Technical Report No. 32-261

Vibration of Thin Circular Rings

Part II. Modal Functions and Eigenvalues of Constrained Semicircular Rings

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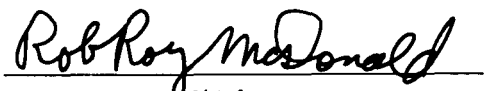

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NOMENCLATURE

a	radius of ring, in.
A	cross-sectional area of ring, in.^{2}
A_{n_i}	constant of modal functions
C_{n_i}	constant of transient solution
e_n	root parameter [Eq. (11)]
E	Young's modulus, lb/in.^{2}
f_n	root parameter [Eq. (11)]
F_R	external force—radial, lb
F_T	external force—tangential, lb
h	thickness of ring, in.
i	$\sqrt{-1}$
I	area moment of inertia of ring cross section, in.^{4}
$M, \mathcal{M}$	in-plane bending moment, in./lb
n	mode number
$N, \mathcal{N}$	circumferential normal force, lb
p	geometric constant
$Q_n(t)$	generalized force
S	arc length
t	time variable, sec
$v(\phi,t)$	tangential displacement, in.
v_0	initial displacement—tangential, in.
$\dot{v}_0$	initial velocity—tangential, in./sec
$\mathcal{V}, \mathcal{V}_n(\phi)$	characteristic or modal function—tangential
$w(\phi,t)$	radial displacement, in.
w_0	initial displacement—radial, in.
$\dot{w}_0$	initial velocity—radial, in./sec
$\mathcal{W}, \mathcal{W}_n(\phi)$	characteristic or modal function—radial
δ_n	root parameter [Eq. (13)]
ϵ_ϕ	extensional strain
$\eta_n(t)$	generalized time variable
λ_{n_i}	root parameter [Eq. (10)]
μ_n	root parameter [Eq. (13)]
ξ_n	characteristic value
ρ	mass density, lb-sec^2/in.
τ	time variable, sec
ϕ	angle coordinate, rad
ϕ^*	central angle of incomplete ring, rad
χ_ϕ	change of curvature, 1/in.
ψ	bending slope, rad
ω_0	natural frequency of pure extensional mode, rad/sec
ω, ω_n	frequency, rad/sec

Subscripts:

i	
m	
n	
ϕ	

ABSTRACT

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The equations of motion of thin circular rings are derived and solved for in-plane inextensional deformation. Modal characteristics are tabulated for the five lowest flexural modes of vibration of semicircular ring segments for the possible combinations of homogeneous boundary conditions that can occur.

I. INTRODUCTION

The solution for the in-plane vibration of thin circular rings involving inextensional deformation is considered further in this Report. In Section V of Part 1 of this Report, an assumption was made that was found to be inconsistent with the inextensional condition, and which led to approximate solutions. To clarify the discrepancy in Part I, the solution of the inextensional equations is derived in Sections III, IV, and V of this part of the Report. Since the tabulated data in Section VII are based upon solutions of the inextensional differential equations, the tabulated data combined with the solution of the equations are complementary information and represent a fundamental analysis of in-plane vibration of thin circular ring segments.

The Tables in Section VII summarize the modal characteristics of semicircular ring segments having the combinations of homogeneous boundary conditions that are admissible.

II. INEXTENSIONAL EQUATIONS OF MOTION

The equations for in-plane motion of an element of a circular ring (see Part 1, Section II for derivation), expressed in terms of the normal stress resultant N and the bending moment M, and neglecting shear deflection and rotary inertia, are

$$\frac{\partial N}{\partial \phi} + \frac{1}{a}\frac{\partial M}{\partial \phi} - \rho a A \frac{\partial^2 v}{\partial t^2} = -F_T(\phi,t) \qquad (1)$$

$$N - \frac{1}{a}\frac{\partial^2 M}{\partial \phi^2} - \rho a A \frac{\partial^2 w}{\partial t^2} = -F_R(\phi,t) \qquad (2)$$

where the variables other than N and M are defined in Fig. 1.

Based upon geometric and elastic considerations, the internal force and moment resultants can be defined in terms of the radial and tangential displacements. First-order approximations of these relationships are

$$N = EA\,\epsilon_\phi = \frac{EA}{a}\left(\frac{\partial v}{\partial \phi} - w\right) \qquad (3)$$

$$M = EI\,\chi_\phi = \frac{EI}{a^2}\left(\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right) \qquad (4)$$

Equations (1)–(4) describe the motion of a ring element including extensional deformation of the ring neutral axis.

However, the lower modes of flexural vibration of thin rings do not involve large extensional deformation of the neutral axis, so that a theory neglecting this effect can be formulated that leads to simpler differential equations. To derive a condition on inextension, first consider the changes in length of a ring neutral axis when deformations occur. Let dS_1 be the length of the undeformed ring element and let dS_2 be the length after deformation. Based upon the configuration shown in Fig. 2, the following equations apply:

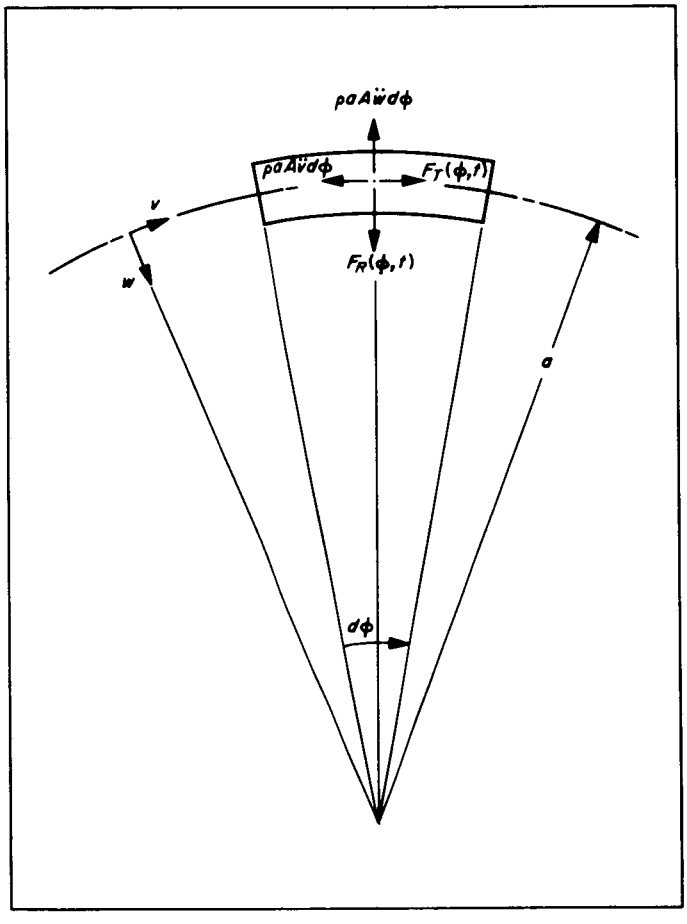

Fig. 1. In-plane forces on a ring element

$$dS_1 = a\, d\phi$$

$$dS_2 = (a - w)\, d\phi + \frac{\partial v}{\partial \phi}\, d\phi$$

But, for inextension $dS_1 = dS_2$ which results in the relation

$$w = \frac{\partial v}{\partial \phi} \tag{5}$$

Equations (1)–(5) represent a dependent set of equations (five equations with four unknowns); however, within the framework of the inextensional theory it is assumed that N is defined by one of the equilibrium equations, Eq. (2), and Eq. (3) is eliminated.

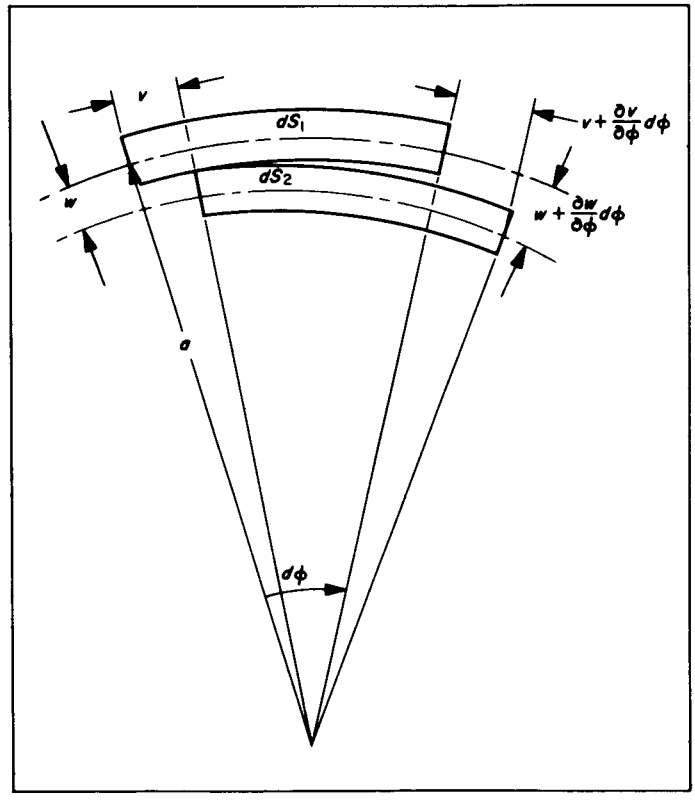

Fig. 2. In-plane deformation of a ring element

III. SOLUTION OF THE HOMOGENEOUS EQUATIONS

In order to evaluate the modal characteristics of a constrained ring, the homogeneous parts of Eq. (1) and (2) may be combined with Eq. (4)–(5), yielding a single sixth-order differential equation in either v or w. The equation is

$$\left(\frac{\partial^6}{\partial\phi^6} + 2\frac{\partial^4}{\partial\phi^4} + \frac{\partial^2}{\partial\phi^2} + \frac{1}{p\omega_0^2}\frac{\partial^4}{\partial\phi^2\partial t^2} - \frac{1}{p\omega_0^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} v \\ w \end{Bmatrix} = 0 \tag{6}$$

where

$$p = \frac{I}{Aa^2}$$

$$\omega_0^2 = \frac{E}{\rho a^2}$$

It can be anticipated that the solutions to Eq. (6) will be an infinite number of modal functions, having corresponding eigenvalues. Therefore, the solution for the nth mode may be assumed of the form

$$v(\phi,t) = \mathcal{V}_n(\phi)\, e^{i\omega_n t} \tag{7}$$

Substituting Eq. (7) into Eq. (6) yields:

$$\frac{d^6\mathcal{V}_n}{d\phi^6} + 2\frac{d^4\mathcal{V}_n}{d\phi^4} + (1-\xi_n)\frac{d^2\mathcal{V}_n}{d\phi^2} + \xi_n\mathcal{V}_n = 0 \tag{8}$$

where

$$\xi_n = \frac{\rho A a^4 \omega_n^2}{EI} \tag{9}$$

Eq. (8) is a linear differential equation with constant coefficients so the standard form of solution for $\mathcal{V}_n$ is

$$\mathcal{V}_n(\phi) = \sum_{i=1}^{6} A_{n_i}\, e^{\lambda_{n_i}\phi} \tag{10}$$

where the λ_{n_i} are the roots of the auxiliary equation. The roots are of three types depending upon the value of ξ_n.

Case I: $\xi_n < 0.1134$

The roots are of the form

$$\lambda_{n_1} = i\,d_n \qquad \lambda_{n_3} = i\,e_n \qquad \lambda_{n_5} = i\,f_n$$
$$\lambda_{n_2} = -i\,d_n \qquad \lambda_{n_4} = -i\,e_n \qquad \lambda_{n_6} = -i\,f_n$$

The modal functions are

$$\begin{aligned}\mathcal{V}_n = {} & A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi \\ & + A_{n_3}\cos e_n\phi + A_{n_4}\sin e_n\phi \\ & + A_{n_5}\cos f_n\phi + A_{n_6}\sin f_n\phi\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{W}_n = \frac{d\mathcal{V}_n}{d\phi} = {} & -A_{n_1}d_n\sin d_n\phi + A_{n_2}d_n\cos d_n\phi \\ & - A_{n_3}e_n\sin e_n\phi + A_{n_4}e_n\cos e_n\phi \\ & - A_{n_5}f_n\sin f_n\phi + A_{n_6}f_n\cos f_n\phi\end{aligned} \tag{12}$$

Case II: $0.1134 < \xi_n < 17.64$

The roots are of the form

$$\lambda_{n_1} = i\,d_n \qquad \lambda_{n_3} = \delta_n + i\,\mu_n \qquad \lambda_{n_5} = -\delta_n - i\,\mu_n$$
$$\lambda_{n_2} = -i\,d_n \qquad \lambda_{n_4} = \delta_n - i\,\mu_n \qquad \lambda_{n_6} = -\delta_n + i\,\mu_n$$

The modal functions are

$$\begin{aligned}\mathcal{V}_n = {} & A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi \\ & + A_{n_3}\cos\mu_n\phi\cosh\delta_n\phi + A_{n_4}\sin\mu_n\phi\cosh\delta_n\phi \\ & + A_{n_5}\cos\mu_n\phi\sinh\delta_n\phi + A_{n_6}\sin\mu_n\phi\sinh\delta_n\phi\end{aligned} \tag{13}$$

$$\begin{aligned}\mathcal{W}_n = {} & -A_{n_1}d_n\sin d_n\phi + A_{n_2}d_n\cos d_n\phi \\ & + (-A_{n_3}\mu_n + A_{n_6}\delta_n)\sin\mu_n\phi\cosh\delta_n\phi \\ & + (A_{n_4}\mu_n + A_{n_5}\delta_n)\cos\mu_n\phi\cosh\delta_n\phi \\ & + (-A_{n_5}\mu_n + A_{n_4}\delta_n)\sin\mu_n\phi\sinh\delta_n\phi \\ & + (A_{n_6}\mu_n + A_{n_3}\delta_n)\cos\mu_n\phi\sinh\delta_n\phi\end{aligned}$$

Case III: $\xi_n > 17.64$

The roots are of the form

$$\lambda_{n_1} = i\,d_n \qquad \lambda_{n_3} = e_n \qquad \lambda_{n_5} = f_n$$
$$\lambda_{n_2} = -i\,d_n \qquad \lambda_{n_4} = -e_n \qquad \lambda_{n_6} = -f_n$$

and the modal functions are

$$\begin{aligned}\mathcal{V}_n &= A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi \\ &+ A_{n_3}\cosh e_n\phi + A_{n_4}\sinh e_n\phi \\ &+ A_{n_5}\cosh f_n\phi + A_{n_6}\sinh f_n\phi\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{W}_n &= -A_{n_1} d_n \sin d_n\phi + A_{n_2} d_n \cos d_n\phi \\ &+ A_{n_3} e_n \sinh e_n\phi + A_{n_4} e_n \cosh e_n\phi \\ &+ A_{n_5} f_n \sinh f_n\phi + A_{n_6} f_n \cosh f_n\phi\end{aligned} \tag{16}$$

The ξ_n and associated A_{n_i} $(i = 1, \cdots\cdots 6)$ can be evaluated only when specific ring configurations (boundary conditions) are defined.

IV. HOMOGENEOUS BOUNDARY CONDITIONS AND ORTHOGONALITY OF THE RING MODES

The natural orthogonality properties of the inextensional ring modal functions can readily be derived from the homogeneous parts of Eq. (1) and (2) plus Eq. (4) and (5). Assuming nth mode solutions for the four dependent variables of the form

$$\begin{aligned}v_n(\phi,t) &= \mathcal{V}_n(\phi)\, e^{i\omega_n t} \\ w_n(\phi,t) &= \mathcal{W}_n(\phi)\, e^{i\omega_n t} \\ M_n(\phi,t) &= \mathcal{M}_n(\phi)\, e^{i\omega_n t} \\ N_n(\phi,t) &= \mathcal{N}_n(\phi)\, e^{i\omega_n t}\end{aligned} \tag{17}$$

the equations reduce to

$$\frac{d\mathcal{N}_n}{d\phi} + \frac{1}{a}\frac{d\mathcal{M}_n}{d\phi} = -\rho aA\,\omega_n^2\,\mathcal{V}_n \tag{18}$$

$$\mathcal{N}_n - \frac{1}{a}\frac{d^2\mathcal{M}_n}{d\phi^2} = -\rho aA\,\omega_n^2\,\mathcal{W}_n \tag{19}$$

$$\frac{a}{EI}\mathcal{M}_n - \frac{1}{a}\frac{d\mathcal{V}_n}{d\phi} - \frac{1}{a}\frac{d^2\mathcal{W}_n}{d\phi^2} = 0 \tag{20}$$

$$\mathcal{W}_n - \frac{d\mathcal{V}_n}{d\phi} = 0 \tag{21}$$

Selecting solutions from another mode (mth) $\mathcal{V}_m$, $\mathcal{W}_m$, $\mathcal{M}_m$, and $\mathcal{N}_m$ and multiplying Eq. (18)–(21) by these quantities in respective order, then integrating over the arc length of the ring and adding, leads to the following equation:

$$\begin{aligned}&\int_0^{\phi^*}\left\{\mathcal{V}_m\left(\frac{d\mathcal{N}_n}{d\phi} + \frac{1}{a}\frac{d\mathcal{M}_n}{d\phi}\right) + \mathcal{W}_m\left(\mathcal{N}_n - \frac{1}{a}\frac{d^2\mathcal{M}_n}{d\phi^2}\right)\right. \\ &\quad + \mathcal{M}_m\left(\frac{a}{EI}\mathcal{M}_n - \frac{1}{a}\frac{d\mathcal{V}_n}{d\phi} - \frac{1}{a}\frac{d^2\mathcal{W}_n}{d\phi^2}\right) \\ &\quad \left. + \mathcal{N}_m\left(\mathcal{W}_n - \frac{d\mathcal{V}_n}{d\phi}\right)\right\} d\phi \\ &= -\omega_n^2\int_0^{\phi^*}\rho aA\,(\mathcal{V}_m\mathcal{V}_n + \mathcal{W}_m\mathcal{W}_n)\,d\phi\end{aligned} \tag{22}$$

Integrating each term of Eq. (22) by parts a number of times corresponding to the order of the derivative in ϕ, and subtracting from this result an equation identical to Eq. (22) except with subscripts reversed, yields

$$\begin{aligned}&(\omega_m^2 - \omega_n^2)\int_0^{\phi^*}\rho aA\,(\mathcal{V}_m\,\mathcal{V}_n + \mathcal{W}_m\,\mathcal{W}_n)\,d\phi \\ &= \left[\mathcal{V}_n\mathcal{N}_m - \mathcal{V}_m\,\mathcal{N}_n - \frac{1}{a}\mathcal{M}_n\left(\mathcal{V}_m + \frac{d\mathcal{W}_m}{d\phi}\right)\right. \\ &\quad + \frac{1}{a}\mathcal{M}_m\left(\mathcal{V}_n + \frac{d\mathcal{W}_n}{d\phi}\right) \\ &\quad \left. - \frac{1}{a}\mathcal{W}_n\frac{d\mathcal{M}_m}{d\phi} + \frac{1}{a}\mathcal{W}_m\frac{d\mathcal{M}_n}{d\phi}\right]_0^{\phi^*}\end{aligned} \tag{23}$$

The right side of Eq. (23) defines the possible types of homogeneous boundary conditions. If a set of these conditions matches the end constraints of a constrained ring, then the orthogonality relation between modes is

$$\int_0^{\phi^*} \rho a A \left(\mathcal{V}_m \mathcal{V}_n + \mathcal{W}_m \mathcal{W}_n\right) d\phi = 0 \qquad m \neq n \tag{24a}$$

Because of the relation expressed by Eq. (5), the above equation can also be written as

$$\int_0^{\phi^*} \rho a A \left(\mathcal{V}_m \mathcal{V}_n + \frac{d\mathcal{V}_m}{d\phi}\frac{d\mathcal{V}_n}{d\phi}\right) d\phi = 0 \qquad m \neq n \tag{24b}$$

When $m = n$, the constants of the modal functions can be adjusted such that

$$\int_0^{\phi^*} \rho a A \left[\mathcal{V}_n^2 + \left(\frac{d\mathcal{V}_n}{d\phi}\right)^2\right] d\phi = \rho a A \phi^* \qquad m = n \tag{25}$$

where $\rho a A \phi^*$ is the total mass of the ring segment.

The possible combinations of terms that make the right side of Eq. (23) equal zero are as follows:

1. Free end

$$\begin{aligned} \frac{d\mathcal{V}}{d\phi} + \frac{d^3\mathcal{V}}{d\phi^3} &= 0 \qquad (\mathcal{M} = 0) \\ \frac{d^2\mathcal{V}}{d\phi^2} + \frac{d^4\mathcal{V}}{d\phi^4} &= 0 \qquad \left(\frac{d\mathcal{M}}{d\phi} = Q = 0\right) \\ (1+\xi)\frac{d\mathcal{V}}{d\phi} - \frac{d^5\mathcal{V}}{d\phi^5} &= 0 \qquad (\mathcal{N} = 0, \text{from Eq. (19)}) \end{aligned} \tag{26}$$

2. Pinned end

$$\begin{aligned} \mathcal{V} &= 0 \\ \frac{d\mathcal{V}}{d\phi} &= 0 \qquad (\mathcal{W} = 0) \\ \frac{d^3\mathcal{V}}{d\phi^3} &= 0 \qquad (\mathcal{M} = 0) \end{aligned} \tag{27}$$

3. Pinned–circumferentially guided end

$$\begin{aligned} \frac{d\mathcal{V}}{d\phi} &= 0 \qquad (\mathcal{W} = 0) \\ \frac{d^3\mathcal{V}}{d\phi^3} &= 0 \qquad (\mathcal{M} = 0) \\ \frac{d^5\mathcal{V}}{d\phi^5} &= 0 \qquad (\mathcal{N} = 0) \end{aligned} \tag{28}$$

4. Pinned–radially guided end

$$\begin{aligned} \mathcal{V} &= 0 \\ \frac{d\mathcal{V}}{d\phi} + \frac{d^3\mathcal{V}}{d\phi^3} &= 0 \qquad (\mathcal{M} = 0) \\ \frac{d^2\mathcal{V}}{d\phi^2} + \frac{d^4\mathcal{V}}{d\phi^4} &= 0 \qquad \left(\frac{d\mathcal{M}}{d\phi} = 0\right) \end{aligned} \tag{29}$$

5. Clamped end

$$\begin{aligned} \mathcal{V} &= 0 \\ \frac{d\mathcal{V}}{d\phi} &= 0 \qquad (\mathcal{W} = 0) \\ \frac{d^2\mathcal{V}}{d\phi^2} &= 0 \\ \left[\text{slope, } \psi = \frac{1}{a}\left(\mathcal{V} + \frac{d\mathcal{W}}{d\phi}\right) = 0\right] \end{aligned} \tag{30}$$

6. Clamped–circumferentially guided end

$$\begin{aligned} \frac{d\mathcal{V}}{d\phi} &= 0 \qquad (\mathcal{W} = 0) \\ \mathcal{V} + \frac{d^2\mathcal{V}}{d\phi^2} &= 0 \qquad (\psi = 0) \\ \frac{d^3\mathcal{V}}{d\phi^3} + \frac{d^5\mathcal{V}}{d\phi^5} &= 0 \qquad (\mathcal{N} = 0) \end{aligned} \tag{31}$$

7. Clamped–radially guided end

$$\begin{aligned} \mathcal{V} &= 0 \\ \frac{d^2\mathcal{V}}{d\phi^2} &= 0 \qquad (\psi = 0) \\ \frac{d^4\mathcal{V}}{d\phi^4} &= 0 \qquad \left(\frac{d\mathcal{M}}{d\phi} = 0\right) \end{aligned} \tag{32}$$

8. Clamped–free end

$$\begin{aligned} \mathcal{V} + \frac{d^2\mathcal{V}}{d\phi^2} &= 0 \qquad (\psi = 0) \\ \frac{d^2\mathcal{V}}{d\phi^2} + \frac{d^4\mathcal{V}}{d\phi^4} &= 0 \qquad \left(\frac{d\mathcal{M}}{d\phi} = 0\right) \\ -\xi\frac{d\mathcal{V}}{d\phi} + \frac{d^3\mathcal{V}}{d\phi^3} + \frac{d^5\mathcal{V}}{d\phi^5} &= 0 \qquad (\mathcal{N} = 0) \end{aligned} \tag{33}$$

In Eq. (26)–(33), where hyphenated words are used to describe the boundary conditions, the first word refers to the condition on rotational motion and the second word refers to translational motion. For example, "clamped–free" means the end of the ring cannot rotate but can translate radially and circumferentially.

V. FORCED-MOTION SOLUTION

Equations (1) and (2) are the coupled forced-motion equations for a ring element. In solving these equations a superposition of normal modes is assumed to represent the deflected shape of the ring for any particular type of forced excitation; therefore, let

$$\begin{aligned} v(\phi,t) &= \sum_{n=1}^{\infty} \mathcal{V}_n(\phi)\,\eta_n(t) \\ w(\phi,t) &= \sum_{n=1}^{\infty} \mathcal{W}_n(\phi)\,\eta_n(t) \\ M(\phi,t) &= \sum_{n=1}^{\infty} \mathcal{M}_n(\phi)\,\eta_n(t) \\ N(\phi,t) &= \sum_{n=1}^{\infty} \mathcal{N}_n(\phi)\,\eta_n(t) \end{aligned} \tag{34}$$

Substituting Eq. (34) into Eq. (1) and (2) yields

$$\sum_{n=1}^{\infty} \left\{ \rho a A \mathcal{V}_n \ddot{\eta}_n - \left[\frac{d\mathcal{N}_n}{d\phi} + \frac{1}{a}\frac{d\mathcal{M}_n}{d\phi} \right] \eta_n \right\} = F_T(\phi,t)$$

$$\sum_{n=1}^{\infty} \left\{ \rho a A \mathcal{W}_n \ddot{\eta}_n - \left[\mathcal{N}_n - \frac{1}{a}\frac{d^2\mathcal{M}_n}{d\phi^2} \right] \eta_n \right\} = F_R(\phi,t)$$

The coefficients of η_n in each equation can be simplified by use of Eq. (18) and (19). In addition, the orthogonality relation of the natural modes, Eq. (24), can be imposed to eliminate the summation in order to solve for each η_n. The result of these considerations is

$$\int_0^{\phi^*} \rho a A \mathcal{V}_n^2\, d\phi\, (\ddot{\eta}_n + \omega_n^2 \eta_n) = \int_0^{\phi^*} F_T(\phi,t)\, \mathcal{V}_n\, d\phi$$

$$\int_0^{\phi^*} \rho a A \mathcal{W}_n^2\, d\phi\, (\ddot{\eta}_n + \omega_n^2 \eta_n) = \int_0^{\phi^*} F_R(\phi,t)\, \mathcal{W}_n\, d\phi$$

Adding these equations results in the standard differential form for solving for forced motion. The equation is

$$\ddot{\eta}_n + \omega_n^2 \eta_n = Q_n(t) \tag{35}$$

where the generalized force $Q_n(t)$ is

$$Q_n(t) = \frac{\int_0^{\phi^*} (F_T(\phi,t)\,\mathcal{V}_n + F_R(\phi,t)\,\mathcal{W}_n)\, d\phi}{\int_0^{\phi^*} \rho a A (\mathcal{V}_n^2 + \mathcal{W}_n^2)\, d\phi} \tag{36}$$

The complete solution, including the transient part, to Eq. (35) is derived in Part 1, Section IV, of this Report, and for the tangential displacement it is

$$v(\phi,t) = \sum_{n=1}^{\infty} \mathcal{V}_n(\phi) \cdot \left[\frac{1}{\omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t-\tau) d\tau + C_{n_1} \sin \omega_n t + C_{n_2} \cos \omega_n t \right] \tag{37}$$

and for the radial displacement

$$w(\phi,t) = \sum_{n=1}^{\infty} \mathcal{W}_n(\phi) \cdot \left[\frac{1}{\omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t-\tau) d\tau + C_{n_1} \sin \omega_n t + C_{n_2} \cos \omega_n t \right] \tag{38}$$

The constants C_{n_1} and C_{n_2} depend upon the initial conditions on displacement and velocity at time $t = 0$. Let

$$\begin{aligned} v(\phi,0) &= v_0 \\ \dot{v}(\phi,0) &= \dot{v}_0 \\ w(\phi,0) &= w_0 \\ \dot{w}(\phi,0) &= \dot{w}_0 \end{aligned}$$

Then

$$C_{n_1} = \frac{\int_0^{\phi^*} [\dot{v}_0\,\mathcal{V}_n + \dot{w}_0\,\mathcal{W}_n]\, d\phi}{\omega_n \int_0^{\phi^*} [\mathcal{V}_n^2 + \mathcal{W}_n^2]\, d\phi} \tag{39}$$

and

$$C_{n_2} = \frac{\int_0^{\phi^*} [v_0\,\mathcal{V}_n + w_0\,\mathcal{W}_n]\, d\phi}{\int_0^{\phi^*} [\mathcal{V}_n^2 + \mathcal{W}_n^2]\, d\phi} \tag{40}$$

Hence the complete solution in integral form for the ring displacement is made up of Eq. (37) or (38) and Eq. (39) and (40). These displacement solutions can be substituted into Eq. (2) or (4), to evaluate the normal force and bending moment.

In Eq. (36)–(40) all of the terms are defined except the modal functions $\mathcal{V}_n(\phi)$ and $\mathcal{W}_n(\phi)$, and the eigenvalues ω_n or more generally ξ_n [see Eq. (9)]. These terms cannot be determined unless a specific ring configuration with known boundary conditions is defined. The characteristic

values ξ_n and the coefficients of the modal functions $\mathcal{V}_n(\phi)$ and $\mathcal{W}_n(\phi)$ change in value with changes in arc length so that a plot of these functions vs arc length or included angle (ϕ^*) would need to be derived for each combination of boundary conditions to define completely the in-plane modal properties of rings. Obtaining plots for all possible combinations of homogeneous boundary conditions would be a large effort that could be justified only if ring segments were used as common structural members as beams are presently.

VI. TABLES OF RING MODAL PROPERTIES

The Tables that follow summarize the modal characteristics for semicircular rings ($\phi^* = \pi$) for the 36 combinations of homogeneous boundary conditions that are possible. By considering symmetry the results are applicable for complete rings when supported at two diametrical points only. For example, a complete ring clamped at one point has symmetric modes corresponding to the modes of a clamped: clamped–radially guided semicircular ring and asymmetric modes corresponding to the modes of a clamped: pinned–circumferentially guided semicircular ring. The natural frequencies of the semicircular ring would also be the natural frequencies of the complete ring. The Tables cannot be used to define modal properties of rings constrained other than at diametrically opposite points, but for these configurations the solutions can be computed using the general equations derived in Sections III and IV.

The five lowest in-plane flexural natural frequencies of any uniform semicircular ring can be computed using Eq. (9) and the values of ξ_n listed in the second column on the left in the Tables. The modal functions given were normalized using the orthogonality integral Eq. (25) such that

$$\int_0^\pi (\mathcal{V}_n^2 + \mathcal{W}_n^2)\,d\phi = \int_0^\pi (\mathcal{V}_n^2 + \mathcal{V}_n'^2)\,d\phi = \pi \qquad (41)$$

where in Eq. (41) and in the Tables a prime ($'$) represents one derivative with respect to ϕ.

The numerical computations were performed on an IBM 7090 digital computer. In computing the characteristic values ξ_n, the iteration process was continued until five significant figures were obtained. These numbers were rounded off to four significant figures prior to listing them in the Tables.

Table 1. Free: free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	3.375	1.764	0.8937	0.4927	—	—
2	28.18	2.587	—	—	1.866	1.100
3	123.5	3.543	—	—	3.085	1.017
4	361.6	4.525	—	—	4.179	1.006
5	838.6	5.516	—	—	5.237	1.002

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$(1 + \xi_n)\,\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad (1 + \xi_n)\,\mathcal{V}_n'(\pi) - \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2040 \cos d_1\phi + 0.5249 \sin d_1\phi - 0.8628 \cos \mu_1\phi \cosh \delta_1\phi + 0.7803 \sin \mu_1\phi \cosh \delta_1\phi + 0.7770 \cos \mu_1\phi \sinh \delta_1\phi - 0.6794 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.2969 \cos d_2\phi + 0.3913 \sin d_2\phi - 0.7772 \cosh e_2\phi + 0.7728 \sinh e_2\phi + 0.3080 \cosh f_2\phi - 0.2891 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2496 \cos d_3\phi + 0.2854 \sin d_3\phi - 0.3624 \cosh e_3\phi + 0.3624 \sinh e_3\phi + 0.03461 \cosh f_3\phi - 0.03757 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2060 \cos d_4\phi + 0.2228 \sin d_4\phi - 0.2548 \cosh e_4\phi + 0.2548 \sinh e_4\phi + 0.01284 \cosh f_4\phi - 0.01179 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1725 \cos d_5\phi + 0.1816 \sin d_5\phi - 0.1982 \cosh e_5\phi + 0.1982 \sinh e_5\phi + 0.004518 \cosh f_5\phi - 0.004923 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3599 \sin d_1\phi + 0.9260 \cos d_1\phi - 0.1820 \sin \mu_1\phi \cosh \delta_1\phi - 1.106 \cos \mu_1\phi \sinh \delta_1\phi + 1.079 \cos \mu_1\phi \cosh \delta_1\phi + 0.3145 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.7679 \sin d_2\phi + 1.012 \cos d_2\phi - 1.450 \sinh e_2\phi + 1.442 \cosh e_2\phi + 0.3387 \sinh f_2\phi - 0.3180 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8844 \sin d_3\phi + 1.011 \cos d_3\phi - 1.118 \sinh e_3\phi + 1.118 \cosh e_3\phi + 0.03521 \sinh f_3\phi - 0.03821 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9320 \sin d_4\phi + 1.008 \cos d_4\phi - 1.065 \sinh e_4\phi + 1.065 \cosh e_4\phi + 0.01291 \sinh f_4\phi - 0.01186 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9515 \sin d_5\phi + 1.002 \cos d_5\phi - 1.038 \sinh e_5\phi + 1.038 \cosh e_5\phi + 0.004529 \sinh f_5\phi - 0.004935 \cosh f_5\phi$$

Table 2. Free: pinned semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.8442	1.454	0.5871	0.5358	—	—
2	13.65	2.246	1.258	0.2478	—	—
3	81.22	3.230	—	—	2.717	1.027
4	269.3	4.227	—	—	3.854	1.008
5	669.2	5.229	—	—	4.933	1.003

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$(1+\xi_n)\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n'''(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.02195 \cos d_1\phi + 0.6169 \sin d_1\phi - 1.018 \cos \mu_1\phi \cosh \delta_1\phi + 0.7287 \sin \mu_1\phi \cosh \delta_1\phi + 1.122 \cos \mu_1\phi \sinh \delta_1\phi - 0.5600 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.3188 \cos d_2\phi + 0.4644 \sin d_2\phi - 0.6717 \cos \mu_2\phi \cosh \delta_2\phi + 1.697 \sin \mu_2\phi \cosh \delta_2\phi + 0.6045 \cos \mu_2\phi \sinh \delta_2\phi - 1.660 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2688 \cos d_3\phi + 0.3181 \sin d_3\phi - 0.4317 \cosh e_3\phi + 0.4316 \sinh e_3\phi + 0.1049 \cosh f_3\phi - 0.06205 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2197 \cos d_4\phi + 0.2404 \sin d_4\phi - 0.2812 \cosh e_4\phi + 0.2812 \sinh e_4\phi - 0.01595 \cosh f_4\phi - 0.01620 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1829 \cos d_5\phi + 0.1939 \sin d_5\phi - 0.2138 \cosh e_5\phi + 0.2138 \sinh e_5\phi + 0.03175 \cosh f_5\phi - 0.006275 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.03192 \sin d_1\phi + 0.8971 \cos d_1\phi + 0.2168 \sin \mu_1\phi \cosh \delta_1\phi + 1.049 \cos \mu_1\phi \cosh \delta_1\phi - 0.1732 \sin \mu_1\phi \sinh \delta_1\phi - 0.8978 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.7161 \sin d_2\phi + 1.043 \cos d_2\phi - 1.922 \sin \mu_2\phi \cosh \delta_2\phi + 1.181 \cos \mu_2\phi \cosh \delta_2\phi + 1.986 \sin \mu_2\phi \sinh \delta_2\phi - 1.257 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8684 \sin d_3\phi + 1.028 \cos d_3\phi - 1.173 \sinh e_3\phi + 1.172 \cosh e_3\phi + 0.1077 \sinh f_3\phi - 0.06372 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9288 \sin d_4\phi + 1.016 \cos d_4\phi - 1.083 \sinh e_4\phi + 1.083 \cosh e_4\phi - 0.01607 \sinh f_4\phi - 0.01632 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9564 \sin d_5\phi + 1.014 \cos d_5\phi - 1.054 \sinh e_5\phi + 1.054 \cosh e_5\phi + 0.03185 \sinh f_5\phi - 0.006294 \cosh f_5\phi$$

Table 3. Free: pinned–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.9067	1.467	0.6022	0.5352	—	—
2	15.75	2.308	1.300	0.1690	—	—
3	86.56	3.275	—	—	2.771	1.025
4	279.9	4.264	—	—	3.895	1.007
5	685.9	5.259	—	—	4.965	1.003

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n'''(\pi) = 0$$

$$(1 + \xi_n)\,\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.06616 \cos d_1\phi + 0.6377 \sin d_1\phi - 1.101 \cos \mu_1\phi \cosh \delta_1\phi + 0.7589 \sin \mu_1\phi \cosh \delta_1\phi + 1.143 \cos \mu_1\phi \sinh \delta_1\phi - 0.7150 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.3100 \cos d_2\phi + 0.4490 \sin d_2\phi - 0.5771 \cos \mu_2\phi \cosh \delta_2\phi + 2.469 \sin \mu_2\phi \cosh \delta_2\phi + 0.5782 \cos \mu_2\phi \sinh \delta_2\phi - 2.470 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2656 \cos d_3\phi + 0.3116 \sin d_3\phi - 0.4180 \cosh e_3\phi + 0.4180 \sinh e_3\phi + 0.05731 \cosh f_3\phi - 0.05712 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2171 \cos d_4\phi + 0.2374 \sin d_4\phi - 0.2767 \cosh e_4\phi + 0.2767 \sinh e_4\phi + 0.01555 \cosh f_4\phi - 0.01549 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1814 \cos d_5\phi + 0.1921 \sin d_5\phi - 0.2115 \cosh e_5\phi + 0.2115 \sinh e_5\phi + 0.006120 \cosh f_5\phi - 0.006098 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.09706 \sin d_1\phi + 0.9356 \cos d_1\phi + 0.1584 \sin \mu_1\phi \cosh \delta_1\phi + 1.094 \cos \mu_1\phi \cosh \delta_1\phi - 0.1547 \sin \mu_1\phi \sinh \delta_1\phi - 1.045 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.7153 \sin d_2\phi + 1.036 \cos d_2\phi - 3.114 \sin \mu_2\phi \cosh \delta_2\phi + 1.169 \cos \mu_2\phi \cosh \delta_2\phi + 3.113 \sin \mu_2\phi \sinh \delta_2\phi - 1.168 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8699 \sin d_3\phi + 1.021 \cos d_3\phi - 1.158 \sinh e_3\phi + 1.158 \cosh e_3\phi + 0.05875 \sinh f_3\phi - 0.05857 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9258 \sin d_4\phi + 1.012 \cos d_4\phi - 1.078 \sinh e_4\phi + 1.078 \cosh e_4\phi + 0.01566 \sinh f_4\phi - 0.01561 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9541 \sin d_5\phi + 1.010 \cos d_5\phi - 1.050 \sinh e_5\phi + 1.050 \cosh e_5\phi + 0.006138 \sinh f_5\phi - 0.006116 \cosh f_5\phi$$

Table 4. Free: pinned–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	2.905	1.723	0.8585	0.5018	—	—
2	27.92	2.582	—	—	1.858	1.101
3	123.3	3.541	—	—	3.083	1.017
4	361.5	4.525	—	—	4.179	1.006
5	838.5	5.516	—	—	5.237	1.002

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) = 0$$

$$(1 + \xi_n)\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2205 \cos d_1\phi + 0.5609 \sin d_1\phi - 0.9864 \cos \mu_1\phi \cosh \delta_1\phi + 0.8035 \sin \mu_1\phi \cosh \delta_1\phi + 0.8446 \cos \mu_1\phi \sinh \delta_1\phi - 0.7730 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.2977 \cos d_2\phi + 0.3939 \sin d_2\phi - 0.7884 \cosh e_2\phi + 0.7835 \sinh e_2\phi + 0.3295 \cosh f_2\phi - 0.2963 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2500 \cos d_3\phi + 0.2858 \sin d_3\phi - 0.3630 \cosh e_3\phi + 0.3630 \sinh e_3\phi + 0.02813 \cosh f_3\phi - 0.03768 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2060 \cos d_4\phi + 0.2229 \sin d_4\phi - 0.2549 \cosh e_4\phi + 0.2549 \sinh e_4\phi + 0.01592 \cosh f_4\phi - 0.01180 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1725 \cos d_5\phi + 0.1816 \sin d_5\phi - 0.1982 \cosh e_5\phi + 0.1982 \sinh e_5\phi + 0.002702 \cosh f_5\phi - 0.004924 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3801 \sin d_1\phi + 0.9667 \cos d_1\phi - 0.1686 \sin \mu_1\phi \cosh \delta_1\phi + 1.128 \cos \mu_1\phi \cosh \delta_1\phi + 0.2660 \sin \mu_1\phi \sinh \delta_1\phi - 1.235 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.7685 \sin d_2\phi + 1.017 \cos d_2\phi - 1.465 \sinh e_2\phi + 1.456 \cosh e_2\phi + 0.3628 \sinh f_2\phi - 0.3262 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8854 \sin d_3\phi + 1.012 \cos d_3\phi - 1.119 \sinh e_3\phi + 1.119 \cosh e_3\phi + 0.02861 \sinh f_3\phi - 0.03833 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9321 \sin d_4\phi + 1.008 \cos d_4\phi - 1.065 \sinh e_4\phi + 1.065 \cosh e_4\phi + 0.01601 \sinh f_4\phi - 0.01187 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9515 \sin d_5\phi + 1.002 \cos d_5\phi - 1.038 \sinh e_5\phi + 1.038 \cosh e_5\phi + 0.002708 \sinh f_5\phi - 0.004936 \cosh f_5\phi$$

Table 5. Free: clamped semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.1894	1.248	0.2525	0.5338	—	—
2	1.891	1.618	0.7610	0.5205	—	—
3	22.18	2.466	—	—	1.662	1.149
4	110.6	3.456	—	—	2.985	1.019
5	338.3	4.455	—	—	4.103	1.006

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$(1 + \xi_n)\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2898 \cos d_1\phi - 0.2431 \sin d_1\phi + 1.378 \cos \mu_1\phi \cosh \delta_1\phi - 0.2604 \sin \mu_1\phi \cosh \delta_1\phi - 0.8196 \cos \mu_1\phi \sinh \delta_1\phi + 0.5699 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.1782 \cos d_2\phi + 0.6100 \sin d_2\phi - 0.6470 \cos \mu_2\phi \cosh \delta_2\phi + 0.8019 \sin \mu_2\phi \cosh \delta_2\phi + 0.9697 \cos \mu_2\phi \sinh \delta_2\phi - 0.7020 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3128 \cos d_3\phi + 0.4188 \sin d_3\phi - 1.049 \cosh e_3\phi + 1.074 \sinh e_3\phi + 0.3974 \cosh f_3\phi - 0.5474 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2564 \cos d_4\phi + 0.2966 \sin d_4\phi - 0.3826 \cosh e_4\phi + 0.3825 \sinh e_4\phi + 0.1138 \cosh f_4\phi - 0.04318 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2104 \cos d_5\phi + 0.2280 \sin d_5\phi - 0.2619 \cosh e_5\phi + 0.2619 \sinh e_5\phi - 0.03738 \cosh f_5\phi - 0.01274 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3617 \sin d_1\phi - 0.3034 \cos d_1\phi - 0.5915 \sin \mu_1\phi \cosh \delta_1\phi - 0.3459 \cos \mu_1\phi \cosh \delta_1\phi + 0.3717 \sin \mu_1\phi \sinh \delta_1\phi + 0.6520 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.2883 \sin d_2\phi + 0.9867 \cos d_2\phi - 0.1975 \sin \mu_2\phi \cosh \delta_2\phi + 1.155 \cos \mu_2\phi \cosh \delta_2\phi + 0.1056 \sin \mu_2\phi \sinh \delta_2\phi - 0.8578 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.7715 \sin d_3\phi + 1.033 \cos d_3\phi - 1.742 \sinh e_3\phi + 1.784 \cosh e_3\phi + 0.4567 \sinh f_3\phi - 0.6291 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.8861 \sin d_4\phi + 1.025 \cos d_4\phi - 1.142 \sinh e_4\phi + 1.142 \cosh e_4\phi + 0.1160 \sinh f_4\phi - 0.04402 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9375 \sin d_5\phi + 1.016 \cos d_5\phi - 1.075 \sinh e_5\phi + 1.075 \cosh e_5\phi - 0.03761 \sinh f_5\phi - 0.01281 \cosh f_5\phi$$

Table 6. Free: clamped–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.4138	1.342	0.4360	0.5376	—	—
2	1.997	1.630	0.7732	0.5186	—	—
3	23.27	2.490	—	—	1.705	1.136
4	115.1	3.487	—	—	3.021	1.019
5	349.3	4.489	—	—	4.140	1.006

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$(1 + \xi_n)\,\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.03644 \cos d_1\phi - 0.3642 \sin d_1\phi + 1.358 \cos \mu_1\phi \cosh \delta_1\phi - 0.4060 \sin \mu_1\phi \cosh \delta_1\phi - 0.7987 \cos \mu_1\phi \sinh \delta_1\phi + 0.9760 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.1537 \cos d_2\phi + 0.5605 \sin d_2\phi - 0.4966 \cos \mu_2\phi \cosh \delta_2\phi + 0.7439 \sin \mu_2\phi \cosh \delta_2\phi + 0.8844 \cos \mu_2\phi \sinh \delta_2\phi - 0.5862 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3125 \cos d_3\phi + 0.4195 \sin d_3\phi - 0.9865 \cosh e_3\phi + 1.001 \sinh e_3\phi + 0.3846 \cosh f_3\phi - 0.4755 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2556 \cos d_4\phi + 0.2942 \sin d_4\phi - 0.3774 \cosh e_4\phi + 0.3773 \sinh e_4\phi + 0.07419 \cosh f_4\phi - 0.04128 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2089 \cos d_5\phi + 0.2262 \sin d_5\phi - 0.2593 \cosh e_5\phi + 0.2593 \sinh e_5\phi - 0.006145 \cosh f_5\phi - 0.01231 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.04892 \sin d_1\phi - 0.4889 \cos d_1\phi - 0.3048 \sin \mu_1\phi \cosh \delta_1\phi - 0.5665 \cos \mu_1\phi \cosh \delta_1\phi + 0.2524 \sin \mu_1\phi \sinh \delta_1\phi + 1.117 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.2505 \sin d_2\phi + 0.9137 \cos d_2\phi - 0.1957 \sin \mu_2\phi \cosh \delta_2\phi + 1.070 \cos \mu_2\phi \cosh \delta_2\phi + 0.1165 \sin \mu_2\phi \sinh \delta_2\phi - 0.6880 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.7780 \sin d_3\phi + 1.045 \cos d_3\phi - 1.682 \sinh e_3\phi + 1.707 \cosh e_3\phi + 0.4371 \sinh f_3\phi - 0.5403 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.8916 \sin d_4\phi + 1.026 \cos d_4\phi - 1.140 \sinh e_4\phi + 1.140 \cosh e_4\phi + 0.07556 \sinh f_4\phi - 0.04205 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9377 \sin d_5\phi + 1.015 \cos d_5\phi - 1.073 \sinh e_5\phi + 1.073 \cosh e_5\phi - 0.006181 \sinh f_5\phi - 0.01238 \cosh f_5\phi$$

Table 7. Free: clamped–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.1916	1.249	0.2556	0.5339	—	—
2	4.573	1.852	0.9669	0.4688	—	—
3	40.55	2.787	—	—	2.154	1.061
4	162.8	3.769	—	—	3.343	1.013
5	449.7	4.761	—	—	4.434	1.005

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$
$$(1 + \xi_n)\,\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2886 \cos d_1\phi - 0.2900 \sin d_1\phi + 1.391 \cos \mu_1\phi \cosh \delta_1\phi - 0.3107 \sin \mu_1\phi \cosh \delta_1\phi - 0.9672 \cos \mu_1\phi \sinh \delta_1\phi + 0.5868 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.2806 \cos d_2\phi + 0.5596 \sin d_2\phi - 0.8065 \cos \mu_2\phi \cosh \delta_2\phi + 0.9101 \sin \mu_2\phi \cosh \delta_2\phi + 0.8020 \cos \mu_2\phi \sinh \delta_2\phi - 0.9135 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2923 \cos d_3\phi + 0.3687 \sin d_3\phi - 0.6010 \cosh e_3\phi + 0.6010 \sinh e_3\phi + 0.1553 \cosh f_3\phi - 0.1556 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2398 \cos d_4\phi + 0.2695 \sin d_4\phi - 0.3308 \cosh e_4\phi + 0.3308 \sinh e_4\phi + 0.02770 \cosh f_4\phi - 0.02779 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1978 \cos d_5\phi + 0.2122 \sin d_5\phi - 0.2392 \cosh e_5\phi + 0.2392 \sinh e_5\phi + 0.009387 \cosh f_5\phi - 0.009421 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3605 \sin d_1\phi - 0.3622 \cos d_1\phi - 0.5931 \sin \mu_1\phi \cosh \delta_1\phi - 0.4131 \cos \mu_1\phi \cosh \delta_1\phi + 0.4369 \sin \mu_1\phi \sinh \delta_1\phi + 0.6690 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.5196 \sin d_2\phi + 1.036 \cos d_2\phi - 0.5051 \sin \mu_2\phi \cosh \delta_2\phi + 1.202 \cos \mu_2\phi \cosh \delta_2\phi + 0.5039 \sin \mu_2\phi \sinh \delta_2\phi - 1.208 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8146 \sin d_3\phi + 1.027 \cos d_3\phi - 1.295 \sinh e_3\phi + 1.295 \cosh e_3\phi + 0.1647 \sinh f_3\phi - 0.1651 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9037 \sin d_4\phi + 1.016 \cos d_4\phi - 1.106 \sinh e_4\phi + 1.106 \cosh e_4\phi + 0.02805 \sinh f_4\phi - 0.02815 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9416 \sin d_5\phi + 1.010 \cos d_5\phi - 1.061 \sinh e_5\phi + 1.061 \cosh e_5\phi + 0.009430 \sinh f_5\phi - 0.009464 \cosh f_5\phi$$

Table 8. Free: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.4197	1.344	0.4391	0.5377	—	—
2	6.133	1.945	1.040	0.4363	—	—
3	42.43	2.813	—	—	2.190	1.057
4	165.1	3.780	—	—	3.356	1.013
5	452.0	4.767	—	—	4.440	1.004

Boundary conditions:

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$(1 + \xi_n)\mathcal{V}_n'(0) - \mathcal{V}_n^{v}(0) = 0 \qquad -\xi_n\mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.03181\cos d_1\phi - 0.4196\sin d_1\phi + 1.379\cos\mu_1\phi\cosh\delta_1\phi - 0.4681\sin\mu_1\phi\cosh\delta_1\phi - 0.9157\cos\mu_1\phi\sinh\delta_1\phi + 0.9995\sin\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.2593\cos d_2\phi + 0.4979\sin d_2\phi - 0.6147\cos\mu_2\phi\cosh\delta_2\phi + 0.9073\sin\mu_2\phi\cosh\delta_2\phi + 0.6936\cos\mu_2\phi\sinh\delta_2\phi - 0.8706\sin\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2879\cos d_3\phi + 0.3596\sin d_3\phi - 0.5761\cosh e_3\phi + 0.5764\sinh e_3\phi + 0.1100\cosh f_3\phi - 0.1432\sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2380\cos d_4\phi + 0.2675\sin d_4\phi - 0.3278\cosh e_4\phi + 0.3278\sinh e_4\phi + 0.04526\cosh f_4\phi - 0.02725\sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1972\cos d_5\phi + 0.2116\sin d_5\phi - 0.2384\cosh e_5\phi + 0.2384\sinh e_5\phi - 0.001705\cosh f_5\phi - 0.009353\sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.04276\sin d_1\phi - 0.5640\cos d_1\phi - 0.3027\sin\mu_1\phi\cosh\delta_1\phi - 0.6538\cos\mu_1\phi\cosh\delta_1\phi + 0.2868\sin\mu_1\phi\sinh\delta_1\phi + 1.143\cos\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.5044\sin d_2\phi + 0.9686\cos d_2\phi - 0.6376\sin\mu_2\phi\cosh\delta_2\phi + 1.118\cos\mu_2\phi\cosh\delta_2\phi + 0.6414\sin\mu_2\phi\sinh\delta_2\phi - 1.019\cos\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8098\sin d_3\phi + 1.012\cos d_3\phi - 1.262\sinh e_3\phi + 1.262\cosh e_3\phi + 0.1163\sinh f_3\phi - 0.1514\cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.8997\sin d_4\phi + 1.011\cos d_4\phi - 1.100\sinh e_4\phi + 1.100\cosh e_4\phi + 0.04583\sinh f_4\phi - 0.02759\cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9402\sin d_5\phi + 1.008\cos d_5\phi - 1.059\sinh e_5\phi + 1.059\cosh e_5\phi - 0.001713\sinh f_5\phi - 0.009395\cosh f_5\phi$$

Table 9. Pinned: pinned semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	5.138	1.888	0.9958	0.4572	—	—
2	47.93	2.886	—	—	2.286	1.050
3	195.4	3.927	—	—	3.522	1.011
4	520.7	4.928	—	—	4.613	1.004
5	1151	5.950	—	—	5.692	1.002

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}'_n(0) = 0 \qquad \mathcal{V}'_n(\pi) = 0$$
$$\mathcal{V}'''_n(0) = 0 \qquad \mathcal{V}'''_n(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.6537 \cos d_1\phi - 0.1161 \sin d_1\phi - 0.6537 \cos \mu_1\phi \cosh \delta_1\phi - 0.7845 \sin \mu_1\phi \cosh \delta_1\phi + 0.5803 \cos \mu_1\phi \sinh \delta_1\phi + 0.7399 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.4625 \cos d_2\phi - 0.08394 \sin d_2\phi + 0.2419 \cosh e_2\phi - 0.2422 \sinh e_2\phi - 0.7043 \cosh f_2\phi + 0.7584 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3495 \cos d_3\phi - 0.04008 \sin d_3\phi + 0.06459 \cosh e_3\phi - 0.06459 \sinh e_3\phi - 0.4141 \cosh f_3\phi + 0.3808 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2814 \cos d_4\phi - 0.03198 \sin d_4\phi + 0.04263 \cosh e_4\phi - 0.04263 \sinh e_4\phi - 0.3240 \cosh f_4\phi + 0.3529 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2353 \cos d_5\phi - 0.01843 \sin d_5\phi + 0.02234 \cosh e_5\phi - 0.02234 \sinh e_5\phi - 0.2576 \cosh f_5\phi + 0.2364 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -1.234 \sin d_1\phi - 0.2192 \cos d_1\phi + 1.036 \sin \mu_1\phi \cosh \delta_1\phi + 0.2192 \cos \mu_1\phi \cosh \delta_1\phi - 1.047 \sin \mu_1\phi \sinh \delta_1\phi - 0.3126 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.335 \sin d_2\phi - 0.2422 \cos d_2\phi + 0.5529 \sinh e_2\phi - 0.5538 \cosh e_2\phi - 0.7392 \sinh f_2\phi + 0.7960 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.372 \sin d_3\phi - 0.1574 \cos d_3\phi + 0.2275 \sinh e_3\phi - 0.2275 \cosh e_3\phi - 0.4185 \sinh f_3\phi + 0.3849 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.387 \sin d_4\phi - 0.1576 \cos d_4\phi + 0.1967 \sinh e_4\phi - 0.1967 \cosh e_4\phi - 0.3253 \sinh f_4\phi + 0.3543 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.400 \sin d_5\phi - 0.1097 \cos d_5\phi + 0.1271 \sinh e_5\phi - 0.1271 \cosh e_5\phi - 0.2581 \sinh f_5\phi + 0.2368 \cosh f_5\phi$$

Table 10. Pinned: pinned–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	5.986	1.937	1.034	0.4394	—	—
2	53.19	2.950	—	—	2.369	1.044
3	202.7	3.960	—	—	3.559	1.010
4	538.8	4.968	—	—	4.655	1.004
5	1170	5.973	—	—	5.716	1.002

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n'''(\pi) = 0$$
$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.6385 \cos d_1\phi - 0.1273 \sin d_1\phi - 0.6385 \cos \mu_1\phi \cosh \delta_1\phi - 0.9432 \sin \mu_1\phi \cosh \delta_1\phi + 0.6392 \cos \mu_1\phi \sinh \delta_1\phi + 0.9398 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.4512 \cos d_2\phi - 0.07192 \sin d_2\phi + 0.1939 \cosh e_2\phi - 0.1939 \sinh e_2\phi - 0.6451 \cosh f_2\phi + 0.6433 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3451 \cos d_3\phi - 0.04323 \sin d_3\phi + 0.06903 \cosh e_3\phi - 0.06903 \sinh e_3\phi - 0.4141 \cosh f_3\phi + 0.4126 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2785 \cos d_4\phi - 0.02837 \sin d_4\phi + 0.03763 \cosh e_4\phi - 0.03763 \sinh e_4\phi - 0.3161 \cosh f_4\phi + 0.3149 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2338 \cos d_5\phi - 0.01998 \sin d_5\phi + 0.02418 \cosh e_5\phi - 0.02418 \sinh e_5\phi - 0.2580 \cosh f_5\phi + 0.2571 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -1.237 \sin d_1\phi - 0.2467 \cos d_1\phi + 1.253 \sin \mu_1\phi \cosh \delta_1\phi + 0.2467 \cos \mu_1\phi \cosh \delta_1\phi - 1.256 \sin \mu_1\phi \sinh \delta_1\phi - 0.2474 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.331 \sin d_2\phi - 0.2121 \cos d_2\phi + 0.4593 \sinh e_2\phi - 0.4593 \cosh e_2\phi - 0.6733 \sinh f_2\phi + 0.6714 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.367 \sin d_3\phi - 0.1712 \cos d_3\phi + 0.2456 \sinh e_3\phi - 0.2456 \cosh e_3\phi - 0.4183 \sinh f_3\phi + 0.4169 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.383 \sin d_4\phi - 0.1409 \cos d_4\phi + 0.1752 \sinh e_4\phi - 0.1752 \cosh e_4\phi - 0.3173 \sinh f_4\phi + 0.3161 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.397 \sin d_5\phi - 0.1193 \cos d_5\phi + 0.1382 \sinh e_5\phi - 0.1382 \cosh e_5\phi - 0.2585 \sinh f_5\phi + 0.2575 \cosh f_5\phi$$

Table 11. Pinned: pinned–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.5637	1.388	0.5019	0.5377	—	—
2	13.12	2.229	1.247	0.2645	—	—
3	81.13	3.229	—	—	2.716	1.027
4	268.8	4.225	—	—	3.851	1.008
5	669.2	5.229	—	—	4.933	1.003

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$

$$\mathcal{V}'_n(0) = 0 \qquad \mathcal{V}'_n(\pi) + \mathcal{V}'''_n(\pi) = 0$$

$$\mathcal{V}'''_n(0) = 0 \qquad \mathcal{V}''_n(\pi) + \mathcal{V}^{iv}_n(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.6946 \cos d_1\phi - 0.04992 \sin d_1\phi - 0.6946 \cos \mu_1\phi \cosh \delta_1\phi - 0.1559 \sin \mu_1\phi \cosh \delta_1\phi + 0.3051 \cos \mu_1\phi \sinh \delta_1\phi + 0.2001 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5625 \cos d_2\phi - 0.1175 \sin d_2\phi - 0.5625 \cos \mu_2\phi \cosh \delta_2\phi - 1.925 \sin \mu_2\phi \cosh \delta_2\phi + 0.6185 \cos \mu_2\phi \sinh \delta_2\phi + 1.857 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4126 \cos d_3\phi - 0.05992 \sin d_3\phi + 0.1297 \cosh e_3\phi - 0.1295 \sinh e_3\phi - 0.5423 \cosh f_3\phi + 0.5308 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3237 \cos d_4\phi - 0.03924 \sin d_4\phi + 0.05876 \cosh e_4\phi - 0.05876 \sinh e_4\phi - 0.3825 \cosh f_4\phi + 0.3891 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2647 \cos d_5\phi - 0.02565 \sin d_5\phi + 0.03304 \cosh e_5\phi - 0.03304 \sinh e_5\phi - 0.2978 \cosh f_5\phi + 0.2962 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.9639 \sin d_1\phi - 0.06927 \cos d_1\phi + 0.4740 \sin \mu_1\phi \cosh \delta_1\phi + 0.06927 \cos \mu_1\phi \cosh \delta_1\phi - 0.2423 \sin \mu_1\phi \sinh \delta_1\phi - 0.2410 \cos \mu_1\phi \sinh \delta_1$$

$$\mathcal{W}_2(\phi) = -1.254 \sin d_2\phi - 0.2620 \cos d_2\phi + 2.465 \sin \mu_2\phi \cosh \delta_2\phi + 0.2620 \cos \mu_2\phi \cosh \delta_2\phi - 2.564 \sin \mu_2\phi \sinh \delta_2\phi - 0.2101 \cos \mu_2\phi \sinh \delta_2$$

$$\mathcal{W}_3(\phi) = -1.332 \sin d_3\phi - 0.1935 \cos d_3\phi + 0.3521 \sinh e_3\phi - 0.3516 \cosh e_3\phi - 0.5570 \sinh f_3\phi + 0.5452 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.368 \sin d_4\phi - 0.1658 \cos d_4\phi + 0.2263 \sinh e_4\phi - 0.2263 \cosh e_4\phi - 0.3854 \sinh f_4\phi + 0.3921 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.384 \sin d_5\phi - 0.1341 \cos d_5\phi + 0.1630 \sinh e_5\phi - 0.1630 \cosh e_5\phi - 0.2987 \sinh f_5\phi + 0.2971 \cosh f_5\phi$$

Table 12. Pinned: clamped semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	10.59	2.143	1.186	0.3341	—	—
2	68.08	3.109	—	—	2.569	1.033
3	252.2	4.163	—	—	3.784	1.008
4	631.2	5.157	—	—	4.857	1.003
5	135.4	6.186	—	—	5.939	1.001

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.5857 \cos d_1\phi - 0.08153 \sin d_1\phi - 0.5857 \cos \mu_1\phi \cosh \delta_1\phi - 0.9749 \sin \mu_1\phi \cosh \delta_1\phi + 0.4218 \cos \mu_1\phi \sinh \delta_1\phi + 1.043 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.4329 \cos d_2\phi - 0.08023 \sin d_2\phi + 0.1879 \cosh e_2\phi - 0.1884 \sinh e_2\phi - 0.6208 \cosh f_2\phi + 0.7100 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3310 \cos d_3\phi - 0.03343 \sin d_3\phi + 0.05076 \cosh e_3\phi - 0.05075 \sinh e_3\phi - 0.3818 \cosh f_3\phi + 0.3285 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2696 \cos d_4\phi - 0.03093 \sin d_4\phi + 0.04014 \cosh e_4\phi - 0.04014 \sinh e_4\phi - 0.3097 \cosh f_4\phi + 0.3533 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2268 \cos d_5\phi - 0.01611 \sin d_5\phi + 0.01924 \cosh e_5\phi - 0.01924 \sinh e_5\phi - 0.2460 \cosh f_5\phi + 0.2136 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -1.255 \sin d_1\phi - 0.1747 \cos d_1\phi + 1.433 \sin \mu_1\phi \cosh \delta_1\phi + 0.1747 \cos \mu_1\phi \cosh \delta_1\phi - 1.297 \sin \mu_1\phi \sinh \delta_1\phi - 0.3462 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.346 \sin d_2\phi - 0.2495 \cos d_2\phi + 0.4827 \sinh e_2\phi - 0.4839 \cosh e_2\phi - 0.6412 \sinh f_2\phi + 0.7334 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.378 \sin d_3\phi - 0.1392 \cos d_3\phi + 0.1920 \sinh e_3\phi - 0.1920 \cosh e_3\phi - 0.3849 \sinh f_3\phi + 0.3312 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.390 \sin d_4\phi - 0.1595 \cos d_4\phi + 0.1950 \sinh e_4\phi - 0.1950 \cosh e_4\phi - 0.3107 \sinh f_4\phi + 0.3544 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.403 \sin d_5\phi - 0.09969 \cos d_5\phi + 0.1143 \sinh e_5\phi - 0.1143 \cosh e_5\phi - 0.2464 \sinh f_5\phi + 0.2139 \cosh f_5\phi$$

Table 13. Pinned: clamped–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.3355	1.314	0.3902	0.5370	—	—
2	10.77	2.149	1.191	0.3294	—	—
3	72.53	3.152	—	—	2.621	1.031
4	259.5	4.190	—	—	3.814	1.008
5	652.3	5.197	—	—	4.899	1.003

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2152 \cos d_1\phi - 0.2192 \sin d_1\phi - 0.2152 \cos \mu_1\phi \cosh \delta_1\phi - 0.6180 \sin \mu_1\phi \cosh \delta_1\phi + 1.589 \cos \mu_1\phi \sinh \delta_1\phi + 1.825 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5911 \cos d_2\phi - 0.08857 \sin d_2\phi - 0.5911 \cos \mu_2\phi \cosh \delta_2\phi - 1.081 \sin \mu_2\phi \cosh \delta_2\phi + 0.4589 \cos \mu_2\phi \sinh \delta_2\phi + 1.145 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4299 \cos d_3\phi - 0.07146 \sin d_3\phi + 0.1623 \cosh e_3\phi - 0.1626 \sinh e_3\phi - 0.5922 \cosh f_3\phi + 0.6322 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3288 \cos d_4\phi - 0.03658 \sin d_4\phi + 0.05519 \cosh e_4\phi - 0.05519 \sinh e_4\phi - 0.3840 \cosh f_4\phi + 0.3609 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2675 \cos d_5\phi - 0.02743 \sin d_5\phi + 0.03545 \cosh e_5\phi - 0.03545 \sinh e_5\phi - 0.3030 \cosh f_5\phi + 0.3153 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.2829 \sin d_1\phi - 0.2882 \cos d_1\phi + 0.8278 \sin \mu_1\phi \cosh \delta_1\phi + 0.2882 \cos \mu_1\phi \cosh \delta_1\phi - 1.094 \sin \mu_1\phi \sinh \delta_1\phi + 0.8961 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.270 \sin d_2\phi - 0.1904 \cos d_2\phi + 1.559 \sin \mu_2\phi \cosh \delta_2\phi + 0.1904 \cos \mu_2\phi \cosh \delta_2\phi - 1.439 \sin \mu_2\phi \sinh \delta_2\phi - 0.3267 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.355 \sin d_3\phi - 0.2252 \cos d_3\phi + 0.4256 \sinh e_3\phi - 0.4263 \cosh e_3\phi - 0.6104 \sinh f_3\phi + 0.6515 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.378 \sin d_4\phi - 0.1533 \cos d_4\phi + 0.2105 \sinh e_4\phi - 0.2105 \cosh e_4\phi - 0.3870 \sinh f_4\phi + 0.3637 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.390 \sin d_5\phi - 0.1426 \cos d_5\phi + 0.1737 \sinh e_5\phi - 0.1737 \cosh e_5\phi - 0.3039 \sinh f_5\phi + 0.3163 \cosh f_5\phi$$

Table 14. Pinned: clamped–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.8113	1.447	0.5787	0.5362	—	—
2	21.13	2.443	—	—	1.616	1.165
3	110.4	3.455	—	—	2.983	1.019
4	341.1	4.464	—	—	4.113	1.006
5	809.5	5.470	—	—	5.188	1.002

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.7644 \cos d_1\phi - 0.1276 \sin d_1\phi - 0.7644 \cos \mu_1\phi \cosh \delta_1\phi - 0.4338 \sin \mu_1\phi \cosh \delta_1\phi + 0.7211 \cos \mu_1\phi \sinh \delta_1\phi + 0.4476 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5306 \cos d_2\phi - 0.09650 \sin d_2\phi + 0.8521 \cosh e_2\phi - 0.8521 \sinh e_2\phi - 1.383 \cosh f_2\phi + 1.384 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3913 \cos d_3\phi - 0.05556 \sin d_3\phi + 0.1063 \cosh e_3\phi - 0.1063 \sinh e_3\phi - 0.4976 \cosh f_3\phi + 0.4992 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3083 \cos d_4\phi - 0.03498 \sin d_4\phi + 0.04998 \cosh e_4\phi - 0.04998 \sinh e_4\phi - 0.3583 \cosh f_4\phi + 0.3596 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2540 \cos d_5\phi - 0.02382 \sin d_5\phi + 0.02997 \cosh e_5\phi - 0.02997 \sinh e_5\phi - 0.2840 \cosh f_5\phi + 0.2851 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -1.106 \sin d_1\phi - 0.1847 \cos d_1\phi + 0.6689 \sin \mu_1\phi \cosh \delta_1\phi + 0.1847 \cos \mu_1\phi \cosh \delta_1\phi - 0.6376 \sin \mu_1\phi \sinh \delta_1\phi - 0.2024 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.296 \sin d_2\phi - 0.2357 \cos d_2\phi + 1.377 \sinh e_2\phi - 1.377 \cosh e_2\phi - 1.610 \sinh f_2\phi + 1.613 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.352 \sin d_3\phi - 0.1920 \cos d_3\phi + 0.3170 \sinh e_3\phi - 0.3170 \cosh e_3\phi - 0.5072 \sinh f_3\phi + 0.5089 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.376 \sin d_4\phi - 0.1561 \cos d_4\phi + 0.2056 \sinh e_4\phi - 0.2056 \cosh e_4\phi - 0.3604 \sinh f_4\phi + 0.3617 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.390 \sin d_5\phi - 0.1303 \cos d_5\phi + 0.1555 \sinh e_5\phi - 0.1555 \cosh e_5\phi - 0.2847 \sinh f_5\phi + 0.2858 \cosh f_5\phi$$

Table 15. Pinned: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.0903	0.6840	—	—	0.3721	1.181
2	1.656	1.588	0.7318	0.5244	—	—
3	23.50	2.495	—	—	1.714	1.134
4	111.9	3.466	—	—	2.996	1.019
5	344.5	4.474	—	—	4.124	1.006

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$\mathcal{V}_n'''(0) = 0 \qquad -\xi_n \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.9273 \cos d_1\phi - 0.8832 \sin d_1\phi - 1.366 \cos e_1\phi + 1.198 \sin e_1\phi + 0.4386 \cos f_1\phi + 0.1343 \sin f_1\phi$$

$$\mathcal{V}_2(\phi) = 0.6326 \cos d_2\phi - 0.1494 \sin d_2\phi - 0.6326 \cos \mu_2\phi \cosh \delta_2\phi - 0.6227 \sin \mu_2\phi \cosh \delta_2\phi + 0.7705 \cos \mu_2\phi \sinh \delta_2\phi + 0.8264 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.5079 \cos d_3\phi - 0.08248 \sin d_3\phi + 0.5501 \cosh e_3\phi - 0.5461 \sinh e_3\phi - 1.058 \cosh f_3\phi + 1.007 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3879 \cos d_4\phi - 0.05748 \sin d_4\phi + 0.1094 \cosh e_4\phi - 0.1094 \sinh e_4\phi - 0.4972 \cosh f_4\phi + 0.5170 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.3069 \cos d_5\phi - 0.03302 \sin d_5\phi + 0.04710 \cosh e_5\phi - 0.04710 \sinh e_5\phi - 0.3540 \cosh f_5\phi + 0.3400 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.6343 \sin d_1\phi - 0.6041 \cos d_1\phi + 0.5082 \sin e_1\phi + 0.4456 \cos e_1\phi - 0.5178 \sin f_1\phi + 0.1585 \cos f_1\phi$$

$$\mathcal{W}_2(\phi) = -1.004 \sin d_2\phi - 0.2373 \cos d_2\phi + 0.9365 \sin \mu_2\phi \cosh \delta_2\phi + 0.2373 \cos \mu_2\phi \cosh \delta_2\phi - 0.8598 \sin \mu_2\phi \sinh \delta_2\phi - 0.02954 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.267 \sin d_3\phi - 0.2058 \cos d_3\phi + 0.9428 \sinh e_3\phi - 0.9358 \cosh e_3\phi - 1.200 \sinh f_3\phi + 1.142 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.344 \sin d_4\phi - 0.1992 \cos d_4\phi + 0.3277 \sinh e_4\phi - 0.3277 \cosh e_4\phi - 0.5067 \sinh f_4\phi + 0.5269 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.373 \sin d_5\phi - 0.1477 \cos d_5\phi + 0.1942 \sinh e_5\phi - 0.1942 \cosh e_5\phi - 0.3561 \sinh f_5\phi + 0.3420 \cosh f_5\phi$$

Table 16. Pinned–circumferentially guided: pinned–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	7.200	2.000	1.082	0.4133	—	—
2	57.60	3.000	—	—	2.433	1.040
3	211.8	4.000	—	—	3.603	1.010
4	553.8	5.000	—	—	4.690	1.004
5	1192	6.000	—	—	5.744	1.002

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'''(\pi) = 0$$
$$\mathcal{V}_n^v(0) = 0 \qquad \mathcal{V}_n^v(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.6325 \cos d_1\phi$$
$$\mathcal{V}_2(\phi) = 0.4472 \cos d_2\phi$$
$$\mathcal{V}_3(\phi) = 0.3430 \cos d_3\phi$$
$$\mathcal{V}_4(\phi) = 0.2774 \cos d_4\phi$$
$$\mathcal{V}_5(\phi) = 0.2325 \cos d_5\phi$$

$$\mathcal{W}_1(\phi) = -1.265 \sin d_1\phi$$
$$\mathcal{W}_2(\phi) = -1.342 \sin d_2\phi$$
$$\mathcal{W}_3(\phi) = -1.372 \sin d_3\phi$$
$$\mathcal{W}_4(\phi) = -1.387 \sin d_4\phi$$
$$\mathcal{W}_5(\phi) = -1.395 \sin d_5\phi$$

Table 17. Pinned–circumferentially guided: pinned–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.5904	1.395	0.5117	0.5377	—	—
2	15.40	2.298	1.294	0.1843	—	—
3	86.31	3.273	—	—	2.768	1.025
4	279.7	4.264	—	—	3.894	1.007
5	685.7	5.259	—	—	4.965	1.003

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$

$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) = 0$$

$$\mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.7245 \cos d_1\phi - 0.4180 \cos \mu_1\phi \cosh \delta_1\phi + 0.04503 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5528 \cos d_2\phi + 0.04639 \cos \mu_2\phi \cosh \delta_2\phi - 0.09159 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4102 \cos d_3\phi + 0.1422 \times 10^{-3} \cosh e_3\phi - 0.01254 \cosh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3219 \cos d_4\phi - 0.2703 \times 10^{-5} \cosh e_4\phi + 0.005072 \cosh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2637 \cos d_5\phi + 0.7115 \times 10^{-7} \cosh e_5\phi - 0.002579 \cosh f_5\phi$$

$$\mathcal{W}_1(\phi) = -1.010 \sin d_1\phi + 0.2478 \sin \mu_1\phi \cosh \delta_1\phi - 0.1897 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.270 \sin d_2\phi - 0.1271 \sin \mu_2\phi \cosh \delta_1\phi + 0.04314 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.343 \sin d_3\phi + 0.3937 \times 10^{-3} \sinh e_3\phi - 0.01286 \sinh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.372 \sin d_4\phi - 0.1053 \times 10^{-4} \sinh e_4\phi + 0.005110 \sinh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.387 \sin d_5\phi + 0.3533 \times 10^{-6} \sinh e_5\phi - 0.002586 \sinh f_5\phi$$

Table 18. Pinned–circumferentially guided: clamped semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.3208	1.309	0.3803	0.5368	—	—
2	11.46	2.174	1.208	0.3116	—	—
3	75.51	3.180	—	—	2.655	1.029
4	259.4	4.190	—	—	3.813	1.008
5	653.4	5.199	—	—	4.901	1.003

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$\mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.6203 \cos d_1\phi + 0.8551 \cos \mu_1\phi \cosh \delta_1\phi + 0.3548 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5885 \cos d_2\phi - 0.1642 \cos \mu_2\phi \cosh \delta_2\phi + 0.08326 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4266 \cos d_3\phi - 0.0003206 \cosh e_3\phi + 0.08128 \cosh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3299 \cos d_4\phi + 0.4693 \times 10^{-5} \cosh e_4\phi - 0.05446 \cosh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2681 \cos d_5\phi - 0.1088 \times 10^{-6} \cosh e_5\phi + 0.04120 \cosh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.8118 \sin d_1\phi - 0.3241 \sin \mu_1\phi \cosh \delta_1\phi + 0.5157 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.279 \sin d_2\phi + 0.1518 \sin \mu_2\phi \cosh \delta_2\phi - 0.1725 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.356 \sin d_3\phi - 0.8512 \times 10^{-3} \sinh e_3\phi + 0.08366 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.382 \sin d_4\phi + 0.1790 \times 10^{-4} \sinh e_4\phi - 0.05489 \sinh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.394 \sin d_5\phi - 0.5332 \times 10^{-6} \sinh e_5\phi + 0.04133 \sinh f_5\phi$$

Table 19. Pinned–circumferentially guided: clamped–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.7404	1.432	0.5594	0.5368	—	—
2	11.86	2.188	1.218	0.3008	—	—
3	78.96	3.210	—	—	2.693	1.028
4	268.5	4.224	—	—	3.850	1.008
5	670.7	5.231	—	—	4.936	1.003

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$

$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n^v(0) = 0 \qquad \mathcal{V}_n'''(\pi) + \mathcal{V}_n^v(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.3652 \cos d_1\phi + 0.7290 \cos \mu_1\phi \cosh \delta_1\phi + 0.5264 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5950 \cos d_2\phi - 0.1194 \cos \mu_2\phi \cosh \delta_2\phi + 0.07521 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4243 \cos d_3\phi - 0.0002141 \cosh e_3\phi + 0.04042 \cosh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3274 \cos d_4\phi + 0.3326 \times 10^{-5} \cosh e_4\phi - 0.02123 \cosh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2663 \cos d_5\phi - 0.8085 \times 10^{-7} \cosh e_5\phi + 0.1334 \cosh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.5229 \sin d_1\phi - 0.09684 \sin \mu_1\phi \cosh \delta_1\phi + 0.6903 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.302 \sin d_2\phi + 0.1275 \sin \mu_2\phi \cosh \delta_2\phi - 0.1227 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.362 \sin d_3\phi - 0.5766 \times 10^{-3} \sinh e_3\phi + 0.04154 \sinh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.383 \sin d_4\phi + 1.281 \times 10^{-5} \sinh e_4\phi - 0.02139 \sinh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.393 \sin d_5\phi - 0.3990 \times 10^{-6} \sinh e_5\phi + 0.01338 \sinh f_5\phi$$

Table 20. Pinned–circumferentially guided: clamped–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.1125	0.5596	—	—	0.5000	1.199
2	1.082	1.500	0.6397	0.5331	—	—
3	23.76	2.500	—	—	1.723	1.131
4	117.0	3.500	—	—	3.035	1.018
5	353.1	4.500	—	—	4.152	1.006

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 1.265 \cos e_1\phi$$
$$\mathcal{V}_2(\phi) = 0.7845 \cos d_2\phi$$
$$\mathcal{V}_3(\phi) = 0.5252 \cos d_3\phi$$
$$\mathcal{V}_4(\phi) = 0.3885 \cos d_4\phi$$
$$\mathcal{V}_5(\phi) = 0.3068 \cos d_5\phi$$

$$\mathcal{W}_1(\phi) = -0.6325 \sin e_1\phi$$
$$\mathcal{W}_2(\phi) = -1.177 \sin d_2\phi$$
$$\mathcal{W}_3(\phi) = -1.313 \sin d_3\phi$$
$$\mathcal{W}_4(\phi) = -1.360 \sin d_4\phi$$
$$\mathcal{W}_5(\phi) = -1.381 \sin d_5\phi$$

Table 21. Pinned–circumferentially guided: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.1545	1.228	0.1920	0.5323	—	—
2	2.224	1.656	0.7974	0.5145	—	—
3	25.69	2.539	—	—	1.790	1.115
4	119.2	3.515	—	—	3.052	1.018
5	355.5	4.507	—	—	4.159	1.006

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$\mathcal{V}_n^{v}(0) = 0 \qquad -\xi_n \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2903 \cos d_1\phi - 1.123 \cos \mu_1\phi \cosh \delta_1\phi - 0.7885 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.6342 \cos d_2\phi + 0.1396 \cos \mu_2\phi \cosh \delta_2\phi + 0.1568 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.5063 \cos d_3\phi + 0.002285 \cosh e_3\phi - 0.04470 \cosh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3846 \cos d_4\phi - 0.4298 \times 10^{-5} \cosh e_4\phi + 0.02100 \cosh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.3056 \cos d_5\phi + 0.3880 \times 10^{-7} \cosh e_5\phi - 0.01248 \cosh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3563 \sin d_1\phi + 0.4463 \sin \mu_1\phi \cosh \delta_1\phi - 0.6352 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.050 \sin d_2\phi + 0.05319 \sin \mu_2\phi \cosh \delta_2\phi + 0.1920 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.286 \sin d_3\phi + 0.004090 \sinh e_3\phi - 0.04985 \sinh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.352 \sin d_4\phi - 1.312 \times 10^{-5} \sinh e_4\phi + 0.02137 \sinh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.377 \sin d_5\phi + 1.614 \times 10^{-7} \sinh e_5\phi - 0.01255 \sinh f_5\phi$$

Table 22. Pinned–radially guided: pinned–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	2.354	1.670	0.8102	0.5122	—	—
2	27.69	2.578	—	—	1.852	1.102
3	123.0	3.539	—	—	3.081	1.017
4	361.4	4.524	—	—	4.178	1.006
5	838.4	5.516	—	—	5.237	1.002

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.3284 \cos d_1\phi + 0.5753 \sin d_1\phi - 0.3284 \cos \mu_1\phi \cosh \delta_1\phi + 1.164 \sin \mu_1\phi \cosh \delta_1\phi + 0.1468 \cos \mu_1\phi \sinh \delta_1\phi - 1.134 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.3055 \cos d_2\phi + 0.3909 \sin d_2\phi - 0.8512 \cosh e_2\phi + 0.8461 \sinh e_2\phi + 0.5457 \cosh f_2\phi - 0.5125 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2519 \cos d_3\phi + 0.2850 \sin d_3\phi - 0.3677 \cosh e_3\phi + 0.3677 \sinh e_3\phi + 0.1158 \cosh f_3\phi - 0.1257 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2064 \cos d_4\phi + 0.2227 \sin d_4\phi - 0.2555 \cosh e_4\phi + 0.2555 \sinh e_4\phi + 0.04917 \cosh f_4\phi - 0.04516 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1726 \cos d_5\phi + 0.1815 \sin d_5\phi - 0.1983 \cosh e_5\phi + 0.1983 \sinh e_5\phi + 0.02575 \cosh f_5\phi - 0.02806 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.5483 \sin d_1\phi + 0.9606 \cos d_1\phi - 0.7506 \sin \mu_1\phi \cosh \delta_1\phi + 0.7153 \cos \mu_1\phi \cosh \delta_1\phi + 0.8683 \sin \mu_1\phi \sinh \delta_1\phi - 0.8469 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.7875 \sin d_2\phi + 1.008 \cos d_2\phi - 1.576 \sinh e_2\phi + 1.567 \cosh e_2\phi + 0.6016 \sinh f_2\phi - 0.5650 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8915 \sin d_3\phi + 1.009 \cos d_3\phi - 1.133 \sinh e_3\phi + 1.133 \cosh e_3\phi + 0.1178 \sinh f_3\phi - 0.1278 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9337 \sin d_4\phi + 1.008 \cos d_4\phi - 1.068 \sinh e_4\phi + 1.068 \cosh e_4\phi + 0.04944 \sinh f_4\phi - 0.04542 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9520 \sin d_5\phi + 1.001 \cos d_5\phi - 1.039 \sinh e_5\phi + 1.039 \cosh e_5\phi + 0.02581 \sinh f_5\phi - 0.02812 \cosh f_5\phi$$

Table 23. Pinned–radially guided: clamped semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	1.766	1.602	0.7459	0.5226	—	—
2	21.37	2.448	—	—	1.627	1.161
3	110.6	3.456	—	—	2.984	1.019
4	337.4	4.453	—	—	4.101	1.006
5	802.8	5.459	—	—	5.177	1.003

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2438 \cos d_1\phi + 0.6024 \sin d_1\phi - 0.2438 \cos \mu_1\phi \cosh \delta_1\phi + 0.9534 \sin \mu_1\phi \cosh \delta_1\phi + 0.5785 \cos \mu_1\phi \sinh \delta_1\phi - 0.8515 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.3341 \cos d_2\phi + 0.4153 \sin d_2\phi - 1.379 \cosh e_2\phi + 1.410 \sinh e_2\phi + 1.044 \cosh f_2\phi - 1.207 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2566 \cos d_3\phi + 0.2964 \sin d_3\phi - 0.3833 \cosh e_3\phi + 0.3832 \sinh e_3\phi + 0.1267 \cosh f_3\phi - 0.05614 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2115 \cos d_4\phi + 0.2275 \sin d_4\phi - 0.2639 \cosh e_4\phi + 0.2639 \sinh e_4\phi + 0.05242 \cosh f_4\phi - 0.1031 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1760 \cos d_5\phi + 0.1858 \sin d_5\phi - 0.2029 \cosh e_5\phi + 0.2029 \sinh e_5\phi + 0.02689 \cosh f_5\phi + 0.01198 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3906 \sin d_1\phi + 0.9651 \cos d_1\phi - 0.5077 \sin \mu_1\phi \cosh \delta_1\phi + 0.9297 \cos \mu_1\phi \cosh \delta_1\phi + 0.4088 \sin \mu_1\phi \sinh \delta_1\phi - 0.6268 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.8179 \sin d_2\phi + 1.017 \cos d_2\phi - 2.243 \sinh e_2\phi + 2.294 \cosh e_2\phi + 1.212 \sinh f_2\phi - 1.401 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8870 \sin d_3\phi + 1.025 \cos d_3\phi - 1.144 \sinh e_3\phi + 1.144 \cosh e_3\phi + 0.1292 \sinh f_3\phi - 0.05723 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9418 \sin d_4\phi + 1.013 \cos d_4\phi - 1.082 \sinh e_4\phi + 1.082 \cosh e_4\phi + 0.05273 \sinh f_4\phi - 0.1037 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9609 \sin d_5\phi + 1.015 \cos d_5\phi - 1.050 \sinh e_5\phi + 1.050 \cosh e_5\phi + 0.02695 \sinh f_5\phi + 0.01201 \cosh f_5\phi$$

Table 24. Pinned–radially guided: clamped–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.1396	1.218	0.1557	0.5315	—	—
2	1.931	1.622	0.7658	0.5198	—	—
3	22.66	2.477	—	—	1.681	1.143
4	115.1	3.487	—	—	3.020	1.019
5	349.0	4.488	—	—	4.139	1.006

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.3140 \cos d_1\phi + 0.4447 \sin d_1\phi - 0.3140 \cos \mu_1\phi \cosh \delta_1\phi + 0.7411 \sin \mu_1\phi \cosh \delta_1\phi - 1.796 \cos \mu_1\phi \sinh \delta_1\phi - 3.673 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.1900 \cos d_2\phi + 0.5410 \sin d_2\phi - 0.1900 \cos \mu_2\phi \cosh \delta_2\phi + 0.8330 \sin \mu_2\phi \cosh \delta_2\phi + 0.5978 \cos \mu_2\phi \sinh \delta_2\phi - 0.6601 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3299 \cos d_3\phi + 0.4168 \sin d_3\phi - 1.204 \cosh e_3\phi + 1.220 \sinh e_3\phi + 0.8737 \cosh f_3\phi - 0.9667 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2566 \cos d_4\phi + 0.2937 \sin d_4\phi - 0.3800 \cosh e_4\phi + 0.3799 \sinh e_4\phi + 0.1234 \cosh f_4\phi - 0.09021 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2095 \cos d_5\phi + 0.2259 \sin d_5\phi - 0.2605 \cosh e_5\phi + 0.2605 \sinh e_5\phi + 0.05093 \cosh f_5\phi - 0.06918 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3824 \sin d_1\phi + 0.5417 \cos d_1\phi - 0.4051 \sin \mu_1\phi \cosh \delta_1\phi + 0.1143 \cos \mu_1\phi \cosh \delta_1\phi + 1.070 \sin \mu_1\phi \sinh \delta_1\phi - 2.001 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.3083 \sin d_2\phi + 0.8777 \cos d_2\phi - 0.4067 \sin \mu_2\phi \cosh \delta_2\phi + 0.8907 \cos \mu_2\phi \cosh \delta_2\phi + 0.3272 \sin \mu_2\phi \sinh \delta_2\phi - 0.4886 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8170 \sin d_3\phi + 1.032 \cos d_3\phi - 2.024 \sinh e_3\phi + 2.051 \cosh e_3\phi + 0.9988 \sinh f_3\phi - 1.105 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.8949 \sin d_4\phi + 1.024 \cos d_4\phi - 1.148 \sinh e_4\phi + 1.147 \cosh e_4\phi + 0.1256 \sinh f_4\phi - 0.09188 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9403 \sin d_5\phi + 1.014 \cos d_5\phi - 1.078 \sinh e_5\phi + 1.078 \cosh e_5\phi + 0.05123 \sinh f_5\phi - 0.06958 \cosh f_5\phi$$

Table 25. Pinned–radially guided: clamped–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	4.181	1.825	0.9450	0.4767	—	—
2	40.25	2.782	—	—	2.148	1.061
3	162.6	3.767	—	—	3.342	1.013
4	449.6	4.761	—	—	4.434	1.005
5	1003	5.757	—	—	5.490	1.002

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.3408 \cos d_1\phi + 0.5572 \sin d_1\phi - 0.3408 \cos \mu_1\phi \cosh \delta_1\phi + 1.206 \sin \mu_1\phi \cosh \delta_1\phi + 0.3381 \cos \mu_1\phi \sinh \delta_1\phi - 1.212 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.2987 \cos d_2\phi + 0.3665 \sin d_2\phi - 0.6325 \cosh e_2\phi + 0.6326 \sinh e_2\phi + 0.3338 \cosh f_2\phi - 0.3347 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2410 \cos d_3\phi + 0.2690 \sin d_3\phi - 0.3335 \cosh e_3\phi + 0.3335 \sinh e_3\phi + 0.09252 \cosh f_3\phi - 0.09284 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.1981 \cos d_4\phi + 0.2121 \sin d_4\phi - 0.2398 \cosh e_4\phi + 0.2398 \sinh e_4\phi + 0.04168 \cosh f_4\phi - 0.04184 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1649 \cos d_5\phi + 0.1727 \sin d_5\phi - 0.1872 \cosh e_5\phi + 0.1872 \sinh e_5\phi + 0.02233 \cosh f_5\phi - 0.02241 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.6221 \sin d_1\phi + 1.017 \cos d_1\phi - 0.9829 \sin \mu_1\phi \cosh \delta_1\phi + 0.8944 \cos \mu_1\phi \cosh \delta_1\phi + 0.9785 \sin \mu_1\phi \sinh \delta_1\phi - 0.8998 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.8312 \sin d_2\phi + 1.020 \cos d_2\phi - 1.359 \sinh e_2\phi + 1.359 \cosh e_2\phi + 0.3543 \sinh f_2\phi - 0.3552 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -0.9078 \sin d_3\phi + 1.013 \cos d_3\phi - 1.114 \sinh e_3\phi + 1.114 \cosh e_3\phi + 0.09371 \sinh f_3\phi - 0.09403 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9430 \sin d_4\phi + 1.010 \cos d_4\phi - 1.063 \sinh e_4\phi + 1.063 \cosh e_4\phi + 0.04187 \sinh f_4\phi - 0.04203 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9494 \sin d_5\phi + 0.9945 \cos d_5\phi - 1.028 \sinh e_5\phi + 1.028 \cosh e_5\phi + 0.02237 \sinh f_5\phi - 0.02246 \cosh f_5\phi$$

Table 26. Pinned–radially guided: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.1401	1.218	0.1570	0.5316	—	—
2	5.895	1.932	1.030	0.4413	—	—
3	42.01	2.807	—	—	2.182	1.058
4	165.0	3.780	—	—	3.356	1.013
5	451.8	4.766	—	—	4.439	1.005

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n'(0) + \mathcal{V}_n'''(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad -\xi_n\mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.3154\cos d_1\phi + 0.4691\sin d_1\phi - 0.3154\cos\mu_1\phi\cosh\delta_1\phi + 0.7679\sin\mu_1\phi\cosh\delta_1\phi - 1.667\cos\mu_1\phi\sinh\delta_1\phi - 3.661\sin\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.2925\cos d_2\phi + 0.4916\sin d_2\phi - 0.2925\cos\mu_2\phi\cosh\delta_2\phi + 1.148\sin\mu_2\phi\cosh\delta_2\phi + 0.3772\cos\mu_2\phi\sinh\delta_2\phi - 1.108\sin\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2952\cos d_3\phi + 0.3572\sin d_3\phi - 0.6109\cosh e_3\phi + 0.6112\sinh e_3\phi + 0.3157\cosh f_3\phi - 0.3485\sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2387\cos d_4\phi + 0.2670\sin d_4\phi - 0.3296\cosh e_4\phi + 0.3296\sinh e_4\phi + 0.09083\cosh f_4\phi - 0.07263\sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1976\cos d_5\phi + 0.2114\sin d_5\phi - 0.2391\cosh e_5\phi + 0.2391\sinh e_5\phi + 0.04148\cosh f_5\phi - 0.05238\sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3842\sin d_1\phi + 0.5715\cos d_1\phi - 0.4070\sin\mu_1\phi\cosh\delta_1\phi + 0.1465\cos\mu_1\phi\cosh\delta_1\phi + 1.007\sin\mu_1\phi\sinh\delta_1\phi - 1.996\cos\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.5653\sin d_2\phi + 0.9499\cos d_2\phi - 1.012\sin\mu_2\phi\cosh\delta_2\phi + 0.8956\cos\mu_2\phi\cosh\delta_2\phi + 1.017\sin\mu_2\phi\sinh\delta_2\phi - 0.7904\cos\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.8286\sin d_3\phi + 1.003\cos d_3\phi - 1.333\sinh e_3\phi + 1.333\cosh e_3\phi + 0.3341\sinh f_3\phi - 0.3687\cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.9024\sin d_4\phi + 1.009\cos d_4\phi - 1.106\sinh e_4\phi + 1.106\cosh e_4\phi + 0.09198\sinh f_4\phi - 0.07355\cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.9420\sin d_5\phi + 1.007\cos d_5\phi - 1.062\sinh e_5\phi + 1.062\cosh e_5\phi + 0.04167\sinh f_5\phi - 0.05262\cosh f_5\phi$$

Table 27. Clamped: clamped semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	19.22	2.398	—	—	1.514	1.208
2	93.16	3.328	—	—	2.834	1.023
3	321.2	4.402	—	—	4.045	1.006
4	757.6	5.385	—	—	5.098	1.003
5	1584	6.425	—	—	6.186	1.001

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}'_n(0) = 0 \qquad \mathcal{V}'_n(\pi) = 0$$
$$\mathcal{V}''_n(0) = 0 \qquad \mathcal{V}''_n(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.4319 \cos d_1\phi + 0.3117 \sin d_1\phi + 3.738 \cosh e_1\phi - 3.675 \sinh e_1\phi - 4.170 \cosh f_1\phi + 3.987 \sinh f_1\phi$$

$$\mathcal{V}_2(\phi) = 0.3613 \cos d_2\phi + 0.2049 \sin d_2\phi + 0.6272 \cosh e_2\phi - 0.6274 \sinh e_2\phi - 0.9885 \cosh f_2\phi + 1.071 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.2545 \cos d_3\phi + 0.1862 \sin d_3\phi + 0.3380 \cosh e_3\phi - 0.3380 \sinh e_3\phi - 0.5925 \cosh f_3\phi + 0.5444 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2138 \cos d_4\phi + 0.1476 \sin d_4\phi + 0.2568 \cosh e_4\phi - 0.2568 \sinh e_4\phi - 0.4706 \cosh f_4\phi + 0.5127 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1697 \cos d_5\phi + 0.1336 \sin d_5\phi + 0.1925 \cosh e_5\phi - 0.1925 \sinh e_5\phi - 0.3622 \cosh f_5\phi + 0.3323 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -1.036 \sin d_1\phi + 0.7475 \cos d_1\phi + 5.659 \sinh e_1\phi - 5.563 \cosh e_1\phi - 5.037 \sinh f_1\phi + 4.815 \cosh f_1\phi$$

$$\mathcal{W}_2(\phi) = -1.202 \sin d_2\phi + 0.6818 \cos d_2\phi + 1.778 \sinh e_2\phi - 1.778 \cosh e_2\phi - 1.012 \sinh f_2\phi + 1.096 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.120 \sin d_3\phi + 0.8196 \cos d_3\phi + 1.368 \sinh e_3\phi - 1.367 \cosh e_3\phi - 0.5963 \sinh f_3\phi + 0.5479 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.151 \sin d_4\phi + 0.7949 \cos d_4\phi + 1.309 \sinh e_4\phi - 1.309 \cosh e_4\phi - 0.4719 \sinh f_4\phi + 0.5141 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.090 \sin d_5\phi + 0.8582 \cos d_5\phi + 1.191 \sinh e_5\phi - 1.191 \cosh e_5\phi - 0.3627 \sinh f_5\phi + 0.3327 \cosh f_5\phi$$

Table 28. Clamped: clamped–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.4659	1.359	0.4615	0.5378	—	—
2	19.45	2.403	—	—	1.528	1.201
3	100.0	3.381	—	—	2.896	1.022
4	328.6	4.425	—	—	4.071	1.006
5	784.0	5.429	—	—	5.144	1.003

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$
$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.01423 \cos d_1\phi - 0.4195 \sin d_1\phi - 0.01423 \cos \mu_1\phi \cosh \delta_1\phi + 0.5307 \sin \mu_1\phi \cosh \delta_1\phi + 0.6172 \cos \mu_1\phi \sinh \delta_1\phi + 0.05079 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.4410 \cos d_2\phi + 0.3065 \sin d_2\phi + 3.571 \cosh e_2\phi - 3.522 \sinh e_2\phi - 4.012 \cosh f_2\phi + 3.867 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3487 \cos d_3\phi + 0.2135 \sin d_3\phi + 0.5923 \cosh e_3\phi - 0.5924 \sinh e_3\phi - 0.9410 \cosh f_3\phi + 0.9727 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2562 \cos d_4\phi + 0.1815 \sin d_4\phi + 0.3392 \cosh e_4\phi - 0.3392 \sinh e_4\phi - 0.5954 \cosh f_4\phi + 0.5739 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2086 \cos d_5\phi + 0.1501 \sin d_5\phi + 0.2497 \cosh e_5\phi - 0.2497 \sinh e_5\phi - 0.4583 \cosh f_5\phi + 0.4688 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.01934 \sin d_1\phi - 0.5702 \cos d_1\phi + 0.03109 \sin \mu_1\phi \cosh \delta_1\phi + 0.5702 \cos \mu_1\phi \cosh \delta_1\phi - 0.08708 \sin \mu_1\phi \sinh \delta_1\phi + 0.02075 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.060 \sin d_2\phi + 0.7367 \cos d_2\phi + 5.455 \sinh e_2\phi - 5.381 \cosh e_2\phi - 4.819 \sinh f_2\phi + 4.644 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.179 \sin d_3\phi + 0.7219 \cos d_3\phi + 1.715 \sinh e_3\phi - 1.716 \cosh e_3\phi - 0.9613 \sinh f_3\phi + 0.9937 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.134 \sin d_4\phi + 0.8033 \cos d_4\phi + 1.381 \sinh e_4\phi - 1.381 \cosh e_4\phi - 0.5991 \sinh f_4\phi + 0.5775 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.132 \sin d_5\phi + 0.8146 \cos d_5\phi + 1.285 \sinh e_5\phi - 1.285 \cosh e_5\phi - 0.4595 \sinh f_5\phi + 0.4700 \cosh f_5\phi$$

Table 29. Clamped: clamped–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	2.545	1.689	0.8280	0.5086	—	—
2	33.12	2.673	—	—	1.995	1.079
3	147.2	3.684	—	—	3.247	1.014
4	423.3	4.694	—	—	4.362	1.005
5	963.5	5.702	—	—	5.432	1.002

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.5724 \cos d_1\phi + 0.3872 \sin d_1\phi - 0.5724 \cos \mu_1\phi \cosh \delta_1\phi - 2.205 \sin \mu_1\phi \cosh \delta_1\phi + 0.5648 \cos \mu_1\phi \sinh \delta_1\phi + 2.229 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.4260 \cos d_2\phi + 0.2572 \sin d_2\phi + 1.258 \cosh e_2\phi - 1.258 \sinh e_2\phi - 1.684 \cosh f_2\phi + 1.687 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3120 \cos d_3\phi + 0.2036 \sin d_3\phi + 0.4789 \cosh e_3\phi - 0.4789 \sinh e_3\phi - 0.7909 \cosh f_3\phi + 0.7936 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2425 \cos d_4\phi + 0.1695 \sin d_4\phi + 0.3102 \cosh e_4\phi - 0.3102 \sinh e_4\phi - 0.5527 \cosh f_4\phi + 0.5547 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1994 \cos d_5\phi + 0.1468 \sin d_5\phi + 0.2344 \cosh e_5\phi - 0.2344 \sinh e_5\phi - 0.4338 \cosh f_5\phi + 0.4354 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.9669 \sin d_1\phi + 0.6540 \cos d_1\phi + 2.137 \sin \mu_1\phi \cosh \delta_1\phi - 0.6540 \cos \mu_1\phi \cosh \delta_1\phi - 2.113 \sin \mu_1\phi \sinh \delta_1\phi + 0.6600 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.139 \sin d_2\phi + 0.6876 \cos d_2\phi + 2.509 \sinh e_2\phi - 2.509 \cosh e_2\phi - 1.817 \sinh f_2\phi + 1.821 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.150 \sin d_3\phi + 0.7499 \cos d_3\phi + 1.555 \sinh e_3\phi - 1.555 \cosh e_3\phi - 0.8022 \sinh f_3\phi + 0.8049 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.138 \sin d_4\phi + 0.7956 \cos d_4\phi + 1.353 \sinh e_4\phi - 1.353 \cosh e_4\phi - 0.5553 \sinh f_4\phi + 0.5573 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.137 \sin d_5\phi + 0.8372 \cos d_5\phi + 1.273 \sinh e_5\phi - 1.273 \cosh e_5\phi - 0.4347 \sinh f_5\phi + 0.4363 \cosh f_5\phi$$

Table 30. Clamped: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.4487	1.354	0.4534	0.5378	—	—
2	3.208	1.750	0.8817	0.4960	—	—
3	36.23	2.723	—	—	2.066	1.070
4	148.3	3.690	—	—	3.254	1.014
5	427.6	4.705	—	—	4.374	1.005

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$\mathcal{V}_n''(0) = 0 \qquad -\xi_n\mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.1289 \cos d_1\phi + 0.4668 \sin d_1\phi - 0.1289 \cos \mu_1\phi \cosh \delta_1\phi - 0.9094 \sin \mu_1\phi \cosh \delta_1\phi - 0.3151 \cos \mu_1\phi \sinh \delta_1\phi + 0.4624 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5725 \cos d_2\phi + 0.3107 \sin d_2\phi - 0.5725 \cos \mu_2\phi \cosh \delta_2\phi - 2.257 \sin \mu_2\phi \cosh \delta_2\phi + 0.6526 \cos \mu_2\phi \sinh \delta_2\phi + 2.353 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4003 \cos d_3\phi + 0.2600 \sin d_3\phi + 1.098 \cosh e_3\phi - 1.097 \sinh e_3\phi - 1.498 \cosh f_3\phi + 1.456 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3122 \cos d_4\phi + 0.2001 \sin d_4\phi + 0.4783 \cosh e_4\phi - 0.4783 \sinh e_4\phi - 0.7906 \cosh f_4\phi + 0.8066 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2396 \cos d_5\phi + 0.1709 \sin d_5\phi + 0.3060 \cosh e_5\phi - 0.3060 \sinh e_5\phi - 0.5456 \cosh f_5\phi + 0.5321 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.1745 \sin d_1\phi + 0.6319 \cos d_1\phi + 0.2790 \sin \mu_1\phi \cosh \delta_1\phi - 0.6320 \cos \mu_1\phi \cosh \delta_1\phi - 0.2429 \sin \mu_1\phi \sinh \delta_1\phi + 0.1902 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.002 \sin d_2\phi + 0.5438 \cos d_2\phi + 2.358 \sin \mu_2\phi \cosh \delta_2\phi - 0.5438 \cos \mu_2\phi \cosh \delta_2\phi - 2.313 \sin \mu_2\phi \sinh \delta_2\phi + 0.6621 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.090 \sin d_3\phi + 0.7078 \cos d_3\phi + 2.267 \sinh e_3\phi - 2.266 \cosh e_3\phi - 1.603 \sinh f_3\phi + 1.558 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.152 \sin d_4\phi + 0.7385 \cos d_4\phi + 1.557 \sinh e_4\phi - 1.557 \cosh e_4\phi - 0.8018 \sinh f_4\phi + 0.8180 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.127 \sin d_5\phi + 0.8039 \cos d_5\phi + 1.339 \sinh e_5\phi - 1.339 \cosh e_5\phi - 0.5482 \sinh f_5\phi + 0.5347 \cosh f_5\phi$$

Table 31. Clamped–circumferentially guided: clamped–circumferentially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	1.095	1.502	0.6424	0.5329	—	—
2	19.79	2.412	—	—	1.548	1.192
3	105.6	3.421	—	—	2.944	1.020
4	338.0	4.454	—	—	4.102	1.006
5	805.7	5.464	—	—	5.182	1.003

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n'(\pi) = 0$$

$$\mathcal{V}_n(0) + \mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n'''(0) + \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.2228 \cos d_1\phi - 0.2212 \sin d_1\phi + 0.9385 \cos \mu_1\phi \cosh \delta_1\phi + 1.379 \sin \mu_1\phi \cosh \delta_1\phi - 0.6272 \cos \mu_1\phi \sinh \delta_1\phi - 1.138 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.4334 \cos d_2\phi + 0.3271 \sin d_2\phi + 2.711 \cosh e_2\phi - 2.669 \sinh e_2\phi - 2.939 \cosh f_2\phi + 2.804 \sinh f_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3189 \cos d_3\phi + 0.2485 \sin d_3\phi + 0.4359 \cosh e_3\phi - 0.4360 \sinh e_3\phi - 0.3914 \cosh f_3\phi + 0.4245 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2360 \cos d_4\phi + 0.2044 \sin d_4\phi + 0.2769 \cosh e_4\phi - 0.2769 \sinh e_4\phi - 0.2439 \cosh f_4\phi + 0.2241 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.1900 \cos d_5\phi + 0.1697 \sin d_5\phi + 0.2062 \cosh e_5\phi - 0.2062 \sinh e_5\phi - 0.1294 \cosh f_5\phi + 0.1410 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.3348 \sin d_1\phi - 0.3323 \cos d_1\phi - 1.231 \sin \mu_1\phi \cosh \delta_1\phi + 0.3323 \cos \mu_1\phi \cosh \delta_1\phi + 1.220 \sin \mu_1\phi \sinh \delta_1\phi - 0.003553 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.045 \sin d_2\phi + 0.7888 \cos d_2\phi + 4.195 \sinh e_2\phi - 4.130 \cosh e_2\phi - 3.503 \sinh f_2\phi + 3.342 \cosh f_2\phi$$

$$\mathcal{W}_3(\phi) = -1.091 \sin d_3\phi + 0.8503 \cos d_3\phi + 1.283 \sinh e_3\phi - 1.283 \cosh e_3\phi - 0.3994 \sinh f_3\phi + 0.4331 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.051 \sin d_4\phi + 0.9106 \cos d_4\phi + 1.136 \sinh e_4\phi - 1.136 \cosh e_4\phi - 0.2454 \sinh f_4\phi + 0.2254 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.038 \sin d_5\phi + 0.9272 \cos d_5\phi + 1.069 \sinh e_5\phi - 1.069 \cosh e_5\phi - 0.1297 \sinh f_5\phi + 0.1413 \cosh f_5\phi$$

Table 32. Clamped–circumferentially guided: clamped–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.4656	1.359	0.4613	0.5378	—	—
2	2.550	1.690	0.8285	0.5085	—	—
3	34.77	2.700	—	—	2.033	1.074
4	153.3	3.718	—	—	3.286	1.014
5	436.5	4.728	—	—	4.398	1.005

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n(0) + \mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n'''(0) + \mathcal{V}_n^{v}(0) = 0 \qquad \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.3909\cos d_1\phi - 0.1853\sin d_1\phi + 1.124\cos\mu_1\phi\cosh\delta_1\phi + 1.306\sin\mu_1\phi\cosh\delta_1\phi - 0.9761\cos\mu_1\phi\sinh\delta_1\phi - 1.424\sin\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5633\cos d_2\phi + 0.3825\sin d_2\phi - 0.6425\cos\mu_2\phi\cosh\delta_2\phi - 2.304\sin\mu_2\phi\cosh\delta_2\phi + 0.6341\cos\mu_2\phi\sinh\delta_2\phi + 2.329\sin\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4025\cos d_3\phi + 0.2917\sin d_3\phi + 0.8960\cosh e_3\phi - 0.8960\sinh e_3\phi - 0.9604\cosh f_3\phi + 0.9626\sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2865\cos d_4\phi + 0.2338\sin d_4\phi + 0.3700\cosh e_4\phi - 0.3700\sinh e_4\phi - 0.3405\cosh f_4\phi + 0.3417\sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2215\cos d_5\phi + 0.1928\sin d_5\phi + 0.2516\cosh e_5\phi - 0.2516\sinh e_5\phi - 0.1934\cosh f_5\phi + 0.1942\sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.5313\sin d_1\phi - 0.2518\cos d_1\phi - 1.262\sin\mu_1\phi\cosh\delta_1\phi + 0.2518\cos\mu_1\phi\cosh\delta_1\phi + 1.127\sin\mu_1\phi\sinh\delta_1\phi - 0.2476\cos\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.9519\sin d_2\phi + 0.6464\cos d_2\phi + 2.256\sin\mu_2\phi\cosh\delta_2\phi - 0.6464\cos\mu_2\phi\cosh\delta_2\phi - 2.232\sin\mu_2\phi\sinh\delta_2\phi + 0.6522\cos\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.087\sin d_3\phi + 0.7876\cos d_3\phi + 1.822\sinh e_3\phi - 1.822\cosh e_3\phi - 1.032\sinh f_3\phi + 1.034\cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.065\sin d_4\phi + 0.8692\cos d_4\phi + 1.216\sinh e_4\phi - 1.216\cosh e_4\phi - 0.3452\sinh f_4\phi + 0.3464\cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.047\sin d_5\phi + 0.9115\cos d_5\phi + 1.107\sinh e_5\phi - 1.107\cosh e_5\phi - 0.1944\sinh f_5\phi + 0.1951\cosh f_5\phi$$

Table 33. Clamped–circumferentially guided: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	1.040	1.492	0.6313	0.5336	—	—
2	3.263	1.755	0.8857	0.4949	—	—
3	37.47	2.741	—	—	2.092	1.067
4	155.0	3.727	—	—	3.296	1.014
5	439.5	4.736	—	—	4.406	1.005

Boundary conditions:

$$\mathcal{V}_n'(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n(0) + \mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$\mathcal{V}_n'''(0) + \mathcal{V}_n^{v}(0) = 0 \qquad -\xi_n\mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.05549\cos d_1\phi - 0.2966\sin d_1\phi + 1.087\cos\mu_1\phi\cosh\delta_1\phi + 1.860\sin\mu_1\phi\cosh\delta_1\phi - 0.8709\cos\mu_1\phi\sinh\delta_1\phi - 1.696\sin\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5843\cos d_2\phi + 0.3150\sin d_2\phi - 0.3602\cos\mu_2\phi\cosh\delta_2\phi - 1.920\sin\mu_2\phi\cosh\delta_2\phi + 0.4487\cos\mu_2\phi\sinh\delta_2\phi + 2.018\sin\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.3821\cos d_3\phi + 0.2884\sin d_3\phi + 0.8343\cosh e_3\phi - 0.8338\sinh e_3\phi - 0.9332\cosh f_3\phi + 0.8935\sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.2857\cos d_4\phi + 0.2312\sin d_4\phi + 0.3649\cosh e_4\phi - 0.3649\sinh e_4\phi - 0.3191\cosh f_4\phi + 0.3365\sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2201\cos d_5\phi + 0.1926\sin d_5\phi + 0.2512\cosh e_5\phi - 0.2512\sinh e_5\phi - 0.2056\cosh f_5\phi + 0.1936\sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.08282\sin d_1\phi - 0.4427\cos d_1\phi - 1.651\sin\mu_1\phi\cosh\delta_1\phi + 0.4427\cos\mu_1\phi\cosh\delta_1\phi + 1.639\sin\mu_1\phi\sinh\delta_1\phi - 0.2188\cos\mu_1\phi\sinh\delta_1\phi$$

$$\mathcal{W}_2(\phi) = -1.025\sin d_2\phi + 0.5528\cos d_2\phi + 1.966\sin\mu_2\phi\cosh\delta_2\phi - 0.5528\cos\mu_2\phi\cosh\delta_2\phi - 1.923\sin\mu_2\phi\sinh\delta_2\phi + 0.6798\cos\mu_2\phi\sinh\delta_2\phi$$

$$\mathcal{W}_3(\phi) = -1.048\sin d_3\phi + 0.7908\cos d_3\phi + 1.745\sinh e_3\phi - 1.744\cosh e_3\phi - 0.9959\sinh f_3\phi + 0.9537\cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -1.065\sin d_4\phi + 0.8618\cos d_4\phi + 1.203\sinh e_4\phi - 1.203\cosh e_4\phi - 0.3234\sinh f_4\phi + 0.3410\cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -1.042\sin d_5\phi + 0.9123\cos d_5\phi + 1.107\sinh e_5\phi - 1.107\cosh e_5\phi - 0.2066\sinh f_5\phi + 0.1945\cosh f_5\phi$$

Table 34. Clamped–radially guided: clamped–radially guided semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	7.200	2.000	1.082	0.4133	—	—
2	57.60	3.000	—	—	2.433	1.040
3	211.8	4.000	—	—	3.603	1.010
4	553.8	5.000	—	—	4.690	1.004
5	1192	6.000	—	—	5.744	1.002

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) = 0$$
$$\mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n''(\pi) = 0$$
$$\mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n^{iv}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.6325 \sin d_1\phi$$
$$\mathcal{V}_2(\phi) = 0.4472 \sin d_2\phi$$
$$\mathcal{V}_3(\phi) = 0.3430 \sin d_3\phi$$
$$\mathcal{V}_4(\phi) = 0.2774 \sin d_4\phi$$
$$\mathcal{V}_5(\phi) = 0.2325 \sin d_5\phi$$

$$\mathcal{W}_1(\phi) = 1.265 \cos d_1\phi$$
$$\mathcal{W}_2(\phi) = 1.342 \cos d_2\phi$$
$$\mathcal{W}_3(\phi) = 1.372 \cos d_3\phi$$
$$\mathcal{W}_4(\phi) = 1.387 \cos d_4\phi$$
$$\mathcal{W}_5(\phi) = 1.395 \cos d_5\phi$$

Table 35. Clamped–radially guided: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	0.4887	1.366	0.4716	0.5378	—	—
2	8.839	2.074	1.137	0.3764	—	—
3	59.69	3.023	—	—	2.462	1.038
4	214.0	4.010	—	—	3.614	1.010
5	556.2	5.005	—	—	4.695	1.004

Boundary conditions:

$$\mathcal{V}_n(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$\mathcal{V}_n^{iv}(0) = 0 \qquad -\xi_n \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) + \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.5453 \sin d_1\phi - 0.4991 \sin \mu_1\phi \cosh \delta_1\phi - 0.4125 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.5817 \sin d_2\phi + 0.01009 \sin \mu_2\phi \cosh \delta_2\phi + 0.06837 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.4392 \sin d_3\phi + 0.6525 \times 10^{-4} \sinh e_3\phi - 0.02890 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.3410 \sin d_4\phi - 0.3736 \times 10^{-6} \sinh e_4\phi + 0.01586 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.2767 \sin d_5\phi + 0.4432 \times 10^{-8} \sinh e_5\phi - 0.01002 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = 0.7449 \cos d_1\phi - 0.4630 \cos \mu_1\phi \cosh \delta_1\phi - 0.01350 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = 1.206 \cos d_2\phi + 0.08151 \cos \mu_2\phi \cosh \delta_2\phi - 0.01426 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = 1.328 \cos d_3\phi + 0.1606 \times 10^{-3} \cosh e_3\phi - 0.03001 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = 1.367 \cos d_4\phi - 1.350 \times 10^{-6} \cosh e_4\phi + 0.01601 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = 1.385 \cos d_5\phi + 2.081 \times 10^{-8} \cosh e_5\phi - 0.01005 \cosh f_5\phi$$

Table 36. Clamped–free: clamped–free semicircular ring

Characteristic values:

n	ξ_n	d_n	δ_n	μ_n	e_n	f_n
1	1.270	1.532	0.6740	0.5305	—	—
2	10.11	2.125	1.173	0.3459	—	—
3	62.13	3.049	—	—	2.494	1.037
4	215.9	4.018	—	—	3.623	1.010
5	559.0	5.011	—	—	4.701	1.004

Boundary conditions:

$$\mathcal{V}_n(0) + \mathcal{V}_n''(0) = 0 \qquad \mathcal{V}_n(\pi) + \mathcal{V}_n''(\pi) = 0$$

$$\mathcal{V}_n''(0) + \mathcal{V}_n^{iv}(0) = 0 \qquad \mathcal{V}_n''(\pi) + \mathcal{V}_n^{iv}(\pi) = 0$$

$$-\xi_n \mathcal{V}_n'(0) + \mathcal{V}_n'''(0) + \mathcal{V}_n^{v}(0) = 0 \qquad -\xi_n \mathcal{V}_n'(\pi) + \mathcal{V}_n'''(\pi) - \mathcal{V}_n^{v}(\pi) = 0$$

Modal functions:

$$\mathcal{V}_1(\phi) = 0.3443 \cos d_1\phi - 0.3117 \sin d_1\phi + 0.9063 \cos \mu_1\phi \cosh \delta_1\phi + 1.053 \sin \mu_1\phi \cosh \delta_1\phi - 0.6983 \cos \mu_1\phi \sinh \delta_1\phi - 0.8386 \sin \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{V}_2(\phi) = 0.1077 \cos d_2\phi - 0.5424 \sin d_2\phi + 0.5280 \cos \mu_2\phi \cosh \delta_2\phi + 1.001 \sin \mu_2\phi \cosh \delta_2\phi - 0.5854 \cos \mu_2\phi \sinh \delta_2\phi - 1.002 \sin \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{V}_3(\phi) = 0.03301 \cos d_3\phi - 0.4291 \sin d_3\phi - 0.07640 \cosh e_3\phi + 0.07633 \sinh e_3\phi + 0.3979 \cosh f_3\phi - 0.3684 \sinh f_3\phi$$

$$\mathcal{V}_4(\phi) = 0.009504 \cos d_4\phi - 0.3392 \sin d_4\phi - 0.01445 \cosh e_4\phi + 0.01445 \sinh e_4\phi + 0.1723 \cosh f_4\phi - 0.1874 \sinh f_4\phi$$

$$\mathcal{V}_5(\phi) = 0.004747 \cos d_5\phi - 0.2758 \sin d_5\phi - 0.006133 \cosh e_5\phi + 0.006133 \sinh e_5\phi + 0.1276 \cosh f_5\phi - 0.1171 \sinh f_5\phi$$

$$\mathcal{W}_1(\phi) = -0.5273 \sin d_1\phi - 0.4774 \cos d_1\phi - 1.046 \sin \mu_1\phi \cosh \delta_1\phi + 0.08772 \cos \mu_1\phi \cosh \delta_1\phi + 1.080 \sin \mu_1\phi \sinh \delta_1\phi + 0.1660 \cos \mu_1\phi \sinh \delta_1\phi$$

$$\mathcal{W}_2(\phi) = -0.2288 \sin d_2\phi - 1.152 \cos d_2\phi - 1.358 \sin \mu_2\phi \cosh \delta_2\phi - 0.3408 \cos \mu_2\phi \cosh \delta_2\phi + 1.377 \sin \mu_2\phi \sinh \delta_2\phi + 0.2730 \cos \mu_2\phi \sinh \delta_2\phi$$

$$\mathcal{W}_3(\phi) = -0.1006 \sin d_3\phi - 1.308 \cos d_3\phi - 0.1905 \sinh e_3\phi + 0.1904 \cosh e_3\phi + 0.4125 \sinh f_3\phi - 0.3819 \cosh f_3\phi$$

$$\mathcal{W}_4(\phi) = -0.03818 \sin d_4\phi - 1.363 \cos d_4\phi - 0.05234 \sinh e_4\phi + 0.05234 \cosh e_4\phi + 0.1740 \sinh f_4\phi - 0.1892 \cosh f_4\phi$$

$$\mathcal{W}_5(\phi) = -0.02379 \sin d_5\phi - 1.382 \cos d_5\phi - 0.02883 \sinh e_5\phi + 0.02883 \cosh e_5\phi + 0.1281 \sinh f_5\phi - 0.1176 \cosh f_5\phi$$

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