24p

RIAS

TECHNICAL REPORT

63-8

APRIL 1963

TOPOLOGICALLY CONVEX SETS AND FIXED POINT THEORY

By

G. Stephen Jones

OTS PRICE

XEROX $ [redacted]

MICROFILM $ [redacted]

TOPOLOGICALLY CONVEX SETS AND FIXED POINT THEORY

by

G. Stephen Jones

April 1963

Research Institute for Advanced Studies (RIAS)
7212 Bellona Avenue
Baltimore 12, Maryland

NASA CR-50,251

Code 1

Topologically Convex Sets and Fixed Point Theory

By

G. Stephen Jones*

In the study of topological invariants associated with convex sets it is important that large classes of sets which are topologically equivalent to convex sets be identified. This is particularly true in investigations which are concerned with the fixed point property associated with continuous functions on convex sets. It is the main purpose of this paper to identify a class of topologically convex sets and to use this identification to obtain several interesting results in the theory of fixed points.

For a linear topological space X and subsets A and B of X, we denote by $\bar{A}$, A^o, $\partial(A)$, and $A \setminus B$ the closure of A, interior of A, boundary of A, and the set of all elements in A not in B respectively. If $S \subset X$ and x and y are elements of X, then S is said to be linearly connected relative to x and y when for each z in S, the set $S \cap P_z$ is connected where P_z is the 2-dimensional plane containing x, y, and z. Our principal result concerning topologically convex sets is embodied in the following theorem.

Theorem 1. Let A and B be two convex sets in a real linear topological space X such that $A \cap B$ is bounded. If relative to some x_1 in $A^o \cap \partial(B)$ and x_o in $B^o \cap \partial(A)$, $A^o \cap \partial(B)$ and $B^o \cap \partial(A)$ linearly connected sets, then there exists a one-to-one continuous function

---

*
This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research, Office of Aerospace Research, under Contract No. AF 49(638)-382, in part by the U. S. Army, Army Ordnance Missile Command under Contract DA-36-034-ORD-3514 RD, in part by the National Aeronautics and Space Administration under Contract No. NASr-103, and Office of Naval Research under contract Nonr-3693(00). Reproduction in whole or in part is permitted for any purpose of the United States Government.

f which maps $\bar{A}$ onto $\bar{A} \setminus B^o$ and f is a homeomorphism on A^o. Furthermore, if p is the support function of $A \cap B$ defined relative to some interior point, then for arbitrary $\epsilon > 0$, f may be defined to be the identity map outside the set $\{x : p(x) < 1 + \epsilon\}$.

We recall that the Tychonoff Fixed Point theorem states that every compact convex subset of a locally convex linear topological space has the fixed point property, [1]. Before proving Theorem 1 let us make a straightforward application of it together with the Tychonoff theorem in proving the three essentially equivalent theorems stated below. For this purpose we define the linear extension of a subset A of a linear topological space X as the smallest linear subspace of X containing A. If $A \subset Y \subset X$, then $(A \sim Y)^o$ and $\partial(A \sim Y)$ denote the interior and boundary of A respectively in the relative topology of Y. We denote a mapping as compact if its range is contained in a compact set.

<u>Theorem 2</u>. Let X be a locally convex linear topological space, A a closed convex subset of X, and F a continuous mapping of A into itself. Let the closed convex hull of $F(A)$ be compact, let B be a convex body in the linear extension Y of A, and relative to some x_1 in $(A \sim Y)^o \cap \partial(B \sim Y)$ and x_o in $(B \sim Y)^o \cap \partial(A \sim Y)$ let $(A \sim Y)^o \cap \partial(B \sim Y)$ and $(B \sim Y)^o \cap \partial(A \sim Y)$ be linearly connected. If $F(\bar{A} \setminus (B \sim Y)^o) \subset \bar{A} \setminus (B \sim Y)^o$, then F has a fixed point in $\bar{A} \setminus (B \sim Y)^o$.

<u>Theorem 3</u>. Let X be a complete locally convex linear topological space, A a closed convex subset of X, and F a compact mapping of A into itself. Let B be a convex body in the linear extension Y of A, and relative to some x_1 in $(A \sim Y)^o \cap \partial(B \sim Y)$ and x_o in $(B \sim Y)^o \cap \partial(A \sim Y)$ let $(A \sim Y)^o \cap \partial(B \sim Y)$ and $(B \sim Y)^o \cap \partial(A \sim Y)$ be linearly connected. If $F(\bar{A} \setminus (B \sim Y)^o) \subset \bar{A} \setminus (B \sim Y)^o$, then F has a fixed point in $\bar{A} \setminus (B \sim Y)^o$.

Theorem 4. Let X be a locally convex linear topological space, A a compact convex subset of X, and F a continuous mapping of A into itself. Let B be a convex body in the linear extension Y of A, and relative to some x_1 in $(A \sim Y)^{o} \cap \partial(B \sim Y)$ and x_o in $(B \sim Y)^{o} \cap \partial(A \sim Y)$ let $(A \sim Y)^{o} \cap \partial(B \sim Y)$ and $(B \sim Y)^{o} \cap \partial(A \sim Y)$ be linearly connected sets. If $F(\bar{A} \setminus (B \sim Y)^{o}) \subset \bar{A} \setminus (B \sim Y)^{o}$, then F has a fixed point in $\bar{A} \setminus (B \sim Y)^{o}$.

Proofs. Let us suppose the hypotheses of Theorem 4. As subsets of the linear space Y, clearly A and B satisfy the hypothesis of Theorem 1. Hence there exists a one-to-one continuous function f which maps A onto $A \setminus (B \sim Y)^{o}$. Since A is compact f is a homeomorphism on A and the mapping $f^{-1}Ff$ is a continuous mapping of A into A. Hence using the Tychonoff theorem we have that there exist x^* in A such that

$$f^{-1}Ff(x^*) = x^*.$$

But, of course, it follows that

$$F(f(x^*)) = f(x^*),$$

where $f(x^*)$ is contained in $A \setminus (B \sim Y)^{o}$, so Theorem 4 is proved.

Now let H denote the closed convex hull of $F(A)$. If X is a complete locally convex linear topological space and $F(A)$ is compact, then the fact that H is compact is proved in [2]. Hence the hypotheses of either Theorem 2 or Theorem 3 imply that H is compact. Replacing A by H we observe easily that the hypotheses of Theorem 4 are satisfied, so the validity of Theorem 2 and Theorem 3 is established.

These theorems bring out the usefulness of Theorem 1 in providing a technique whereby a set may be partitioned and the question of fixed points considered on its component parts. As an example of a situation calling for such a technique, let us suppose that the zero element 0 of X is contained in the set A, and it is important to know whether or not F has a nontrivial fixed point in A. If a neighborhood of 0 with the properties of B can be constructed, then it is clear from our results that such a fixed point exists. Situations of this type associated with establishing the existence of nontrivial periodic solutions of functional-differential equations are discussed in [3].

Let us now use Theorem 1 to prove a fixed point theorem which shows that, in a rather general sense, all the fixed points of a mapping of a convex set into itself can not be isolated boundary points and repulsive.

Theorem 5. Let S be a closed convex subset of a complete locally convex linear topological space X, let F be a compact mapping of S into X with $F(S) \subset S$, and let $\Delta = \{x_i : i = 1,2, \dots, n\}$ be a finite set of fixed points under F contained in $\partial(S)$. Let $\mathcal{N} = \{N_i : i = 1,2,\dots, n\}$ be a set of bounded convex neighborhoods of the origin and let $p_1, p_2, \dots, p_n$ denote their respective support functions defined relative to the origin. F has a fixed point in $S \setminus \Delta$ if one of the following conditions is satisfied.

(a) There exist $\delta > 0$ such that for each x_i in Δ, $p_i(F(x) - F(x_i)) \geqq p_i(x - x_i)^{1-\delta}$ when $p_i(x - x_i)$ is sufficiently small.

(b) There exist $\lambda_1 > 0$ such that for $0 < \lambda \leqq \lambda_1$ and N_i in $\mathcal{N}$, $(\lambda N_i)^o \cap \partial(S)$ is linearly connected relative to x_i and a point in $\partial(\lambda N_i) \cap S^o$, and for each x_i in Δ, $p_i(F(x) - F(x_i)) \geqq p_i(x - x_i)$ when $p_i(x - x_i)$ is sufficiently small.

In preparation to proving Theorem 5, Theorem 1, and other results to follow, it is convenient to introduce some additional notations. If x is an element of a linear topological space X and A is a subset of X, then $C(x, A)$ denotes the cone with vertex at x generated by A. That is,

$$C(x, A) = \{y : y = (1 - \lambda)x + \lambda z, \quad z \text{ in } A, \quad \lambda > 0\}.$$

For x and y in X,

$$\widehat{xy} = \{z : z = (1 - \lambda)x + \lambda y, \quad 0 < \lambda < 1\},$$

$$\overline{xy} = \text{the closure of } \widehat{xy},$$

$$\ell(xy) = \{z : z = (1 - \lambda)x + \lambda y, \quad -\infty < \infty < \infty\},$$

and

$$r(xy) = \{z : z = (1 - \lambda)x + \lambda y, \quad \lambda \geqq 0\}.$$

The following lemma will be useful in the proof of Theorem 5 when it is supposed that condition (a) is satisfied.

Lemma 1. Let A be a convex set contained in a locally convex linear topological space X and let K be a bounded convex neighborhood of the origin. Let x_o be contained in $\partial(A)$ and suppose x_1 is a point in $\partial(x_o + K) \cap A^o$. For all $\eta > 0$ let $x(\eta) = \partial(x_o + \eta K) \cap \widehat{x_o x_1}$. Then there exist positive constants η_1 and ϵ such that

$$C(x(\eta), x_o + \epsilon\eta K) \cap A \subset (x_o + \eta K) \cap A,$$

for all η in $(0, \eta_1]$.

Proof. Let us decribe the ray $r(x_1 x_o)$ by the formula

$$\zeta(\lambda) = (1 - \lambda)x_1 + \lambda x_o, \quad \lambda \geqq 0.$$

Since A is convex, obviously $\zeta(2) = 2x_o - x_1 = x_2$ is contained in $X \setminus \bar{A}$. Let p denote the support function of K defined relative to the origin, and for arbitrary x in X and $a > 0$ let

$$N(x, a) = \{y : y = x + u, \quad p(u) < a\}.$$

Clearly there exist a constant r such that $N(x_1, r) \subset A$ and $N(x_2, r) \subset X \setminus \bar{A}$. Let $0 < \eta < p(x_1 - x_o)$ and let c be a constant such that $c \geq 6(p(x_1 - x_o) + p(x_o - x_1))/r$. Let $x_3 = \mu x_1 + (1 - \mu)x_o$ and $x_4 = \mu x_2 + (1 - \mu)x_o$ where $\mu p(x_1 - x_o) = \eta$. Let $y = x_3 + u$ when $p(u) \leq \mu r$. Then

$$y = x_3 + u = \mu(x_1 + \frac{1}{\mu}u) + (1 - \mu)x_o,$$

and $z = x_1 + \frac{1}{\mu}u$ is contained in $N(x_1, r)$. Hence y is contained in $\widehat{z\,x_o}$ and consequently in A. It follows that $N(x_3, \mu r) \subset A$, and in a similar fashion it may be shown that $N(x_4, \mu r) \subset X \setminus \bar{A}$. Furthermore, it is clear that

$$V_o = \{x : x = (1 - \lambda)x_o + \lambda y,\ y \text{ in } N(x_3, \mu r),\ 0 < \lambda \leq 1\} \quad A^o,$$

and

$$V_1 = \{x : x = (1 - \lambda)x_o + \lambda y,\ y \text{ in } N(x_4, \mu r),\ 0 < \lambda \leq 1\} \subset X \setminus \bar{A}.$$

Let $C_o = C(x_o, N(x_4, \mu r))$, $C_1 = C(x_3, N(x_o, \eta/c)$ and

$$V_2 = \{x : x = (1 - \lambda)x_3 + \lambda y,\ y \text{ in } N(x_o, \eta/c),\ \lambda \geq 5/4\}.$$

Clearly $c > 5\, p(x_1 - x_o)/\, r$ implies

$$\frac{\lambda}{c} < \frac{(\lambda - 1)r}{p(x_1 - x_o)}$$

for $\lambda \geq 5/4$, and considering an arbitrary element $x = (1 - \lambda)x_3 + \lambda y$ in V_2, we have

$$x = (1 - \lambda)(-x_4 + 2x_o) + \lambda(x_o + y_1) = (\lambda - 1)x_4 + (2 - \lambda)x_o + \lambda y_1,$$

where $p(y_1) \leq \eta/c$. Letting $\rho = \lambda - 1$, we have $x = \rho x_4 + (1 - \rho)x_o + \lambda y_1$. Hence if $p(\lambda y_1) \leq \rho\mu r$, then x is contained in C_o. But by definition $p(y_1) \leq \eta/c$, so for $\lambda \geq 5/4$ it follows that

$$p(\lambda y_1) \leq \lambda\eta/c < (\lambda - 1)r\eta/p(x_1 - x_o) = \rho\mu r,$$

and consequently $V_2 \subset C_o$. Let ω be an arbitrary point in $\partial(C_o) \cap \partial(C_1)$. Then $\omega = (1 - \lambda)x_3 + \lambda(x_o + y_2)$ where $1 - \eta/c < \lambda < \frac{5}{4}$ and $p(y_2) \leq \eta/c$. Hence we have

$$\omega = x_o + (1 - \lambda)\mu(x_1 - x_o) + \lambda y_2,$$

where

$$\begin{aligned} p((1 - \lambda)\mu(x_1 - x_o) + \lambda y_2) &\leq (1-\lambda)\mu p(x_1 - x_o) + \lambda p(y_2) \\ &\leq (1-\lambda)\eta + \lambda\eta/c < \eta. \end{aligned}$$

Hence every ray $r(x_3 y)$, y in $N(x_o, \eta/c)$, intersects $C_o \cap N(x_o, \eta)$. Since it is easily varified that such rays must remain in C_o once they have entered, it follows that they intersect $\partial(N(x_o, \eta))$ outside A which, of course, implies that $C_1 \cap A \subset N(x_o, \eta) \cap A$. It is clear that $N(x_o, \eta/c) = x_o + \frac{\eta}{c} K$, $N(x_o, \eta) = x_o + \eta K$, and $x_3 = x(\eta) = \partial(x_o + \eta K) \cap \widehat{x_o x_1}$. Therefore, letting $\epsilon = 1/c$, $\eta_1 = p(x_1 - x_o)$, we have

$$C(x(\eta), x_o + \epsilon\eta K) \cap A \subset (x_o + \eta K) \cap A,$$

and our proof is complete.

<u>Proof of Theorem 5.</u> Preceding directly now with our varification of Theorem 5, we assume condition (a) is satisfied. Clearly there exist

$\beta > 0$ such that the sets $N_i^* = \beta N_i$, $i = 1, 2, \dots, n$ are mutually disjoint and do not cover S. Let s be an element in $S^o \setminus \bigcup_{i=1}^{n} \overline{N_i^*}$ and for $i = 1, 2, \dots, n$, let $u_i = \partial(x_i + N_i^*) \cap \widehat{sx_i}$ and $v_i(\eta) = \partial(x_i + \eta N_i^*) \cap \widehat{u_i x_i}$. From Lemma 1 it is clear that for η sufficiently small there exists $\epsilon > 0$ such that

$$C(v_i(\eta),\ x_i + \epsilon\eta\ N_i^*) \cap S \subset (x_i + \eta\ N_i^*) \cap S \tag{1}$$

for $i = 1, 2, \dots, n$. Using condition (a) we know that there exist $\eta_1 > 0$ such that if $\eta \leqq \eta_1$ and $p_i(x - x_i) \leqq \eta$, then

$$p_i(F(x) - F(x_i)) \geqq \frac{1}{\epsilon} p_i(x - x_i). \tag{2}$$

Hence defining q_i, $i = 1, 2, \dots, n$ to be the support functions of the sets $C(v_i(\eta_1),\ x_i + \eta_1 N_i)$ it is clear from (1) and (2) that for $q_i(x - x_i)$ sufficiently small,

$$q_i(F(x) - F(x_i)) \geqq q_i(x - x_i).$$

On the other hand we observe that the sets $C(v_i(\eta_1),\ x_i + \eta_1 N_i^*)$ are linearly connected relative to the points x_i and $v_i(\eta_1)$. Hence replacing the sets N_i, $i = 1, 2, \dots, n$ in our hypothesis of Theorem 5 by the sets $C(v_i(\eta_1),\ x_i + \eta_1 N_i^*)$ we have that condition (b) is met. Therefore, Theorem 5 will be proved once it is shown to be valid when condition (b) is satisfied.

Let H denote the closed convex hull of $F(S)$, and let Y denote the linear extension of H. As we have observed previously the compactness of $F(S)$ implies H is compact and obviously $F(H) \subset H$. Assuming condition (b) is satisfied it is clear that we may chose ϵ_1 to be such that $p_i(x - x_i) \leq \epsilon_1$ for each x_i in Δ implies

$$p_i(F(x) - F(x_i)) = p_i(F(x) - x_i) \geq p_i(x - x_i),$$

and the sets $(B_i \sim Y)^o \cap \partial(H \sim Y)$ have mutually disjoint closures and are linearly connected relative to x_i and points in $\partial(B_i \sim Y) \cap (H \sim Y)^o$, when

$$B_i = \{x : p_i(x - x_i) < \frac{\epsilon_1}{2}\}.$$

Since the compactness of H implies H is bounded, we can easily verify that the linear connectivity of $(B_i \sim Y)^o \cap \partial(H \sim Y)$ implies the linear connectivity of $(H \sim Y)^o \cap \partial(B_i \sim Y)$ relative to the same points. Letting

$$A_i = \{x : \epsilon_1/2 \leq p_i(x - x_i) \leq \epsilon_1, \quad x \text{ in } H\},$$

it is clear that $F(A_i) \subset H \setminus B_i$, $i = 1, 2, \ldots, n$.

Now let F^* be the mapping defined on $H \setminus \bigcup_{i=1}^{n} B_i$, in the following way: $F^*(x) = F(x)$ if $F(x)$ is contained in $H \setminus \bigcup_{i=1} B_i$, and

$$F^*(x) = \partial(B_i \sim Y) \cap \widehat{xF(x)}$$

if $F(x)$ is contained in B_i. Since the B_i's are convex it is clear that F^* is well defined and the fact that $F(A_i) \subset H \setminus B_i$ implies F^* has the same fixed points in $H \setminus \bigcup_{i=1}^{n} B_i$ as F. We observe that for each p_i and arbitrary x in $H \setminus \bigcup_{i=1}^{n} B_i$ that

$$\lim_{\epsilon \to 0} \{\widehat{yF(y)} : p_i(y - x) \leq \epsilon, \quad y \text{ in } H\} = \widehat{xF(x)}.$$

Suppose $F(x)$ is contained in B_i. Then for an arbitrary neighborhood $N(F^*(x), \epsilon)$ defined relative to p_i there exist a neighborhood $N(x, \nu)$ defined relative to p_i such that

$$M = \{\widehat{yF(y)} : p_i(y - x) \leqq \nu, \ y \text{ in } H\} \cap \partial(B_i \sim Y) \subset N(F^*(x), \epsilon).$$

Since $F^*(N(x, \nu) \cap H)$ is obviously contained in M it follows that

$$F^*(N(x, \nu) \cap H) \subset N(F^*(x), \epsilon).$$

Hence it is clear that F^* is continuous on $H \setminus \bigcup_{i=1}^{n} B_i$.

Now let $N_1, N_2, \ldots, N_n$ be open convex neighborhoods of B_1, $B_2, \ldots, B_n$ respectively with mutually disjoint closures. By Theorem 1 for each B_i, there exists a homeomorphism f_i on H in Y which maps H onto $H \setminus B_i$ and f_i is the identity map outside N_i. If $g = f_1 f_2 \ldots f_n$, then clearly g is a homeomorphism of H onto $H \setminus \bigcup_{i=1}^{n} B_i$, and $g^{-1}F^*g$ is a continuous mapping of H onto H. Hence using the Tychonoff theorem we have that there exists x^* in H such that

$$g^{-1}F^*g(x^*) = x^*.$$

But, of course, this implies $F^*(g(x^*)) = g(x^*)$, and since $g(x^*)$ is in $H \setminus \bigcup_{i=1}^{n} B_i$ we also have that

$$F(g(x^*)) = g(x^*).$$

Therefore, $g(x^*)$ is a fixed point under F contained in $H \setminus \Delta$, and our theorem is established.

A set $S \subset X$ is called a star body if there exists a point x_o in S^o with respect to which S is a star set and if each ray $r(x_o y)$ intersects $\partial(S)$ in at most one point. The notion of a star body will be used in proving some of the lemmas to follow which are preparatory to proving Theorem 1.

Lemma 1. Let X be a locally convex linear topological space and let $S \subset X$ be a bounded star body with respect to a point x_1 in S^o.

Then there exists a unique positive continuous function λ on $X \setminus \{x_1\}$ and a unique continuous mapping $\omega : X \setminus \{x_1\} \to \partial(S)$ such that for each x in $X \setminus \{x_1\}$,

$$x = (1 - \lambda(x))x_1 + \lambda(x)\omega(x). \tag{3}$$

Proof. Consider an arbitrary point x in $X \setminus \{x_1\}$. Since the ray $r(x_1 x)$ must intersect $\partial(S)$ at a unique point, we may denote this point by $\omega(x)$. Obviously then x may be expressed by the formula

$$x = (1 - \lambda(x))x_1 + \lambda(x)\omega(x),$$

where $\lambda(x)$ is a unique positive number. Hence to prove our lemma we have only to show that λ and ω are continuous at x.

We may assume without loss of generality that $x_1 = 0$, and our formula reduces to $x = \lambda(x)\omega(x)$. Let $N(0)$ be an arbitrary bounded neighborhood of the origin and let $N_1(0)$ contained in $N(0)$ be a convex neighborhood of the origin such that $\omega(x)$ is not in $N_1(0)$. Let the neighborhood $N_2(0)$ of the origin be convex and such that $\overline{N_2(0)} \subset N_1(0)$. We let x_2 in $r(0\,\omega(x))$ be such that $x_2 + a\,N_2(0)$ is contained in $\omega(x) + N_2(0)$ and is disjoint from $\bar{S}$ for some fixed $a > 0$. Consider $C(0, \omega(x) + b\,N_2(x))$, $b > 0$, and $C(\omega(x), 2\omega(x) + a\,N_2(0))$. Now x in $C(0, \omega(x) + b\,N_2(0))$ implies $x = \xi[\omega(x) + by_1]$, where $\xi > 0$ and y_1 is contained in $N_2(0)$. Hence

$$x = (2 - \xi)\omega(x) + (1 - \xi)(2\omega(x)) + \xi b y_1$$

$$= (1 - \rho)\omega(x) + \rho(2\omega(x)) + \xi b y_1,$$

where $\rho = \xi - 1$. Now by boundedness there exist $\mu > 0$ such that

$$(1 - \eta)\omega(x) + \eta(2\omega(x)) + \xi b\,N_2(0) \subset \omega(x) + N_2(0)$$

for all η such that $|\eta| \leq \mu$ and $|b| \leq \mu$. Hence letting $b \leq \min\{\mu, a\mu/\xi\}$ we have that $x = (1-\rho)\omega(x) + \rho(x_2 + \frac{\xi b}{\rho} y_1)$ which is contained in $C(\omega(x), 2\omega(x) + a\, N_2(0))$ whenever $\rho > \mu$ which, of course, implies $r(0(\omega(x) + by_1))$ intersects $\partial(N_1(0))$ inside $C(\omega(x), 2\omega(x) + a\, N_2(0)) \cap (X \setminus \bar{S})$. Therefore $C(0, \omega(x) + b\, N_2(0))$ must intersect $\partial(S)$ inside $N_1(0)$. It follows that $\omega(y)$ not contained in $\omega(x) + N(0)$ implies $\omega(y)$ is not contained in $C(0, \omega(x) + b\, N_2(0))$ and consequently $y = \lambda(y)\omega(y)$ is not contained in $C(0, \omega(x) + b\, N_2(0))$. Clearly there must exist $c > 0$ such that $x + cb\, N_2(0) \subset C(0, \omega(x) + b\, N_2(0))$, so y is not in $x + cb\, N_2(0)$. Hence y in $x + cb\, N_2(0)$ implies $\omega(y)$ is in $\omega(x) + N(0)$ and the continuity of ω follows.

Now let $N_3(0) \subset S^o$ be a convex neighborhood of the origin and let p denote the support function of $N_3(0)$ defined relative to 0. For arbitrary $\delta > 0$ we may choose $N_4(0) \subset (\frac{\delta}{2} N_3(0)) \cap (-\frac{\delta}{2} N_3(0))$ such that y in $x + N_4(0)$ implies $\omega(y)$ is contained in $(\omega(x) + \frac{\delta}{2\lambda(x)} N_3(0)) \cap (\omega(x) - \frac{\delta}{2\lambda(x)} N_3(0))$. We observe that

$$\begin{aligned} y - x &= \lambda(y)\omega(y) - \lambda(x)\omega(x) \\ &= (\lambda(y) - \lambda(x))\omega(y) + \lambda(x)(\omega(y) - \omega(x)) \end{aligned}$$

and

$$(\lambda(y) - \lambda(x))\omega(y) = y - x + \lambda(x)(\omega(x) - \omega(y)).$$

Hence

$$\begin{aligned} |\lambda(y) - \lambda(x)| &< |\lambda(y) - \lambda(x)|\, p(\omega(y)) \\ &\leq \max\{p(\pm[(y-x) + \lambda(x)(\omega(x) - \omega(y))])\}, \end{aligned}$$

and it follows that for y in $x + N_4(0)$

$$|\lambda(y) - \lambda(x)| < \delta.$$

Therefore λ is also continuous and the proof of our lemma is complete.

Lemma 2. Let X be a locally convex linear topological space. Every closed and bounded star body contained in X is topologically equivalent to a closed and bounded convex body.

Proof. Let S be a closed and bounded star body in X with respect to a point x_1 in S^o. Since X is locally convex there exist a closed and bounded convex body K such that $S \subset K^o$. By Lemma 1 each x in $X \setminus \{x_1\}$ may be uniquely expressed as

$$x = (1 - \lambda(x))x_1 + \lambda(x)\omega(x),$$

where λ is a continuous positive functional defined on $X \setminus \{x_1\}$ and ω is a continuous mapping of $X \setminus \{x_1\}$ onto $\partial(K)$. For each x in S we define

$$\varphi(x) = x_1 + \frac{x - x_1}{\alpha(\omega(x))},$$

where x is in $r(x_1\omega(x))$ and $\alpha(\omega(x))$ is the positive functional such that

$$(1 - \alpha(\omega(x)))x_1 + \alpha(\omega(x))\omega(x) = \partial(S) \cap r(x_1\omega(x)).$$

Since K is convex and S is a star body it follows that φ and φ^{-1} are well defined. We easily observe that $\varphi(S) = K$. Since $\alpha(\omega(x))$ is bounded away from zero and continuous by virtue of Lemma 1 it follows that φ is continuous. φ^{-1} is given by the formula

$$\varphi^{-1}(y) = x_1 + \alpha(\omega(y))(y - x_1),$$

for y in K, and its continuity follows from the continuity of α and ω. Hence we have that φ is a homeomorphism mapping S onto K and our proof is complete.

One observes that Lemma 2 yields an immediate yet perhaps useful corollary to the Tychonoff Fixed Point Theorem which may be stated as follows.

Theorem 6. Every compact subset of a locally convex linear topological space which is a star body in its linear extension has the fixed point property.

Proof. Since Lemma 2 implies every compact subset which is a star body in its linear extension is topologically equivalent to a compact convex subset, this theorem follows trivially from the Tychonoff theorem.

Lemma 3. Let A and B be two convex subsets of a real linear topological space X such that $A \cap B$ is bounded and let $\Delta = \partial(A) \cap \partial(B)$. Relative to x_1 in $A^o \cap \partial(B)$ and x_o in $B^o \cap \partial(A)$ let $A^o \cap \partial(B)$ and $B^o \cap \partial(A)$ be linearly connected and let $C = C(x_1, \bar{B} \cap \partial(A))$. Then there exist a one-to-one continuous function φ which maps $\bar{A} \setminus C^o$ onto $\bar{A} \setminus B^o$ and which maps $\bar{A} \setminus (C^o \cup \Delta)$ onto $\bar{A} \setminus (B^o \cup \Delta)$ topologically.

Proof. For each y in Δ let

$$C_y = \{x : \ x = (1 - \lambda)x_1 + \lambda z, \ \ a \ \text{in} \ x_o y, \ \lambda \geq 0\},$$

and let P_y denote the two dimensional plane containing C_y. We shall show that

$$C = \cup \{C_y : y \text{ in } \Delta\}. \tag{4}$$

We begin by letting P_y^- and P_y^+ denote the two half planes composing $P_y \setminus \ell(x_o y)$ where P_y^- contains x_1. Clearly C_y, P_y, P_y^+, and P_y^- are convex. Let z be an arbitrary point in $\widehat{x_o y}$. Since $r(x_1 z)(\tau) = (1 - \tau)x_1 + \tau z$ is contained in $A^o \cap B$ for $\tau_1 < 1$ and $\partial(B) \cap P_y \cap A^o \subset P_y^-$, it is clear that for some unique $\tau \geq 1$, $r(x_1 z)(\tau_1)$ is contained in $\partial(A)$. Since $r(x_1 z)(\tau)$ for $\tau \geq 1$ must be contained in $\overline{P_y^+}$ and $(\partial(A) \cap P_y) \setminus \bar{B} \subset P_y^-$, we have that $r(x_1 z)(\tau_1)$ is contained in $\partial(A) \cap B^o$. Hence $r(x_1 z) \subset C$ and we may conclude therefore that $C_y \subset C$ for all y in Δ. That is, $\cup \{C_y : y \text{ in } \Delta\} \subset C$.

Now let x be an arbitrary point of C. For some $\eta > 0$, $x = (1 - \eta)x_1 = \eta u$, where u is contained in $\partial(A) \cap B^o$. If u is contained in $C(x_o, \Delta)$, then obviously x is contained in C_y for some y in Δ. If u is not contained in $C(x_o, \Delta)$, let P_u be the two dimensional plane determined by x_o, x_1, and u. Clearly there is a unique arc in $\partial(A) \cap P_u \cap B$ from x_o to a point y in Δ which contains u, and $\overline{x_1 u}$ must intersect $\ell(x_o y)$. Since $\widehat{x_1 u} \subset A^o$ and $A^o \cap \ell(x_o y) = \widehat{x_o y}$ it follows that $\overline{x_1 u}$ intersects $\widehat{x_o y}$ which, of course, implies u and consequently x is contained in C_y. Thus we have $C \subset \cup \{C_y : y \text{ in } \Delta\}$ which together with the reverse inclusion obtained in the previous paragraph establishes (4).

Our next step is to construct a one-to-one mapping φ which maps $\bar{A} \setminus B^o$ onto $\bar{A} \setminus C^o$. To this end we consider an arbitrary but fixed point x_2 in $\widehat{x_o x_1}$, an arbitrary element v in $A \cap \partial(B)$ and let P_v be the two dimensional plane determined by x_2, x_1, and v. Since there is a unique arc in $\partial(B) \cap P_v \cap A$ from x_1 to a point y in Δ which contains v, we have $P_v = P_y$. Let P_y^* and P_y^{**} denote the two half planes composing $P_y \setminus \ell(x_1 y)$ where x_2 is contained in P_y^*. Since v is contained in $\bar{P}_y^{**}$ it follows that $r(x_2 v)$ must intersect $\ell(x_1 y)$ once and only once. It is clear that the correspondence $y \longleftrightarrow C_y$ is one-to-one, so we have that each ray $r(x_2 v)$ must intersect the set $\partial(C) \cap \bar{A}$ once and only once. Now let K be a closed and bounded convex set such that $\overline{A \cap B}$ is contained in K^o. It is clear from Lemma 1 that that each x in $X \setminus \{x\ \}$ may be uniquely expressed as

$$x = (1 - \lambda(x))x_2 + \lambda(x)\omega(x),$$

where λ is a continuous positive functional defined on $X \setminus \{x_2\}$ and ω is a continuous mapping of $X \setminus \{x_2\}$ onto $\partial(K \cap A)$. For each x in $(\bar{A} \setminus C^o) \cap K$ we define

$$\varphi(x) = \omega(x) + \frac{(1 - \alpha(\omega(x)))}{(1 - \beta(\omega(x)))}\,(x - \omega(x)), \tag{5}$$

where x is in $r(x_2(\omega(x))$ and $\alpha(\omega(x))$ and $\beta(\omega(x))$ are positive functional such that

$$(1 - \alpha(\omega(x)))x_2 + \alpha(\omega(x))\omega(x) = \partial(B) \cap r(x_2\omega(x))$$

and

$$(1 - \beta(\omega(x)))x_2 + \beta(\omega(x))\omega(x) = \partial(C) \cap r(x_2\omega(x)).$$

On $(\bar{A} \cap (X \setminus K))$ we define $\varphi(x) = x$. Clearly the fact that φ and φ^{-1} are well defined follows immediately from the convexity of B and K and the unique point of intersection property we have established between $\partial(C)$ and any ray $r(x_2\omega(x))$. We also easily observe that $\varphi(\bar{A} \setminus C^o) = \bar{A} \setminus B^o$.

Now consider the functional γ defined by the formula

$$\gamma(x) = \frac{1 - \alpha(\omega(x))}{1 - \beta(\omega(x))}, \tag{6}$$

for x in $\bar{A} \setminus (C^o \cup \Delta)$. We have that $A \cap B$ is a bounded convex body and by (4) it is clear that $\bar{A} \cap C$ is a bounded star body. Hence using Lemma 1 we have that α, β, and ω are continuous. Choosing x in $\bar{A} \setminus (B^o \cup \Delta)$ there exists a neighborhood $N(0)$ of the origin such that for y in $x + N(0)$, $\beta(\omega(y)) < 1 - \delta$ when δ is come positive constant. Thus for y in $x + N(0)$ we have

$$|\gamma(y) - \gamma(x)| = \left| \frac{(1-\beta(\omega(x)))(\alpha(\omega(x)) - \alpha(\omega(y))) + (1-\alpha(\omega(x)))(\beta(\omega(y))-\beta(\omega(x)))}{(1 - \beta(\omega(y)))\,(1 - \beta(\omega(x)))} \right|$$

$$\leq \frac{1}{\delta}|\alpha(\omega(x)) - \alpha(\omega(y))| + \frac{1}{\delta^2}|\beta(\omega(x)) - \beta(\omega(y))|.$$

Hence it is clear that the continuity γ at x follows from the continuity of α, β, and ω. We have, therefore, that γ is continuous on $\bar{A} \setminus (C^O \cap \Delta)$, and since $\gamma(x)$ does not vanish on this domain, the functional $\zeta(x) = 1 / \gamma(x)$ is also continuous. In addition, we observe that

$$0 \leq \gamma(x) < 1,$$

on $A \setminus (C^O \cup \Delta)$, since $1 > \alpha(\omega(x)) > \beta(\omega(x)) > 0$.

Returning to our mapping φ we have that the continuity of φ on $(\bar{A} \setminus (C^O \cup \Delta)) \cap K^O$ follows immediately from the continuity of ω and γ. Also φ^{-1} on $(\bar{A} \setminus (B^O \cup \Delta)) \cap K^O$ is expressed by the formula

$$\varphi^{-1}(y) = \omega(y) + \zeta(y)(y - \omega(y)), \tag{7}$$

so the continuity of φ^{-1} on $(\bar{A} \setminus (C^O \cup \Delta)) \cap K^O$ follows from the continuity of ω and ζ. Now consider an arbitrary point x in $\Delta \cup (\bar{A} \cap \partial(K))$ and let y be an arbitrary point in $\bar{A} \setminus C^O$. We have

$$\begin{aligned} \varphi(y) - \varphi(x) &= \omega(y) + \gamma(y)(y - \omega(y)) - \omega(x) \\ &= (1 - \gamma(y))(\omega(y) - \omega(x)) + \gamma(y)(y - x), \end{aligned} \tag{8}$$

if y is in $(\bar{A} \setminus C^O) \cap K$, and

$$\varphi(y) - \varphi(x) = y - x,$$

if y is in $\bar{A} \cap (X \setminus K)$. Since $0 \leqq \gamma(y) < 1$, it is clear that the continuity of φ at x follows immediately from the continuity of ω at x. Hence we have that φ is continuous on $(\bar{A} \setminus C^o) \cap K$. The extension of the continuity of φ to the remainder of $\bar{A} \setminus C^o$ follows trivially, so we have that φ maps $\bar{A} \setminus C^o$ onto $\bar{A} \setminus B^o$ continuously.

Now let y be contained in $\bar{A} \cap \partial(K)$ and consider the transformation φ^{-1}. For arbitrary z in $\bar{A} \setminus B^o$ we have

$$\begin{aligned}\varphi^{-1}(z) - \varphi^{-1}(y) &= \omega(z) + \zeta(z)(z - \omega(z)) - \omega(y) \\ &= (1 - \zeta(z))(\omega(z) - \omega(y)) + \zeta(z)(z - y),\end{aligned}$$

if y is in $(\bar{A} \setminus (B^o \cup \Delta)) \cap K$, and

$$\varphi^{-1}(z) - \varphi^{-1}(y) = z - y,$$

if y is in $\bar{A} \cap (X \setminus K)$. It is easily verified that there exist a neighborhood in y on which ζ is bounded so it is clear that the continuity of φ^{-1} at y follows from the continuity of ω at y. Thus φ^{-1} is continuous on $(\bar{A} \setminus (B^o \cup \Delta)) \cap K$ and the extension to the remainder of $\bar{A} \setminus (B^o \cup \Delta)$ follows trivially. We may conclude, therefore, that φ is a homeomorphism on $\bar{A} \setminus (C^o \cup \Delta)$ and the proof of our lemma is complete.

Lemma 4. Let X, A, B, and C be defined as in Lemma 2. There existsa homeomorphism ψ mapping $\bar{A} \setminus C^o$ onto $\bar{A}$.

Proof. Let x_o, Δ, C_y, and P_y be defined as in the proof of Lemma 2. Let x_3 be a point in $(A^o \setminus \bar{B}) \cap \ell(x_1 x_o)$ and let $C^* = C(x_3, B \cap \partial(A))$. By the boundedness of $A \cap B$ and the convexity of A, obviously every ray in C^* must intersect $\partial(A)$ at a unique point. For arbitrary u in $B^o \cap \partial(A)$ we consider the segment $\overline{x_3 u}$

in C^*. It was established in the proof of Lemma 2 that u must be contained in P_y for some y in Δ. Since $\widehat{x_o x_1}$ is contained in P_y it follows that $r(x_3 u)$ is contained in P_y. Since x_3 is an exterior point of C_y and u is an interior point it follows that $\overline{x_2 u}$ must intersect $\partial(C)$ at a unique point.

By Lemma 1 we may express each x in $C^* \setminus \{x_3\}$ uniquely as

$$x = (1 - \mu(x))x_2 + \mu(x)\nu(x),$$

where μ is a continuous positive functional on $C^* \setminus \{x_3\}$ and ν is a continuous mapping of $C^* \setminus \{x_3\}$ onto $\bar{B} \cap \partial(A)$. We define $\psi(x)$ on $C^* \setminus C^o$ by the formula

$$\psi(x) = x_2 + \frac{x - x_2}{\eta(\nu(x))}, \tag{9}$$

where x is in $r(x_2\nu(x))$ and $\eta(\nu(x))$ is the positive continuous functional such that

$$(1 - \eta(\nu(x)))x_2 + \eta(\nu(x))\nu(x) = \partial(C) \cap r(x_2\nu(x)).$$

On $\bar{A} \setminus C^*$ we define $\psi(x) = x$. Clearly ψ and ψ^{-1} are well defined by virtue of the unique point of intersection property established between $\overline{x_2\nu(x)}$ and $\partial(C)$ in the previous paragraph. Also we can easily verify that $\psi(\bar{A} \setminus C^o) = \bar{A}$.

Since $\eta(\nu(x))$ is continuous, bounded, and bounded away from zero on $C^* \setminus C^o$ it is clear from (9) that ψ and ψ^{-1} are continuous on $C^{*o} \setminus C^o$ and $C^{*o} \cap \bar{A}$ respectively. Let x be an arbitrary point in $\partial(C^*) \cap \bar{A}$ and let y be arbitrary in $\bar{A} \setminus C^o$. We have

$$\psi(y) - \psi(x) = x_2 + \frac{y - x_2}{\eta(\nu(y))} + x$$

$$= \left(1 - \frac{1}{\eta(\nu(y))}\right) (x_2 - y) + (y - x),$$

if y is in C^*, and

$$\psi(y) - \psi(x) = y - x,$$

if y is in $\bar{A} \setminus C^*$. Hence it is clear that the continuity of ψ at x follows from the continuity of η and ν and the fact that $\eta(\nu(x)) = 1$. Hence ψ is continuous on $C^* \setminus C^o$ and it follows trivially that ψ is continuous on all of $\bar{A} \setminus C^o$. In a completely analogous fashion we may verify the continuity of ψ^{-1} on all of $\bar{A}$. Hence ψ is a homeomorphism mapping $\bar{A} \setminus C_o$ onto $\bar{A}$ and our proof is complete.

Theorem 1. Let A and B be two convex sets in a real linear topological space X such that $A \cap B$ is bounded. If relative to some x_1 in $A^o \cap \partial(B)$ and x_o in $B^o \cap \partial(A)$, $A^o \cap \partial(B)$ and $B^o \cap \partial(A)$ are linearly connected sets, then there exists a one-to-one continuous function f which maps $\bar{A}$ onto $\bar{A} \setminus B^o$ and f is a homeomorphism on A^o. Furthermore if p is the support function of $A \cap B$ defined relative to some interior point, then for arbitrary $\epsilon > 0$, f may be defined to be the identity map outside the set $\{x : p(x) < 1 + \epsilon\}$.

Proof. Let C be defined as in Lemma 3. Clearly K, as specified in the proof of Lemma 3, may be chosen such that

$$K = \{x : p(x) \leq 1 + \epsilon\}.$$

Hence φ as constructed in the proof of Lemma 3 is a one-to-one continuous mapping of $\bar{A} \setminus C^o$ onto $\bar{A} \setminus B^o$ which is a homeomorphism on A^o and is the identity map outside K. Also ψ as constructed in the proof of Lemma 4 is a homeomrophism mapping $\bar{A} \setminus C^o$ onto $\bar{A}$ and in the identity map outside K. Therefore, $f = \varphi\psi^{-1}$ is a one-to-one continuous mapping of $\bar{A}$ onto $\bar{A} \setminus B^o$ which is a homeomorphism on A^o and the identity map on $\bar{A} \setminus K$, so the proof of our theorem is complete.

As a final remark we mention that Theorem 1 is one of several similar theorems concerned with topologically convex sets which have very interesting applications in the theory of fixed points. For example, a theorem of the same type is presented in [4] and used to establish an asymptotic fixed point theorem which is very useful when investigating periodic systems in Banach space.

References

[1] Tychonoff, A., Ein Fixpunktsatz. Math. Ann. 111, 767-776 (1935).

[2] Bourbaki, N., Espaces vectoriels topologiques, livre 5, Hermann, Paris, 80-81 (1953).

[3] Jones, G. S., Periodic motions in Banach space and applications to functional-differential equations, Contributions to Differential Equations, in press.

[4] Jones, G. S., Asymptotic fixed point theorems and periodic systems in Banach space, in preparation.