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REMARKS ON HILL'S LUNAR THEORY. PART I

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SUMMARY

G. W. Hill, by neglecting solar parallax and the eccentricity of the solar orbit, presented a particularly simple form of the differential equation of the restricted three-body problem for the motion of a massless satellite. In the present paper, the Hill equation is modified to give a third order differential equation for r—the planetocentric distance of the satellite. And this equation can be solved by iteration (if the satellite orbit is considered to be a disturbed Kepler ellipse). This new solution is not only suitable for computer applications but it is independent of the coordinate system and is valid for both a fixed and a rotating system whereas Hill's solution was limited to simple-periodic orbits in a rotating system. The discussion of Hill's lunar problem provides a simple example of a method applicable also to more difficult problems such as the restricted three-body problem, in which conic-section orbits can be considered good approximate solutions.

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REMARKS ON HILL'S LUNAR THEORY. PART I

by

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INTRODUCTION

By discounting the solar parallax, G. W. Hill[†] gave a particularly simple form to the differential equation of the restricted three-body problem for the motion of a massless satellite around a planet in a circular orbit about the sun. This relatively simple problem offers tempting considerations and experiments, which will be discussed here. In this paper, it will be shown that the Hill Equation can be modified so that a differential equation of the third order for r —the planetocentric distance of the satellite—results, in which (besides r and its derivations through the third order) only the Jacobian constant occurs. And this equation can be solved by iteration if the satellite orbit is considered to be a disturbed Kepler ellipse.

HILL'S DIFFERENTIAL EQUATIONS

Hill's differential equations can be written as:

$$\left.\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= -\xi\left(\frac{1}{r^3} - 3\right) , \\ \ddot{\eta} + 2\dot{\xi} &= -\frac{\eta}{r^3} , \end{aligned}\right\} \quad (1)$$

where

$$r^2 = \xi^2 + \eta^2 .$$

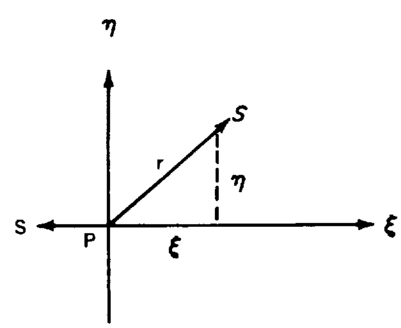

Figure 1—Geometric representation of a massless satellite *S* of a planet P.

Here ξ, η are the right-angle planetocentric coordinates of the assumed massless satellite *S* in a coordinate system (Figure 1)

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†Hill, G. W., "Researches in the Lunar Theory," in *The Collected Works of George William Hill, Vol. I,* Washington, D. C.: Carnegie Institution, 1905.

which revolves with the constant angular velocity $n = 1$. And we have

$$n = k\sqrt{\frac{m_1 + m}{a^3}} = 1 \tag{2}$$

for the mean motion of the planet P in its circular orbit around the sun S, which is to be assumed at a great distance, a, along the negative ξ-axis.

The units of mass are to be so selected that Equation 2 is valid. This is the case when we set $k = 1$ and $a^3 = m_1 + m$ where m_1 is the mass of the sun and m the mass of the planet. In addition, the unit of mass is selected so that $m = 1$. In the case of the motion of a satellite, or moon, about the earth, $m_1 \approx 330{,}000$ and the unit of time is equal to $1/k = 58.^d13244$. It also follows that $a = \sqrt[3]{m_1 + m} \approx 69$. Thus Equation 1 is obtained if, in the differential equation of the restricted three-body problem, m_1 and a are allowed to go towards infinity so far that $m_1/a^3 \to 1$, but m_1/a^5, m_1/a^7, $\cdots \to 0$. This simplification of the problem means that the sun, as a disturbing body, is removed to any great distance desired (so that its parallax at the earth-moon distance becomes unnoticeably small), but that its disturbing effect upon the satellite's motion (which is proportional $m_1 a^{-3}$) about the earth remains unchanged.

Hill's equations (Equation 1) cannot simply be transformed into the equations of the undisturbed motion of the satellite, since the mass of the disturbing body (the sun) does not occur explicitly in them, but rather is implicitly contained in the definition of the unit of length. The equations can, however, be given a form which makes it possible to combine the disturbed and undisturbed motion. If we write

$$\left.\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi\left(1 - \frac{1}{r^3} + 2k\right), \\ \ddot{\eta} + 2\dot{\xi} &= \eta\left(1 - \frac{1}{r^3} - k\right), \\ \text{where} \qquad \xi^2 + \eta^2 &= r^2 \end{aligned}\right\} \tag{3}$$

then these equations assume the form of Equation 1 for $k = 1$, whereas letting $k = 0$ yields the equations of undisturbed motion (in a coordinate system revolving with the angular velocity $n = 1$).

Equations 3 have the Jacobian integral

$$\dot{\xi}^2 + \dot{\eta}^2 = r^2(1-k) + \frac{2}{r} + 3k\xi^2 - C . \tag{4}$$

THE DIFFERENTIAL EQUATION FOR r(t)

The two differential equations of the second order (Equation 3) form a fourth-order system. In place of them, a single fourth-order differential equation can be written for a suitable variable — perhaps $r = \sqrt{\xi^2 + \eta^2}$. With the help of the Jacobian integral, the order of this equation can be decreased by one, so that we have a relation of the form

$$f(\dddot{r},\ \ddot{r},\ \dot{r},\ r;\ C) = 0\ . \tag{5}$$

This same consideration applies *mutatis mutandis* for the complicated *problème restreint,* which we will discuss, but first we will make this experiment with the simple Hill problem.

If we differentiate

$$r^2 = \xi^2 + \eta^2 \tag{6}$$

three times, and if we eliminate $\ddot{\xi}$, $\ddot{\eta}$ by using Equation 3 and $\dot{\xi}^2 + \dot{\eta}^2$ by using Equation 4, we obtain

$$\left.\begin{aligned} r\,\dot{r} &= \xi\dot{\xi} + \eta\dot{\eta}\ , \\ r\,\ddot{r} + \dot{r}^2 &= 2(\xi\dot{\eta} - \eta\dot{\xi}) + 2r^2(1-k) + \frac{1}{r} + 6k\,\xi^2 - C\ , \\ r\,\dddot{r} + 3\dot{r}\,\ddot{r} &= 6k(2\xi\dot{\xi} - \xi\eta) - \frac{\dot{r}}{r^2} - 4k\,r\,\dot{r}\ . \end{aligned}\right\} \tag{7}$$

Equations 6, 7 and 4 form a system

$$\phi_0 = r^2 = \xi^2 + \eta^2\ , \tag{8a}$$

$$\phi_1 = r\,\dot{r} = \xi\dot{\xi} + \eta\dot{\eta}\ , \tag{8b}$$

$$\phi_2 = r\,\ddot{r} + \dot{r}^2 - \frac{1}{r} - 2r^2(1-k) + C = 2(\xi\dot{\eta} - \eta\dot{\xi}) + 6k\xi^2\ , \tag{8c}$$

$$\phi_3 = r\,\dddot{r} + 3\dot{r}\,\ddot{r} + \frac{\dot{r}}{r^2} + 4kr\,\dot{r} = 6k(2\xi\dot{\xi} - \xi\eta)\ , \tag{8d}$$

$$\phi_4 = r^2(1-k) + \frac{2}{r} - C = \dot{\xi}^2 + \dot{\eta}^2 - 3k\xi^2\ , \tag{8e}$$

in which five functions ϕ_0, $\cdots$ ϕ_4 of r, $\dot{r}$, $\ddot{r}$, $\dddot{r}$, and C appear on the left and appear on the right as functions of the four quantities ξ, η, $\dot{\xi}$, $\dot{\eta}$. Therefore, to obtain the desired equation in the form of Equation 5, it suffices to eliminate the latter four quantities from Equation 8.

ELIMINATION

It follows from Equations 8b and 8c, and by taking Equation 8a, that

$$2\phi_0\dot{\xi} = 2\phi_1\,\xi - \phi_2\,\eta + 6k\xi^2\eta \tag{9a}$$

$$2\phi_0\dot{\eta} = 2\phi_1\eta + \phi_2\,\xi - 6k\xi^3 \ . \tag{9b}$$

If we multiply Equation 9a by $\dot{\xi}$, Equation 9b by $\dot{\eta}$, and add, we get

$$2\phi_0\left(\dot{\xi}^2 + \dot{\eta}^2\right) = 2\phi_1^2 + (\xi\,\dot{\eta} - \eta\,\dot{\xi})\left(\phi_2 - 6k\,\xi^2\right)$$

or, since from Equation 8c it follows that $\xi\,\dot{\eta} - \eta\,\dot{\xi} = \frac{1}{2}(\phi_2 - 6k\xi^2)$,

$$\phi_0\left(\dot{\xi}^2 + \dot{\eta}^2\right) = \phi_1^2 + \frac{1}{4}\left(\phi_2 - 6k\xi^2\right)^2 \ . \tag{10}$$

If Equations 9 and 10 are substituted into Equations 8d and 8e, there results

$$\phi_0\phi_3 = 6k\left[2\phi_1\xi^2 - \xi\eta\left(\phi_0 + \phi_2 - 6k\xi^2\right)\right] \ , \tag{11a}$$

$$\phi_0\phi_4 = \phi_1^2 + \frac{1}{4}\left(\phi_2 - 6k\xi^2\right)^2 - 3k\phi_0\,\xi^2 \ , \tag{11b}$$

and from Equation 8a

$$\phi_0\xi^2 = \xi^4 + (\xi\,\eta)^2 \ . \tag{11c}$$

The $6k\xi^2$ and the $6k\xi\eta$ terms can easily be eliminated from Equations 11 by letting $x = 6k\xi^2$, $y = 6\kappa\xi\eta$, for if we set

$$\phi_0 + \phi_2 - x = \lambda \ , \tag{12}$$

from Equation 11b we get

$$4\left(\phi_0\phi_4 - \phi_1^2\right) = (\lambda - \phi_0)^2 - 2\phi_0\left(\phi_0 + \phi_2 - \lambda\right) = \lambda^2 + \phi_0^2 - 2\phi_0\left(\phi_0 + \phi_2\right) \ .$$

The above equation means that

$$\lambda^2 = \phi_0^2 + 2\phi_0\left(\phi_2 + 2\phi_4\right) - 4\phi_1^2 \tag{13}$$

is dependent only upon ϕ_0, ϕ_1, ϕ_2, ϕ_4; i.e., upon r, $\dot{r}$, $\ddot{r}$, C.

From Equations 11a, 11c, and 12 we have

$$y\lambda = 2\phi_1(\phi_0 + \phi_2 - \lambda) - \phi_0\phi_3 ,$$

$$y^2 = -(\phi_0 + \phi_2 - \lambda)^2 + 6k\phi_0(\phi_0 + \phi_2 - \lambda) ,$$

from which, we obtain a fourth-order equation in λ by eliminating y:

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0 \tag{14}$$

where the coefficients of λ are dependent only upon $\phi_0, \cdots \phi_4$:

$$\left.\begin{aligned} \alpha_1 &= 2\left[\phi_0(3k-1) - \phi_2\right] , \\ \alpha_2 &= 4\phi_1^2 - 9k^2\phi_0^2 + \frac{1}{4}\alpha_1^2 , \\ \alpha_3 &= 4\phi_1\left[\phi_0\phi_3 - 2\phi_1(\phi_0 + \phi_2)\right] , \\ \alpha_4 &= \left[\phi_0\phi_3 - 2\phi_1(\phi_0 + \phi_2)\right]^2 . \end{aligned}\right\} \tag{15}$$

It is worthy of note that $\ddot{r}$ occurs only in ϕ_3, specifically in the combination $\phi_0\phi_3 - 2\phi_1(\phi_0 + \phi_2)$, which, in turn, appears only in α_3 and α_4. Therefore, if we set

$$f = \phi_0\phi_3 - 2\phi_1(\phi_0 + \phi_2) , \tag{16}$$

we can also write Equation 14 as

$$f^2 + 4\phi_1\lambda f + \lambda^2(\alpha_2 + \alpha_1\lambda + \lambda^2) = 0 .$$

The above equation has the following solution:

$$f = -2\phi_1\lambda \pm \lambda\sqrt{4\phi_1^2 - (\alpha_2 + \alpha_1\lambda + \lambda^2)} ;$$

or, if α_1, α_2 are expressed by Equation 15,

$$f = -2\phi_1\lambda \pm \lambda\sqrt{9k^2\phi_0^2 - \left[\lambda + \phi_0(3k-1) - \phi_2\right]^2} . \tag{17}$$

Equation 17, together with Equations 13 and 16, provides the formal solution of the problem.

APPLICATION

If we consider the motion of the satellite as a disturbed Kepler motion, it is practical to recall that for $k = 0$ the third of Equations 7 transforms into the relation

$$r\dddot{r} + 3\dot{r}\ddot{r} + \frac{\dot{r}}{r^2} = 0 , \tag{18}$$

which can be integrated in a closed form. For, according to Equation 8d, the disturbed motion is

$$r\dddot{r} + 3\dot{r}\ddot{r} + \frac{\dot{r}}{r^2} = \phi_3 - 4k\phi_1 .$$

But from Equation 16,

$$\phi_0(\phi_3 - 4k\phi_1) = f + 2\phi_1(\phi_0 + \phi_2 - 2k\phi_0) ;$$

therefore

$$\phi_0\left(\dot{r}\ddot{r} + 3\dot{r}\ddot{r} + \frac{\dot{r}}{r^2}\right) = 2\phi_1(\phi_0 + \phi_2 - \lambda - 2k\phi_0) \pm \lambda\sqrt{9k^2\phi_0^2 - [\phi_0 + \phi_2 - \lambda - 3k\phi_0]^2} \tag{19}$$

The right side of this equation becomes zero for $k = 0$, since, according to Equation 12, $\phi_0 + \phi_2 - \lambda = x = 6k\xi^2$ also contains k as a factor.

For the case in which the satellite's orbit can, to first approximation, be considered a Kepler ellipse, the right side of Equation 19 is small, i.e., of the order of the perturbation. We then have, for $k = 1$:

$$\left.\begin{aligned} &\phi_0\left(r\dddot{r} + 3\dot{r}\ddot{r} + \frac{\dot{r}}{r^2}\right) = 2\phi_1(\phi_2 - \phi_0 - \lambda) \pm \lambda\sqrt{9\phi_0^2 - (\phi_2 - 2\phi_0 - \lambda)^2} \\ &\text{with} \\ &\lambda^2 = \phi_0^2 + 2\phi_0(\phi_2 + 2\phi_4) - 4\phi_1^2 . \end{aligned}\right\} \tag{20}$$

The sign of the square root is determined from the initial conditions of the problem. λ itself is certainly positive if the motion of the satellite is direct, for, by definition,

$$\lambda = \phi_0 + \phi_2 - 6k\xi^2 = r^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) .$$

On the other hand, if, in the fixed xy-system, $x\dot{y} - y\dot{x} = r^2\dot{\psi}$ is the velocity at a given instant,

$$\xi\dot{\eta} - \eta\dot{\xi} = r^2(\dot{\psi} - 1) ,$$

so that

$$\lambda = r^2(2\dot{\psi} + 1).$$

Therefore λ is positive even for $\dot{\psi} > -1/2$.

The advantage of the differential equation (Equation 20) is that it contains only one variable, so that the expression for the perturbation (on the right side) also depends only upon r, $\dot{r}$, $\ddot{r}$ and C. In the application of the methods of numerical integration, it suffices, therefore, to set up a single difference table. This step could be of particular importance when working with electronic computers. The relatively complicated structure of the disturbing function does not constitute an appreciable barrier for the computers.

It must also be mentioned that Equation 20 gives the function $r(t)$ independently of the coordinate system selected, and that therefore its solution is as valid in a fixed as in a revolving system. It is here that this solution differs from that which Hill gave for the system*. Hill's solution was limited to simple-periodic orbits in a revolving ξ, η system. Although the Hill experiments yield – of the known periodic inequalities – only the so-called "variation" which is periodic in the revolving system, a similar analysis of the differential equation (Equation 20) would also yield the "large inequality" of the moon motion (largest periodic term of elliptical motion), the "evection," and some other features. Likewise, considering the secular perturbations, we find the apsidal motion, but it is inherent in the problem that the terms due to the solar parallax, the eccentricity and inclination of the earth's orbit do not appear.

Provided that we have succeeded in determining $r = r(t)$ – either by numerical integration of Equation 20 or by a theoretical formula (developed perhaps through a sufficiently converging series)– it is always possible to determine the corresponding coordinate $\phi(t)$ by means of a simple quadrature. By letting $\xi = r \cos \phi$ and $\eta = r \sin \phi$, Hill's equations (Equation 1) take the form:

$$\left(\ddot{r} - r\dot{\phi}^2 - 2r\dot{\phi}\right)\cos\phi - \left(2\dot{r}\dot{\phi} + r\ddot{\phi} + 2\dot{r}\right)\sin\phi = \left(3 - \frac{1}{r^3}\right) r\cos\phi ,$$

$$\left(\ddot{r} - r\dot{\phi}^2 - 2r\dot{\phi}\right)\sin\phi + \left(2\dot{r}\dot{\phi} + r\ddot{\phi} + 2\dot{r}\right)\cos\phi = -\frac{1}{r^3} r\sin\phi$$

from which

$$\left.\begin{aligned} \ddot{r} - r\dot{\phi}^2 - 2r\dot{\phi} &= 3r\cos^2\phi - \frac{1}{r^2} , \\ 2\dot{r}\dot{\phi} + r\ddot{\phi} + 2\dot{r} &= -3r\sin\phi\cos\phi = -\frac{3}{2} r\sin 2\phi \end{aligned}\right\} \quad (21)$$

*Hill, G. W., "Researches in the Lunar Theory," in *The Collected Works of George William Hill, Vol. I,* Washington, D. C.: Carnegie Institution, 1905.

follow. Hill has shown from the second equation of Equation 21 that $r = r(t)$ is found by quadrature if $\phi(t)$ is known, for it follows from

$$2\frac{\dot{r}}{r}(\dot{\phi}+1) = -\left(\ddot{\phi} + \frac{3}{2}\sin 2\phi\right)$$

that, with k constant,

$$r = \frac{k}{\sqrt{\dot{\phi}+1}} \exp -\frac{3}{4}\int \frac{\sin 2\phi}{\dot{\phi}+1}\, dt \ .$$

In the same manner we also find from the first equation of Equation 21 that

$$\dot{\phi}^2 + 2\dot{\phi} = \frac{\ddot{r}}{r} + \frac{1}{r^3} - 3\cos^2\phi$$

or

$$\dot{\phi} + 1 = \sqrt{\frac{\ddot{r}}{r} + \frac{1}{r^3} + 1 - 3\cos^2\phi}\ , \tag{22}$$

from which it follows, by integration, that

$$t - t_0 = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\frac{\ddot{r}}{r} + \frac{1}{r^3} + 1 - 3\cos^2\phi} - 1}\ . \tag{23}$$

The above relation yields $t = t(\phi)$ and, therefore $\phi = \phi(t)$.

If $\psi = \phi_0 + nt = \phi_0 + t$ is the true length of the satellite in a fixed coordinate system, then

$$\dot{\psi} = \dot{\phi} + 1 = \frac{\sqrt{p}}{r^2}\ ,$$

for undisturbed motion, if p denotes the parameter of the orbit ellipse; we also have

$$r^2\left(r\,\ddot{r} + \frac{1}{r}\right) = p\ ,$$

therefore

$$\dot{\phi} + 1 = \sqrt{\frac{\ddot{r}}{r} + \frac{1}{r^3}}\ . \tag{24}$$

This differential equation can therefore be considered as an approximation for Equation 22 as long as the term $(1 - 3\cos^2\phi)$ in the radicand of the square root remains small by comparison with the other terms. For the earth's moon, which is of particular interest here, this condition is still satisfied, for, since the unit of length is so selected that the mean earth-sun distance is about 69, the average value of r (normally 390 times smaller) is about $3/17$; the expression $(\ddot{r}/r) + (1/r^3)$ is therefore of the order p/r^4 or $1/r^3$, or about 180, whereas the attachment term varies from -2 to $+1$.

INTRODUCTION OF A NEW INDEPENDENT VARIABLE

The form of the differential equation (Equation 20) suggests a solving process which is a modification of that used in the undisturbed problem (Equation 18). If a new variable q is introduced instead of time t by means of $\dot{q} = 1/r$ and the initial condition $q(t_0) = 0$ so that q disappears simultaneously and increases monotonically with $(t - t_0)$, we find that*

$$\left.\begin{aligned} \dot{r} &= r'\dot{q} = \frac{r'}{r}, \\ \ddot{r} &= \frac{r''}{r^2} - \frac{r'^2}{r^3}, \\ \dddot{r} &= \frac{r'''}{r^3} - 4\frac{r'r''}{r^4} + 3\frac{r'^3}{r^5}. \end{aligned}\right\} \qquad (25)$$

With the above equations, the undisturbed equation of motion (Equation 18) assumes the form

$$\frac{1}{r^2}\left(r''' + \frac{1 - r''}{r}\, r'\right) = 0 . \qquad (26)$$

But since

$$\frac{d}{dq}\left(\frac{1 - r''}{r}\right) = -\frac{1}{r}\left(r''' + \frac{1 - r''}{r}\, r'\right) = 0 , \qquad (27)$$

$$\alpha^2 = \frac{1 - r''}{r}$$

is constant, and Equation 26 goes over into the integrable form:

$$r''' + \alpha^2 r' = 0 . \qquad (28)$$

*The derivatives of q are denoted by primes.

On the other hand, in the disturbed problem (Equation 20) we have

$$r''' + \frac{1 - r''}{r} r' = \phi(q) = \phi(r, r', r'', C)$$

or

$$r''' + \alpha^2 r' = \phi(q) , \tag{29}$$

where $\phi(q)$ represents the expression for the perturbation (right side of Equation 20) as a function of q. And because of Equation 27,

$$\frac{d}{dq}(\alpha^2) = -\frac{\phi(q)}{r}$$

or

$$\alpha^2 = \alpha_0^2 - \int_0^q \frac{\phi(q)}{r(q)} dq . \tag{30}$$

Therefore, if we set

$$g(q) = \phi + r' \int_0^q \frac{\phi}{r} dq , \tag{31}$$

we get

$$r''' + \alpha_0^2 r' = g(q) , \tag{32}$$

where $\alpha_0^2 = (1 - r_0'')/r_0$ is a constant which is known from the initial conditions.

If $g(q)$ is a sufficiently small perturbation in the interval $|q| < Q$, the integration method described in a companion paper* could be utilized. If, for $s = \alpha_0 q$, we set $c_0(s^2) = \cos s$, $c_1(s^2) = (\sin s)/s$, $c_2(s^2) = (1 - \cos s)/s^2$ and $c_3(s^2) = (s - \sin s)/s^3$, $\cdots$, then the recurrence formula

$$\frac{1}{n!} = c_n + s^2 c_{n+2} \tag{33}$$

*Stumpff, K., "Calculation of Ephemerides from Initial Values," NASA Technical Note D-1415, 1962.

and the differential equation

$$\frac{d}{dq}\left(c_{n+1}\ q^{n+1}\right) \quad = \quad c_n q^n \ . \tag{34}$$

are applicable. Therefore the Taylor development

$$r(q) \quad = \quad r_0 + \frac{1}{1!}\ r_0'\ q + \frac{1}{2!}\ r_0''\ q^2 + \frac{1}{3!}\ r_0'''\ q^3 + \cdots \ ,$$

can, by replacing the reciprocal factorials with the expressions given by Equation 33, be written

$$r(q) \quad = \quad r_0 + r_0'\left(c_1 + s^2 c_3\right)q + r_0''\left(c_2 + s^2 c_4\right)q^2 + \cdots \ .$$

If we again set $s^2 = \alpha^2 q^2$ and arrange according to powers of q, we have

$$r(q) \quad = \quad r_0 + r_0'\ c_1 q + r_0''\ c_2 q^2 + \left(r_0''' + \alpha_0^2\ r_0'\right) c_3 q^3 + \left(r_0^{iv} + \alpha_0^2\ r_0''\right) c_4 q^4 + \cdots \ ,$$

a series which, because of Equation 32, becomes

$$r(q) \quad = \quad r_0 + r_0'\ c_1 q + r_0''\ c_2 q^2 + g_0 c_3 q^3 + g_0' c_4 q^4 + g_0''\ c_5 q^5 + \cdots \ . \tag{35}$$

Now we set

$$r_0 q \quad = \quad z\tau \ , \tag{36}$$

where $\tau = k\left(t - t_0\right)$ denotes the intermediate time expressed in units of $1/k$ days; we introduce the expressions $\gamma_1,\ \gamma_2,\ \gamma_3,\ \cdots$ (which are invariable against coordinate transformations) through

$$r_0' \ = \ r_0^2\gamma_1, \qquad r_0'' \ = \ r_0^3\gamma_2, \qquad g_0 \ = \ r_0^4\gamma_3, \qquad g_0' \ = \ r_0^5\gamma_4, \qquad \cdots \ ,$$

and we set

$$\gamma_1\tau \ = \ \eta_1, \qquad \gamma_2\tau^2 \ = \ \eta_2, \qquad \cdots \qquad \gamma_n\tau^n \ = \ \eta_n \ ,$$

Equation 35 now assumes the form

$$r(q) \quad = \quad r_0\left[1 + c_1\eta_1 z + c_2\eta_2 z^2 + c_3\eta_3 z^3 + \cdots\right] , \tag{37}$$

a series, the terms of which are small above the third order, and which, in practical cases and for moderate intermediate times, converges very rapidly. The association between τ and q is provided by the differential equation

$$d\tau = r\,dq = r_0 \left[1 + c_1 \eta_1 z + c_2 \eta_2 z^2 + \cdots\right] dq \; . \tag{38}$$

But

$$r_0 \int c_n \eta_n z^n \, dq = r_0 \gamma_n \int c_n (z\tau)^n \, dq = r_0^{n+1} \gamma_n \int c_n q^n \, dq \; ;$$

and therefore, according to Equation 34,

$$r_0 \int c_n \eta_n z^n \, dq = r_0^{n+1} \gamma_n c_{n+1} q^{n+1} = \gamma_n c_{n+1} z^{n+1} \tau^{n+1} = \tau \eta_n c_{n+1} z^{n+1} \; .$$

If we divide the integral of Equation 38 by τ, we obtain the *main equation*

$$1 = z + c_2 \eta_1 z^2 + c_3 \eta_2 z^3 + c_4 \eta_3 z^4 + \cdots \; . \tag{39}$$

For undisturbed motion $(g = 0)$, it limits itself to the first three terms, since η_3, η_4, $\cdots$ disappear. The c-functions

$$c_n(\lambda^2) = \frac{1}{n!} - \frac{\lambda^2}{(n+2)!} + \frac{\lambda^4}{(n+4)!} - \cdots$$

have the always-real argument

$$\lambda^2 = (\alpha_0 q)^2 = \alpha_0^2 \frac{z^2 \tau^2}{r_0^2} \; ;$$

or, since

$$\alpha_0^2 = \frac{1 - r_0''}{r_0} = \frac{1 - r_0^3 \gamma_2}{r_0} \, r_0^2 \left(\frac{1}{r_0^3} - \gamma_2\right) \; ,$$

$$\lambda^2 = \left(\frac{1}{r_0^3} - \gamma_2\right) \tau^2 z^2 \; .$$

CLOSING COMMENTS AND OUTLOOK

The aforementioned integration method replaces the numerical integration method of Hill's differential equation (Equation 1) for the case where we can, as a first approximation, consider the satellite motion as a Keplerian ellipse with any desired eccentricity. This method permits the determination of $r(t)$ directly through iteration — at least under some conditions, for a certain fairly extended time interval in the vicinity of the initial time t_0. The iteration process itself is limited to the solution of the transcendental *main equation* (Equation 39), since the quantities g_0, g_0', $\cdots$ can be derived from the initial conditions.

Here we have attempted to demonstrate this method on a problem — Hill's moon problem — which perhaps has little practical importance, but is simple enough that its solution may be written explicitly and understandably. Thus it serves as an example for other, more difficult problems, which can be handled with the same general principles and solved in the same manner. This method can be used to attack certain special cases of the *problème restreint*, in which conic-section orbits can be considered good approximate solutions. For example, we could consider the case of the sufficiently close orbit of an infinitesimal body around one of two finite masses; i.e., either the actual satellite problem in which the massless body moves around the lesser mass (planet) or the problem of disturbed planetoid orbits, when the motion occurs about the greater mass (sun) and the distance from the disturbing planet remains sufficiently large.

In these cases it is also possible to apply the aforementioned method rationally, since we are again concerned with a problem of motion of the fourth order, which can be reduced to the third order by applying the Jacobian integral. But here the difficulties are considerably greater, although not insurmountable. The elimination process described becomes much more complicated because the quantities to be eliminated $(\xi, \eta, \dot{\xi}, \dot{\eta})$ appear not only in the distance and velocity of the body in reference to its central mass, but also in the distance and velocity in reference to the disturbing mass. In Hill's problem, the last step of this process leads to the elimination of λ from the two algebraic equations (Equations 13 and Equation 14) which are of the second and fourth degrees. The resultant equation (Equation 15), which solves the problem, is therefore obtained in the form of a six-row Sylvester determinant which is set equal to zero. The same process applied to the *problème restreint* leads to a correspondingly-to-be-determined quantity λ that is to be eliminated from the two algebraic equations of the 10th and 14th degree, so that the final solution appears as a 24-row Sylvester determinant which is equated to zero, the strict solution of which would be hopelessly complicated. However, as long as we restrict ourselves to those cases in which the motion of the satellite or planetoid can be viewed as an only slightly disturbed Kepler orbit, the elimination can be accomplished with the help of a rapidly converging approximation process. The demonstration of this must be reserved for another treatise.

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