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INTERACTION BETWEEN THE SOLAR WIND AND THE GEOMAGNETIC FIELD

by

James Hurley

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Project Report

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NEW YORK UNIVERSITY
College of Engineering
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Project Report

Approved A Beiser
Arthur Beiser
Project Director

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ABSTRACT

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There is now considerable evidence for the existence of an interaction between the solar wind and the geomagnetic field. The solar wind is a highly ionized plasma and as such may well influence the earth's magnetic field. It is this interaction that we have studied.

We have limited ourselves to a two dimensional model to permit an exact solution to be obtained. The plasma wind is considered as impinging upon an arbitrarily oriented two dimensional magnetic dipole. We find that the field is compressed by the plasma and essentially confined to form a cavity in the plasma wind. The thickness of the boundary layer within which the plasma and field interact is found to be small. We have determined the equation of this boundary surface and the extent to which the field is changed by such a confinement.

We have also considered the stability of such a system. We find that there are certain regions of instability, but for the part of the surface facing the wind it is stable.

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INTRODUCTION

Recently a great deal of work has been devoted to the problem of the interaction of the solar wind with the earth's magnetic field. The solar wind consists of a totally ionized hydrogen plasma continuously streaming from the sun. The velocity and density in the neighborhood of the earth are approximately 10^7 cm/sec. and 100 cm^{-3} respectively. These values have been estimated by Biermann and others by observing the deflection of comet tails in the solar wind. Further evidence of its existence is the geomagnetic storms accompanying increased solar activity. The mechanism for this disturbance will be made clear in the work that follows.

The energy density of this plasma is $1/2\ Nmv^2 = 0.83 \times 10^{-8}$ ergs/cm^3 where N is the number density of the plasma particles and m the proton mass. If we approximate the earth's field by a simple dipole the magnetic field in the equatorial plane is $B = 0.35(R_o/R)^3$ where R_o is the radius of the earth and 0.35 the earth's field in gauss at the equator. The energy density of the magnetic field then is $B^2/8\pi = (.35R_o{}^3/R^3)^2/8\pi$. We see that the energy density of the field and the energy density of the plasma wind are equal at approximately 9.1 earth's radii. If the plasma wind is to have any effect on the earth's field it should be felt at 9.1 earth's radii and beyond.

There is direct experimental evidence for such an interaction. Rocket studies[1] indicate that the earth's magnetic field behaves roughly like a dipole out to 12-13 earth's radii where it sharply decreases to the field

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[1] A Radial Rocket Survey of the Distant Geomagnetic Field, C.P. Sonett, D.L. Judge, A.R. Sims, and J.M. Kelso, Space Technology Laboratories Report 7320,2-13.

of interplanetary space (5×10^{-5} gauss).

We wish to describe the nature of the interaction between the plasma wind and the earth's field. In particular we shall show that the wind compresses the field so that a cavity is formed in the wind. This compression may be seen physically from either macroscopic or microscopic models.

First let us consider the macroscopic model. The solar plasma has a high conductivity. Parker[1] has estimated the mean free path to be of the order of 10^6 km. Now a good conductor behaves diamagnetically. Any magnetic field which exists initially in the conductor will tend to persist. If the field is zero initially it will remain zero. This phenomenon of diamagnetism may be explained by Lenz's law. If the magnetic field does change there will be an induced electric field. The induced currents will flow in such a direction so as to oppose the change in the field. The greater the conductivity the greater the opposition to change. Thus if we imagine a plasma streaming toward the dipole from infinity where the magnetic field is zero the magnetic field must remain zero for all time in the limit of infinite conductivity, i.e. the magnetic field in the plasma may not change.

We may also envisage this exclusion of the field from the plasma from a microscopic picture. Since the mean free paths are so enormous a particle description is certainly better suited to describe the interaction. Charged particles moving in a magnetic field themselves create magnetic fields. Imagine a uniform magnetic field in the negative half

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[1] E.N. Parker, Interaction of the Solar Wind with the Geomagnetic Field, Phys. of Fluids, 1, 171 (1958).

space $y < 0$ and zero in the positive half space $y > 0$. Consider a particle incident on the field-vacuum interface along the y-axis from $y = +\infty$. It will be curved by the magnetic field moving in a semicircular path with center of curvature on the positive or negative x-axis depending on the sign of the charge. In either case the particle will decrease the field within its orbit (i.e. within the semicircle) and increase the field outside. We may now generalize to consider the problem of a plasma incident on the interface. For sufficiently large numbers of particles the magnetic field may be materially reduced within the region where the particles are turning, and increased beyond this region. Thus we may speak of a compression of the magnetic field by the plasma stream. We shall study this problem in more detail later.

In our particle picture we may describe the interaction as follows. The plasma particles move in straight lines (because of their long mean free paths), penetrate a short distance into the field, and are then deflected out.

A more rigorous justification of these statements will follow in the next section.

DEPTH OF PENETRATION

The problem to be described now has been considered by Dungey[1] and Rosenbluth[2]. Our reason for including it here is partly for completeness and partly because we feel that an important feature of the problem

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[1] J.W. Dungey, Cosmic Electrodynamics (Cambridge University Press, New York 1958), Sec. 8.3.

[2] M. Rosenbluth, in Magnetohydrodynamics, edited by R. Landshoff (Stanford University Press, Stanford, California, 1957).

has been ignored.

We would like to know how far the plasma will penetrate into the magnetic field. To this end we shall investigate the following two dimensional problem. The magnetic field is constant in the half space $x < 0$ and approaches 0 as x approaches infinity. The field is everywhere parallel to the z-axis. The plasma is incident on the field from $x = +\infty$. We assume that all the particles are moving parallel to the x-axis before encountering the field. We expect the path of the particles to be curved as they enter the field and be deflected out. The protons having the greater momentum will penetrate further than the electrons, thereby building up a space charge. The electric field of this space charge will tend to pull the electrons in with the protons. The paths of the particles are roughly described in Fig. 1. We observe that the particle current is responsible for the decay of the magnetic field in the x-direction. The discontinuity in the magnetic field between $x = 0$ and $x = \infty$ is equal to the total particle current in the y-direction.

The solution of this problem is determined by a self-consistent solution of the equations of motions of the particles and Maxwell's equations. The boundary conditions are that $B = B_0$ at $x = 0$ and $B \to 0$ as $x \to \infty$. The electric field must be 0 at $x = 0$ and approach zero as $x \to +\infty$. Let $(u_p(x), v_p(x), 0)$ be the (x,y,z) components of the outgoing protons. From symmetry we see that the velocity components for the incoming protons are $(-u_p, v_p, 0)$. The corresponding velocity components of the electrons are $(u_e, v_e, 0)$ and $(-u_e, v_e, 0)$. Let $n_p(x)$ and $n_e(x)$ be the number density of the incoming and outgoing protons and electrons respectively. The magnetic field is represented by a vector potential having only a y-component $A(x)$, and the electric field by a scalar potential $\phi(x)$. The number density and velocity at $x = \infty$ are N and U.

We may integrate the y-component of the equations of motion

$$v_p + \frac{eA}{m_p c} = v_e - \frac{eA}{m_e c} = 0 \ . \tag{1}$$

The energy equation gives

$$u_p^{\ 2} + v_p^{\ 2} + \frac{2e\phi}{m_p} = u_e^{\ 2} + v_e^{\ 2} - \frac{2e\phi}{m_e} = U^2 \ . \tag{2}$$

Particle conservation requires

$$n_p u_p = n_e u_e = NU \tag{3}$$

Maxwell's equations are

$$\frac{d^2 A}{dx^2} = \frac{-8\pi e}{c} (n_p v_p - n_e v_e) \tag{4}$$

and

$$\frac{d^2 \phi}{dx^2} = -8\pi e(n_p - n_e) \ . \tag{5}$$

(The currents and charge densities have been doubled to include both incoming and outgoing particles.)

The above system of equations apply in region II only. If the electron terms are dropped we obtain the equations appropriate to region I.

It goes without saying that these equations cannot be solved exactly. We can however find a valuable first integral. (It is at this point that we depart from Dungey's analysis.)

These six simultaneous equations can be reduced to two simultaneous equations relating the vector and scalar potentials. They are

$$\frac{d^2A}{dx^2} = \frac{8\pi e^2 NU}{c^2} A \left[\frac{\frac{1}{m_p}}{\sqrt{U^2 - \frac{e^2A^2}{m_p^2c^2} - \frac{2e\phi}{m_p}}} + \frac{\frac{1}{m_e}}{\sqrt{U^2 - \frac{e^2A^2}{m_e^2c^2} + \frac{2e\phi}{m_e}}} \right]$$

and

$$\frac{d^2\phi}{dx^2} = -8\pi eNU \left[\frac{1}{\sqrt{U^2 - \frac{e^2A^2}{m_p^2c^2} - \frac{2e\phi}{m_p}}} - \frac{1}{\sqrt{U^2 - \frac{e^2A^2}{m_e^2c^2} + \frac{2e\phi}{m_e}}} \right]$$

We may simplify the right hand sides

$$\frac{d^2A}{dx^2} = -8\pi NU \left[m_p \frac{\partial}{\partial A} \sqrt{U^2 - \frac{e^2A^2}{m_p^2c^2} - \frac{2e\phi}{m_p}} + m_e \frac{\partial}{\partial A} \sqrt{U^2 - \frac{e^2A^2}{m_e^2c^2} + \frac{2e\phi}{m_e}} \right]$$

$$\frac{d^2\phi}{dx^2} = 8\pi NU \left[m_p \frac{\partial}{\partial \phi} \sqrt{U^2 - \frac{e^2A^2}{m_p^2c^2} - \frac{2e\phi}{m_p}} + m_e \frac{\partial}{\partial \phi} \sqrt{U^2 - \frac{e^2A^2}{m_e^2c^2} + \frac{2e\phi}{m_e}} \right]$$

If we multiply the first equation by $-dA/dx$ and the second by $d\phi/dx$ and add them, the right-hand side becomes an exact differential:

$$\frac{d\phi}{dx}\frac{d^2\phi}{dx^2} - \frac{dA}{dx}\frac{d^2A}{dx^2} = 8\pi NU \left[m_p \frac{d}{dx} \sqrt{U^2 - \frac{e^2A^2}{m_p^2c^2} - \frac{2e\phi}{m_p}} \right.$$

$$\left. + m_e \frac{d}{dx} \sqrt{U^2 - \frac{e^2A^2}{m_e^2c^2} + \frac{2e\phi}{m_e}} \right]$$

Integrating we obtain

$$\frac{E^2}{8\pi} - \frac{H^2}{8\pi} = 2NU(m_p u_p + m_e u_e) + \text{constant.}$$

To evaluate the constant we let $x \to \infty$ where $E = H = 0$ and $u_p = u_e = U$. Therefore

$$\frac{E^2}{8\pi} - \frac{H^2}{8\pi} = 2NU(m_p u_p + m_e u_e) - 2NU^2(m_p + m_e). \tag{6}$$

At the interface $(x = 0)$ $E = 0$, $H = H_o$, and $u_p = u_e = 0$. Hence

$$H_o^{\,2} = 16\pi NU^2(m_p + m_e).$$

Thus we have found what magnetic field strength is required to hold back the plasma without obtaining a complete solution to the problem.

We might note that the integral obtained in equation (6) could be found more simply using the Maxwell stress tensor. Consider the volume of plasma contained in a parallelpiped of unit cross sectional area. The face a is located a distance x from the plasma-field interface, and the face b at ∞. (See Fig. 2.) The net force acting on the plasma in this volume must equal the time rate of change of the particles within. The force on face a is $H^2/8\pi - E^2/8\pi$. The force on face b is zero since the field vanishes at ∞. The forces on the remaining four faces cancel by pairs. We shall equate this force to the time rate of change of the momentum of the particles. In the initial configuration the total x-component of momentum is zero since there are as many particles moving to the right as to the left. The momentum change which occurs in the time dt will then be equal to the total momentum of the system after a lapse of time dt.

(We have omitted the motion in the y-direction as this would only complicate the picture and is of no interest in calculating the change in the x-component of momentum.) Those particles which were at the surface a and moving to the left are at the surface a' after a time dt. Those moving to the right are located at a" . b' and b" are defined in a similar way. The change in momentum then is the total momentum of those particles between a' and b" which were between a and b at time t. There will be other particles between a' and b" which moved in during the time dt. These must not be counted. The momentum of the particles between a" and b' is zero. The momentum of the particles between a' and a" is $-2u_p dt n_p m_p u_p$ and the momentum of those between b' and b" , $2UdtNm_pU$. We must add similar expressions for the electrons. Equating F and dp/dt,

$$\frac{E^2}{8\pi} - \frac{H^2}{8\pi} = 2n_p m_p u_p{}^2 + 2n_e m_e u_e{}^2 - 2N(m_p + m_e)U^2,$$

but since $n_p u_p = n_e u_e = NU$

$$\frac{E^2}{8\pi} - \frac{H^2}{8\pi} = 2NU(m_p u_p + m_e u_e) - 2NU^2(m_p + m_e)$$

which we see is eq. (6).

It is not possible to proceed any further with an exact solution. We would however like to estimate the rate of decay of the magnetic field, i.e., how far from the vacuum-plasma interface must we go before the magnetic field becomes a small fraction of the vacuum field. It will be important in our subsequent work that this distance is small compared to the dimensions of the cavity.

Dungey has presented an approximate solution to this problem. He assumes that the electron density is everywhere approximately equal to the proton density. He has justified the approximation using the full set of equations (1,2,3,4,5). In doing so he has overlooked the possibility of an absolute change separation such as would occur in region I. In this region the electron terms must be omitted from the equations and hence are not to be included in an estimate of the relative densities.

We will first find an upper limit on the thickness of region I and then verify that Dungey's solution is valid in region II. Having studied both regions we may then estimate the thickness of the current sheath.

We return to the integral we found in eq. (6).

$$E^2 = H^2 + 16\pi NU(m_p u_p + m_e u_e) - 16\pi NU^2(m_p + m_e). \tag{7}$$

Now at the interface between region I and II

$$u_e = 0$$
$$u_p < U$$
$$H < H_o$$

and

$$E = 8\pi e \int_o^{x_o} n_p \, dx \gg 8\pi e \int_o^{x_o} N \, dx = 8\pi e N x_o ,$$

where x_o is the distance between the plasma-vacuum interface and the interface between I and II. Substituting into eq. (7) we find

$$(8\pi e N x_o)^2 \ll E^2 = H^2 + 16\pi NU m_p u_p - 2NU^2(m_p + m_e) <$$
$$< H_o^2 + 16\pi NU^2 m_p - 16\pi NU^2(m_p + m_e)$$
$$= 16\pi NU^2 m_p .$$

So that

$$x_o^{\,2} \ll \frac{m_p U^2}{4\pi e^2 N} \tag{8}$$

or

$$x_o \ll 8 \text{ meters.}$$

Where 8 m. is a large distance in laboratory experiments, it is quite small in comparison with the dimensions of the cavity, $10^7 - 10^8$ m.

We might obtain a better physical insight into this upper limit on the charge separation by considering how much energy is available in the plasma stream for charge separation. Let us imagine a block of plasma of unit cross sectional area and width a moving to the left with a velocity U. The plasma block now runs into an army of Maxwell demons who instantaneously stop all the electrons and allow the protons to pass. We shall simplify the calculation by assuming that the protons move as a block, and we ask how far this block will move before coming to rest. When the protons are at rest the energy stored in the electric field must equal the initial kinetic energy. If the protons are displaced a distance x_o (see fig. 3.) from the electrons the electric field is

$$E = 4\pi e N x_o .$$

Equating the field energy to the kinetic energy

$$E^2/8\pi = \frac{16\pi^2 e^2 N^2 x_o^{\,2}}{8\pi}\, a = 1/2\; N m_p U^2$$

so that

$$x_o^{\,2} = \frac{m_p U^2}{4\pi N e^2}$$

which is identical to the upper limit for the change separation obtained in eq. (8). (We have tacitly assumed that $b \gg x_o$.) The fact that the numerical coefficient is the same is of course fortuitous.

Now let us consider region II. Here we shall show that Dungey's solution is quite good. He assumes that the electron and ion charge distributions are approximately equal. This assumption is to be tested a posteriori. From eq. (3) we see that

$$u_e = u_p .$$

The y-components of the electron and proton velocities are related by eq. (1)

$$m_p v_p = -m_e v_e$$

so that $|v_p| \ll |v_e|$. We may therefore neglect the proton current in eq. (4) so that

$$\frac{d^2A}{dx^2} = \frac{8\pi eNU}{c} \frac{v_e}{u_e} . \tag{9}$$

If we now eliminate ϕ from eq. (2) and substitute for v_p and v_e from eq. (1) we find that

$$m_p u_p{}^2 + \frac{e^2A^2}{m_e c^2} = m_p U^2 \tag{10}$$

neglecting terms of order m_e/m_p. We now solve eq. (10) for u_p and eq. (1) for v_e and substitute in eq. (9) remembering that $u_p = u_e$,

$$\frac{d^2A}{dx^2} = \frac{8\pi eNU}{c} \frac{\frac{eA}{m_e c}}{\sqrt{U^2 - \frac{e^2A^2}{m_e m_p c^2}}} . \tag{11}$$

Let

$$\alpha = A/A_o$$

where

$$A_o^{\,2} = \frac{m_e m_p c^2 U^2}{e^2}$$

and

$$x' = x/x_o$$

where

$$x_o^{\,2} = \frac{m_e c^2}{8\pi N e^2}$$

Eq. (11) then becomes

$$\frac{d^2\alpha}{dx'^2} = \frac{\alpha}{\sqrt{1 - \alpha^2}}$$

Clearly $\alpha \leqq 1$ so we set $\alpha = \sin \theta$. Integrating

$$x' = - \ln(\tan \theta/4) - 2 \cos(\theta/2) + \ln(\sqrt{2} - 1) + \sqrt{2}$$

For small θ

$$x' = - \ln \theta/4$$

or

$$\theta = 4e^{-x'}$$

so that

$$H \propto \frac{dA}{dx'} \propto e^{-x'}$$

The field then will decrease by a factor $1/e$ when $x' = 1$, i.e.

$$x = \sqrt{\frac{m_e c^2}{8\pi N e^2}} \approx 500 \text{ m.}$$

for $N = 100 \text{ cm}^{-3}$. This is again much less than the dimensions of the cavity.

We must now justify the assumption made earlier that the electron and proton densities are approximately equal. We may do this by calculating

$$\frac{d^2\phi}{dx^2} = -8\pi e(n_p - n_e).$$

Now

$$\phi = \frac{m_e}{2e}\left(u_e^{\,2} - v_e^{\,2} - U^2\right)$$

and from eq. (10)

$$u_e^{\,2} = U^2 - \frac{e^2A^2}{m_e m_p c^2} = U^2 - U^2\frac{A^2}{A_o^{\,2}} = U^2\cos^2\theta \tag{12}$$

Also

$$v_e^{\,2} = \frac{e^2A^2}{m_e^{\,2}c^2} = \frac{m_p}{m_e}U^2\frac{A^2}{A_o^{\,2}} = \frac{m_p^{\,2}}{m_e^{\,2}}U^2\sin^2\theta$$

so that

$$\phi = \frac{m_eU^2}{2e}\left(\cos^2\theta + \frac{m_p^{\,2}}{m_e^{\,2}}\sin^2\theta - 1\right) \tag{13}$$

We observe that if m_e were to equal m_p the potential would be zero and consequently no charge separation would occur. This is to be expected. If the particles have the same mass they must penetrate to equal depths.

Differentiating eq. (13) twice we find

$$\frac{d^2\phi}{dx^2} = (m_p - m_e)\ \frac{8\pi NeU^2}{m_ec^2}\left[2 - 2\cos\theta - \frac{\sin^2\theta}{\cos\theta}\right]$$

so that

$$n_e - n_p = \frac{(m_p - m_e)NU^2}{m_ec^2}\left[2 - 2\cos\theta - \frac{\sin^2\theta}{\cos\theta}\right].$$

We see that the plasma is far from neutrality at $\theta = \pi/2$, i.e. at the point where $u_e = 0$ $(n_e = NU/u_e \to \infty)$. Although n_e and n_p differ considerably near $x = 0$ $(\theta = \pi/2)$ the approximation may still be a good one. We must go back to our calculation of the field decay and investigate what is a reasonable measure of large deviation from neutrality.

In calculating the field from eq. (9) we substituted for u_e the value obtained from eq. (10) for u_p assuming that $u_p = u_e$. Let us see how far n_e and n_p must differ before this becomes a poor approximation. Let $n_p - n_e = \delta$. Now

$$n_p u_p = n_e u_e$$

or

$$u_e = \frac{n_p}{n_e} = \left(1 + \frac{\delta}{n_e}\right) u_p .$$

We require that

$$\frac{n_p - n_e}{n_e} \ll 1.$$

Now

$$n_e = \frac{NU}{u_e} \approx U/\cos\theta$$

since

$$u_e = U\cos\theta$$

from eq. (12). So that

$$\frac{n_e - n_p}{n_e} = \frac{(m_p - m_e)U^2}{m_e c^2}(2\cos\theta - 2\cos^2\theta - \sin^2\theta)$$

$$\approx \frac{m_p U^2}{m_e c^2}(2\cos\theta - \cos^2\theta - 1) < .002 \text{ or } .2\%.$$

This completes our discussion of the rate of decay of the field in the stream. We have shown that the region where the field differs appreciably from zero is much smaller than the dimensions of the cavity. (We shall, in the future, refer to this region as the current sheath.) Since the mean free path is so large the plasma constituents will behave as free particles until they reach the cavity boundary where they are reflected. We shall now consider the problem of the reflection of particles with an arbitrary angle of incidence.

REFLECTION FROM THE CAVITY BOUNDARY

We wish to show now that if the current sheath is small the particles will be specularly reflected from the sheath. We shall consider charged particles incident on a magnetic field as above except their velocity components at infinity are now (U, V, W). The magnetic and electric fields, $\vec{B} = B(z)\,\hat{x}$ and $\vec{E} = E(z)\,\hat{z}$, approach zero as $z \to \infty$. We shall show that after reflection the velocity components will be (U, V, -W) so that the angle of incidence is equal to the angle of reflection. (See Fig. 4.)

The fields may be expressed in terms of a vector and scalar potential $A = A_y(z)$ and $\phi = \phi(z)$. The path of the particle is determined by the conservation laws,

$$u = U \tag{14}$$

$$v + \frac{eA(z)}{mc} = V \tag{15}$$

$$v^2 + w^2 + \frac{e\phi}{mc} = V^2 + W^2 \tag{16}$$

where (u, v, w) are the velocity components at some intermediate point of the motion. The constants of the motion in eqs. (14,15,16) have been evaluated at $t = -\infty$. We now look at $t = +\infty$. Assuming that the field has sufficient depth that the particle is reflected and not transmitted then $z \to +\infty$ as $t \to +\infty$. Eq. (15) requires then that $v \to V$ as $t \to +\infty$. Substituting $u = U$ and $v = V$ in eq. (16) and observing that $\phi(\infty) = 0$

$$w^2 = W^2$$

so that

$$w\Big|_{t=+\infty} = +W$$

$$w\Big|_{t=-\infty} = -W\ .$$

We might generalize this conclusion somewhat to include curved fields with the proper symmetry. However, for our purposes it will be sufficient to require that the field may be approximated locally by a plane field whose principle gradient is perpendicular to the field (i.e. within the orbit of a particle, the field does not vary appreciable along the field lines).

DEFINITION OF THE PROBLEM

Let us now return to the problem of physical interest, a plasma wind blowing on the magnetic field of a dipole. As we have demonstrated , the plasma constituents will move in straight lines up to the cavity boundary where they are specularly reflected. The region of interaction (current sheath) separates the plasma from the field. Since the sheath thickness is small we may treat it as a boundary surface separating the two domains. In the plasma domain we have streaming particles making elastic collisions with the boundary and in the field domain we have a static magnetic field confined to a cavity by the boundary surface. The equilibrium shape of the cavity surface will be determined by a condition of pressure balance. The magnetic pressure within the cavity must balance the pressure of the plasma wind from without. We may then formulate our idealized problem as follows:

$$\nabla \times \vec{B} = 0 \tag{17}$$

everywhere inside the cavity except at the dipole where

$$\vec{B} \rightarrow - M/r^3 \sin\theta\,\hat{\theta} - 2M/r^3 \cos\theta\,\hat{r}$$

as $r \rightarrow 0$.

$$\nabla \cdot \vec{B} = 0 \tag{18}$$

inside the cavity,

$$B_n = 0 \tag{19}$$

on the surface, and

$$B^2/8\pi = 2NmU^2 \cos^2\chi \tag{20}$$

on the surface. Here χ is the angle between the normal to the surface and the direction of the incident wind.

In eq. (17) we neglect all currents within the cavity such as atmospheric currents and currents in the Van Allen belts, or else we lump them in with the dipole field. The boundary condition (eq. (19)) is imposed since the field is zero outside the cavity and the normal component must be continuous. Eq. (20) expresses the condition of pressure balance, i.e. we are looking for a steady state solution.

This problem differs in an essential way from the usual boundary value problem. If the magnetic field were expressed as the gradient of a scalar field, the problem would be determined by finding a scalar solution to Laplace's equation having the appropriate singularity at the origin and satisfying two boundary conditions. Normally one is given the boundary and one boundary condition. Here the second boundary condition must determine the boundary.

Unfortunately, even with these simplifications, the problem is still much too difficult to handle. The existing theoretical work on the shape of the cavity boundary in three dimensions consists entirely of free hand drawings. There is one exception. David Beard[1] has presented an approximate solution to the problem. The validity of the approximation is questionable and there exists no test of its accuracy. We shall present here the exact solution of two analogous two dimensional problems, namely, the interaction of a plasma wind with the magnetic field of an infinite line current and with the field of two lines carrying currents in opposite directions (i.e. a two dimensional dipole). We do not propose that these two dimensional problems will approximate the three dimensional problem. We present these problems firstly because they can be solved exactly; secondly, they are of interest in themselves, giving us physical insight into the nature of the interaction between plasmas and fields; thirdly they can be approximated experimentally so that the theory may be tested; and lastly they will serve as a test for approximate theoretical solutions to the three dimensional problem. Regarding this last category, we shall apply Beard's approximation method to the two dimensional problems and demonstrate explicitly the validity of the method.

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[1] David B. Beard, The Interaction of the Terrestrial Magnetic Field with the Solar Corpuscular Radiation, J. Geophys. Research, 65, 3559, 1960.

INTERACTION BETWEEN A PLASMA WIND AND THE FIELD OF AN INFINITE LINE CURRENT

We have developed two methods for the solution of the problem-type defined by eqs. (17,18,19,20). The first, while offering considerable advantages in algebraic simplicity, is restricted in the class of problems it can handle. In particular it may not be used on the two dimensional dipole. To present this first method we consider a plasma wind incident on the field of an infinite line current. (See Fig. 5.) The plasma velocity is everywhere perpendicular to the current. The field inside the cavity boundary must satisfy Maxwell's equations

$$\nabla \cdot \vec{B} = 0$$

everywhere

$$\nabla \times \vec{B} = 0$$

except at the wire, and

$$\vec{B} \rightarrow \frac{2I}{r} \vec{e}_\phi$$

near the wire, where I is the current in the wire.

At the cavity surface we require that B be tangent to the surface, that is,

$$\frac{B_y}{B_x} = \frac{dy}{dx}$$

and the magnetic pressure must balance the pressure of the stream, so that

$$1/8\pi(B_x{}^2 + B_y{}^2) = p_o \cos^2\theta$$

where

$$p_o = 2m\, v_o{}^2\, n.$$

We now reformulate the problem in complex notation. Since

$$2\,\frac{\partial B(z,z^*)}{\partial z} = \nabla \cdot \vec{B} + i\,|\nabla \times \vec{B}| = 0,$$

B must be a function of z* where

$$B = B_x + iB_y.$$

The singularity at the wire required that

$$B \rightarrow 2Ii/z^* \text{ as } z \rightarrow 0.$$

To formulate the boundary conditions consider

$$B^*dz = \vec{B} \cdot \vec{ds} + i\,|\vec{B} \times \vec{ds}|.$$

The imaginary part must be zero (tangency condition). The real part is

$$(8\pi p_o)^{\frac{1}{2}} \cos \theta \, ds$$

from the condition of pressure balance. Since $\cos \theta = dy/ds$ we may state both boundary conditions in the single equation

$$B^*dz = (8\pi p_o)^{\frac{1}{2}}\, dy.$$

Our problem then is to find a complex function B such that

$$B = B(z^*) \tag{21}$$

$$B \rightarrow 2Ii/z^* \quad \text{as} \quad z \rightarrow 0 \tag{22}$$

and

$$B^*dz = (8\pi p_o)^{\frac{1}{2}}\, dy \tag{23}$$

on the surface.

We now use the unknown field B to generate a conformal mapping.[1]

Let

$$w(z) = 1/iB^*$$

[1] J.D. Cole and J.H. Huth, Physics of Fluids, 2, 624 (1959).

where

$$w = u + iv.$$

(The choice of the mapping function is determined by the simplicity of the resulting boundary value problem.) Putting this in eq. (23)

$$dz = (8\pi p_o)^{\frac{1}{2}}\, dy\, iw \tag{24}$$

on the surface. Taking the imaginary part

$$u = (8\pi p_o)^{-\frac{1}{2}} = u_o,$$

a constant. We see that in the w plane the problem has been reduced to one of known boundary and is thereby considerably simplified.

The other boundary condition contained in eq. (24) might be obtained by taking the real part. Instead we proceed by considering the quantity

$$\frac{1}{w}\frac{dz}{dw} = i(8\pi p_o)^{\frac{1}{2}}\frac{dy}{dw}$$

on the surface. But on the surface $dw = i\, dv$ and is pure imaginary. Therefore

$$\mathrm{Im}(1/w \;\; dz/dw) = 0$$

on the surface, i.e. when $w = u_o + iv$.

The singularity condition near the current is that

$$B \to 2Ii/z^*.$$

Therefore

$$z \to 2Iw$$

and

$$1/w \;\; dz/dw \to 2I/w \quad \text{as} \quad w \to 0.$$

These condisions will be satisfied by taking

$$1/w \;\; dz/dw = 2i(1/w - 1/w_o)$$

where (see Fig. 6)

$$w_o = w - 2u_o .$$

We find then for z

$$z = -4I\, u_o\, \ell n\left(\frac{w - 2u_o}{a}\right) .$$

The constant of integration a is determined by the singularity condition. Let $a = -2u_o$. Then

$$z = -4Iu_o\, \ell n(1 - w/2u_o) \tag{25}$$

and approaches 2Iw as it should.

If we evaluate w in eq. (25) on the surface ($w = u_o + iv$) we determine the surface

$$z_s = -4Iu_o\, \ell n\left(1 - \frac{u_o + iv}{2u_o}\right) \tag{26}$$

so that

$$x_s = -4Iu_o\, \ell n\, \frac{(u_o{}^2 + v^2)^{\frac{1}{2}}}{2u_o} \tag{27}$$

$$y_s = 4Iu_o\, \tan^{-1} v/u_o . \tag{28}$$

which are parametric equations for the boundary surface. These results are plotted in Fig. 7.

Eq. (25), besides giving us the shape of the cavity boundary, can be inverted to give us also the magnetic field. Solving for w

$$w = 2u_o\left(1 - e^{-z/4Iu_o}\right) .$$

But

$$w = \frac{1}{iB^*} .$$

Therefore

$$B_x = -\frac{1}{2u_o}\,\frac{e^{-x/4Iu_o}\sin y/4Iu_o}{\left(1 - e^{-x/4Iu_o}\cos y/4Iu_o\right)^2 + e^{-2x/4Iu_o}\sin^2 y/4Iu_o}$$

and

$$B_y = \frac{1}{2u_o}\,\frac{1 - \cos y/4Iu_o}{\left(1 - e^{-x/4Iu_o}\cos y/4Iu_o\right)^2 + e^{-2x/4Iu_o}\sin^2 y/4Iu_o}$$

We observe that B_y does not approach zero as y approaches zero but instead $1/4u_o$ or $\sqrt{\frac{\pi}{2}\,p_o}$. Thus the field is increased on the windward side of the line current and decreased on the leeward side. One might explain this phenomenon by imagining that the cavity boundary has compressed the field lines between the cavity and the line thus strengthening the field. The field on the windward side points generally in the positive y-direction. We expect then that the surface currents on the boundary will produce a field in the y-direction near the line current.

INTERACTION BETWEEN A PLASMA WIND AND THE MAGNETIC FIELD OF A TWO DIMENSIONAL DIPOLE

This problem is formulated in exactly the same way as the previous problem except that now the singularity is that of a dipole instead of a line current. We take the dipole to be perpendicular to the plasma flow. This change alters the problem radically. It is not possible to use the previous technique here. The reason will be made clear shortly.

We begin again with the relevant equations,

$$\nabla \times B = 0$$

everywhere except at the dipole where

$$\vec{B} \to 2M\left(\frac{2xy}{r^2}\ \hat{e}_x - \frac{x^2 - y^2}{r^4}\ \hat{e}_y\right)$$

as $r \to 0$. Here M is the dipole moment.

$$\nabla \cdot \vec{B} = 0$$

everywhere.

The field must be tangent to the cavity surface so that

$$\frac{B_y}{B_x} = \frac{dy}{dx}$$

and the magnetic pressure must balance the pressure of the stream,

$$\frac{1}{8\pi}\left(B_x^{\,2} + B_y^{\,2}\right) = p_o \cos^2\chi$$

where

$$p_o = 2Nmv^2.$$

Again we find a considerable simplification by expressing these conditions in complex notation. The two Maxwell's equations become

$$B = B(z^*) \tag{29}$$

and the two boundary conditions

$$B^*\,dz = \vec{B} \cdot \vec{ds} + i\,\vec{B} \times \vec{ds} = (\mp)\sqrt{8\pi p_o}\ \ dy. \tag{30}$$

The singularity at the dipole is

$$B^* \to \frac{2mi}{z^2} \quad \text{as} \quad z \to 0. \tag{31}$$

The reason for the $(\mp)$ in eq. (30) can be seen from the geometry of the field. We have made a rough sketch of this in Figure 8.

If $\vec{ds}$ is a vector tangent to the surface and the sense is defined in the usual way (moving along the surface we keep the internal domain on our left) we see that $\vec{B}\cdot\vec{ds}$ (i.e. the real part of $B^* \, dz$) is negative between the two neutral points (N) and positive outside.

The magnetic field reverses direction at the neutral points and is therefore zero there. Since the magnetic pressure is zero the particle pressure, and hence the slope, must also be zero.

It is this reversal of sign that prevents us from using the previous method of attack on this problem. If we again used the field to define a transformation, the transformed boundary shape would no longer be a plane. This plane was defined by the condition that $u = 1/\sqrt{8\pi p_o} = u_o$. This condition becomes in the present case $u = \pm u_o$ depending on whether we are inside or outside of the neutral points and the transformed geometry would become

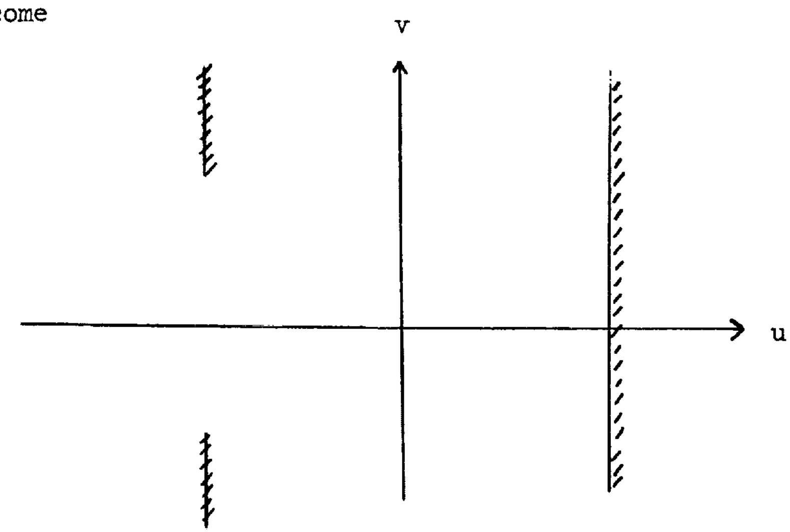


Fig. 9.

The geometry is complicated further by the fact that the boundary lies on a double sheeted Riemanian surface. (This follows from the fact that $w \to z^2$ as $z \to 0$, so that where an argument of 2π described the z-plane an argument of 4π is needed to describe the w-plane.)

For these reasons we must develop a new method of solution. In brief we may describe this method as follows. We assume there exists a mapping $w = w(z)$ which transforms the boundary onto a circle. The boundary conditions then define a set of requirements which this transformation must satisfy. We may solve these equations for $w(z)$. Then knowing the transformation we can map the circle in the w-plane back onto the z-plane to find the boundary shape in the physical space.

ANALYSIS

We introduce a potential function Φ such that

$$B^* = \frac{d\Phi}{dz} \; . \tag{32}$$

The conditions required of Φ are that

$$\Phi = \Phi(z).$$

This follows from eq. (29). The singularity at the origin requires that

$$\Phi \to \frac{2M}{iz} \quad \text{as} \quad z \to 0. \tag{33}$$

And finally the boundary conditions become

$$\frac{d\Phi}{dz}\, dz = (\mp)\ \sqrt{8\pi p_o}\ dy \; . \tag{34}$$

We now assume that there exists a complex transformation $w = w(z)$ which maps the unknown boundary in the z-plane onto a circle of radius a in the w-plane. We further require of the mapping that it be an identity transformation near the origin, i.e. $w \to z$ as $z \to 0$. We shall see that this added restriction does not overdetermine the problem. There are many transformations which map the surface onto a circle and of these we choose that one which will preserve as nearly as possible the original geometry. If for example $w \to z^2$ near the origin we would be dealing with a double sheeted surface.

Our problem has now reduced to finding a function $\Phi(z(w)) = \Phi(w)$ such that

$$\Phi \to \frac{2M}{iw} \quad \text{as} \quad w \to 0 \tag{35}$$

and

$$d\Phi = (\mp) \sqrt{8\pi p_0}\, dy \tag{36}$$

on the surface of a circle of radius a. It must be recalled that the (-) in equation (36) is to be applied at points on the boundary within the neutral points and the (+) to points outside the neutral points. Now the transformation will determine where these points are mapped on the w-plane. We shall place one further restriction on the transformation. We require that the neutral points are mapped to the top and bottom of the circle (i.e. $\phi = \pi/2$ and $\phi = -\pi/2$). We have now placed a number of restrictions on the transformation. It may be that there are too many and the transformation is overdetermined. We shall show that it is not by exhibiting a transformation which satisfies all these requirements.

From eq. (36) we see that $\mathrm{Im}(d\Phi) = 0$ on the boundary, i.e. on the circle. This implies that

$$\mathrm{Im}\,\Phi = \text{constant on the circle.} \tag{37}$$

We can see this condition physically in this way. Let $\Phi = \phi + i\psi$. Then

$$B^* = \frac{d\phi}{dz} + i\,\frac{d\psi}{dz}$$

Since $\Phi(z)$ is analytic we may set $dz = dx$ so that

$$B^* = \frac{\partial\phi}{\partial x} + i\,\frac{\partial\psi}{\partial x}$$

or

$$B_x = \frac{\partial\phi}{\partial x}$$

and

$$B_y = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y}$$

where we have used the Cauchy-Riemann equations. Therefore

$$\vec{B} = B_x\,\hat{e}_x + B_y\,\hat{e}_y = \nabla\phi\ .$$

The magnetic field lines then are the family of orthogonal trajectories to the equipotential surfaces ϕ = constant. But we know that since ϕ and ψ are conjugate functions that ψ = constant are the family of lines orthogonal to ϕ = constant. Therefore the magnetic field lines are coincident with the family of lines ψ = constant. Now on the cavity surface the magnetic field lines must be tangential so that the surface is itself a field line. Therefore $\psi = \text{Im}\,\Phi$ = constant on the surface. We shall choose this constant to be zero.

We shall attempt now to construct a trial solution for the potential. Let

$$\Phi = \frac{2M}{iw} - \frac{2M}{ia^2}\,w\ . \tag{38}$$

This choice clearly fulfills the requirement of eq. (35) since $w \to z$ as $z \to 0$. It also satisfies eq. (37) since on the circle $w = a\,e^{i\phi}$ and

$$\Phi = \frac{2M}{ia} e^{-i\phi} - \frac{2M}{ia} e^{i\phi} = - \frac{4M}{a} \sin \phi$$

so that Im $\Phi = 0$ on the circle.

There remains only to investigate the real part of eq. (36), i.e.

$$- \frac{4M}{a} \cos\phi d\phi = \mp \sqrt{8\pi p_o} \; dy.$$

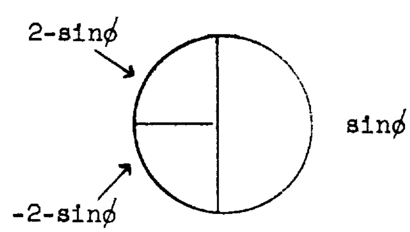
This equation is readily integrated,

$$y' = \begin{cases} -2 - \sin \phi & -\pi < \phi < -\pi/2 \\ \sin \phi & -\pi/2 < \phi < \pi/2 \\ 2 - \sin \phi & \pi/2 < \phi < \pi \end{cases}$$

where

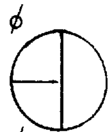
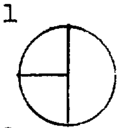
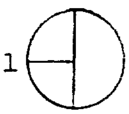
$$y' = \frac{a \sqrt{8\pi p_o}}{4M} y .$$

We shall find it more convenient to write this equation in this form

$y' =$  (40)

We wish now to find a function $z' = z'(w)$ such that z' is proportional to w near the origin and whose imaginary part satisfies eq. (40) on the boundary. This function will define the transformation from the z-plane to the w-plane.

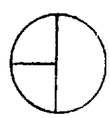
We shall construct this function piecemeal. Let us break up the boundary value problem into two parts

$y' =$ $2 - \sin\phi$  $-2 - \sin\phi$ $\sin\phi = 2 \Big\{$ 1  -1 $0 \Big\} + \sin\phi \Big\{ -1$  $1 \Big\}$

Consider the function

$$F_1 = \frac{1}{2\pi} \ \ell n \ \frac{w_1}{w_2^{\,2} w_3^{\,2}} \quad .$$

It is not difficult to show that

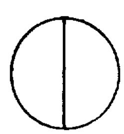
$\mathrm{Im}\, F_1 \doteq$ 1 -1  0

when evaluated on the boundary. (The variables are defined in Fig. 10.)

Also, if

$$F_2 = \frac{1}{\pi} \ \ell n \ \frac{w_2^{\,2}}{w_3^{\,2}} + 2i$$

then

$\mathrm{Im}\, F_2 =$ -1  $+1$

when evaluated on the boundary.

Finally, let

$$F_3 = i/2(a/w - w/a)$$

then

$$F_3 = \sin\phi$$

on the circle.

Let us now evaluate the imaginary part of

$$F_3F_2 + 2F_1 + \text{real constant}$$

on the boundary. We see that it is just y'. (The addition of a real constant will not affect this result.) We therefore set

$$z' = \frac{1}{2\pi}\left(\frac{a}{w} - \frac{w}{a}\right)\left(\ell n\, \frac{w_2^{\,2}}{w_3^{\,2}} + 2\pi i\right) + \frac{1}{\pi}\,\ell n\, \frac{w_1^{\,4}}{w_2^{\,2} w_3^{\,2}} + \frac{2}{\pi} \quad . \tag{41}$$

We have chosen the constant so that $z' \to 0$ as $w \to 0$. In particular

$$z' \to \frac{4w}{\pi a}$$

as $w \to 0$, or

$$z \to \frac{16M}{\pi a^2 \sqrt{8\pi p_o}}\, w \qquad \text{as} \qquad w \to 0 \ .$$

Now we assumed that $z \to w$ as $w \to 0$. We therefore choose the radius <u>a</u> so that the coefficient of w is one, i.e.

$$a^2 = \frac{16M}{\pi\sqrt{8\pi p_o}} \quad . \tag{42}$$

The real part of eq. (41) evaluated on the boundary is

$$x' = \frac{1}{\pi}\,\ell n\, \frac{(1 + \cos\phi)^2}{\cos^2\phi} + \frac{2}{\pi} + \frac{1}{\pi}\sin\phi\ \ell n\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right) \tag{43}$$

Eqs. (39) and (43) are parametric equation for the cavity boundary. This has been plotted in Fig. 12.

There are several interesting features of this shape. First we notice that the neutral points lie to the windward side of the dipole. We believe that this result may be carried over to the three dimensional problem. We have considered several simplified cavity shapes and in all cases the neutral points lay on the side of greatest confinement. As a simple example consider a dipole field confined by two planes intersecting at right angles as shown in Fig. 13. We wish to show that the neutral points lie to the right of the dipole. The boundary conditions are that the magnetic field must be everywhere tangential to the surface. This condition is satisfied by placing three image dipoles as indicated in the figure. If these are two dimensional dipoles the field lines are circles. There is one field line which is common to all four dipoles and this is the circle with center at the right angle. It is clear that the field is zero at the two points where the circle intersects the planes. These are the neutral points and again they lie on that side of the dipole where the field confined by the planes the neutral points would be altered slightly but would still lie to the right of the dipole.

Another interesting feature is the distance from the nose of the cavity to the dipole, $1.35\sqrt{2M/\sqrt{8\pi p_o}}$. This distance is often estimated by locating that point on the earth-sun line where the energy density of the magnetic field of the dipole in free space would equal the energy density of the plasma wind. Where the two energies are comparable we expect that the plasma will interact strongly with the field. We equate then

$$B^2/8\pi = \frac{1}{2}\, Nmv^2$$

with

$$B = \frac{2M}{r^2}$$

and find for the cut off distance

$$r = 1.41\sqrt{2M/\sqrt{8\pi p_o}} \quad .$$

We see that the estimate is good to within 5%.

The magnetic field within the cavity may be determined from eqs. (32,38,41). It is of particular interest to investigate the field near the dipole. We find that at the commencement of a magnetic storm (i.e. at a time of increased solar activity) the magnetic field at the surface of the earth at latitudes below the auroral zone is increased. We shall now show that the magnetic field due to the currents in the current sheath of the cavity boundary will enhance the field of the dipole in the equitorial zone.

If we expand eq. (41) in a power series we find

$$z = w - \frac{1}{3}\frac{w^2}{a} + \frac{1}{3}\frac{w^3}{a^2} + \ldots \quad .$$

Inverting this series we obtain

$$w = z + \frac{1}{3}\frac{z^2}{a} - \frac{1}{9}\frac{z^3}{a^2} \, .$$

Now

$$\Phi = \frac{2M}{i}\left(\frac{1}{w} - \frac{w}{a^2}\right)$$

or

$$\Phi = \frac{2M}{i}\left(\frac{1}{z} - \frac{1}{3a} - \frac{7}{9}\frac{z}{a^2} + \ldots\right)$$

and

$$B^* = \frac{d\Phi}{dz} = 2Mi\left(\frac{1}{z^2} + \frac{7}{9}\frac{1}{a^2} + \ldots\right) \quad .$$

We see that we have a constant field superimposed on the dipole near the origin. This field is

$$By = -2M \; \frac{7}{9a^2}$$

such a field will enhance the field in the equitorial region and diminish the field at the poles (see Fig. 14).

We should point out here that we have neglected the temperature of the plasma wind. Certainly this assumption is valid when the slope of the cavity wall is far from zero. The pressure of the wind on the boundary is determined by the normal component of the plasma velocity. When the slope is small the normal component of the velocity of mass flow may be comparable with the thermal velocities. We expect then that the leeward side of the cavity boundary should be pinched down when the thermal pressure becomes comparable with the magnetic pressure.

ARBITRARY ANGLE OF INCIDENCE

Certainly the earth's dipole is not in general perpendicular to the solar wind. It is of interest therefore to investigate the effect of orientation on the cavity surface.

We might also mention at this point that the problem discussed in the preceding section has been solved independently by Dungey[1]. Our solutions agree. The methods are however different. Dungey points out that his method cannot be used except for the particular symmetry of perpendicular incidence. There is one other orientation which possesses a similar symmetry; the dipole pointing directly into the wind. But even here his method is not adequate to handle the problem. We shall show now that by using the technique developed in the previous section that we may solve the problem for arbitrary orientation.

We assume that the dipole makes an angle α with the vertical. The procedure is exactly as before. The magnetic field near the origin

$$B^* \to \frac{2Mie^{-i\alpha}}{z^2}$$

so that

$$\Phi \to \frac{2Me^{-i\alpha}}{iz}$$

The boundary condition on Φ is that

$$d\Phi = \mp \sqrt{8\pi p_o} \;\; dy$$

on the surface. Again we assume that there exists a transformation $w = w(z)$ that maps the cavity surface onto a circle of radius $\underline{a}$. We also require that the domain close to the dipole is not distorted, i.e.

$$w \to z \quad \text{as} \quad z \to 0$$

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[1] J. W. Dungey, Penn. State Univ., Scientific Report No. 135, July 1960.

We take as a trial solution

$$\Phi = \frac{2M}{i} \left(\frac{e^{-i\alpha}}{w} + \frac{e^{i\alpha} w}{a^2} \right)$$

This satisfies the required singularity condition near the dipole and the imaginary part of the boundary condition since

$$\Phi = \frac{-4M}{a} \sin(\phi + \alpha)$$

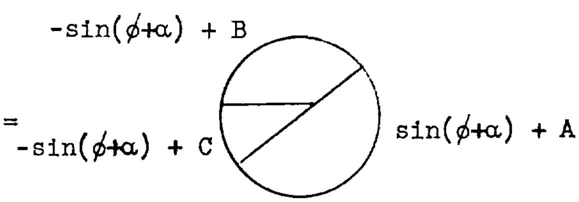
on the surface. It is clear that such a choice for the potential requires that the neutral points map onto the points $\phi = \frac{\pi}{2} - \alpha$ and $\phi = -\frac{\pi}{2} - \alpha$, since at these points $\frac{d\Phi}{d\phi}$ is zero, so that the field is zero. (see Fig. 15)

Now the real part of the boundary condition is

$$y = \pm \frac{4M}{a\sqrt{8\pi p_o}} \sin(\phi + \alpha) + \text{const.}$$

on the surface. We choose to express this equation in this way

$y' =$



where we have set

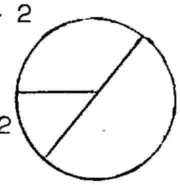
$$y' = \frac{a\sqrt{8\pi p_o}}{4M} y .$$

Now since y' is continuous at N and N'

$$B = A + 2 \qquad \text{and} \qquad C = A - 2$$

so that

$y' =$



We cannot determine A except to say that $A = 0$ when $\alpha = 0$ and $A = 1$ when $\alpha = \pi/2$. We shall leave A arbitrary and determine it later from the condition that $z \to w$ as $w \to 0$.

Also it is not clear where the point $x' = -\infty$ will map on the circle. It is obvious that when $\alpha = 0$ or $\pi/2$, $x = -\infty$ will map onto $\phi = \pi$. We shall assume that this is true of all dipole orientations. We shall show that this assumption is compatible with the requirement of the transformation. We might instead leave it arbitrary, letting $x = -\infty$ map onto $\phi = \pi - \beta$. We suspect that the condition at the origin would become

$$z \to we^{i\beta}$$

so that $\beta = 0$. However this would complicate the analysis considerably and in as much as a solution exists for $\beta = 0$ we may infer from the uniqueness of the solution that β must be zero.

We then solve this boundary value problem for y as before and find that

$$z' = \frac{ia}{\pi}\left(\frac{e^{-i\alpha}}{w} - \frac{we^{i\alpha}}{a^2}\right)\left(\ln\frac{w_1}{w_2} + i\pi\right) - \frac{2i\alpha}{\pi} + \frac{2}{\pi}\ln\frac{w_3^2}{w_1 w_2} + iA + c \; . \tag{44}$$

where w_1, w_2 and w_3 are defined in Fig. 16.

Expandi..g about the origin we find

$$z' \to \frac{4w}{\pi a} - \frac{2}{\pi} + \frac{2i\alpha}{\pi} + iA + c$$

so that $A = \frac{-2\alpha}{\pi}$ and $c = \frac{\pi}{2}$. Then

$$z = \frac{4M}{a\sqrt{8\pi p_o}} \qquad z' \to \frac{16M}{\pi a^2 \sqrt{8\pi p_o}} \; w$$

so that

$$a^2 = \frac{16M}{\pi\sqrt{8\pi p_o}}$$

as before.

Evaluating eq. (44) on the circle we find the parametric equations for the cavity boundary,

$$\frac{\pi}{2}\, x_s' = -(1 - \sin(\phi+\alpha))\, \ln\left(\frac{\cos(\phi+\alpha)}{1+\sin(\phi+\alpha)}\right) + \ln\left(\frac{1+\cos\phi}{1+\sin(\phi+\alpha)}\right) + 1$$

and

$$-\sin(\phi+\alpha) + 2 - \frac{2\alpha}{\pi}$$

$$y_s' = -\sin(\phi+\alpha) - 2 - \frac{2\alpha}{\pi} \qquad \sin(\phi+\alpha) - \frac{2\alpha}{\pi}$$

When $\alpha = 0$ these equations reduce to eqs. (39) and (43) as they must. We have plotted these equations in Figs. 17 and 18 for $\alpha = 37^{\circ}$ and $\alpha = 90^{\circ}$.

AN APPROXIMATION METHOD

Recently Beard[1] has presented an approximate solution to the problem of the interaction between the scalar wind and the earth's dipole field. As a test of the method we shall apply it to our two dimensional problem and compare it with the above solution.

The essential feature of Beard's method is the way in which he approximates the field. Imagine a plane surface brought in from infinity compressing the field in front of it. (see Fig. 19) This problem is easily solved by placing an image dipole at O'. If the plane is inclined at an angle α with the dipole at O then the image dipole at O' is inclined at an angle 2α. It is clear that the resultant magnetic field at the plane is just twice the tangential component of a single dipole.

Beard assumes that this will be approximately true for a curved surface as well. The dipole field is

$$\vec{B} = \frac{2M \sin \theta}{r^2} \hat{e}_r - \frac{2M \cos \theta}{r^2} \hat{e}_\theta$$

---

[1] Op. cit.

(The geometry is defined in Fig. 20.) The tangential component of the field then is

$$-\frac{2M}{r^2} \cos (\psi + \theta)$$

The pressure balance condition at the boundary is

$$\frac{B_t^2}{8\pi} = p_o \cos^2 \chi$$

where

$$B_t = \mp \frac{4M}{r^2} \cos(\psi + \theta)$$

so that

$$\cos \chi = \cos(\theta - \psi) = \mp \frac{4M \cos(\psi + \theta)}{\sqrt{8\pi p_o r^2}} \tag{45}$$

The upper sign is to be applied outside the nodal points and the lower within. Let

$$r_o^{\ 2} = \frac{4M}{\sqrt{8\pi p_o}}$$

so that eq. (45) becomes

$$\cos \theta + \sin \theta \frac{1}{r} \frac{dr}{d\theta} = \frac{(\mp 1)}{r^2} \left(\cos \theta - \sin \theta \frac{1}{r} \frac{dr}{d\theta} \right)$$

where r is now measured in units of r_o. We have made use of the fact that

$$\tan \psi = \frac{1}{r}\frac{dr}{d\theta} \quad .$$

Between the nodal points the solution is

$$r = 1$$

and outside the nodal points

$$r \sin \theta = 2 - \frac{\sin \theta}{r} \quad .$$

This boundary has been plotted in Fig. 12.

It is interesting to note that if there were equal plasma winds streaming from the right and left the approximate solution would be a circle of radius r=1. This is also the exact solution. (The reason for the agreement is that the field of a two dimensional dipole confined to a cylinder has a magnitude on the surface which is twice the tangential component of the dipole alone. This is just Beard's assumption.) If the wind from the left is cut off, the magnetic field will expand into this region and the magnetic pressure on the right will drop. The cavity surface on the right will therefore fall toward the dipole and reach equilibrium at the configuration given by the exact boundary plotted in Fig. 12.

STABILITY

The stability of the cavity surface is a very important question and also a very difficult one. If one is to make any progress at all with the analysis one is forced to make some rather drastic assumptions.

Considerable effort has been devoted to problems of stability. Before proceeding with the special features of our problem we shall describe an important stability requirement for thermal plasmas confined by a magnetic field. (In our problem there is a net mass flow in the plasma wind. We are now considering a plasma at rest.) This problem has been investigated by Kruskal and Schwarzschild[1], Rosenbluth[2], Grad[3] and many others. One common feature of these treatments is that the surface is stable when the center of curvature of the boundary lies in the magnetic field and unstable when it lies in the plasma. We may give qualitative evidence for this conclusion in this way. We consider first the case where the center of curvature lies in the plasma. (see Fig. 21) The magnetic field is directed into the page. In the equilibrium configuration (solid line) the magnetic pressure just balances the plasma pressure, i.e. $B^2/8\pi = p$. The magnetic field increases as we approach the boundary. Thus if the boundary is perturbed as indicated by the dotted line the field at b will increase and the field at a decrease. If we adjust the deformation so that the plasma volume is kept constant the plasma

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[1] M. Druskal and M. Schwarzschild, Proc. Roy. Soc. A223, 348, 1954.

[2] M. N. Rosenbluth and C. L. Longmire, Annals of Phys. 1, 120, 1957.

[3] H. Grad, 2d Geneva Conf., 31:190.

pressure will remain the same. It is clear then that the perturbation will grow. Let us now consider the other case (Fig. 22). The field now increases as we move away from the boundary so that in the perturbed state the field pressure is increased at a and decreased at b. Thus the surface is stable.

Since it is the latter case that prevails in our problem we might expect the surface to be stable. There is however a difference. Our plasma is a wind and not stationary. Further the mean free paths are too long to apply the continuum plasma equations. Also the magnetic field is itself imbedded in a plasma. (The earth's atmosphere is certainly highly ionized at 9 earth's radii.)

Parker[1] has considered the stability of the interface between the solar wind and the earth's field and concludes that it is everywhere unstable. We do not feel that this conclusion necessarily follows from his analysis. We shall conclude now with a more detailed consideration of Parker's model.

The cavity surface is assumed to be locally plane with a plasma stream incident on it at an angle θ_o. See Fig. 23. This system is assumed to be in equilibrium with the magnetic pressure of the wind. The magnetic field below the interface is immersed in an infinitely conducting, incompressible, invisid plasma. It is further assumed that this plasma has a scalar pressure and scalar conductivity. The surface is now perturbed by a traveling wave. The equation of the surface is

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[1] Op. cit.

$z = \eta(y,t) = Ae^{i(wt+ky)}$. The equations of motion together with the boundary conditions determine a dispersion relation $\omega = \omega(k)$. If a solution exists for ω complex the surface will be unstable if Im is negative since $e^{i\omega t}$ will now grow exponentially. Parker states that "the field density B remains uniform" under such a perturbation. This would be true only if the field is directed into the page. Although such a relative orientation of field lines and wind direction does occur in the geomagnetic problem (in the equitorial plane) it does not occur in our two dimensional problem. However, even in the equitorial plane of the geomagnetic field, we would suspect that the curvature of the boundary would have a considerable stabilizing influence.

We shall show now that even if one accepts Parker's model the cavity surface surrounding the two dimensional dipole will for the most part be stable. We shall show that there is a critical angle of incidence above which the surface is stable and below which the surface is unstable.

Because the magnetic field in our two dimensional problem lies in the plane of incidence of the streaming plasma particles we must generalize the geometry considered by Parker. In his analysis the field is perpendicular to the plane of incidence. We shall investigate the following problem. (see Fig. 23) The equilibrium boundary lies in the x-y plane. The incident plasma particles have polar angles θ_o, ϕ_o. We assume that the surface is perturbed by a traveling wave moving in the y-direction. The equation of the perturbed surface is $z = \eta(y,t)$. We shall first consider the case where the magnetic field is perpendicular to the direction of propagation (in the x-direction) and secondly parallel to the direction of propagation (y-direction).

Parker has assumed that the plasma beneath the boundary is incompressible, invisid, and infinitely conducting. In the unperturbed state $u = 0$, $p = p_o$ and $B = B_o$. In the perturbed state we assume that all of these quantities differ only slightly from their equilibrium values. Let the increments be denoted by u, δp, and δB. We shall also find it convenient to introduce a displacement vector $\vec{\xi}$ such that $u = \frac{\partial \vec{\xi}}{\partial t}$.

The equations that apply in such a plasma are[1]

$$\rho \frac{d\vec{u}}{dt} = -\nabla p + \frac{1}{4\pi} (\nabla\times\vec{B})\times\vec{B}$$

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B})$$

$$\nabla\cdot\vec{u} = 0$$

$$\nabla\cdot\vec{B} = 0$$

where ρ is the plasma density, $\vec{u}$ the fluid velocity, p the plasma pressure and d/dt the mobile operator $\frac{\partial}{\partial t} + \vec{u}\cdot\nabla$. Neglecting terms of second order the plasma equations become

$$\rho \frac{\partial^2\vec{\xi}}{\partial t^2} = -\nabla\delta p + \frac{1}{4\pi}(\nabla\times\delta B) \times \vec{B}_o \tag{46}$$

$$\delta\vec{B} = \nabla\times(\vec{\xi}\times\vec{B}_o) \tag{47}$$

$$\begin{aligned} \nabla\cdot\vec{\xi} &= 0 \\ \nabla\cdot\delta\vec{B} &= 0 \end{aligned} \tag{48}$$

[1] S. Chandrasekhar, Plasma Physics (University of Chicago Press, 1960).

Expanding eq. (47) we find that

$$\delta\vec{B} = (\vec{B}_o\cdot\nabla)\vec{\xi} - \vec{B}_o(\nabla\cdot\vec{\xi}) = (\vec{B}_o\cdot\nabla)\vec{\xi} \tag{49}$$

Since B_o has only a x-component

$$\delta\vec{B} = B_o \frac{\partial\vec{\xi}}{\partial x}$$

But $\vec{\xi} = \vec{\xi}\,(y,z,t)$ so that

$$\delta\vec{B} = 0$$

The magnetic field is unchanged to first order.

It follows then from eq. (46) by taking the curl of both sides that

$$\nabla \times \vec{\xi} = 0$$

We see then that the fluid motion is irrotational. We may therefore represent the fluid motion by a scalar potential such that $\vec{u} = -\nabla\Phi$. Eq. (45) becomes

$$-\rho\nabla\dot{\Phi} = -\nabla\delta p$$

Integrating

$$\rho\dot{\Phi} = \delta p. \tag{50}$$

Also since the motion is incompressible

$$\nabla^2\Phi = 0\ . \tag{51}$$

Before one can complete the solution of eqs. (50) and (51) for the plasma motion one must specify the boundary conditions. The boundary in our problem is a free boundary and is determined by the condition that the pressure on the boundary from the plasma wind must balance the pressure of the plasma below. (The pressure from the plasma below the boundary is in part due to the particle pressure and in part due to the magnetic field pressure.) We must now investigate the effect of the perturbation on the plasma wind pressure.

We consider a small segment of the perturbed boundary as illustrated in Fig. 24. The normal to the perturbed surface makes an angle $\alpha(y,t)$ with the z-axis. We assume that α is small. Let U be the normal component of the velocity of the surface. Parker points out that in the local frame of reference moving with the fluid the electric field will vanish so that in this frame the particles are specularly reflected. The pressure then of this moving surface will be

$$p = 2Mm(V_n - U)^2$$

where V_n is the normal component of the particle velocity and so $V_n - U$ is the velocity relative to the surface. The change in pressure then is

$$\delta p = p - p_o = 2Nm\left[(V_n - u)^2 - V_{n_o}^2\right] \cong 4Nm\left[V_{n_o}(V_n - V_{n_o}) - V_{n_o}u\right]$$

to first order in small quantities. V_{n_o} is the normal velocity component to the unperturbed surface. The change in the wind pressure is then

$$\delta p = 4NmV\left[V \cos \Theta_o \sin \Theta_o \sin \phi_o \sin \alpha + U \cos \Theta_o\right]$$

and since the magnetic pressure does not change this must equal the change in the plasma pressure below the boundary as given by eq. (50).

We now let

$$\delta p = A_1 \; e^{i(\omega t+ky)+kz}$$

$$\Phi = A_2 \; e^{i(\omega t+ky)+kz}$$

and

$$\eta(y,t) = A_3 \; e^{i(\omega t+ky)}$$

The z-dependence e^{kz} has been choosen so that $\nabla^2\Phi = 0$. The three constants are determined by two boundary conditions and the equation of motion of the fluid. The two boundary conditions are that the pressure below the surface must equal the pressure above and that the boundary must move with the fluid, i.e.

$$\frac{\partial\eta}{\partial t} = -\frac{\partial\Phi}{\partial z}$$

on the surface. The equation of motion (eq. (49)) gives

$$A_1/\rho = i \; A_2$$

and the boundary conditions become

$$i\omega A_3 = -kA_2 \text{ and } \rho A_1 = 4NmV^2\left[-ik \sin \theta_o \cos \theta_o \sin \phi_o A_3 - \frac{k \cos \theta_o}{V} A_2\right] \tag{52}$$

where we have used the fact that

$$\alpha = - \frac{\partial \eta}{\partial y}$$

The determinant of the coefficients must be zero so that

$$i\omega^2 + \frac{4Vk}{\epsilon} \cos \theta_o \omega - \frac{4V^2k^2}{\epsilon} \sin \theta_o \cos \theta_o \sin \phi_o = 0$$

where we have set the ratio of the densities $\rho/NM = \epsilon$. Solving for $i\omega$ we find

$$i\omega = - \frac{2Vk}{\epsilon} \cos \theta_o \left[1 \mp \frac{1}{\sqrt{2}} \sqrt{\sqrt{1 + \epsilon^2 \tan^2\theta_o \sin^2\phi_o} + 1} \mp \frac{1}{\sqrt{2}} \sqrt{\sqrt{1 + \epsilon^2 \tan^2\theta_o \sin^2\phi_o} - 1}\right]$$

We see that $i\omega$ will have a positive real part unless

$$\epsilon \tan \theta_o \sin \phi_o = 0$$

or

$$\cos \theta_o = 0$$

So that the surface wave will grow except when

$$\theta_o = \begin{cases} 0 \\ \pi/2 \end{cases}$$

or

$$\phi_o = \begin{cases} 0 \\ \pi \end{cases}$$

Fortunately $\phi_o = 0$ or π in both our two dimensional problems, so that the surface is stable with respect to this mode.

Let us now turn to the mode which propagates along the magnetic field lines. We therefore take the field to be $\vec{B} = B_o \hat{y}$. The change in the magnetic field δB is no longer zero. From eq. (49)

$$\delta\vec{B} = (\vec{B}_o \cdot \nabla)\vec{\xi} = B_o \frac{\partial\vec{\xi}}{\partial y}(y,z,t)$$

Now

$$\nabla\times\delta\vec{B} = B_o \frac{\partial}{\partial y}(\nabla\times\xi)$$

and

$$(\nabla\times\delta\vec{B})\times B_o = -B_o \frac{\partial}{\partial y}\left[\vec{B}_o\times(\nabla\times\vec{\xi})\right] = -B_o \frac{\partial}{\partial y}\left[\nabla(\vec{B}_o\cdot\vec{\xi}) - (\vec{B}_o\cdot\nabla)\vec{\xi}\right]$$

$$= -B_o \frac{\partial}{\partial y}\left[\nabla(\vec{B}_o\cdot\vec{\xi}) - B_o \frac{\partial\vec{\xi}}{\partial y}\right]$$

so that

$$\nabla \times \left[(\nabla \times \delta B) \times \vec{B}_o\right] = B_o^{\,2} \frac{\partial^2}{\partial y^2} (\nabla \times \vec{\xi})$$

But from eq. (45) this is also equal to $\rho \nabla \times \ddot{\xi}$ and so

$$\frac{1}{4\pi} B_o^{\,2} \frac{\partial^2}{\partial y^2} (\nabla \times \vec{\xi}) = \rho \nabla \times \ddot{\xi}$$

If we again assume that the disturbance is a traveling wave

$$\frac{\partial^2}{\partial y^2} = -k^2$$

and

$$\frac{\partial^2}{\partial t^2} = -\omega^2$$

so that

$$\omega^2 (\nabla \times \vec{\xi}) = k^2 \frac{B_o^{\,2}}{4\pi\rho} (\nabla \times \xi)$$

We see then that either

$$\frac{\omega}{R} = \frac{B_o}{\sqrt{4\pi\rho}} \tag{53}$$

or

$$\nabla \times \vec{\xi} = 0$$

Eq. (53) is just the dispersion relation for an Alfven wave. If such a solution exists, i.e. if we can satisfy the boundary conditions with such a mode, it will be stable since ω is real. As we are looking for possible unstable modes we shall take $\nabla \times \vec{\xi} = 0$, so that we are again dealing with irrotational flow. The situation is then as before except that $(B^2 - B_o^{\,2})/8\pi$, the change in the magnetic pressure, is not zero. Here

$$\delta\vec{B} = (\vec{B}_o \cdot \nabla)\vec{\xi} = B_o \left(\frac{\partial \xi_y}{\partial y} \hat{y} + \frac{\partial \xi_z}{\partial y} \hat{z} \right)$$

so that

$$\frac{(B^2 - B_o^{\,2})}{8\pi} = \frac{\vec{B}_o \cdot \delta\vec{B}}{4\pi} = \frac{B_o^{\,2}}{4\pi} \frac{\partial \xi_y}{\partial y} = \frac{B_o^{\,2}}{4\pi i \omega} \frac{\partial \dot{\xi}_y}{\partial y} = - \frac{B_o^{\,2}}{4\pi i \omega} \frac{\partial^2 \Phi}{\partial y^2} = \frac{B_o^{\,2} k^2 A_2}{4\pi i \omega}$$

In the equation of pressure balance we must now add a change in magnetic pressure so that eq. (51) becomes

$$A_1 + \frac{B_o^{\,2} k^2}{4\pi i \omega} A_2 = 4NmV^2 \left[\sin \theta_o \cos \theta_o \sin \phi_o \frac{k^2}{\omega} A_2 - \frac{kA_2}{V} \cos \theta_o \right]$$

Eliminating the A's and solving for ω we find

$$i\omega = - \frac{4Vk}{\epsilon} \cos \theta_o \left[1 \mp \frac{\sqrt{1-\epsilon}}{\sqrt{2}} \left(\sqrt{\sqrt{1+x^2} + 1} + i \sqrt{\sqrt{1+x^2} - 1} \right) \right]$$

where

$$x = \frac{\epsilon}{1-\epsilon} \tan \theta_o \sin \phi_o$$

The real part of $i\omega$ is positive, and hence the surface is unstable when

$$\frac{\sqrt{1-\epsilon}}{\sqrt{2}} \sqrt{\sqrt{1+x^2} + 1} > 1$$

or when

$$\tan \theta_o |\sin \phi_o| > \frac{2}{\sqrt{\epsilon}}$$

In our problem $\phi_o = \pi/2$ so that instability occurs when θ_o is greater than the critical angle $(\theta_o)_c$ where

$$\tan (\theta_o)_c = 2/\sqrt{\epsilon}$$

We might say then that for all angles less than $(\theta_o)_c$ the surface will be stable and for all angles greater than $(\theta_o)_c$ the surface will be stable or unstable depending on the stabilizing influence of the curvature of the surface which has been neglected.

Unfortunately there is no reliable estimates for the plasma density as far out as 9 earth's radii. We would suspect however that the density is quite low so that ϵ is small and the critical angle ~ 90^o.

On the basis of the above calculation we might draw some tentative conclusions concerning the stability of the surface surrounding the earth's dipole field. In the azimuthal plane passing through the sun the surface will be stable over a wide range. In the equitorial plane however we feel that the stability question can not be answered by the above analysis. The curvature of the surface will exert a stabilizing influence which should not be neglected.

SUMMARY

We have shown that a plasma incident on the magnetic field of an infinite line current or a two dimensional dipole will confine the field so that it forms a cavity in the wind. The plasma particles move in straight lines (assuming long mean free paths) and penetrate a short distance into the field to form a current sheath within which they are deflected out of the field. We have shown that the thickness of this sheath is small compared to the dimensions of the cavity.

In the steady state the particles will be specularly reflected from the surface. They will therefore exert a pressure on the surface which is proportional to the square of the normal velocity component. The cavity boundary is determined then by the condition that the magnetic pressure from within the cavity balance the particle pressure from without. We have calculated the shape of this boundary separating the plasma wind from the magnetic field of an infinite line current and a two dimensional dipole. The analysis also determines the magnetic field within the cavity.

The cavity boundary has been shown to be stable except for those regions where the cavity wall is almost parallel to the plasma wind direction.

REFERENCES

C.P. Sonett, D.L. Judge, A.R. Sims, and J.M. Kelso, A Radial Rocket Survey of the Distant Geomagnetic Field, Space Technology Laboratories Report 7320, 2-13.

E.N. Parker, Interaction of the Solar Wind with the Geomagnetic Field, Phys. of Fluids, 1, 171 (1958).

David B. Beard, The Interaction of the Terrestrial Magnetic Field with the Solar Corpuscular Radiation, J. Geophys. Research, 65, 3559, 1960.

J.D. Cole and J.H. Huth, Physics of Fluids, 2, 624 (1959).

J.W. Dungey, Penn. State Univ., Scientific Report No. 135, July 1960.

S. Chandrasekhar, Plasma Physics (University of Chicago Press, 1960).

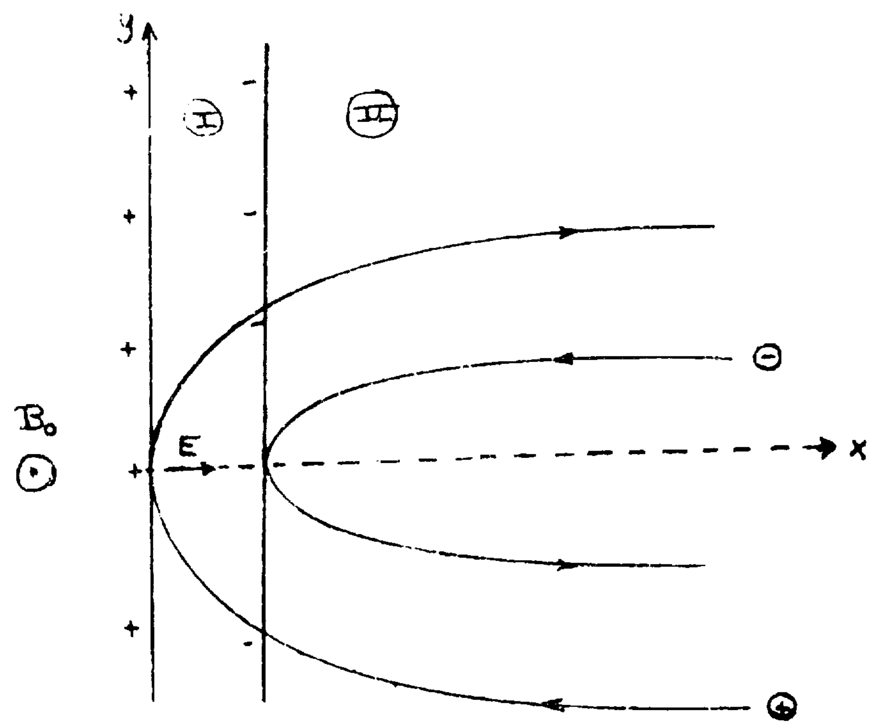

Fig. 1 Particle trajectories in the magnetic field.

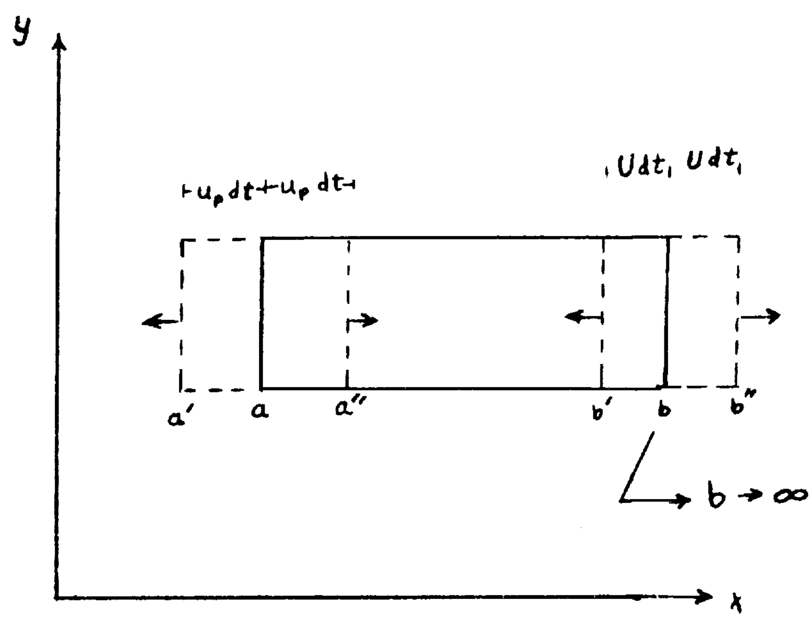

Fig. 2 Motion of plasma block in a time dt.

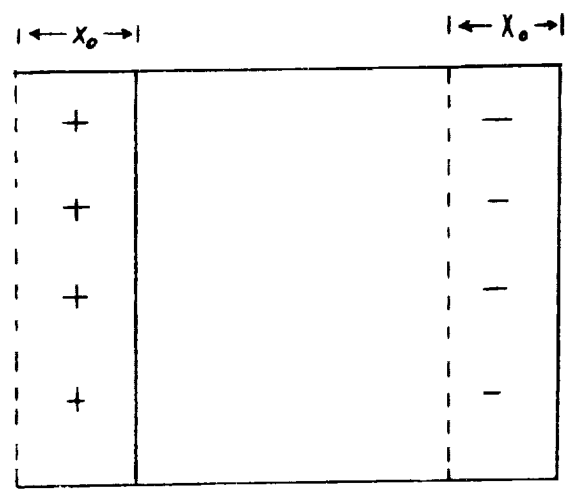

Fig. 3 Maximum charge separation compatible with energy conservation.

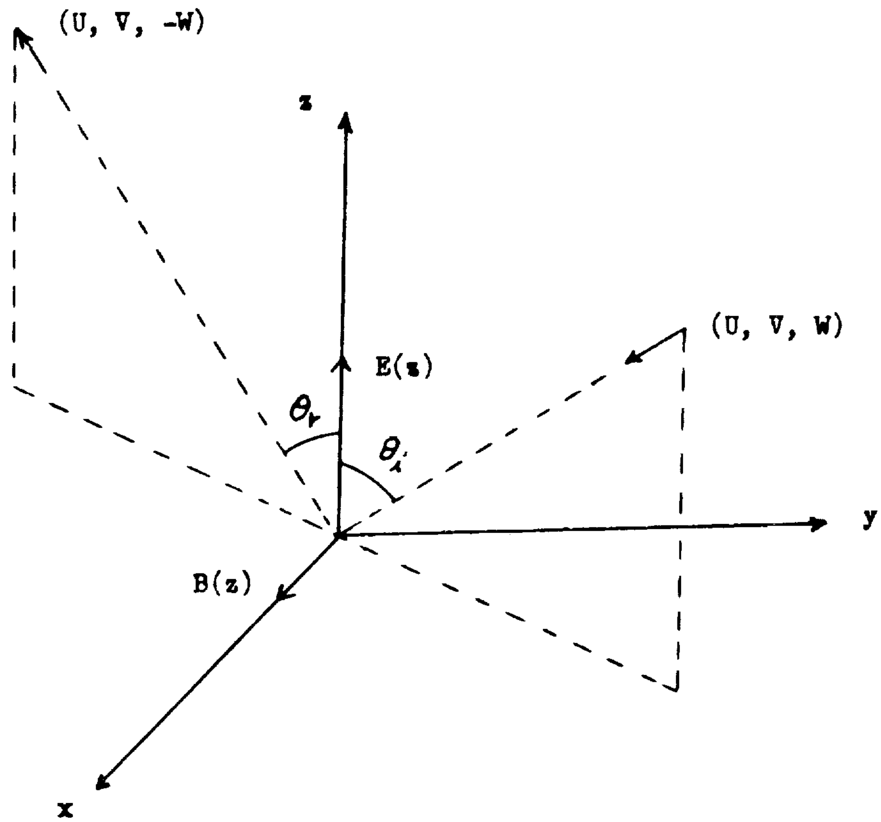

Fig. 4 Specular reflection in current sheath.

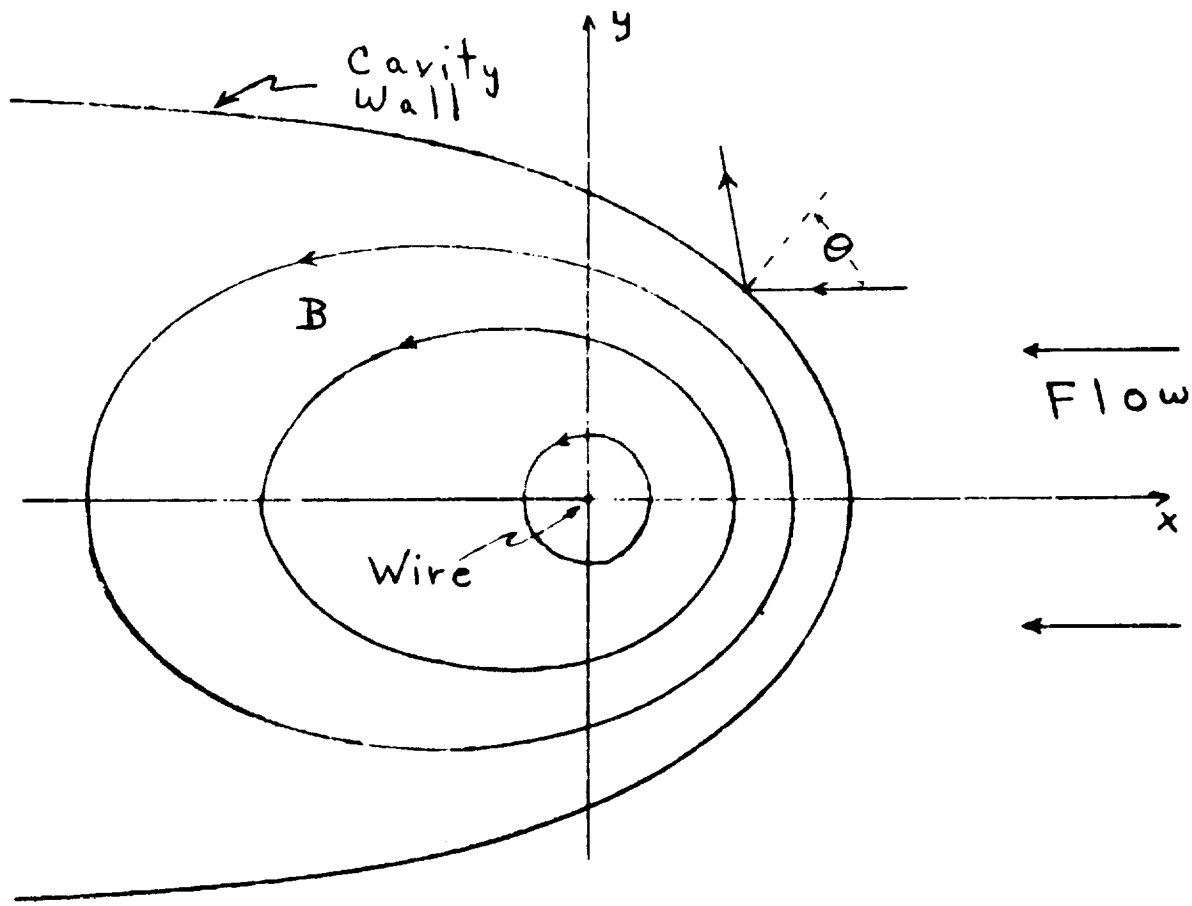

Fig. 5 Flow of plasma past a line current.

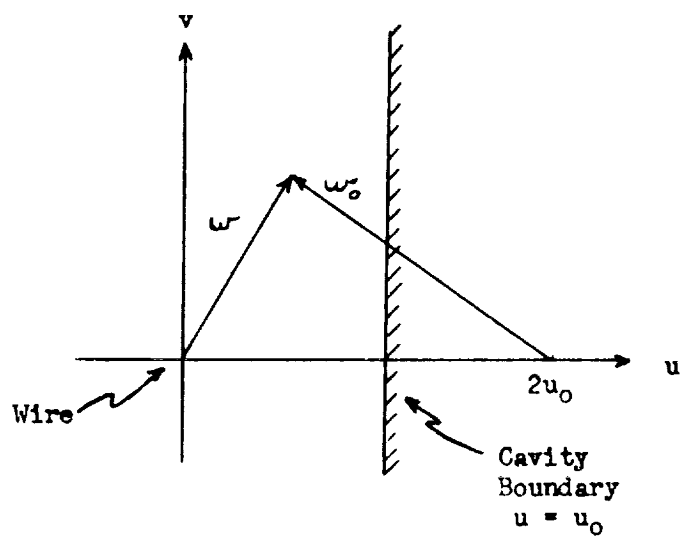

Fig. 6 Geometry as transformed by setting $w = 1/iB^*$.

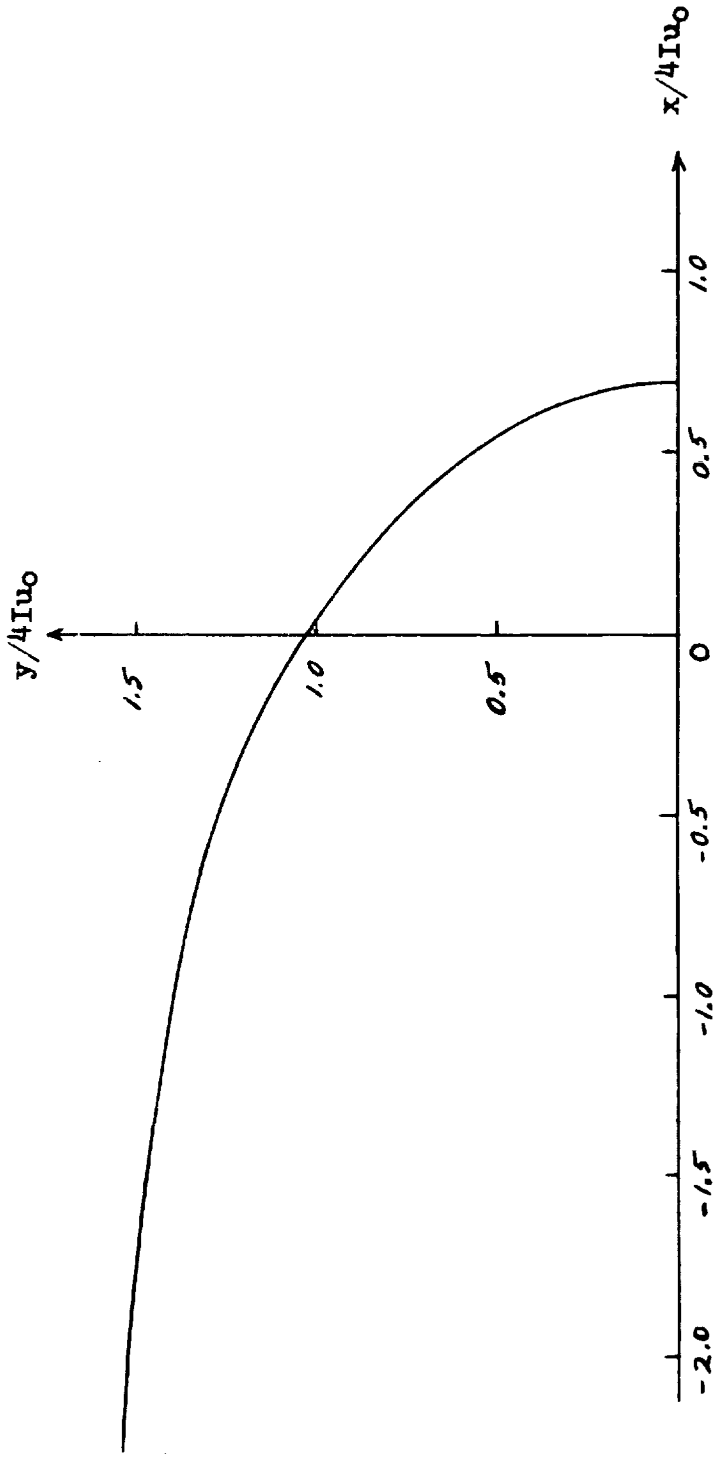

Fig. 7 Cavity surface as determined from Eq. (26).

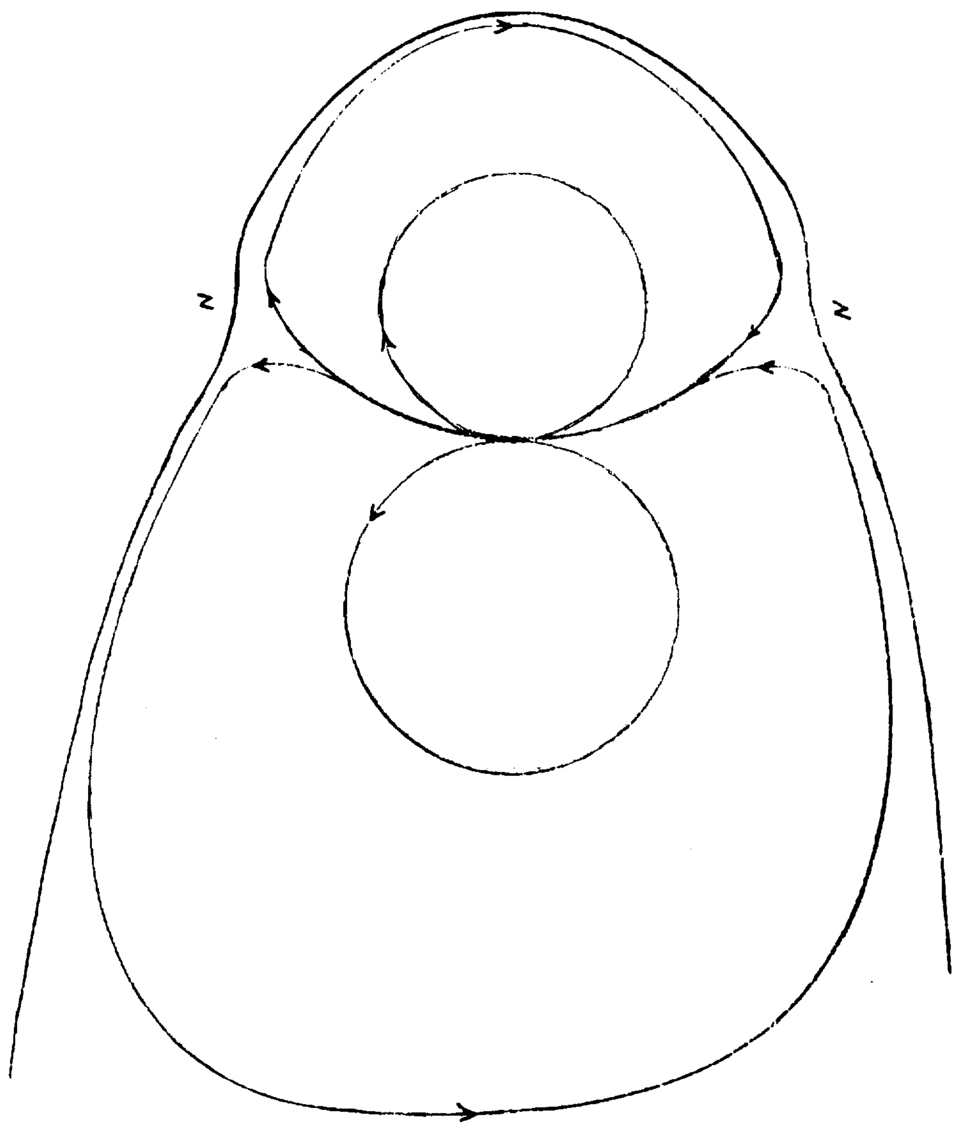

Fig. 8 Expected cavity shape and field configuration.

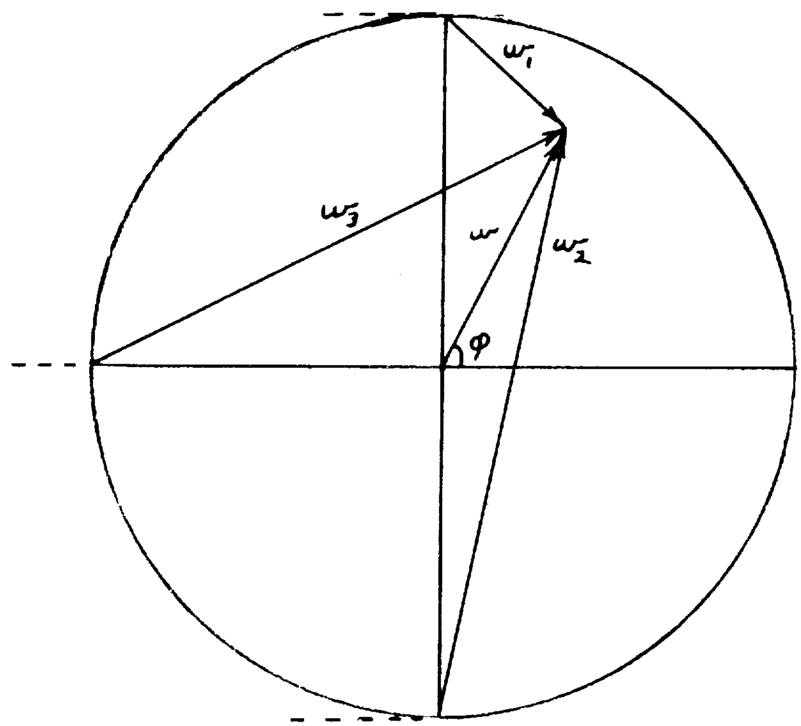

Fig. 10 Dipole cavity boundary mapped onto a circle. The dotted lines are branch cuts.

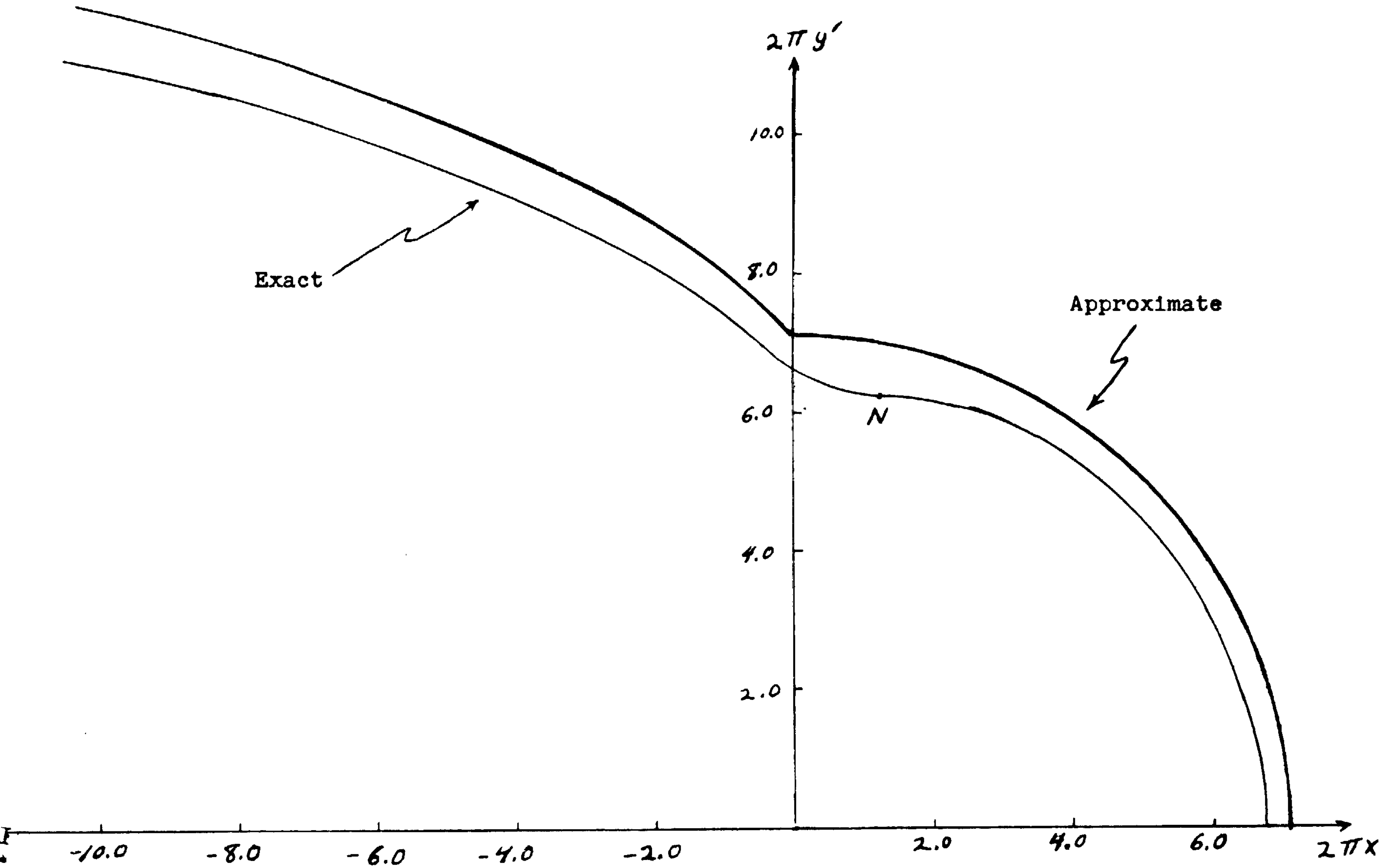

Fig. 12 Exact cavity surface as determined from Eq. (41) and Beard's approximation.

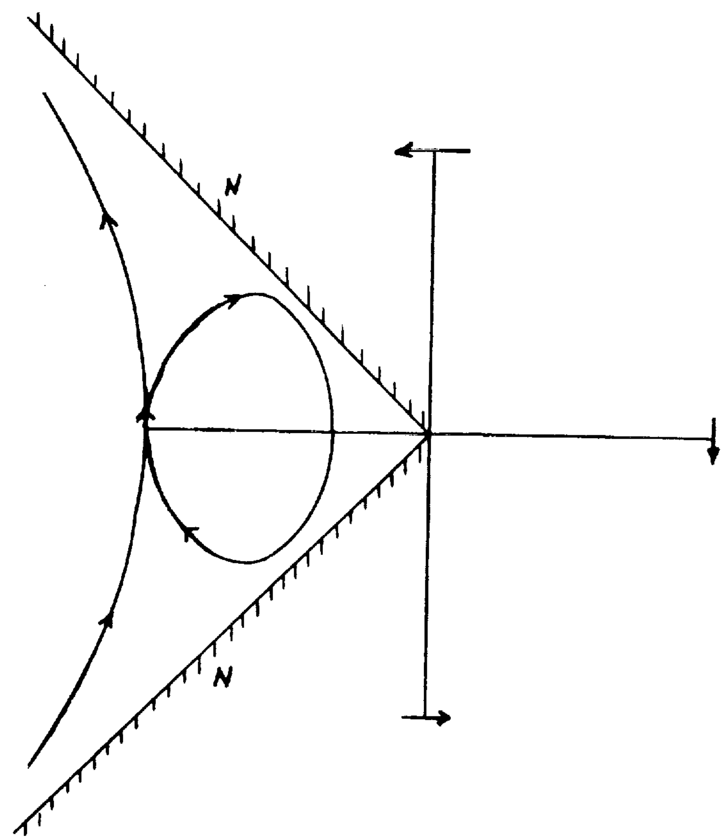

Fig. 13 Dipole field confined by two infinite planes intersecting at right angles.

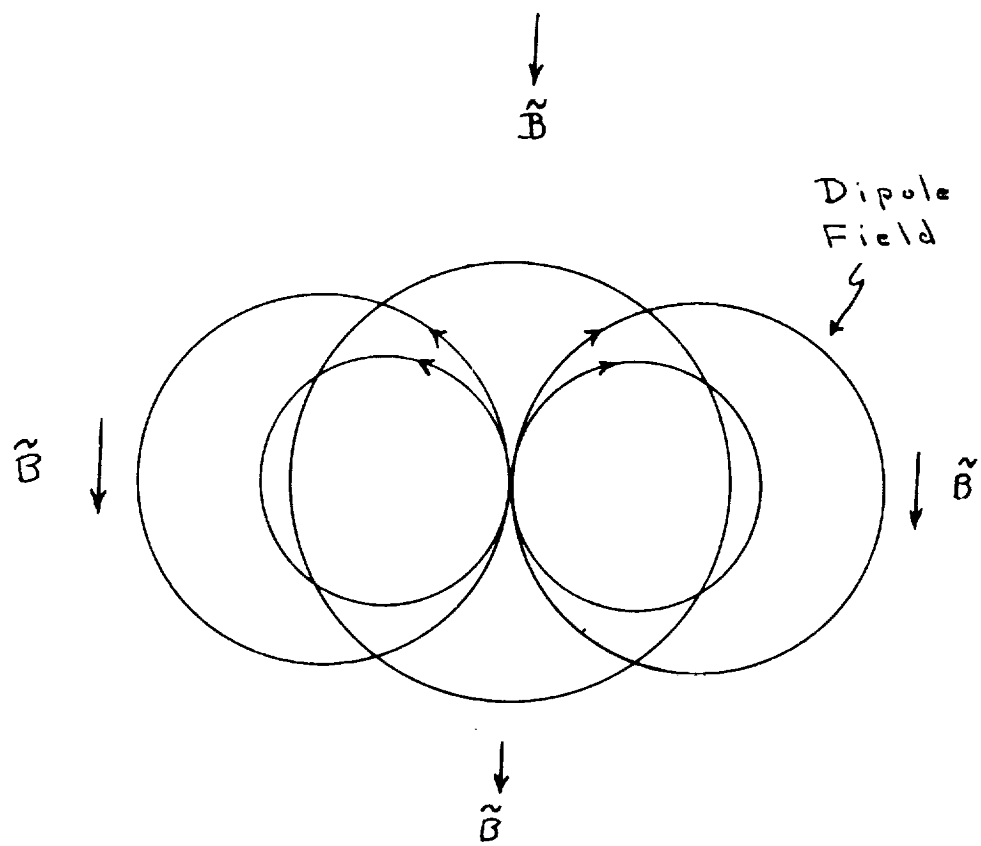

Fig. 14 The effect of the field due to the current sheath (B) will be to increase the dipole field on the equator and decrease it at the poles.

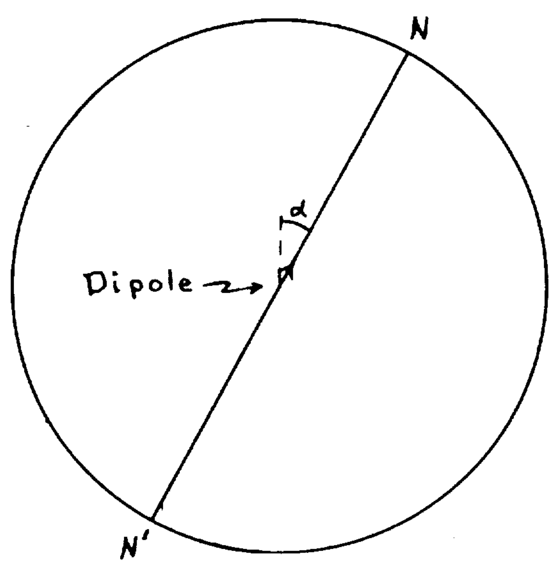

Fig. 15 Cavity boundary transformed to a circle.

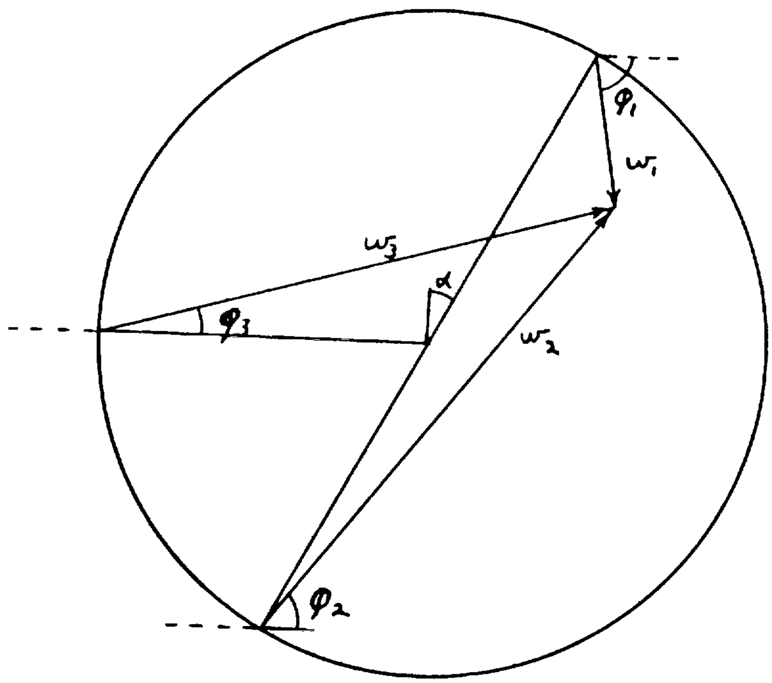

Fig. 16 Definition of polar vectors used in Eq. (44). The dotted lines are branch cuts. φ_1 varies between $-\alpha$ and $+\alpha-\pi$, φ_2 between $-\alpha$ and $\pi-\alpha$ and φ_3 between $-\pi/2$ and $+\pi/2$.

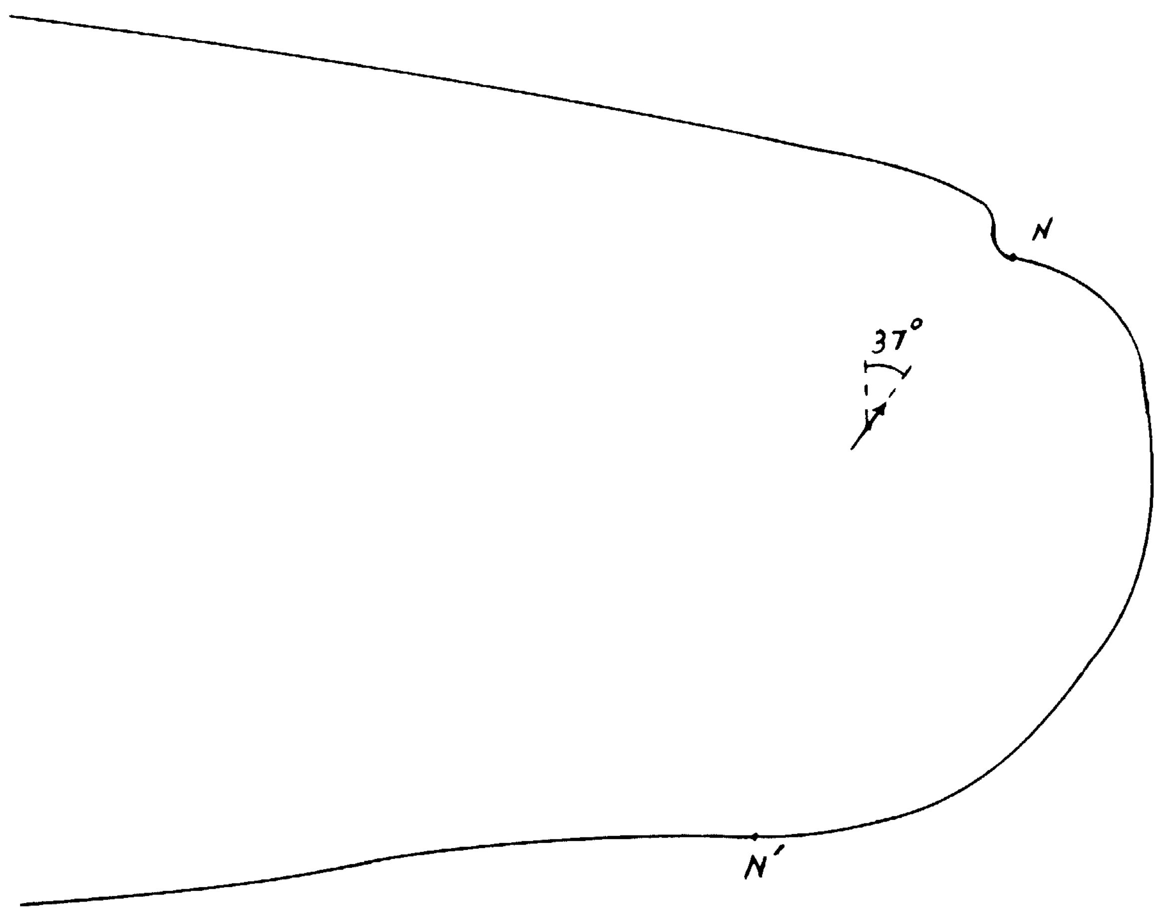

Fig. 17 Cavity surface for $\alpha = 37^{o}$.

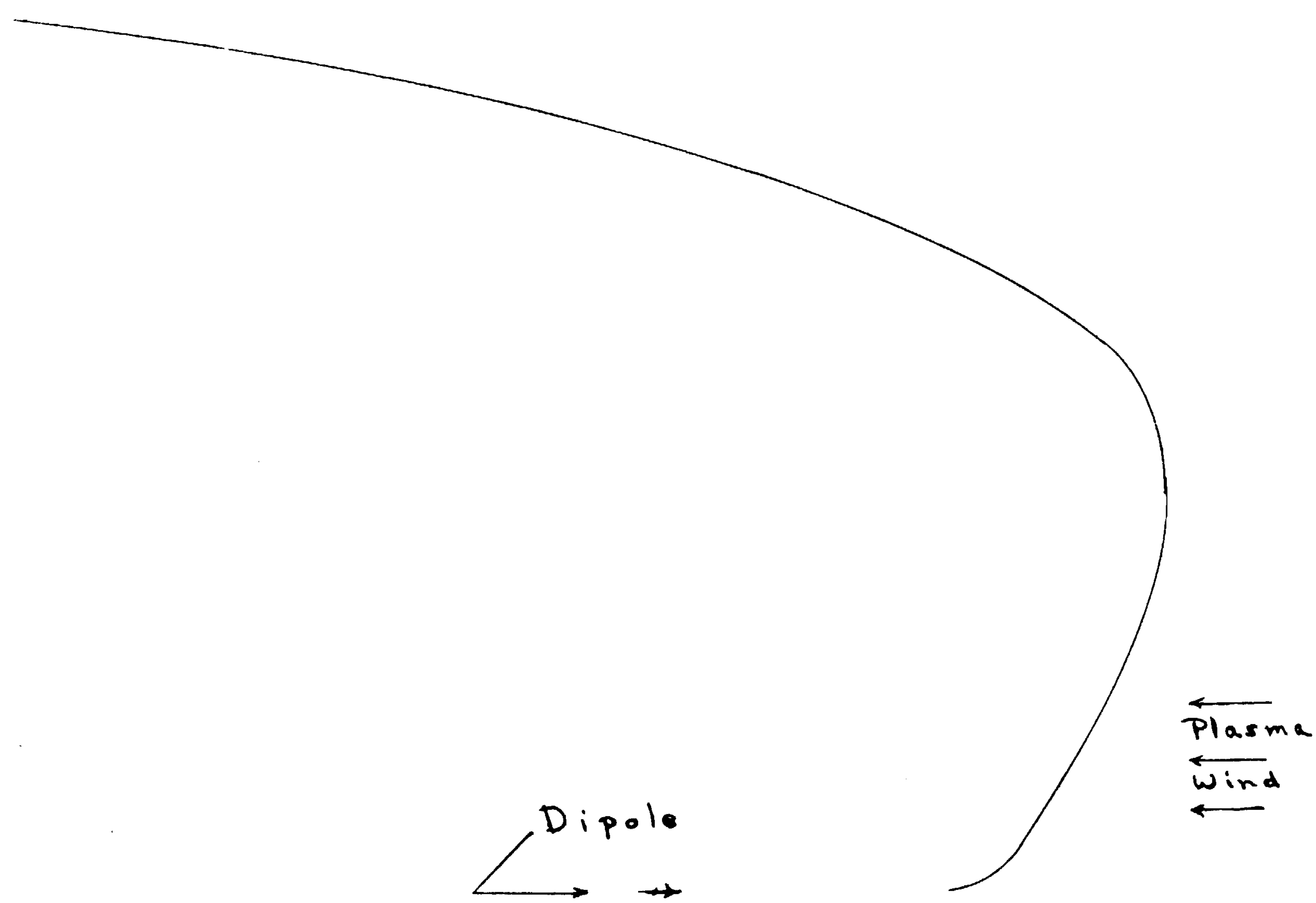

Fig. 18 Cavity surface for α = 90°.

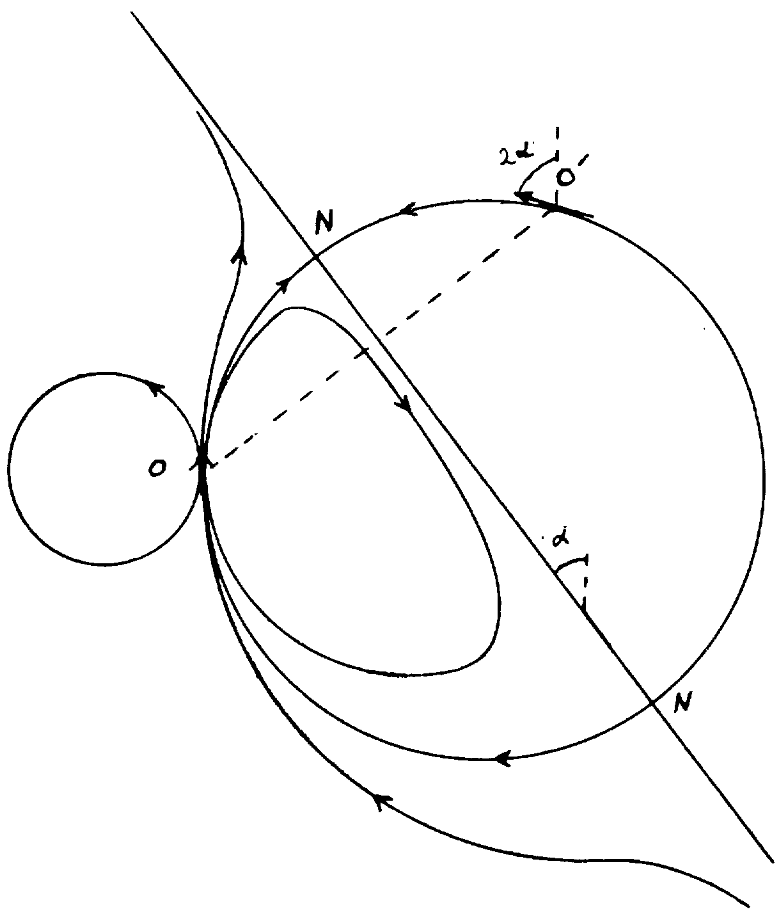

Fig. 19 Dipole field confined by an infinite plane.

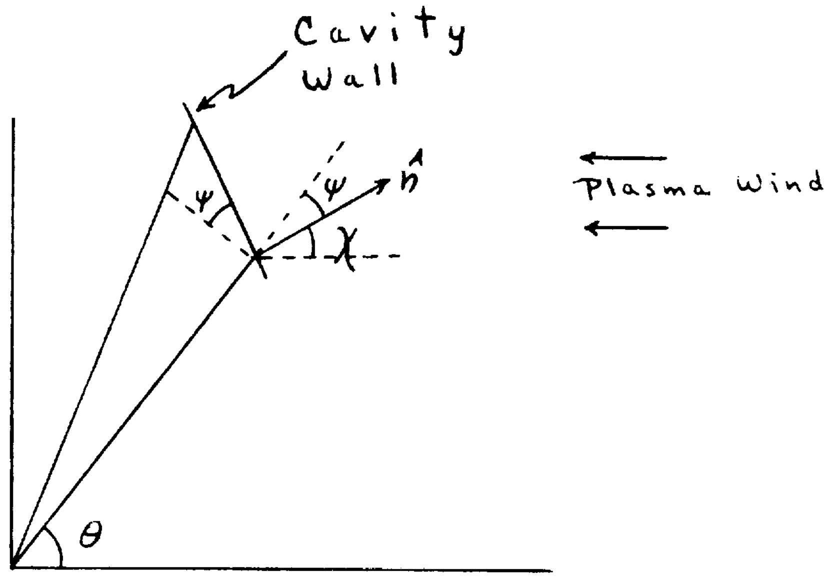

Fig. 20 Segment of cavity boundary. $\hat{n}$ is a unit vector normal to the surface.

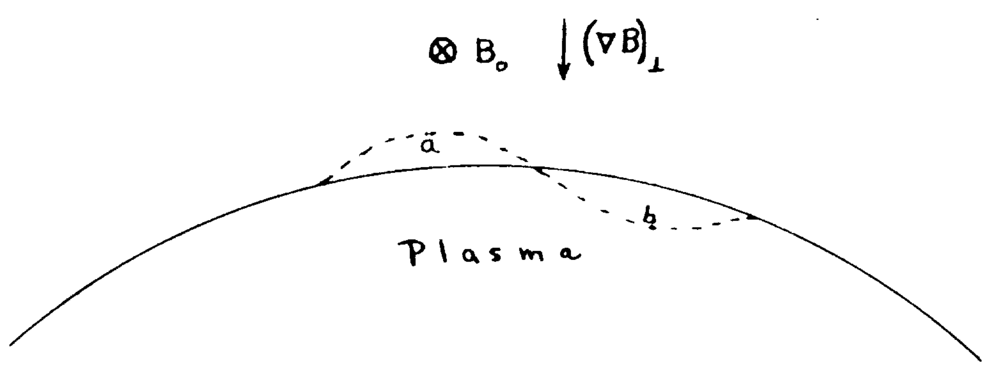

Fig. 21. Perturbation of equilibrium boundary surface when the center of curviture lies in the plasma domain.

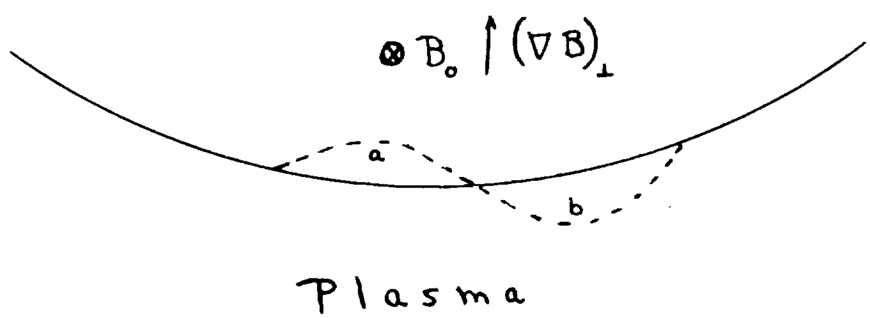

Fig. 22 Perturbation of equilibrium boundary surface when the center of curviture lies in the field domain.

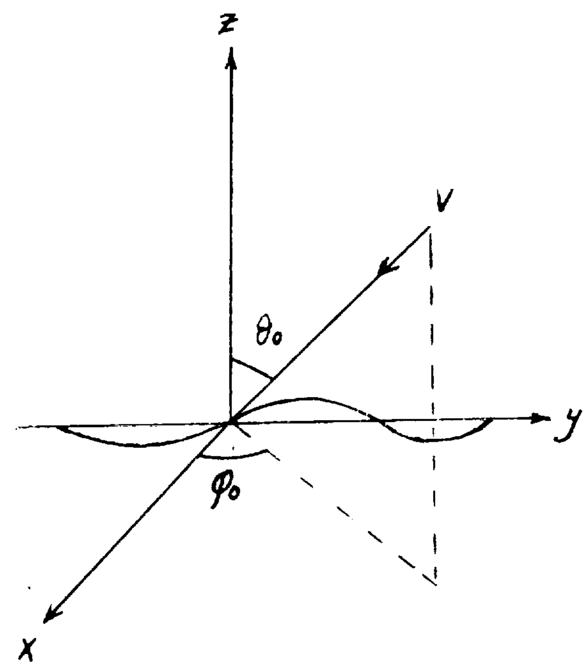

Fig. 23 Plasma wind blowing on a surface wave propagating in the y-direction.

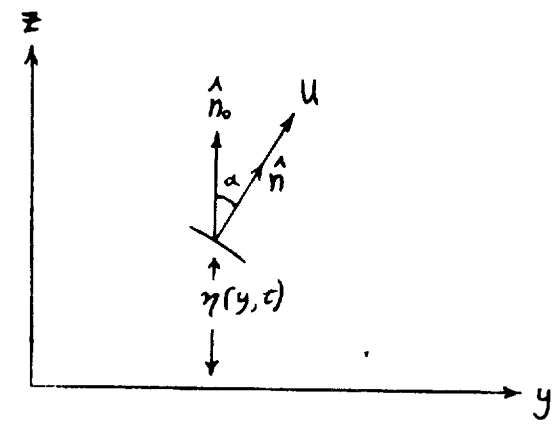

Fig. 24 Perturbed element of boundary surface. $\hat{n}$ is a unit vector normal to the perturbed surface and n_o a unit vector normal to the unperturbed surface. η is the displacement of the perturbed surface element from the unperturbed surface.