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A SATELLITE ORBIT COMPUTATION PROGRAM FOR IZSAK'S SECOND-ORDER SOLUTION OF VINTI'S DYNAMICAL PROBLEM

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SUMMARY

This report extends the results of Vinti and Izsak and presents a computational procedure designed specifically for Izsak's second-order solution of Vinti's dynamical problem. With this procedure, the coordinates and velocity of an unretarded satellite can be obtained from a knowledge of its initial conditions r and v.

In this procedure, the derivation is given for the complete set of six canonical constants from initial conditions. Three of these have been determined by Vinti and the remaining three by the author. All six of them are assumed known in Izsak's solution.

This report includes an adaption of a Newton-Raphson iteration scheme specifically designed to solve a certain system of nonlinear equations introduced by Vinti for the purpose of numerically factoring a certain quartic equation. The solution by this method can be used instead of certain infinite series to obtain Izsak's elements a and e. An example is included to illustrate how these elements may be obtained by the Newton-Raphson method.

Appendix B gives the derivation of exact expressions for the components of velocity in Vinti's accurate intermediary satellite orbit using Izsak's orbital elements. The derivation is one of the necessary steps in comparing such a method with others.

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A SATELLITE ORBIT COMPUTATION PROGRAM FOR IZSAK'S SECOND-ORDER SOLUTION OF VINTI'S DYNAMICAL PROBLEM

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INTRODUCTION

This report provides a computational procedure for determining the orbit of an artificial satellite in the earth's gravitational field. The procedure is based on Izsak's second-order solution of Vinti's dynamical problem (Reference 1). This computing procedure differs from many other methods in that the potential function is included in an analytic solution of the equations of motion. This is advantageous because the difficulties associated with the slow convergence or divergence of some series expansions used in orbit calculations are avoided; also the problem of small divisors is avoided. Another advantage of this procedure is that it does not involve several multiplications of Fourier series, a task common to certain satellite programs. Although Fourier series are well adapted to numerical computation, it is certainly desirable from the standpoint of machine storage and computing time to minimize the total number of such series. In many satellite theories, Fourier series are used from the very beginning to obtain successive approximations of different orders to the solution. The use of Vinti's potential minimizes the use of perturbation theory; Izsak (Reference 1) states that the oblateness perturbations which are not accounted for by Vinti's potential can be treated by a first-order method, that is, without multiplications of Fourier series.

As was pointed out by Izsak (Reference 2) it is advantageous for several practical purposes to have satellite orbits with very small eccentricities. Since the eccentricity never appears as a divisor, this procedure is valid for arbitrarily small values of e or $e = 0$. However, we must avoid polar orbits and orbits which have inclinations of less than 2 degrees.

Vinti (Reference 3) found an axially symmetric solution of Laplace's equation in oblate spheroidal coordinates which may be used as the gravitational potential about an oblate planet. This potential, which leads to separability of the Hamilton-Jacobi equation, is a remarkable approximation to the actual gravitational field of the earth in that it fits the zeroth and second zonal harmonics exactly and accounts for over half of the fourth zonal harmonic. Naturally, the oblateness perturbations are only a part of the factors which affect the satellite motion. Other perturbations not accounted for in this procedure are the effects of the odd harmonics, the residual fourth harmonic, the lunar-solar forces, and aerodynamic and electromagnetic drag.

MATHEMATICAL PROBLEM

In Hamiltonian form, the equations of motion of a dynamical system of n degrees of freedom assume the forms

$$\left.\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}, \\ \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}, \end{aligned}\qquad i = 1, 2, \cdots, n \right\} \tag{1}$$

where $H(q_1, q_2, \cdots, q_n; p_1, p_2, \cdots, p_n; t)$ is the Hamiltonian function (in which time appears explicitly) of the system with n generalized coordinates $q_1, q_2, \cdots, q_n$ and the conjugate momenta $p_1, p_2, \cdots, p_n$.

Solving the Hamilton-Jacobi equation

$$\frac{\partial \hat{W}}{\partial t} + H\left(q_1, q_2, \cdots, q_n; \frac{\partial \hat{W}}{\partial q_1}, \frac{\partial \hat{W}}{\partial q_2}, \cdots, \frac{\partial \hat{W}}{\partial q_n}\right) = 0,$$

where $\hat{W}$ is Hamilton's characteristic function, is equivalent to solving the Hamiltonian equations of motion (Equation 1). If it is possible to separate the variables in the Hamilton-Jacobi equation, then the solution can always be reduced to quadratures.

Vinti's dynamical system belongs to a class of systems which are scleronomic, conservative, and holonomic. Furthermore, it belongs to a class of dynamical systems which are said to be of Stäckel's type. The separability properties of the Hamilton-Jacobi equation of the form solved by Vinti follow from certain conditions determined by Stäckel. The separability of the variables occurs only in certain coordinate systems.

The oblate spheroidal coordinate system is related to the geocentric rectangular coordinate system by

$$x = \sqrt{(\rho^2 + c^2)}\sqrt{1 - \sigma^2}\cos\alpha ,$$

$$y = \sqrt{\rho^2 + c^2}\sqrt{1 - \sigma^2}\sin\alpha ,$$

$$z = \rho\sigma ,$$

$$r = \sqrt{\rho^2 + c^2(1 - \sigma^2)} ,$$

where x, y, and z are rectangular coordinates; r is the geocentric distance of the satellite; ρ, σ, and α are the coordinates in the oblate spheroidal system; and c is a constant defined by Vinti's expression

$$c^2 = J_2 a_E^2 , \tag{2}$$

where J_2 is the coefficient of the second-degree Legendre polynomial in the earth's force function F. The quantity F is expressed as

$$F = +\frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n\left(\frac{a_E}{r}\right)^n P_n(\sin\delta)\right] ,$$

where δ is declination of the satellite, a_E is the earth's equatorial radius, and μ is the product GM where G is the gravitational constant and M is the earth's mass.

The potential which Vinti obtained in oblate spheroidal coordinates is

$$\hat{V} = -\frac{\mu\rho}{\rho^2 + c^2\sigma^2} .$$

Similarly, the Hamiltonian and Lagrangian are

$$H = \frac{1}{2}U^2 - \frac{\mu\rho}{\rho^2 + c^2\sigma^2} ,$$

$$L = \frac{1}{2}U^2 + \frac{\mu\rho}{\rho^2 + c^2\sigma^2} ,$$

where the speed of the satellite U is found from

$$U^2 = \frac{\rho^2 + c^2\sigma^2}{\rho^2 + c^2}\dot{\rho}^2 + \frac{\rho^2 + c^2\sigma^2}{1 - \sigma^2}\dot{\sigma}^2 + \left(\rho^2 + c^2\right)\left(1 - \sigma^2\right)\dot{\alpha}^2 .$$

The generalized momenta are defined by

$$\left.\begin{aligned} P_\rho &= \frac{\partial L}{\partial \dot{\rho}} , \\ P_\sigma &= \frac{\partial L}{\partial \dot{\sigma}} , \\ P_\alpha &= \frac{\partial L}{\partial \dot{\alpha}} , \end{aligned}\right\}$$

The Hamiltonian does not contain the time explicitly, so the Hamilton-Jacobi equation is

$$\frac{1}{2\left(\rho^2 + c^2\sigma^2\right)}\left[\left(\rho^2 + c^2\right)\left(\frac{\partial \hat{W}}{\partial \rho}\right)^2 + \left(1 - \sigma^2\right)\left(\frac{\partial \hat{W}}{\partial \sigma}\right)^2 + \left(\frac{1}{1 - \sigma^2} - \frac{c^2}{\rho^2 + c^2}\right)\left(\frac{\partial \hat{W}}{\partial \alpha}\right)^2\right] - \frac{\mu\rho}{\rho^2 + c^2\sigma^2} = \hat{h} ,$$

where, in the limit as $c^2 \to 0$ of Keplerian motion, $\hat{h}$ is the total energy in the orbit; $\hat{h}$ is always negative.

The implicit equations of motion for the determination of ρ, σ, and α are (Reference 1)

$$\left.\begin{aligned} \frac{\partial \hat{W}}{\partial \hat{h}} &= \int_{\rho_1}^{\rho} \frac{\rho^2\, d\rho}{\sqrt{P(\rho)}} + c^2 \int_0^{\sigma} \frac{\sigma^2\, d\sigma}{\sqrt{Q(\sigma)}} = t - \hat{t} , \\ \frac{\partial \hat{W}}{\partial \hat{c}} &= -\hat{c} \int_{\rho_1}^{\rho} \frac{d\rho}{\sqrt{P(\rho)}} + \hat{c} \int_0^{\sigma} \frac{d\sigma}{\sqrt{Q(\sigma)}} = \hat{\omega} , \\ \frac{\partial \hat{W}}{\partial \hat{G}} &= c^2 \hat{G} \int_{\rho_1}^{\rho} \frac{d\rho}{\left(\rho^2 + c^2\right)\sqrt{P(\rho)}} - \hat{G} \int_0^{\sigma} \frac{d\sigma}{\left(1 - \sigma^2\right)\sqrt{Q(\sigma)}} + \alpha = \hat{\Omega} , \end{aligned}\right\}$$

where

$$
\left.\begin{aligned}
P(\rho) &= 2\hat{h}\,\rho^4 + 2\mu\,\rho^3 - \left(\hat{c}^2 - 2c^2\,\hat{h}\right)\rho^2 + 2c^2\mu p - c^2\left(\hat{c}^2 - \hat{G}^2\right) , \\
Q(\sigma) &= -2c^2\,\hat{h}\sigma^4 - \left(\hat{c}^2 - 2c^2\,\hat{h}\right)\sigma^2 + \left(\hat{c}^2 - \hat{G}^2\right) .
\end{aligned}\right\} \tag{3}
$$

The symbols $\hat{h}$, $\hat{c}$, $\hat{G}$, $-\hat{t}$, $\hat{\omega}$, and $\hat{\Omega}$ are a canonical set of constants of integration. In the limit as $c^2 \to 0$ of Keplerian motion, the canonical constants have the following meanings:

$\hat{h}$ total energy in the orbit, $\hat{h}$ is always negative;

$\hat{c}$ total angular momentum;

$\hat{G}$ z component of the angular momentum, $\hat{G}$ is positive or negative according as the motion is direct or retrograde;

$-\hat{t}$ time of perigee passage;

$\hat{\omega}$ argument of perigee; and

$\hat{\Omega}$ right ascension of the ascending node.

Exact expressions for three canonical constants α_1, α_2, and α_3, denoted by Izsak as $\hat{h}$, $\hat{c}$, and $\hat{G}$, respectively, are determined from initial values of the coordinates and their derivatives (Reference 4). Numerical values of these α's are used to determine a certain set of orbital constants a_0, e_0, and i_0 (the initial values of the semimajor axis of the orbit, the eccentricity of the orbit, and the angle of inclination, respectively). These are used to find the ρ_1, ρ_2, A, and B (the perigee of the orbit, the apogee of the orbit, and coefficients in the quartic polynomial $F(\rho)$ — see Appendix B — respectively) necessary to factor

$$
F(\rho) = -2\alpha_1 \left(\rho - \rho_1\right)\left(\rho_2 - \rho\right)\left(\rho^2 + A\rho + B\right) , \tag{4}
$$

where $\rho_1 = a(1-e)$ and $\rho_2 = a(1+e)$. Similarly, this same quartic designated by Izsak as $P(\rho)$, Equation 3, can be factored into a form which is equivalent to that of $F(\rho)$, Equation 4. That is,

$$
P(\rho) = -2\hat{h}\left(\rho_2 - \rho\right)\left(\rho - \rho_1\right)\left[\left(\rho - a\kappa\right)^2 + a^2\lambda^2\right] ,
$$

and we find that

$$
\rho^2 + A\rho + B = (\rho - a\kappa)^2 + a^2\lambda^2 .
$$

The values for $\rho_1 + \rho_2$, $\rho_1 \rho_2$, A, and B are determined initially by solving the following system of nonlinear equations:

$$\left.\begin{aligned}
\rho_1 + \rho_2 - A &= -\mu \alpha_1^{-1} = 2a_0 , \\
B + \rho_1 \rho_2 - (\rho_1 + \rho_2) A &= c^2 - \frac{1}{2} \alpha_2^2 \alpha_1^{-1} = c^2 + a_0 p_0 , \\
(\rho_1 + \rho_2) B - \rho_1 \rho_2 A &= -\mu \alpha_1^{-1} c^2 = 2a_0 c^2 , \\
\rho_1 \rho_2 B &= -\frac{1}{2} c^2 \left(\alpha_2^2 - \alpha_3^2\right) \alpha_1^{-1} = a_0 p_0 c^2 \sin^2 i_0 .
\end{aligned}\right\}$$

Vinti (Reference 4) has given a second-order solution of this system by a method of successive approximations. However, if higher order accuracy is desired, it is first necessary to obtain additional terms in the series solutions; this is a laborious task. A numerical method to obtain the solution is given in the next section.

NEWTON-RAPHSON ITERATION SCHEME

The solution of a set of nonlinear algebraic equations usually involves a great deal more work than that needed for linear systems. When n, the number of equations, is large, the solution of linear systems entails considerable computation time even on high-speed computers; the solution of nonlinear systems may often be almost prohibitive.

The Newton-Raphson method (References 5, 6, and 7) can easily be applied when a solution is required for only a few equations.

To solve a system of nonlinear equations such as

$$(\rho_1 + \rho_2) - A = 2a_0 , \tag{5}$$

$$B + \rho_1 \rho_2 - (\rho_1 + \rho_2) A = c^2 + a_0 p_0 , \tag{6}$$

$$(\rho_1 + \rho_2) B - \rho_1 \rho_2 A = 2a_0 c^2 , \tag{7}$$

$$\rho_1 \rho_2 B = a_0 p_0 c^2 \sin^2 i_0 , \tag{8}$$

with unknowns $(\rho_1 + \rho_2)$, $\rho_1\rho_2$, A, and B by the Newton-Raphson iteration we begin with a trial vector

$$X^{(k+1)} = X^{(k)} - J^{-1} F\left(X^{(k)}\right) ,$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \rho_1 + \rho_2 \\ \rho_1 \rho_2 \\ A \\ B \end{pmatrix} = \begin{pmatrix} 2a_0 \\ a_0 p_0 \\ 0 \\ 0 \end{pmatrix} . \tag{9}$$

Denoting Equations 5, 6, 7, and 8 by f_1, f_2, f_3, and f_4, respectively, that is,

$$f_1(x_1, x_2, x_3, x_4) = (\rho_1 + \rho_2) - A - 2a_0 ,$$

$$f_2(x_1, x_2, x_3, x_4) = B + \rho_1 \rho_2 - (\rho_1 + \rho_2) A - c^2 - a_0 p_0 ,$$

$$f_3(x_1, x_2, x_3, x_4) = (\rho_1 + \rho_2) B - \rho_1 \rho_2 A - 2a_0 c^2 ,$$

$$f_4(x_1, x_2, x_3, x_4) = \rho_1 \rho_2 B - a_0 p_0 c^2 \sin^2 i_0 ,$$

we introduce the usual Jacobian matrix

$$J \equiv \left(\frac{\partial (f_1, f_2, f_3, f_4)}{\partial (x_1, x_2, x_3, x_4)}\right) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -A & 1 & -(\rho_1 + \rho_2) & 1 \\ B & -A & -\rho_1 \rho_2 & \rho_1 + \rho_2 \\ 0 & B & 0 & \rho_1 \rho_2 \end{pmatrix} . \tag{10}$$

It will be noted that for the initial vector, the Jacobian determinant can be written

$$|J| = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2a_0 & 1 \\ 0 & 0 & -a_0 p_0 & 2a_0 \\ 0 & 0 & 0 & a_0 p_0 \end{vmatrix} = -a_0^2 p_0^2 = -a_0^4 \left(1 - e_0^2\right)^2 .$$

The condition that $|J|$ is not close to zero will be satisfied provided e_0 is not close to unity.

Next, we determine the exact inverse of Equation 10; only the final result is given here:

$$J^{-1} = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{pmatrix},$$

where

$$Z_{11} = 1 - \left[\frac{(\rho_1 \rho_2 - B)(B - A^2) - BA(A + \rho_1 + \rho_2)}{\Delta} \right],$$

$$Z_{12} = \left[\frac{\rho_1 \rho_2 A + (\rho_1 + \rho_2)B}{\Delta} \right],$$

$$Z_{13} = \frac{\rho_1 \rho_2 - B}{\Delta},$$

$$Z_{14} = -\frac{A + \rho_1 + \rho_2}{\Delta},$$

$$Z_{21} = A - \left\{ \frac{(\rho_1 \rho_2)(A + \rho_1 + \rho_2)(B - A^2) - BA(A + \rho_1 + \rho_2)^2 + BA\left[(\rho_1 \rho_2 - B) + A(A + \rho_1 + \rho_2)\right]}{\Delta} \right\},$$

$$Z_{22} = 1 - \left\{ \frac{(\rho_1 \rho_2 - B)\left[-A(A + \rho_1 + \rho_2) + B\right] - B(A + \rho_1 + \rho_2)^2}{\Delta} \right\},$$

$$Z_{23} = -(\rho_1 \rho_2) Z_{14} = \frac{(\rho_1 \rho_2)(A + \rho_1 + \rho_2)}{\Delta},$$

$$Z_{24} = \frac{-(A + \rho_1 + \rho_2)^2 + (\rho_1 \rho_2 - B) + A(A + \rho_1 + \rho_2)}{\Delta},$$

$$Z_{31} = (Z_{11} - 1) = -\left[\frac{(\rho_1 \rho_2 - B)(B - A^2) - BA(A + \rho_1 + \rho_2)}{\Delta} \right],$$

$$Z_{32} = Z_{12} = \frac{(\rho_1 \rho_2)A + (\rho_1 + \rho_2)B}{\Delta},$$

$$Z_{33} = Z_{13} = \frac{(\rho_1 \rho_2 - B)}{\Delta},$$

$$Z_{34} = Z_{14} = \frac{-(A + \rho_1 + \rho_2)}{\Delta},$$

$$Z_{41} = BZ_{32} = \frac{B^2(\rho_1 + \rho_2) + BA(\rho_1 \rho_2)}{\Delta},$$

$$Z_{42} = BZ_{13} = \frac{B(\rho_1 \rho_2 - B)}{\Delta} ,$$

$$Z_{43} = BZ_{34} = \frac{-B(A + \rho_1 + \rho_2)}{\Delta} ,$$

$$Z_{44} = -Z_{33} + AZ_{34} = -\left[\frac{(\rho_1 \rho_2 - B) + A(A + \rho_1 + \rho_2)}{\Delta}\right] ,$$

where

$$\Delta = -\left[(\rho_1 \rho_2 - B)^2 + A(\rho_1 \rho_2 - B)(A + \rho_1 + \rho_2) + B(A + \rho_1 + \rho_2)^2\right] .$$

The Newton-Raphson iteration can now be written

$$\begin{pmatrix} X_1^{(k+1)} \\ X_2^{(k+1)} \\ X_3^{(k+1)} \\ X_4^{(k+1)} \end{pmatrix} = \begin{pmatrix} X_1^{(k)} \\ X_2^{(k)} \\ X_3^{(k)} \\ X_4^{(k)} \end{pmatrix} - \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} .$$

A solution will have been obtained when

$$\operatorname*{Max}_i \left| \left(X_i^{(k+1)} - X_i^{(k)} \right) \right| \leq \epsilon ,$$

where ϵ is any tolerance sufficiently small to obtain the degree of accuracy desired.

The first-order solution through k_0 is obtained in one iteration by beginning with the zero-order solution of Equation 9. It is expected that four iterations will be sufficient to obtain the solution through $O(k_0^3)$.

Table 1 gives the solution to the system as computed on the IBM 7090 using a single precision floating point Fortran program. The solution of Equations 5 through 8 was obtained accurately to seven significant digits in four iterations with $\epsilon = 10^{-7}$.

The initial values of the unknowns, together with other necessary constants were computed from orbital data of Explorer XI (1961 ν).

Table 1

Numerical Results Obtained with the Initial Conditions

$$\rho_1 + \rho_2 = 2a_0 = 15.0588664,$$

$$\rho_1\rho_2 = a_0 p_0 = 56.2660106,$$

$$A = 0.0,$$

$$B = 0.0,$$

$$c^2 = J_2 a_E^2 = 0.044029034,$$

$$\sin i_0 = 0.474484778,$$

$$\epsilon = 1.0 \times 10^{-7}.$$

Iteration Number	$\rho_1 + \rho_2$	$\rho_1\rho_2$	A	B
1	15.0497347	56.1626263	-0.00913084351	+0.00991251186
2	15.0497213	56.1624885	-0.00914439194	+0.00993078329
3	15.0497213	56.1624877	-0.00914439194	+0.00993078336
4	15.0497213	56.1624877	-0.00914439209	+0.00993078336

We immediately obtain the values of the elements a and e from

$$a = \frac{\rho_1 + \rho_2}{2},$$

$$e = \sqrt{1 - \frac{4\rho_1 \rho_2}{(\rho_1 + \rho_2)^2}} .$$

DETERMINATION OF CANONICAL CONSTANTS FROM INITIAL CONDITIONS

If the initial conditions (denoted by zero subscripts) t_0, x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, and $\dot{z}_0$ are known, we can determine a complete set of canonical constants a, e, S, $-\hat{t}$, $\hat{\Omega}$, and ω essential to Izsak's second-order solution. The canonical constants have the following meanings:

- a semimajor axis of the orbit,
- e eccentricity of the orbit,
- S sine of the inclination of the orbit,
- $-\hat{t}$ in the limit as $c^2 \to 0$ of Keplerian motion $-\hat{t}$ is the time of perigee passage,
- $\hat{\Omega}$ in the limit as $c^2 \to 0$ of Keplerian motion $\hat{\Omega}$ is the right ascension of the ascending node,
- ω a constant of integration.

We first give the method of determining a, e, and S. The following expressions are computed:

$$r_0^2 = x_0^2 + y_0^2 + z_0^2 ,$$

$$v_0^2 = \dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 ,$$

$$\rho_0^2 = \frac{r_0^2 - c^2}{2}\left[1 + \sqrt{1 + \left(\frac{2cz_0}{r_0^2 - c^2}\right)^2}\right], \tag{11}$$

$$\sigma_0^2 = \frac{2z_0^2}{r_0^2 - c^2}\left[1 + \sqrt{1 + \left(\frac{2cz_0}{r_0^2 - c^2}\right)^2}\right]^{-1} ,$$

$$r_0 \dot{r}_0 = x_0 \dot{x}_0 + y_0 \dot{y}_0 + z_0 \dot{z}_0 ,$$

$$\dot{\rho}_0 = \frac{\rho_0 r_0 \dot{r}_0 + c^2 \sigma_0 \dot{z}_0}{\rho_0^2 + c^2 \sigma_0^2} ,$$

$$\dot{\sigma}_0 = \frac{-\sigma_0 r_0 \dot{r}_0 + \rho_0 \dot{z}_0}{\rho_0^2 + c^2 \sigma_0^2} ,$$

where σ_0 takes the sign of z_0. Next, the following expressions are computed:

$$\alpha_1 = \frac{1}{2} v_0^2 - \frac{\mu \rho_0}{\left(\rho_0^2 + c^2 \sigma_0^2\right)} , \quad \alpha_1 < 0 ,$$

$$\alpha_3 = x_0 \dot{y}_0 - y_0 \dot{x}_0 ,$$

$$\alpha_2^2 = \frac{\alpha_3^2 + \left(-\sigma_0 r_0 \dot{r}_0 + \rho_0 \dot{z}_0\right)^2}{1 - \sigma_0^2} - 2\alpha_1 c^2 \sigma_0^2 ,$$

$$y^2 = \frac{\alpha_3^2}{\alpha_2^2} = \cos^2 i_0 ,$$

$$\sqrt{1 - y^2} = \sin i_0 ,$$

$$P_0 = \frac{\alpha_2^2}{\mu},$$

$$K_0 = \frac{c^2}{P_0^2},$$

$$\rho_1 + \rho_2 = 2P_0 x^{-2}\left[1 - K_0 x^2 y^2 - K_0^2 x^2 y^2 \left(2x^2 - 3x^2 y^2 - 4 + 8y^2\right) + \cdots\right],$$

$$\rho_1 \rho_2 = P_0{}^2 x^{-2}\left[1 + K_0 y^2 \left(x^2 - 4\right) - K_0^2 y^2 \left(12x^2 - x^4 - 20x^2 y^2 - 16 + 32y^2 + x^4 y^2\right) + \cdots\right],$$

$$a = \frac{1}{2}\left(\rho_1 + \rho_2\right),$$

$$1 - e^2 = \frac{4\rho_1 \rho_2}{\left(\rho_1 + \rho_2\right)^2}, \tag{12}$$

$$e = \frac{\left(\rho_2 - \rho_1\right)}{\rho_1 + \rho_2} = \sqrt{1 - \left(1 - e^2\right)},$$

$$\eta_0 = \left(\sin i_0\right)\left[1 - \frac{1}{2} K_0 x^2 y^2 + \frac{1}{8} K_0^2 x^4 y^2 \left(7y^2 - 4\right) + \cdots\right],$$

where $\eta_0 \approx \sin I = S$. Alternately η_0 may be computed from

$$\eta_0^{-2} = \frac{\alpha_2^2 - 2\alpha_1 c^2}{2\left(\alpha_2^2 - \alpha_3^2\right)}\left[1 + \sqrt{1 + \frac{8\alpha_1 c^2 \left(\alpha_2^2 - \alpha_3^2\right)}{\left(\alpha_2^2 - 2\alpha_1 c^2\right)^2}}\right].$$

We now have determined Izsak's elements a, e, and S from initial conditions; these elements are accurate through $O\left(k_0^2\right)$ and are used as input to the orbit computation procedure.

We are now ready to determine the remaining canonical constants - $\hat{t}$, $\hat{\Omega}$, and ω. We set $\phi = E = 0$ whenever $e = 0$. If $e \neq 0$, we determine E from

$$E = \tan^{-1}\left[\frac{\rho_0 r_0 \dot{r}_0 + c^2 \sigma_0 \dot{z}_0}{\sqrt{-2\hat{h}}\ \sqrt{\left(\rho - a\kappa\right)^2 + a^2 \lambda^2}\ \left(a - \rho_0\right)}\right],$$

since

$$\cos E = \frac{a - \rho_0}{ae},$$

$$\sin E = \frac{\rho_0 r_0 \dot{r}_0 + c^2 \sigma_0 \dot{z}_0}{ae\sqrt{-2\hat{h}}\sqrt{(\rho - a\kappa)^2 + a^2\lambda^2}},$$

where ρ_0^2 is given by Equation 11, c is given by Equation 2, and

$$a^2\lambda^2 = a^2\left[\nu^2 S^2 - \frac{\nu^4}{(1-e^2)^2}(1-S^2)(1-5S^2) + \cdots\right], \tag{13}$$

$$\kappa = \frac{\nu^2(1-S^2)}{1-e^2} + \frac{\nu^4(1-S^2)}{(1-e^2)^3}(1-4S^2-e^2) + \cdots, \tag{14}$$

$$-2\hat{h} = \frac{\mu}{a(1+\kappa)}. \tag{15}$$

The angle ϕ is completely determined by

$$\cos\phi = \frac{\cos E - e_*}{1 - e_* \cos E}, \tag{16}$$

$$\sin\phi = \frac{\sqrt{1-e_*^2}\,\sin E}{1 - e_* \cos E}, \tag{17}$$

where e_* is given by

$$e_* = e\left\{1 + \frac{\nu^2}{1-e^2}(1-2S^2) + \frac{\nu^4}{(1-e^2)^3}\left[(3 - 16S^2 + 14S^4) - 2(1-S^2)^2 e^2\right] + \cdots\right\}. \tag{18}$$

We next determine the angle ψ from

$$\sin\psi = \frac{\sigma_0}{S}, \quad S \neq 0, \tag{19}$$

$$\cos\psi = \frac{-\sigma_0 r_0 \dot{r}_0 + \rho_0 \dot{z}_0}{a\sqrt{-2\hat{h}}\sqrt{1-e^2}\sqrt{\frac{\kappa^2+\lambda^2}{\nu^2}}\sqrt{1-\ell^2\sin^2\psi}}, \tag{20}$$

where $(1 - e^2)$ is given by Equation 12 and

$$\ell^2 = \frac{\nu^2 S^2}{1 - e^2} - \frac{4\nu^4 S^2}{(1-e^2)^3}(1 - S^2) + \cdots , \tag{21}$$

$$\frac{\kappa^2 + \lambda^2}{\nu^2} = S^2 + \frac{4\nu^2 S^2}{(1-e^2)^2}(1 - S^2) + \frac{4\nu^4 S^2}{(1-e^2)^4}(1 - S^2)\left[(1 - 3S^2) - (1 + S^2)e^2\right] + \cdots . \tag{22}$$

Next we compute a "mean anomaly" $\hat{M}$ from

$$\hat{M} = E - K_1 e \sin E - K_2 \phi - K_3 \sin\phi + K_4 \sin 2\phi + K_5 \psi - K_6 \sin 2\psi + K_7 \sin 4\psi ,$$

where

$$K_1 = 1 - \frac{\nu^2 (1 - S^2)}{1 - e^2} + \frac{\nu^4 (1 - S^2)}{(1-e^2)^3} S^2 (3 + e^2) , \tag{23}$$

$$K_2 = \frac{\nu^2 \sqrt{1 - e^2}}{2(1 - e)} S^2 - \frac{\nu^4 \sqrt{1 - e^2}}{16(1-e^2)^3}\left[(24 - 96S^2 + 78S^4) - (8 - 11S^2) S^2 e^2\right] , \tag{24}$$

$$K_3 = \frac{\nu^4 \sqrt{1 - e^2}}{4(1-e^2)^3}(4 - 5S^2) S^2 e , \tag{25}$$

$$K_4 = \frac{3\nu^4 \sqrt{1 - e^2}\, S^4 e^2}{32(1-e^2)^3} , \tag{26}$$

$$K_5 = \left\{\frac{\nu^2 \sqrt{1 - e^2}}{2(1-e^2)} - \frac{\nu^4 \sqrt{1 - e^2}}{16(1-e^2)^3}\left[(24 - 27S^2) - (8 - 11S^2)e^2\right]\right\} S^2 . \tag{27}$$

$$K_6 = \left\{\frac{\nu^2 \sqrt{1 - e^2}}{4(1-e^2)} - \frac{\nu^4 \sqrt{1 - e^2}}{8(1-e^2)^3}\left[(6 - 7S^2) - (2 - 3S^2)e^2\right]\right\} S^2 , \tag{28}$$

$$K_7 = \frac{\nu^4 \sqrt{1 - e^2}}{64(1-e^2)^2} S^4 . \tag{29}$$

We can now compute $-\hat{t}$ as follows:

$$-\hat{t} = \frac{\hat{M}}{\hat{n}} - t ,$$

where

$$\hat{n} = \sqrt{\frac{\mu}{a^3}}\left\{1 - \frac{3\nu^2(1-S^2)}{2(1-e^2)} + \frac{3\nu^4(1-S^2)}{8(1-e^2)^3}\left[(1+11S^2) - (1-5S^2)e^2\right] - \cdots\right\} . \tag{30}$$

The right ascension of the satellite α is determined from

$$\cos\alpha = \frac{x_0}{\sqrt{\rho_0^2 + c^2}\ \sqrt{1-\sigma_0^2}} ,$$

$$\sin\alpha = \frac{y_0}{\sqrt{\rho_0^2 + c^2}\ \sqrt{1-\sigma_0^2}} .$$

When the right ascension α is known, the right ascension of the ascending node $\hat{\Omega}$ is computed from

$$\hat{\Omega} = \alpha - \tan^{-1}\left(\sqrt{1-S^2}\tan\psi\right) + R_1\psi - R_2\sin 2\psi + R_3\phi$$

$$+ R_4\sin\phi + R_5\sin 2\phi - R_6\sin 3\phi - R_7\sin 4\phi ,$$

where

$$R_1 = \frac{\nu^2\sqrt{1-S^2}}{2(1-e^2)} - \frac{\nu^4\sqrt{1-S^2}}{16(1-e^2)^3}\left[(30-35S^2) + (2+3S^2)e^2\right] , \tag{31}$$

$$R_2 = \frac{3\nu^4\sqrt{1-S^2}}{32(1-e^2)^2} S^2 , \tag{32}$$

$$R_3 = \frac{\nu^2\sqrt{1-S^2}}{2(1-e^2)^2}(2+e^2) + \frac{\nu^4\sqrt{1-S^2}}{16(1-e^2)^4}\left[(24-56S^2) - (4+64S^2)e^2 - (2+3S^2)e^4\right] , \tag{33}$$

$$R_4 = \left\{\frac{2\nu^2\sqrt{1-S^2}}{(1-e^2)^2} + \frac{\nu^4\sqrt{1-S^2}}{4(1-e^2)^4}\left[(4-28S^2) - (6+7S^2)e^2\right]\right\} e , \tag{34}$$

$$R_5 = \left\{\frac{\nu^2\sqrt{1-S^2}}{4(1-e^2)^2} - \frac{\nu^4\sqrt{1-S^2}}{8(1-e^2)^4}\left[11 + (1+S^2)e^2\right]\right\} e^2 , \tag{35}$$

$$R_6 = \frac{\nu^4 \sqrt{1 - S^2}}{4\left(1 - e^2\right)^4}\left(2 - S^2\right) e^3 , \tag{36}$$

$$R_7 = \frac{\nu^4 \sqrt{1 - S^2}}{64\left(1 - e^2\right)^4}\left(2 + S^2\right) e^4 . \tag{37}$$

We next compute W and V, which are analogous to the argument of latitude in Keplerian motion and the true anomaly in Keplerian motion, respectively, from

$$W = \psi - M_1 \sin 2\psi + 3M_2 \sin 4\psi ,$$

and

$$V = \phi + L_1 \sin 2\phi + L_2 \sin 4\phi , \tag{38}$$

where

$$M_1 = \frac{\ell^2}{8}\left(1 + \frac{\ell^2}{2}\right) ,$$

$$M_2 = \frac{\ell^4}{256} ,$$

$$L_1 = \frac{-k^2}{8}\left(1 + \frac{k^2}{2}\right) ,$$

$$L_2 = \frac{3k^4}{256} .$$

The mean argument of perigee $\bar{\omega}$ is given by

$$\bar{\omega} = W - V .$$

The constant of integration ω is given by

$$\omega = W - \left(1 + \epsilon\right)V ,$$

where

$$\epsilon = \frac{\nu^2}{4\left(1 - e^2\right)^2}\left(12 - 15S^2\right) + \frac{\nu^4}{64\left(1 - e^2\right)^4}\left[\left(288 - 1296S^2 + 1035S^4\right) - \left(144 + 288S^2 - 510S^4\right)e^2\right] + \cdots . \tag{39}$$

ORBIT COMPUTATION PROCEDURE

With the exception of the velocity formulation, the computational procedure developed here makes use of the unmodified expressions of Izsak (Reference 1).

Input

The 11 inputs are: a, e, S, $-\hat{t}$, $\hat{\Omega}$, ω, J_2, a_E, μ, Δt, and T_f. The six inputs a, e, S, $-\hat{t}$, $\hat{\Omega}$, and ω are constants of integration (see definitions on page 10). The other inputs have the following meanings:

J_2 the coefficient of the second-degree Legendre polynomial in the earth's gravitational potential,

a_E the earth's equatorial radius,

μ the product GM where G is the gravitational constant and M is the earth's mass,

Δt time interval of integration,

T_f final time.

The following values of μ, J_2, and a_E determined by W. M. Kaula (Reference 8) were used in the computations:

$\mu = 3.986032 \times 10^2$ megameters^3 ksec^{-2},

$J_2 = 1.0823 \times 10^{-3}$,

$a_E = 6.378165$ megameters.

Equations and Fundamental Constants

From Vinti's expression (Equation 2) and the input constants determined by Kaula, we have $c = 0.20983097$ megameters. In addition to $\alpha^2\lambda^2$ (Equation 13), κ (Equation 14), $-2h$ (Equation 15), $\cos\phi$ (Equation 16), $\sin\phi$ (Equation 17), e_* (Equation 18), ℓ^2 (Equation 21), $(\kappa^2+\lambda^2)/\nu^2$ (Equation 22), $\hat{n}$ (Equation 30), V (Equation 38), and ϵ (Equation 39), the following equations are used in the computation:

$$\nu^2 = \frac{c^2}{a^2},$$

$$k^2 = \frac{\nu^2 e^2}{(1-e^2)^2} S^2 - \frac{\nu^4 e^2}{(1-e^2)^4}\left[\left(1 - 10S^2 + 11S^4\right) + S^4 e^2\right] + \cdots ,$$

and the generalized Kepler equation

$$E - K_1 e \sin E = \hat{n}(t - \hat{t}) + K_2 \phi + K_3 \sin\phi - K_4 \sin 2\phi - K_5 \psi + K_6 \sin 2\psi - K_7 \sin 4\psi \cdot \quad (40)$$

where the K_i are given by Equations 23 through 29.

The right ascension α is computed from

$$\alpha = \hat{\Omega} + \tan^{-1}\left(\sqrt{1 - S^2}\tan\psi\right) - R_1 \psi + R_2 \sin 2\psi - R_3 \phi - R_4 \sin\phi$$

$$- R_5 \sin 2\phi + R_6 \sin 3\phi + R_7 \sin 4\phi ,$$

where the R_i are given by Equations 31 through 37.

The argument of latitude ψ is computed from the following equations:

$$W = (1 + \epsilon)V + \omega ,$$

$$\psi = W + M_1 \sin 2W + M_2 \sin 4W + \cdots .$$

The mean argument of perigee $\bar{\omega}$ is computed from

$$\bar{\omega} = \epsilon V + \omega .$$

The anomalistic mean motion n_ϕ is computed from

$$n_\phi = \hat{n}\left[1 - \frac{3\nu^4 \sqrt{1 - e^2}}{16\left(1 - e^2\right)^3}\left(8 - 32S^2 + 25S^4\right) + \cdots\right] .$$

The motion of the node η is computed from

$$\eta = -\frac{3\nu^2 \sqrt{1 - S^2}}{2\left(1 - e^2\right)^2} + \frac{3\nu^4 \sqrt{1 - S^2}}{16\left(1 - e^2\right)^4}\left[\left(18 - 13S^2\right) + 24S^2 e^2\right] - \cdots .$$

The oblate spheroidal coordinate σ is computed from

$$\sigma = s \sin\psi .$$

The z component of the angular momentum $\hat{G}$ is computed from

$$\hat{G}^2 = -2\hat{h}a^2\left(1 - S^2\right)\left[\left(1 - e^2\right)\left(\frac{\kappa^2 + \lambda^2}{\nu^2 S^2}\right) - \nu^2\right].$$

The oblate-spheroidal coordinate ρ is computed from

$$\rho = a\left(1 - e\cos E\right).$$

Initially for $t = t_0$ we start with values ϕ and ψ determined from Equations 16, 17, 19, and 20 to solve the generalized Kepler equation given by Equation 40 using a Newton-Raphson iteration scheme. We test $|E(\phi_{i+1}, \psi_{i+1}) - E(\phi_i, \psi_i)| < \epsilon$, where $\epsilon > 0$ was chosen to be 10^{-7}. In general, only two or three iterations are required before sufficiently accurate values of E, ϕ, and ψ are obtained. The oblate-spheroidal coordinates ρ, σ, and α are then computed; ρ, σ, and α are then used to calculate x, y, and z.

OUTPUT

This program generates position and velocity for equally spaced intervals of time. Oblate-spheroidal coordinates are defined by the equations

$$x = \sqrt{\rho^2 + c^2}\sqrt{1 - \sigma^2}\cos\alpha,$$

$$y = \sqrt{\rho^2 + c^2}\sqrt{1 - \sigma^2}\sin\alpha,$$

$$z = \rho\sigma,$$

$$r = \sqrt{\rho^2 + c^2\left(1 - \sigma^2\right)}.$$

The formulas for velocity are given in Appendix B, they are

$$\dot{x} = -\dot{\alpha}y + x\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\sigma\dot{\sigma}}{1 - \sigma^2}\right),$$

$$\dot{y} = +\dot{\alpha}x + y\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\sigma\dot{\sigma}}{1 - \sigma^2}\right),$$

$$\dot{z} = \rho\dot{\sigma} + \sigma\dot{\rho},$$

where

$$\dot{\rho} = \frac{\sqrt{-2\hat{h}}}{\rho^2 + c^2\sigma^2}\, ae \left(\sqrt{(\rho - a\kappa)^2 + a^2\lambda^2}\right)\sin E ,$$

$$\dot{\sigma} = \frac{\sqrt{-2\hat{h}}}{\rho^2 + c^2\sigma^2}\, a\sqrt{1 - e^2}\sqrt{\frac{\kappa^2 + \lambda^2}{\nu^2}}\sqrt{1 - \ell^2\sin^2\psi}\,\cos\psi ,$$

$$\dot{\alpha} = \frac{\hat{G}}{\left(\rho^2 + c^2\right)\left(1 - \sigma^2\right)} .$$

REMARKS

The computational procedure as it exists in this report was programmed by the author in single-precision floating-point Fortran for an IBM 7090 computer at the Goddard Space Flight Center. All machine results were compared with hand calculations and the practicality of the method was confirmed. The procedure is presently being compared with both single and double precision numerical integration.

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Appendix A

List of Symbols

A	coefficient in the quartic polynomial $F(\rho)$. See Appendix B.
a	canonical constant, one of Izsak's elements, semimajor axis of the orbit.
a_E	the earth's equatorial radius.
a_0	initial value of the canonical constant a.
B	coefficient in the quartic polynomial $F(\rho)$. See Appendix B.
c	a constant defined by Vinti's expression $c^2 = J_2 a_E^2$.
$\hat{c}$	a canonical constant; in the limit as $c^2 \to 0$ of Keplerian motion $\hat{c}$ is the total angular momentum.
E	angle corresponding to the eccentric anomaly.
$E(\phi_i, \psi_i)$	the ith value of the eccentric anomaly.
$E(\phi_{i+1}, \psi_{i+1})$	the $(i+1)$th value of the eccentric anomaly.
e	canonical constant, one of Izsak's elements, eccentricity of the orbit.
e_0	initial value of the canonical constant e.
e_*	second eccentricity.
F	the earth's force function.
$F(\rho)$	quartic polynomial fundamental to Vinti's theory.
f_1, f_2, f_3, f_4	representation of a set of four equations to be solved by the Newton-Raphson method.
G	the gravitational constant.
$\hat{G}$	a canonical constant; in the limit as $c^2 \to 0$ of Keplerian motion $\hat{G}$ is the z component of the angular momentum. $\hat{G}$ is positive or negative accordingly as the motion is direct or retrograde.

H the Hamiltonian.

$H(q_1, q_2, \cdots, q_n; p_1, p_2, \cdots, p_n; t)$ the Hamiltonian function (in which time appears explicitly) of a dynamical system of n degrees of freedom with n generalized coordinates $q_1, q_2, \cdots, q_n$ and the conjugate momenta $p_1, p_2, \cdots, p_n$.

$\hat{h}$ a canonical constant; in the limit as $c^2 \to 0$ of Keplerian motion $\hat{h}$ is the total energy in the orbit and always negative.

I one of Izsak's elements, inclination of the orbit.

i angle of inclination.

i_0 initial angle of inclination.

J the Jacobian matrix of the Newton-Raphson method.

$|J|$ the Jacobian determinant.

J_2 the coefficient of the second-degree Legendre polynomial in the earth's gravitational potential.

K_i notation used for the coefficients in Kepler equation.

K_0 the value $\frac{c^2}{P_0^2}$.

k modulus appearing in elliptic integral of the first kind.

ℓ modulus appearing in elliptic integral of the first kind.

L the Lagrangian.

M the earth's mass.

$\hat{M}$ "mean anomaly".

n_ϕ the anomalistic mean motion.

$\hat{n}$ a constant used in the generalized Kepler equation, the auxiliary mean motion.

p_0	the value $\frac{\alpha_2^2}{\mu}$.
$P(\rho)$	quartic polynomial fundamental to Vinti's theory.
p_ρ, p_σ, p_α	the generalized momenta.
$Q(\sigma)$	quartic polynomial fundamental to Vinti's theory.
R_i	notation used for the coefficients in the equation for right ascension of the ascending node.
r	the geocentric distance of the satellite.
r_0	the initial geocentric distance of the satellite.
S	canonical constant, one of Izsak's elements, sine of the inclination of the orbit.
T_f	final time.
t	time.
t_0	initial time.
Δt	time interval of integration.
$-\hat{t}$	a canonical constant; in the limit as $c^2 \to 0$ of Keplerian motion $-\hat{t}$ is the time of perigee passage.
U	the speed of the satellite.
V	a "true anomaly" analogous to that in Keplerian motion.
$\hat{V}$	the potential which Vinti obtained in oblate spheroidal coordinates.
v	velocity of the satellite.
v_0	initial velocity of the satellite.
W	"argument of latitude" analogous to that in Keplerian motion.
$\hat{W}$	Hamilton's characteristic function.
X	a trial vector for the solution of a set of nonlinear equations by the Newton-Raphson method.
x, y, z	coordinates in the rectangular system.
x_0, y_0, z_0	the initial values of the coordinates in the rectangular system.

$\dot{x}, \dot{y}, \dot{z}$	the velocity coordinates in the rectangular system.
$\dot{x}_0, \dot{y}_0, \dot{z}_0$	the initial value of the velocity coordinates in the rectangular system.
Z_{ij}	element of the inverse Jacobian matrix.
α	the right ascension of the satellite.
α_1, α_2, and α_3	Vinti's canonical constants denoted by Izsak as $\hat{h}$, $\hat{c}$, and $\hat{G}$ respectively.
δ	the declination of the satellite.
ϵ	the motion of perigee.
ϵ	an arbitrarily chosen small positive real number (used as a tolerance in the Newton-Raphson method).
η	the motion of the node.
$\eta_0 \approx \sin I = s$	
κ	a series used in the computation: defined by Equation 14.
μ	the product GM where G is the gravitational constant and M is the earth's mass.
ν^2	a dimensionless parameter of the order 10^{-3} in the case of the earth.
ρ, σ, α	coordinates in the oblate spheroidal system.
$\rho_0, \sigma_0, \alpha_0$	the initial condition of the coordinates in the oblate spheroidal system.
$\dot{\rho}, \dot{\sigma}, \dot{\alpha}$	the velocity coordinates in the oblate spheroidal system.
$\dot{\rho}_0, \dot{\sigma}_0, \dot{\alpha}_0$	the initial conditions of the velocity coordinates in the oblate spheroidal system.
ρ_1	perigee of the orbit.
ρ_2	apogee of the orbit.
ϕ	"true anomaly".
ψ	"argument of latitude".
$\hat{\Omega}$	a canonical constant; in the limit as $c^2 \to 0$ of Keplerian motion $\hat{\Omega}$ is the right ascension of the ascending node.

ω	a canonical constant, one of Izsak's constants, a constant of integration.
$\hat{\omega}$	a canonical constant; in the limit as $c^2 \to 0$ of Keplerian motion $\hat{\omega}$ is the argument of perigee.
$\bar{\omega}$	the mean argument of perigee.

Appendix B

Derivation of the Velocities in Vinti's Accurate Intermediary Orbit of an Artificial Satellite

Introduction

Izsak (Reference B1) has given an analytic solution for Vinti's intermediary orbit, with both periodic and secular terms correct through the second order in a certain oblateness parameter $\gamma = c/a$ (to be defined later). His solution giving the position vector of the satellite makes extensive use of Jacobian elliptic functions, linear transformations, and mappings in the complex plane. Vinti (Reference B2) also has given an analytic solution to this problem of satellite motion using rapidly converging infinite series instead of Jacobian elliptic functions. His solution not only gives the periodic terms correct to the second order, but also the secular terms to an arbitrarily high order.

This appendix presents the derivation of the velocity vector through the use of equations from both Vinti and Izsak. However, the orbital elements used in this derivation were introduced by Izsak.

Determination of Velocity

The oblate spheroidal coordinates ρ, σ, and α are defined by

$$x = \sqrt{\rho^2 + c^2} \quad \sqrt{1 - \sigma^2} \cos \alpha \ , \tag{B1}$$

$$y = \sqrt{\rho^2 + c^2} \quad \sqrt{1 - \sigma^2} \sin \alpha \ , \tag{B2}$$

$$z = \rho \sigma \ , \tag{B3}$$

$$r = \sqrt{\rho^2 + c^2 \left(1 - \sigma^2\right)} \ , \tag{B4}$$

where α is the right ascension of a satellite; r is the geocentric distance; and c is a constant defined by $c^2 = J_2 a_E^2$. The quantity J_2 is the coefficient of the second-degree Legendre polynomial in the earth's gravitational potential

$$V = -\frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{a_E}{r}\right)^n P_n(\sin\delta)\right] \tag{B5}$$

where δ is the declination of the satellite, a_E is the equatorial radius of the earth, and $\mu = GM$, where G is the gravitational constant and M the mass of the earth.

Differentiating Equations B1-B3 with respect to time we find

$$\dot{x} = -\dot{\alpha} y + x\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\sigma\dot{\sigma}}{1 - \sigma^2}\right), \tag{B6}$$

$$\dot{y} = +\dot{\alpha} x + y\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\sigma\dot{\sigma}}{1 - \sigma^2}\right), \tag{B7}$$

$$\dot{z} = \rho\dot{\sigma} + \sigma\dot{\rho} . \tag{B8}$$

Squaring and adding Equations B6-B8 we obtain

$$\begin{aligned} U^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= \left(\frac{\rho^2 + c^2\sigma^2}{\rho^2 + c^2}\right)\dot{\rho}^2 + \left(\frac{\rho^2 + c^2\sigma^2}{1 - \sigma^2}\right)\dot{\sigma}^2 + \left(\rho^2 + c^2\right)\left(1 - \sigma^2\right)\dot{\alpha}^2 . \end{aligned} \tag{B9}$$

The expressions for $\dot{\rho}$, $\dot{\sigma}$, and $\dot{\alpha}$ can be obtained from the following equations, which define the generalized momenta:

$$P_\rho = \frac{\partial L}{\partial \dot{\rho}} = h_1^2 \dot{\rho} = \frac{\partial S}{\partial \rho} = \pm \frac{\sqrt{P(\rho)}}{\rho^2 + c^2}, \tag{B10}$$

$$P_\sigma = \frac{\partial L}{\partial \dot{\sigma}} = h_2^2 \dot{\sigma} = \frac{\partial S}{\partial \sigma} = \pm \frac{\sqrt{Q(\sigma)}}{1 - \sigma^2}, \tag{B11}$$

$$P_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = h_3^2 \dot{\alpha} = \frac{\partial S}{\partial \alpha} = \hat{G}^{*}, \tag{B12}$$

*The caret above h, c, and G (that is, $\hat{h}$, $\hat{c}$, and $\hat{G}$) indicates canonical constants, referred to by Izsak (Reference B1), where $\hat{h}$ is the total energy in the orbit and always negative, $\hat{c}$ is the total angular momentum, and $\hat{G}$ is the z component of the angular momentum, positive for direct motion.

where

$$h_1^2 = \frac{\rho^2 + c^2\sigma^2}{\rho^2 + c^2}, \quad h_2^2 = \frac{\rho^2 + c^2\sigma^2}{1 - \sigma^2}, \quad h_3^2 = \left(\rho^2 + c^2\right)\left(1 - \sigma^2\right), \tag{B13}$$

$$P(\rho) = 2\hat{h}\rho^4 + 2\mu\rho^3 - \left(\hat{c}^2 - 2c^2\hat{h}\right)\rho^2 + 2c^2\mu\rho - c^2\left(\hat{c}^2 - \hat{G}^2\right), \tag{B14}$$

$$Q(\sigma) = -2c^2\hat{h}\sigma^4 - \left(\hat{c}^2 - 2c^2\hat{h}\right)\sigma^2 + \left(\hat{c}^2 - \hat{G}^2\right), \tag{B15}$$

$S = S(\rho, \sigma, \alpha)$ is the action function, and L is the Lagrangian given by $L = T - V$, where

$$T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2$$

$$= \frac{1}{2}\left(h_1^2\dot{\rho}^2 + h_2^2\dot{\sigma}^2 + h_3^2\dot{\alpha}^2\right),$$

$$V = V(\rho, \sigma) = \frac{-\mu\rho}{\rho^2 + c^2\sigma^2}.$$

Here ds/dt is the speed along the path and $V(\rho, \sigma)$ is the potential function introduced by Vinti (Reference B3).

The radicand in Equation B10 can be written in the form

$$P(\rho) = -2\hat{h}\left(\rho - \rho_1\right)\left(\rho_2 - \rho\right)\left(\rho - \rho_3\right)\left(\rho - \rho_4\right), \tag{B16}$$

where ρ_1, ρ_2, ρ_3, and ρ_4 are the zeros of $P(\rho)$. Izsak (Reference B1) has given the zeros in the form

$$\rho_1 = a(1 - e), \quad \rho_2 = a(1 + e), \quad \rho_3 = a(\kappa - i\lambda), \quad \rho_4 = a(\kappa + i\lambda). \tag{B17}$$

The orbital elements a and e are the semimajor axis and the eccentricity of the orbit, respectively, even though it is not an exact ellipse. They are defined by the first two of Equations B17. The quantity i is the imaginary unit $\left(\sqrt{-1}\right)$.

If we substitute ρ_3 and ρ_4 from Equations B17 into Equation B16:

$$P(\rho) = -2\hat{h}\left(\rho - \rho_1\right)\left(\rho_2 - \rho\right)\left[\rho^2 - 2a\kappa\rho + a^2\left(\kappa^2 + \lambda^2\right)\right].$$

The quantities κ and $\kappa^2 + \lambda^2$ are given in Reference B1 in terms of a, e, and $s = \sin I$, where I is the inclination of the orbit:

$$\kappa = \frac{\gamma^2\left(1 - s^2\right)\left(1 - e^2 - \gamma^2 s^2\right)}{\left(1 - e^2 - \gamma^2\right)\left(1 - e^2 - \gamma^2 s^2\right) + 4\gamma^2 s^2}, \tag{B18}$$

$$\kappa^2 + \lambda^2 = \frac{\gamma^2 s^2 \left[\left(1 - e^2 - \gamma^2\right)\left(1 - e^2 - \gamma^2 s^2\right) + 4\gamma^2\right]}{\left(1 - e^2 - \gamma^2\right)\left(1 - e^2 - \gamma^2 s^2\right) + 4\gamma^2 s^2} , \tag{B19}$$

where $\gamma = c/a$, a small dimensionless parameter.

The quartic $Q(\sigma)$ contains only even powers of σ and can be written

$$Q(\sigma) = -2c^2\hat{h}\left(\sigma_1^2 - \sigma^2\right)\left(\sigma_2^2 - \sigma^2\right) ,$$

where the four real zeros of $Q(\sigma) = 0$ are $\pm\sigma_1$ and $\pm\sigma_2$: $0 \le \sigma_1 < 1$, $\sigma_2 \gg 1$. As pointed out by Izsak (Reference B1), σ oscillates between the values $-\sigma_1$ and $+\sigma_1$. Therefore, σ_1 is a convenient parameter to use as the sine of the inclination I of the orbit.

When we introduce Izsak's formulas,

$$\rho_1 = a(1 - e), \; \rho_2 = a(1 + e) ,$$

$$\rho = a(1 - e\cos E) ,$$

$$\sigma_1 = s = \sin I ,$$

$$\sigma = s\sin\psi ,$$

$$\frac{\sigma_1^2}{\sigma_2^2} = \frac{s^2}{\sigma_2^2} = l^2 ,$$

$$c^4 s^2 \frac{s^2}{l^2} = a^4\left(1 - e^2\right)\left(\kappa^2 + \lambda^2\right) ,$$

where E is the eccentric anomaly and ψ is the argument of latitude, and several of the aforementioned relations into Equations B10 and B11 we obtain

$$P_\rho = \frac{\sqrt{-2\hat{h}}}{\rho^2 + c^2}\, ae \sqrt{\rho^2 - 2a\kappa\rho + a^2\left(\kappa^2 + \lambda^2\right)} \sin E , \tag{B20}$$

$$P_\sigma = \frac{\sqrt{-2\hat{h}}}{1 - \sigma^2}\frac{a^2}{c} \sqrt{1 - e^2} \sqrt{\kappa^2 + \lambda^2} \sqrt{1 - l^2\sin^2\psi} \cos\psi . \tag{B21}$$

The coefficients and the roots of the quartic equation $P(\rho) = 0$ can be related to those of $Q(\sigma) = 0$ in the following manner:

$$2a(1 + \kappa) = -\frac{\mu}{\hat{h}} , \tag{B22}$$

$$a^2\left[\left(1 - e^2\right) + 4\kappa + \left(\kappa^2 + \lambda^2\right)\right] = -\frac{\hat{c}^2 - 2c^2\hat{h}}{2\hat{h}}$$

$$= c^2\left(s^2 + \frac{s^2}{l^2}\right) , \tag{B23}$$

$$2a^3\left[\left(1 - e^2\right)\kappa + \left(\kappa^2 + \lambda^2\right)\right] = -c^2\frac{\mu}{\hat{h}} , \tag{B24}$$

$$a^4\left(1 - e^2\right)\left(\kappa^2 + \lambda^2\right) = -c^2\frac{\hat{c}^2 - \hat{G}^2}{2\hat{h}}$$

$$= c^4 s^2 \frac{s^2}{l^2} . \tag{B25}$$

Consider the following expression for l^2 given by Izsak (Reference B1)

$$l^2 = \frac{\gamma^2 s^2}{1 - e^2}\left[\frac{\left(1 - e^2 - \gamma^2\right)\left(1 - e^2 - \gamma^2 s^2\right) + 4\gamma^2 s^2}{\left(1 - e^2 - \gamma^2\right)\left(1 - e^2 - \gamma^2 s^2\right) + 4\gamma^2}\right] . \tag{B26}$$

If we substitute c/a for γ and solve for s^2/l^2 we obtain

$$\frac{s^2}{l^2} = \frac{a^2\left(1 - e^2\right)}{c^2}\left\{\frac{\left[a^2\left(1 - e^2\right) - c^2\right]\left[a^2\left(1 - e^2\right) - c^2 s^2\right] + 4a^2c^2}{\left[a^2\left(1 - e^2\right) - c^2\right]\left[a^2\left(1 - e^2\right) - c^2 s^2\right] + 4a^2c^2s^2}\right\} . \tag{B27}$$

Next we introduce a parameter $p = a\left(1 - e^2\right)$, the semilatus rectum, which Vinti (Reference B2) uses in his oblateness parameter $k = c^2/p^2$. It is clear that Equation B27 can be written

$$\frac{s^2}{l^2} = \frac{ap}{c^2}\left[\frac{\left(ap - c^2\right)\left(ap - c^2 s^2\right) + 4a^2c^2}{\left(ap - c^2\right)\left(ap - c^2 s^2\right) + 4a^2c^2s^2}\right] . \tag{B28}$$

From Equation 4.13 of Reference B2:

$$B = c^2\eta_0^2 \frac{\left(ap - c^2\right)\left(ap - c^2\eta_0^2\right) + 4a^2c^2}{\left(ap - c^2\right)\left(ap - c^2\eta_0^2\right) + 4a^2c^2\eta_0^2} , \tag{B29}$$

where $\eta_0 = s = \sin I$, we see that $c^4 s^4 / l^2 = \mathrm{Bap}$. Solving for $c^2 s^2 / l^2$ from Equation B23 and inserting it into Equation B25, we obtain

$$\hat{G}^2 = \left(1 - s^2\right)\left(\hat{c}^2 + 2c^2 \hat{h} s^2\right) . \tag{B30}$$

From Equation B23,

$$\frac{-\hat{c}^2}{2\hat{h}} = -c^2\left(1 - s^2\right) + c^2 \frac{s^2}{l^2} . \tag{B31}$$

Multiplying Equation B28 by c^2 to obtain $c^2 s^2 / l^2$ and inserting it into Equation B31 we find

$$\frac{-\hat{c}^2}{2\hat{h}} = -c^2\left(1 - s^2\right) + ap\left[\frac{\left(ap - c^2\right)\left(ap - c^2 s^2\right) + 4a^2c^2}{\left(ap - c^2\right)\left(ap - c^2 s^2\right) + 4a^2c^2s^2}\right] . \tag{B32}$$

Vinti has given an identical result in Equation 4.15 of Reference B2:

$$\frac{-\alpha_2^2}{2\alpha_1} \equiv a_0 p_0 = -c^2\left(1 - \eta_0^2\right) + ap\left[\frac{\left(ap - c^2\right)\left(ap - c^2\eta_0^2\right) + 4a^2c^2}{\left(ap - c^2\right)\left(ap - c^2\eta_0^2\right) + 4a^2c^2\eta_0^2}\right] , \tag{B33}$$

where a_0 is a semimajor axis, p_0 the semilatus rectum. Since $\hat{h} = \alpha_1$, $\hat{c} = \alpha_2$, and $\hat{G} = \alpha_3$, we can easily rewrite Equation B30, using Equation B33, to obtain the final result for $\hat{G}$,

$$\hat{G} = \hat{c}\sqrt{\left(1 - \frac{c^2 s^2}{a_0 p_0}\right)\left(1 - s^2\right)} . \tag{B34}$$

Equation B34 is equivalent to Vinti's Equation 4.15a of Reference B2,

$$\alpha_3 = \alpha_2\left(1 - \frac{c^2\eta_0^2}{a_0 p_0}\right)^{\frac{1}{2}} \cos I .$$

Using Equation B34 we obtain, as in Equation B12,

$$P_\alpha = \hat{G} . \tag{B35}$$

It should be noted that the following formulas relate κ and $\kappa^2 + \lambda^2$ to Vinti's A and B:

$$a^2\left(\kappa^2 + \lambda^2\right) = B , \tag{B36}$$

$$-2a\kappa = A , \tag{B37}$$

where A and B are given by Equations 4.12 and 4.13 of Reference B2:

$$\left.\begin{aligned} A &= \frac{-2ac^2\left(1-\eta_0^2\right)\left(ap-c^2\eta_0^2\right)}{\left(ap-c^2\right)\left(ap-c^2\eta_0^2\right)+4a^2c^2\eta_0^2}, \\ B &= c^2\eta_0^2\,\frac{\left(ap-c^2\right)\left(ap-c^2\eta_0^2\right)+4a^2c^2}{\left(ap-c^2\right)\left(ap-c^2\eta_0^2\right)+4a^2c^2\eta_0^2}. \end{aligned}\right\} \tag{B38}$$

From Vinti's Equation 4.16 (Reference B2) we have

$$\eta_2^{-2} = \frac{c^2}{ap}\,\frac{\left(ap-c^2\right)\left(ap-c^2\eta_0^2\right)+4a^2c^2\eta_0^2}{\left(ap-c^2\right)\left(ap-c^2\eta_0^2\right)+4a^2c^2}\Bigg] = c^4\eta_0^2\,(Bap)^{-1}, \tag{B39}$$

where $\eta_2 = \sigma_2$. Using Equations B36-B38, together with the values for $P(\rho)$ and $Q(\sigma)$, we find that in Vinti's notation the quartics $P(\rho)$ and $Q(\sigma)$ can be factored in the form

$$F(\rho) = -2\alpha_1\left(\rho_2-\rho\right)\left(\rho-\rho_1\right)\left(\rho^2+A\rho+B\right), \tag{B40}$$

$$G(\eta) = -2\alpha_1 c^2\left(\eta_0^2-\eta^2\right)\left(\eta_2^2-\eta^2\right), \tag{B41}$$

where $\eta = \sigma$.

The following equations for $\dot{\rho}$, $\dot{\sigma}$, and $\dot{\alpha}$ are easily obtained from Equations B10-B13, B20, B21, and B35:

$$\dot{\rho} = \frac{\sqrt{-2\hat{h}}}{\rho^2+c^2\sigma^2}\,ae\,\sqrt{\rho^2-2a\kappa\rho+a^2\left(\kappa^2+\lambda^2\right)}\,\sin E, \tag{B42}$$

$$\dot{\sigma} = \frac{\sqrt{-2\hat{h}}}{\rho^2+c^2\sigma^2}\,\frac{a^2}{c}\,\sqrt{1-e^2}\,\sqrt{\kappa^2+\lambda^2}\,\sqrt{1-l^2\sin^2\psi}\,\cos\psi, \tag{B43}$$

$$\dot{\alpha} = \frac{\hat{G}}{\left(\rho^2+c^2\right)\left(1-\sigma^2\right)}. \tag{B44}$$

If we write the equations for $\dot{\rho}$, $\dot{\sigma}$, and $\dot{\alpha}$ in Vinti's notation, we obtain

$$\dot{\rho} = \frac{\sqrt{-2\alpha_1}}{\rho^2+c^2\eta^2}\,ae\,\sqrt{\rho^2+A\rho+B}\,\sin E, \tag{B45}$$

$$\dot{\eta} = \frac{\sqrt{-2\alpha_1}}{\rho^2 + c^2\eta^2} c\eta_0\eta_2 \sqrt{1 - q^2 \sin^2\psi} \cos\psi, \tag{B46}$$

$$\dot{\phi} = \frac{\alpha_3}{\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)}, \tag{B47}$$

where $q = \eta_0/\eta_2$.

Now, substituting Equations B42-B44 into Equations B6-B8 we obtain the following:

$$\dot{x} = \frac{-\hat{G}y}{\left(\rho^2 + c^2\right)\left(1 - \sigma^2\right)} + \frac{xa\sqrt{-2\hat{h}}}{\rho^2 + c^2\sigma^2}\left[\frac{\rho e\sqrt{\rho^2 - 2a\kappa\rho + a^2\left(\kappa^2 + \lambda^2\right)}}{\rho^2 + c^2} \sin E - \frac{\sigma\sqrt{1 - e^2}}{1 - \sigma^2}\frac{a}{c}\sqrt{\kappa^2 + \lambda^2}\sqrt{1 - l^2 \sin^2\psi} \cos\psi\right], \tag{B48}$$

$$\dot{y} = \frac{+\hat{G}x}{\left(\rho^2 + c^2\right)\left(1 - \sigma^2\right)} + \frac{ya\sqrt{-2\hat{h}}}{\rho^2 + c^2\sigma^2}\left[\frac{\rho e\sqrt{\rho^2 - 2a\kappa\rho + a^2\left(\kappa^2 + \lambda^2\right)}}{\rho^2 + c^2} \sin E - \frac{\sigma\sqrt{1 - e^2}}{1 - \sigma^2}\frac{a}{c}\sqrt{\kappa^2 + \lambda^2}\sqrt{1 - l^2 \sin^2\psi} \cos\psi\right], \tag{B49}$$

$$\dot{z} = \frac{a\sqrt{-2\hat{h}}}{\rho^2 + c^2\sigma^2}\left[\frac{a}{c}\rho\sqrt{1 - e^2}\sqrt{\kappa^2 + \lambda^2}\sqrt{1 - l^2 \sin^2\psi} \cos\psi + \sigma e\sqrt{\rho^2 - 2a\kappa\rho + a^2\left(\kappa^2 + \lambda^2\right)} \sin E\right]. \tag{B50}$$

The velocity components given above are now being used in an orbit determination program formulated by the author.

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