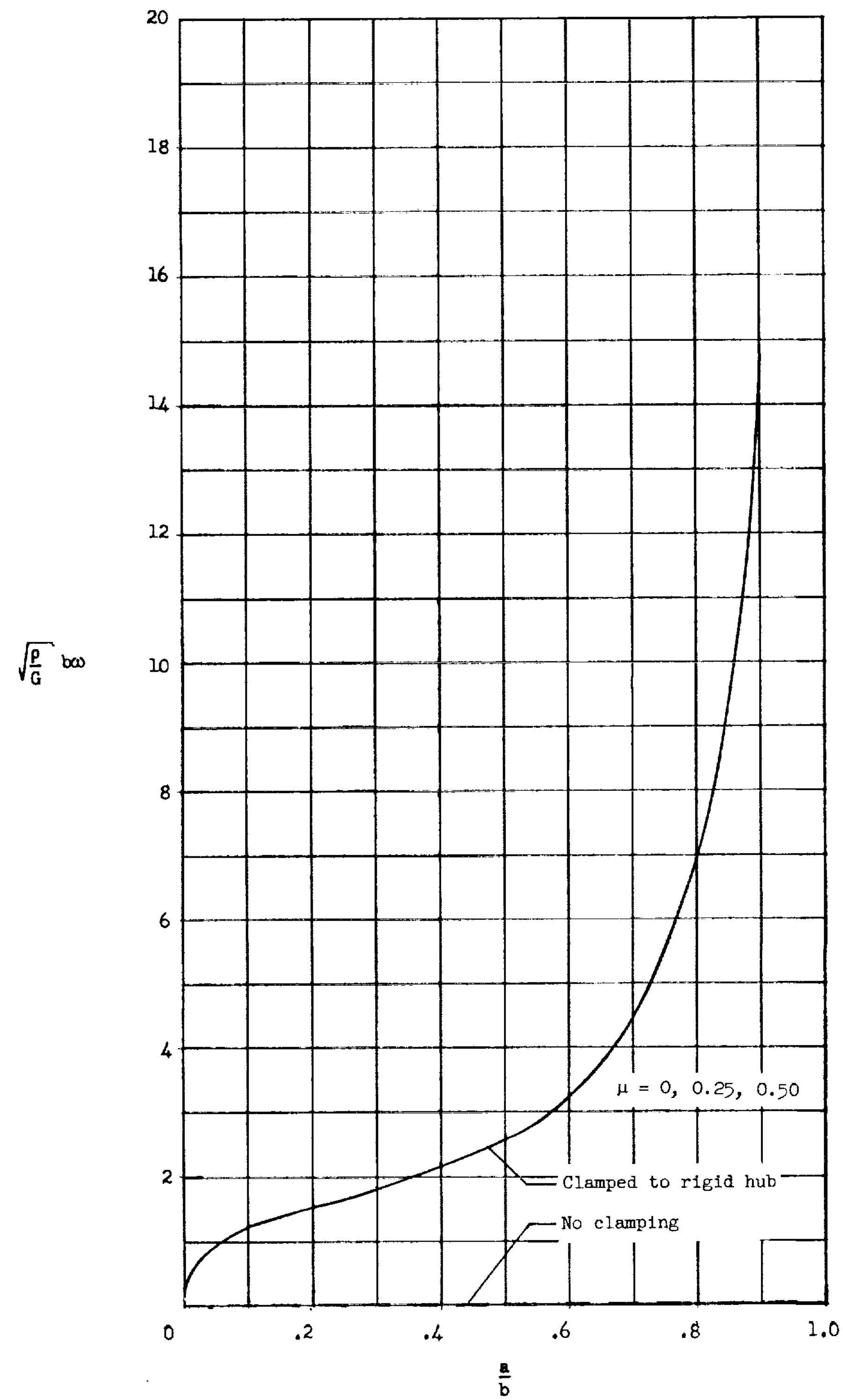

Figure 8.- Lowest natural frequency for $m = 1$.

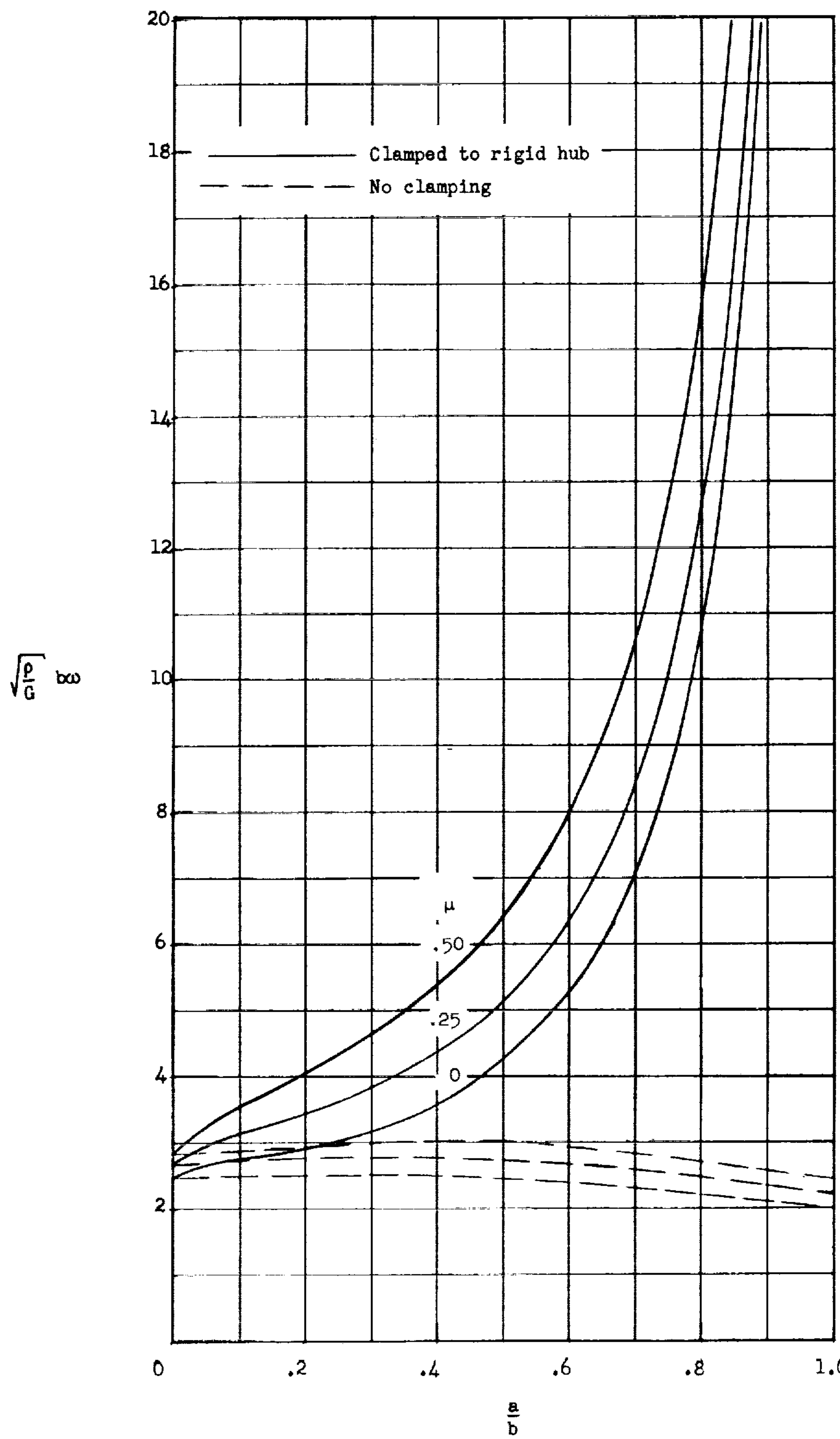

Figure 9.- Second lowest natural frequency for $m = 1$.

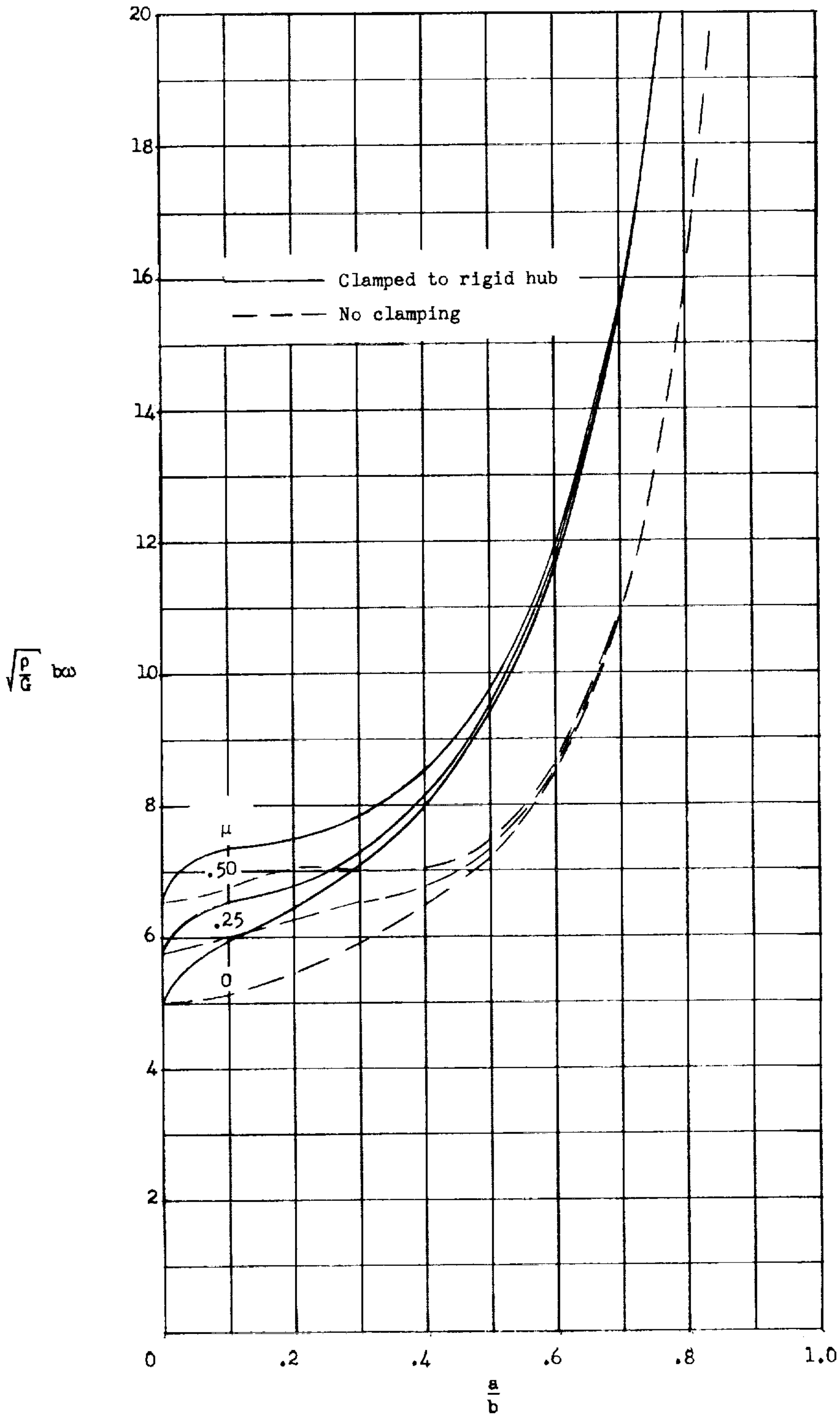

Figure 10.- Third lowest natural frequency for $m = 1$.

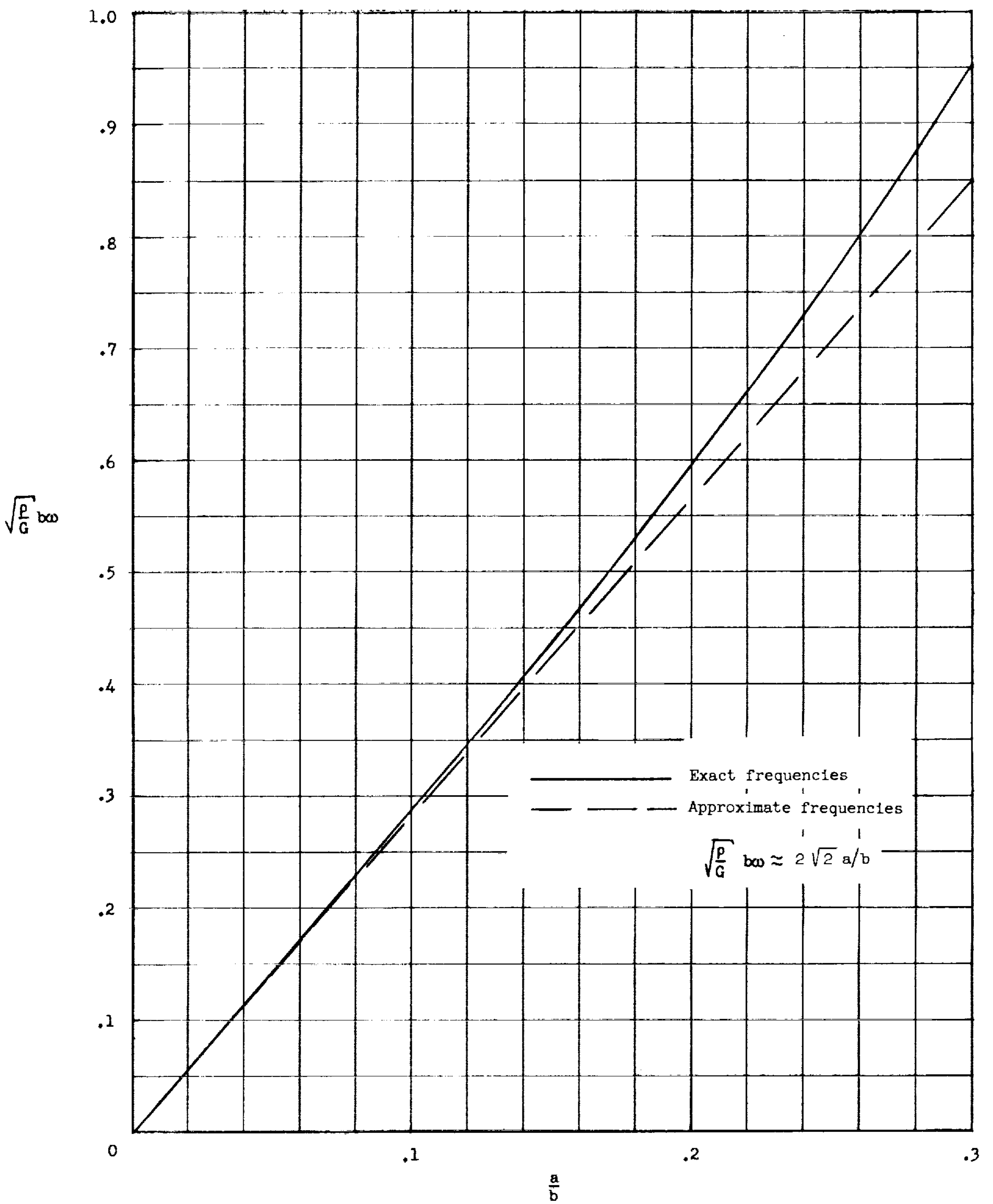

Figure 11.- Comparison of exact and approximate values of lowest natural torsional frequency for the clamped disk.

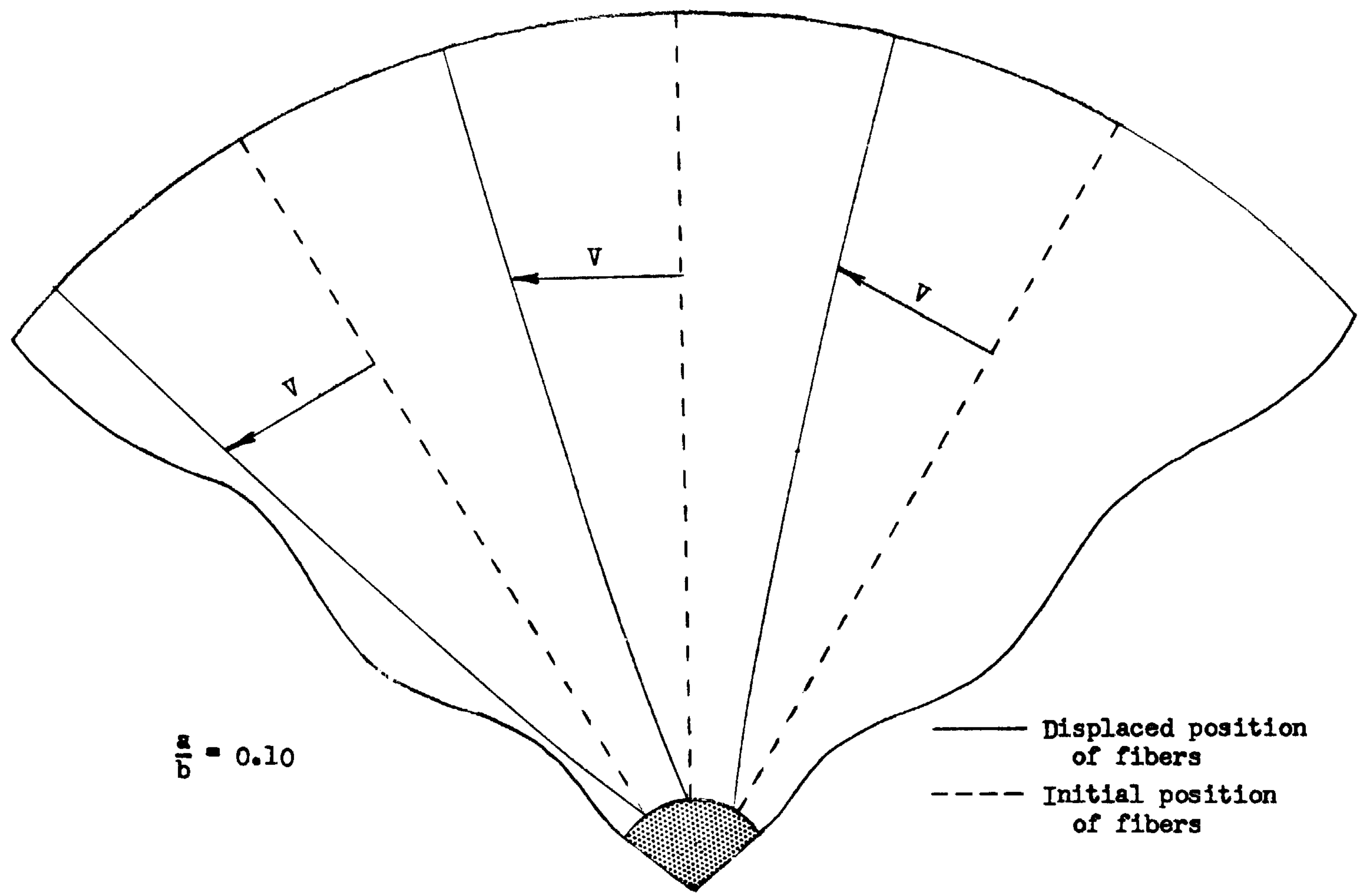

Figure 12.- Shape of lowest torsional mode of disk clamped to rigid hub for small value of $\frac{a}{b}$ (amplitude of displacement exaggerated for clarity).

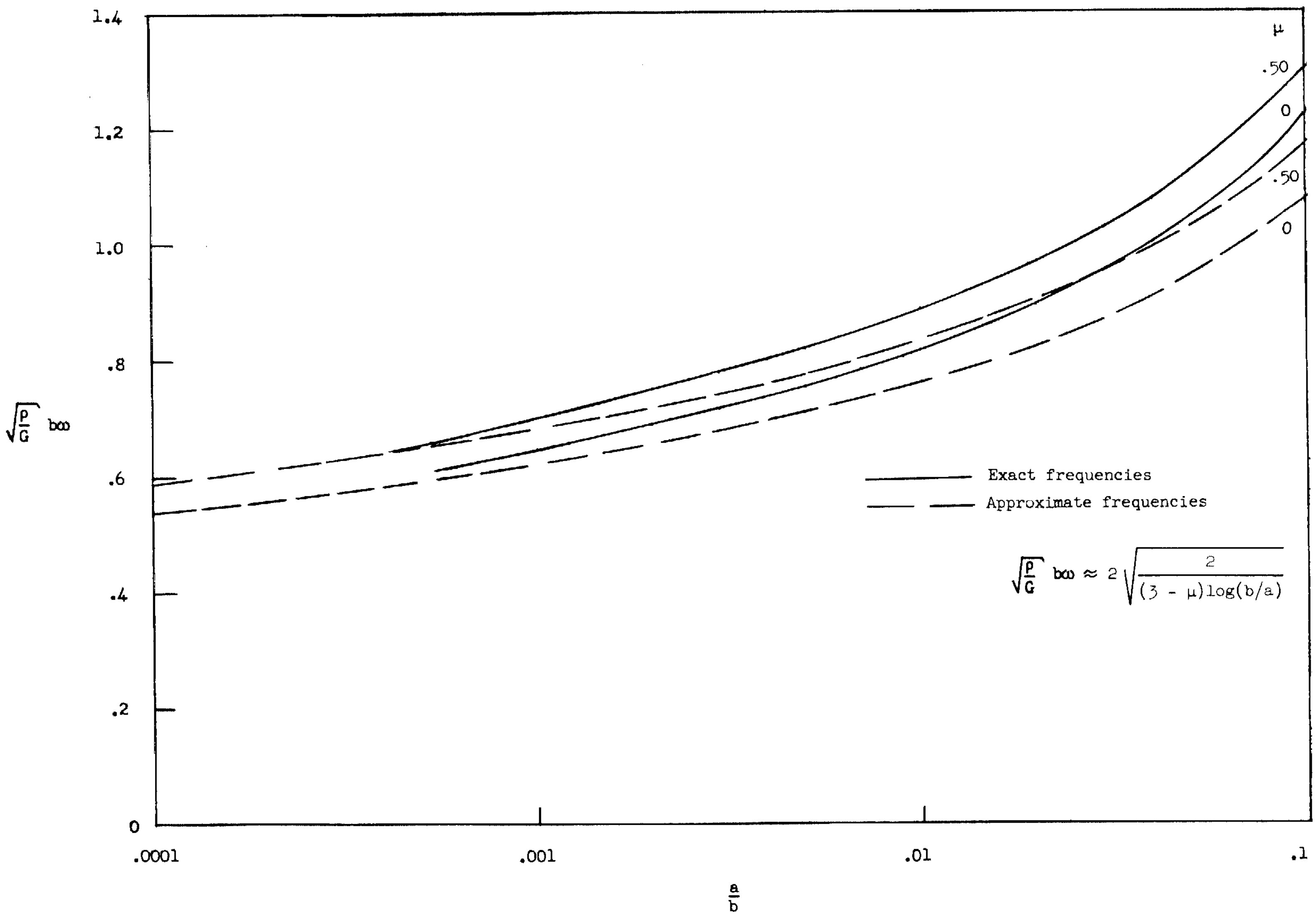

Figure 13.- Comparison of exact and approximate values of lowest natural frequency of the clamped disk in the $m = 1$ mode.