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ON ISOMETRIC DEFORMATION OF SCREW SURFACES

By

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SUMMARY

A study of strain-free deformation mechanisms of surfaces that are capable of folding into a flat annular disk of vanishing engulfed volume is presented. The approach used follows a technique of isometric mapping.

INTRODUCTION

The use of packageable and deployable structural devices is of interest for a variety of space applications where relatively large surfaces and/or enclosed volumes are required, but where payload size limitations in available boosters prevent their egress from terrestrial atmosphere in expanded condition. Another source of interest for foldable structural devices rests in the need for containers of variable volume which are useful for expulsion of liquids, particularly of rocket propellants. In either case, reduction of enclosed volume, frontal area, and package size to a minimum is desired.

The problem of finding specific, packageable surfaces has been approached by defining a particular surface in a folded, packaged configuration and, then, by defining possible strain-free expanded geometrical configurations of the same surface for which continuous and topologically permissible isometric transformations exist. Thus, the problem of foldability has been inverted by defining the original surface in the desirable, folded configuration first, and then by generating geometrical shapes that can conform to the desired pattern (reference 1).

SYMBOLS

c	constant
ds	line element
f	radius vector function at original surface
n	integer
p, q, r, s	deformation parameters
S	surface
w	auxiliary function
x, y, z	Euclidian coordinates
α, β, γ	functions describing deformed surfaces
ξ, φ	surface parameters

Plain subscripts refer to partial derivatives, subscripts in parentheses indicate the variable of the function, superscript stars indicate functions referring to the deformed surface.

ANALYSIS

A surface S which can be deformed by bending without stretching, compressing, or tearing, is said to permit an isometric deformation. The necessary and sufficient condition for an isometric deformation is that the line element of S remains unchanged during the deformation process.

It is known that, apart from the group of rigid displacements relative to the Euclidian space, certain surfaces, including sphere and torus, cannot be deformed isometrically without violating some differentiability conditions on the radius vector describing the surface (reference 2). In loosening these constraints and admitting a certain folding mechanism, however, the class of possible isometric deformations of a given surface can be extended considerably. This class may cover the solution to the problem where the volume enclosed or engulfed by the deformed surface is as small as desired.

Let (x, y, z) denote an Euclidian system of coordinates. Consider the surface S described by the parameter equations

$$\left.\begin{aligned} x &= x_{(\xi,\varphi)} = f_{(\xi)} \cos\varphi \\ y &= y_{(\xi,\varphi)} = f_{(\xi)} \sin\varphi \\ z &= z_{(\xi,\varphi)} = 0 \end{aligned}\right\} \qquad (1)$$

where ξ and φ are the surface parameters. The function $f_{(\xi)}$ is assumed to be a continuous, piecewise, differentiable and non-negative function of ξ within the interval $\xi_o \le \xi \le \xi_N$, $\xi_o < \xi_N$, which vanishes identically outside this interval. The surface S is a planar rotational surface generated by a straight line which covers once or several times certain parts of the positive x-axis , according to the particular choice of $f_{(\xi)}$. If it is further assumed that none of the meridians φ and $\varphi + 2\pi n$, n = 0, 1, ... , coincide, then S represents a winding surface with zero advancing rate in z-direction. The volume engulfed by S is apparently zero. A realization of this type of surface is shown in figure 1.

The line element of S is given by the first fundamental form (references 2 and 3)

$$ds^2 = f'^2_{(\xi)}\, d\xi^2 + f^2_{(\xi)}\, d\varphi^2 \qquad (2)$$

where f' denotes the derivative $\frac{df}{d\xi}$.

It is useful to choose the parameter ξ as the arc length of the meridians φ = Constant . From equation (1) it follows then that $f'_{(\xi)} = \pm 1$ for this case. The function $f_{(\xi)}$ can be defined more precisely as follows:

$$f_{(\xi)} = \left\{\begin{array}{ll} c_i + \operatorname{sign}(\Delta\xi_i) \cdot \xi & \text{for} \quad \xi_{i-1} \le \xi \le \xi_i \\ & \qquad i = 1, 2, \ldots, N \\ 0 & \text{for} \quad \xi < \xi_o , \ \xi > \xi_N \end{array}\right\} \qquad (3)$$

Here, ξ_i denote those points of the interval $\xi_o \le \xi \le \xi_N$ where the derivative $f'_{(\xi)}$ is discontinuous; the values of ξ_i may satisfy the inequality $\xi_o < \xi_1 < \xi_2 < \dots < \xi_{N-1} < \xi_N$. sign $(\Delta\xi_i)$ stands for plus or minus one, and $\Delta\xi_i = \xi_i - \xi_{i-1}$. Since $f_{(\xi)}$ is continuous, the constants c_i are subject to the conditions

$$c_i + \mathrm{sign}(\Delta\xi_i) \cdot \xi_i = c_{i+1} + \mathrm{sign}(\Delta\xi_{i+1}) \cdot \xi_i \; , \; i = 1, 2, \dots , N-1 \tag{4}$$

Consider now another surface S^* described by

$$\left.\begin{aligned} x &= x^*_{(\xi, \varphi;\, p, q, \dots)} \\ y &= y^*_{(\xi, \varphi;\, p, q, \dots)} \\ z &= z^*_{(\xi, \varphi;\, p, q, \dots)} \end{aligned}\right\} \tag{5}$$

where the functions x^* , y^* , and z^* depend not only on the variables ξ and φ , but also on some additional deformation parameters $p, q, \dots$. The line element of S^* is given by

$$\begin{aligned} ds^{*2} = (x_\xi^{*2} + y_\xi^{*2} + z_\xi^{*2})\, d\xi^2 &+ 2(x_\xi^* x_\varphi^* + y_\xi^* y_\varphi^* + z_\xi^* z_\varphi^*)\, d\xi\, d\varphi \\ &+ (x_\varphi^{*2} + y_\varphi^{*2} + z_\varphi^{*2})\, d\varphi^2 \end{aligned} \tag{6}$$

The subscripts indicate the partial derivatives of the coordinates x, y, z with respect to the indicated parameters ξ and φ .

The surface S^* is isometric to the surface S if the line elements ds^2 and ds^{*2} are identical, that is if

$$\left.\begin{aligned} x_\xi^{*2} + y_\xi^{*2} + z_\xi^{*2} &= f'^2_{(\xi)} = 1 \\ x_\xi^* x_\varphi^* + y_\xi^* y_\varphi^* + z_\xi^* z_\varphi^* &= 0 \\ x_\varphi^{*2} + y_\varphi^{*2} + z_\varphi^{*2} &= f^2_{(\xi)} \end{aligned}\right\} \tag{7}$$

The second equation of equations (7) is a result of the orthogonality of the parameter lines φ = Constant and ξ = Constant .

The surface S^* is obtained by an isometric deformation of the surface S if equations (7) are satisfied, and a set of parameter values $p_o, q_o, \ldots$ exists such that the equations

$$\left.\begin{aligned} x^*(\xi, \varphi; p_o, q_o, \ldots) &= f_{(\xi)} \cos \varphi \\ y^*(\xi, \varphi; p_o, q_o, \ldots) &= f_{(\xi)} \sin \varphi \\ z^*(\xi, \varphi; p_o, q_o, \ldots) &= 0 \end{aligned}\right\} \tag{8}$$

hold. The problem of finding isometric deformations of the given surface S is thus reduced to solving the system of partial differential equations (7) and satisfying equations (8).

SOLUTIONS

A class of isometric deformations of the original surface S is obtained by letting

$$\left.\begin{aligned} x^* &= \alpha_{(\xi)} \cos p\varphi + \beta_{(\xi)} \sin p\varphi \\ y^* &= \alpha_{(\xi)} \sin p\varphi - \beta_{(\xi)} \cos p\varphi \\ z^* &= \gamma_{(\xi)} + q \cdot \varphi \end{aligned}\right\} \tag{9}$$

with the "twist" and "stretch" deformation parameters p and q respectively, and with still arbitrary functions α , β , and γ , which depend on ξ only. The line element is given by

$$\begin{aligned} ds^{*2} = {} & (\alpha'^2 + \beta'^2 + \gamma'^2)\, d\xi^2 + 2\,[\, p(\beta\alpha' - \alpha\beta') + q\gamma' \,]\, d\xi\, d\varphi \\ & + [\, p^2(\alpha^2 + \beta^2) + q^2 \,]\, d\varphi^2 \end{aligned} \tag{10}$$

The functions $\alpha_{(\xi)}$, $\beta_{(\xi)}$, and $\gamma_{(\xi)}$ have to satisfy the conditions for isometry

$$\left.\begin{aligned} \alpha'^2_{(\xi)} + \beta'^2_{(\xi)} + \gamma'^2_{(\xi)} &= f'^2_{(\xi)} \\ p(\beta_{(\xi)} \cdot \alpha'_{(\xi)} - \alpha_{(\xi)} \cdot \beta'_{(\xi)}) + q\,\gamma'_{(\xi)} &= 0 \\ p^2(\alpha^2_{(\xi)} + \beta^2_{(\xi)} + q^2 &= f^2_{(\xi)} \end{aligned}\right\} \qquad (11)$$

Solutions of equations (11) can be obtained by letting

$$\left.\begin{aligned} \alpha &= \frac{1}{p}\sqrt{f^2 - q^2}\,\cos w \\ \beta &= \frac{1}{p}\sqrt{f^2 - q^2}\,\sin w \end{aligned}\right\} \qquad (12)$$

with the generating function $w_{(\xi)}$. These forms for α and β satisfy apparently the third equation of equations (11). Differentiation leads to

$$\left.\begin{aligned} \alpha' &= \frac{1}{p}\left\{\frac{f\,f'}{\sqrt{f^2 - q^2}}\cos w - w'\sqrt{f^2 - q^2}\,\sin w\right\} \\ \beta' &= \frac{1}{p}\left\{\frac{f\,f'}{\sqrt{f^2 - q^2}}\sin w + w'\sqrt{f^2 - q^2}\,\cos w\right\} \end{aligned}\right\} \qquad (13)$$

Equations (12) and (13) are now to be inserted into the second equation of equations (11) and furnish

$$\gamma = \frac{1}{pq} \cdot (f^2 - q^2) \cdot w' \qquad (14)$$

Inserting equations (13) and (14) into the first equation of equations (11) results in

$$\frac{1}{p^2}\left[\frac{f^2\,f'^2}{f^2 - q^2} + (f^2 - q^2)\,w'^2\right] + \frac{1}{p^2 q^2}(f^2 - q^2)^2\,w'^2 = f'^2 \qquad (15a)$$

or, after solving for w'^2 ,

$$w' = \pm \frac{q}{f(f^2 - q^2)} \sqrt{(p^2 - 1) f^2 - p^2 q^2} \cdot f' \tag{15b}$$

Integration yields

$$w_{(\xi)} = \pm q \int_{f_o}^{f} \frac{\sqrt{(p^2 - 1) f^2 - p^2 q^2}}{f(f^2 - q^2)} \, df + r \tag{16}$$

where r is an integration constant representing a rigid body rotation of the surface around the z-axis.

Inserting expressions (15a) into (14), it follows after integration

$$\gamma_{(\xi)} = \pm \frac{1}{p} \int_{f_o}^{f} \frac{\sqrt{(p^2 - 1) f^2 - p^2 q^2}}{f} \, df + s \tag{17}$$

where s is also an integration constant representing a rigid body translation of the surface along the z-axis.

The surface S^* represented by equations (9) is now given by the parameter representation

$$\left.\begin{aligned}
x^* &= \frac{1}{p} \sqrt{f^2 - g^2} \cdot \cos \left[p\varphi \pm q \int_{f_o}^{f} \frac{\sqrt{(p^2 - 1) f^2 - p^2 q^2}}{f(f^2 - q^2)} \, df - r \right] \\
y^* &= \frac{1}{p} \sqrt{f^2 - q^2} \cdot \sin \left[p\varphi \pm q \int_{f_o}^{f} \frac{\sqrt{(p^2 - 1) f^2 - p^2 q^2}}{f(f^2 - q^2)} \, df - r \right] \\
z^* &= \pm \frac{1}{p} \int_{f_o}^{f} \frac{\sqrt{(p^2 - 1) f^2 - p^2 q^2}}{f} \, df + q \cdot \varphi + s
\end{aligned}\right\} \tag{18}$$

Letting here $p = 1$, $q = r = s = 0$, the original surface S is obtained. Hence, the four-parametric family of surfaces S^* results from isometric deformations of S .

DISCUSSION

In discussing the surfaces S^* , it is observed that the lines ξ = Constant (which include f = Constant) are spirals advancing in z-direction with constant advancing rate q. Their projections into the x, y-plane are circles of radius $\rho_{(\xi)} = \frac{1}{p}\sqrt{f^2_{(\xi)} - q^2}$. Hence, these curves lie on circular cylinders of radius $\rho_{(\xi)}$. Another special feature results from the fact that either the upper or the lower sign of the integrals in equations (18) may be chosen. If $f_{(\xi)}$ is a continuous piecewise differentiable function of ξ as considered earlier, with derivatives $f'_{(\xi)} = +1$ for $\xi_o \le \xi \le \xi_1$, $f'_{(\xi)} = -1$ for $\xi_1 < \xi \le \xi_2$, then

$$df = \begin{Bmatrix} +d\xi \text{ for } \xi_o \le \xi \le \xi_1 \\ \\ -d\xi \text{ for } \xi_1 < \xi \le \xi_2 \end{Bmatrix} \tag{19}$$

Since $f_{(\xi)}$ is positive within the entire interval $\xi_o \le \xi \le \xi_2$, the sign of the integrals may be chosen in such a manner that $\pm df = -d\xi$ for all ξ . Letting $f_{(\xi_o)} = f_o$ and $f_{(\xi_2)} = f_2$, the z-component in equations (18) can be written in the form

$$z^* = -\int_{\xi_o}^{\xi} \frac{\sqrt{(p^2 - 1)\, f^2_{(\xi)} - p^2 q^2}}{f_{(\xi)}}\, d\xi + q \cdot \varphi + S \tag{20}$$

From this equation it can be deduced that the lines $\xi = \xi_o$ and $\xi = \xi_2$ may coincide by choosing the parameters p and q properly. More precisely, the points of the space curves $z = z^*_{(\xi_o, \varphi)}$ and $z = z^*_{(\xi_2, \varphi + 2\pi)}$ may coincide. The condition for that is

$$\int_{\xi_o}^{\xi_2} \frac{\sqrt{(p^2 - 1) f_{(\xi)}^2 - p^2 q^2}}{f_{(\xi)}} d\xi = 2\pi q \tag{21}$$

A realization of such a surface is shown in figure 2. This surface has the topological character of a cylinder, and can maintain this character throughout the deformation process. This requires, however, that a relative sliding motion of the joint $\xi = \xi_o \equiv \xi_2$ can occur.

It should be noted that the surface S^* is real only if the radicand $(p^2 - 1 f_{(\xi)}^2 - p^2 q^2$ is non-negative. Let $f_{(\xi^*)} = f^*$ be the minimum of f within the interval $\xi_o \leq \xi \leq \xi_2$. The reality condition, then, is

$$(p^2 - 1) f^{*2} - p^2 q^2 \leq 0 \tag{22}$$

It results that the "twist parameter p" has to be greater than or equal to unity:

$$p^2 \leq 1 \tag{23}$$

and the advancing rate q is restricted by

$$0 \leq q^2 \leq \left(\frac{p^2 - 1}{p^2}\right) \cdot f^{*2} \tag{24}$$

Finally, it will be observed that isometric surfaces of forms other than those given by equations (9) are possible. A realization of such a surface in which the advance rate is not constant is shown in figure 3.

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Santa Barbara, California
August 29, 1962

REFERENCES

1. Schuerch, H. U., and Schindler, G. M.: A Contribution to the Theory of Folding Deformations in Expandable Structures with a Particular Application to Toroidal Shells. Astro Research Corporation, 20 July 1961 (NASA accession number N62-13366).

2. Blaschke, W.: Differential Geometrie. Springer, Berlin, 1935.

3. Struick, D. J.: Differential Geometry. Addison Wellesley Press, 1950.

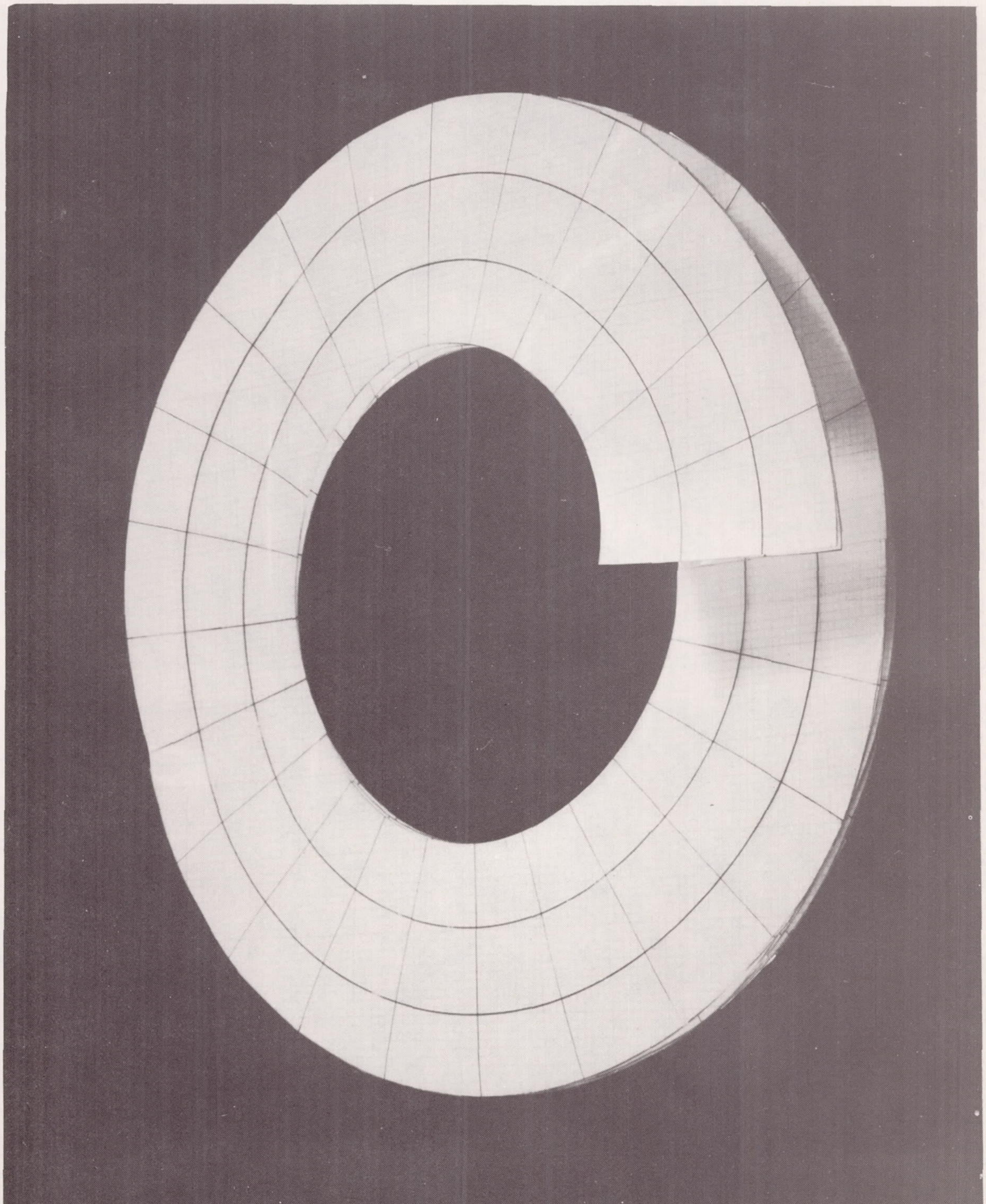

Figure 1. - Folded Configuration of Two-Leaved Screw Surface.

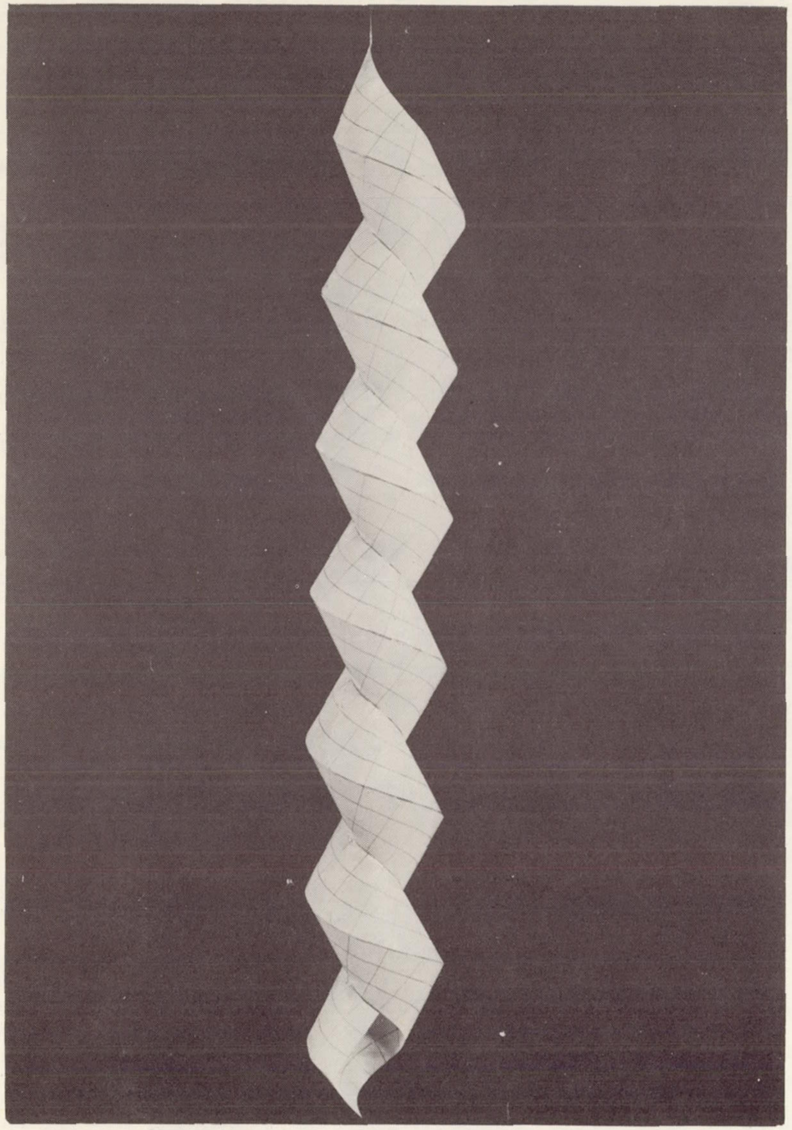

Figure 2. - Expanded Configuration of Two-Leaved Screw Surface Satisfying $\xi_o \equiv \xi_2$.

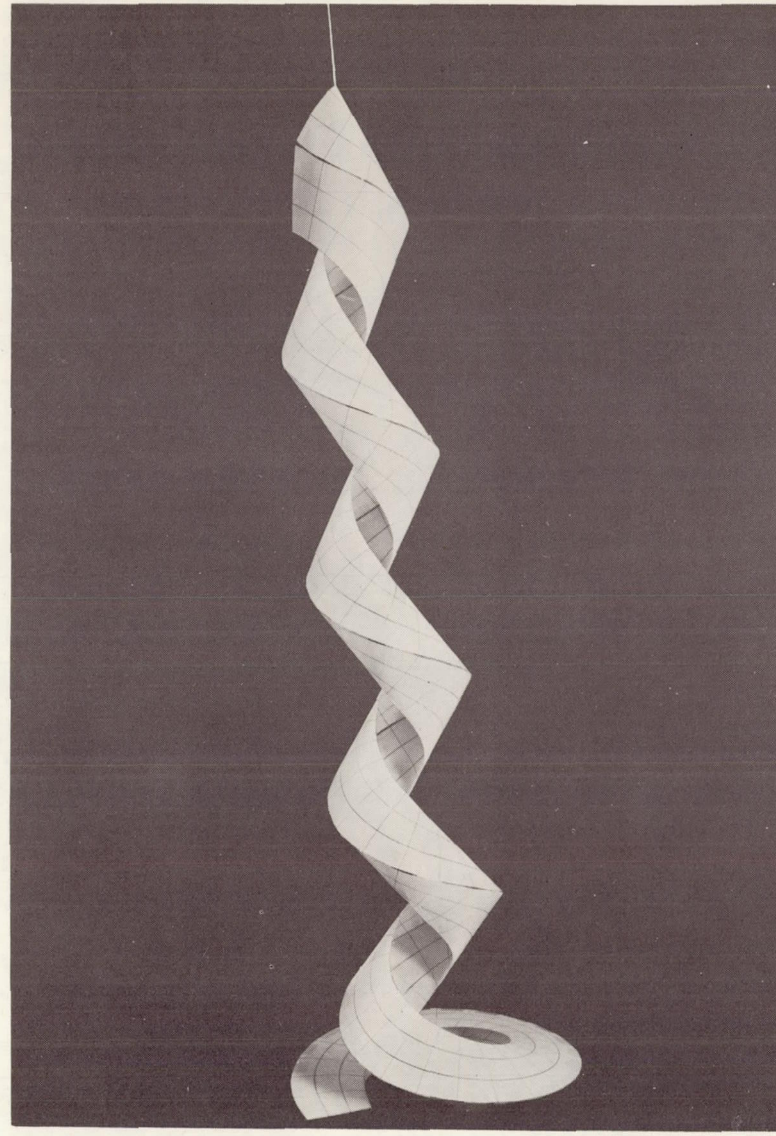

Figure 3. - General Configuration of Two-Leaved Screw Surface.