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ON THE MOTION OF A 24-HOUR SATELLITE

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SUMMARY

The theory and conditions for stability of a satellite with a 24-hour period are given. Bohlin's resonance theory was applied to obtain the solution. It is shown that the integrals of the problem can be represented in series form, with respect to the small parameter w, which is proportional to the mean motion of the critical argument in a nonresonance case. Expressions for the period of libration and the mean motion of the critical argument in the unstable case are also given. A system of formulas is presented which can be used to compute any particular case.

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INTRODUCTION

In this paper the authors have investigated the stability conditions of a satellite with a period of revolution approximately equal to one day. The criteria for stability are obtained in a form valid for large inclinations—provided that the eighth power of the eccentricity is negligible.

THE DISTURBING FUNCTION

The disturbing function consists of a secular part produced by the zonal harmonics k_2 and k_4, and a periodic part produced by the ellipticity of the earth's equator. The periodic disturbing function has the form

$$F_1 = \frac{3}{2}\frac{\mu A_{22}}{r^5}\left(x^2 - y^2\right) \cdot \tag{1}$$

The x-axis is directed along the semi-major axis of the equator, and the z-axis is directed along the axis of rotation of the earth. Substituting

$$\frac{x}{r} = \frac{1+\cos i}{2}\cos(f+\omega+\Omega-n't) + \frac{1-\cos i}{2}\cos(f+\omega-\Omega+n't) ,$$

$$\frac{y}{r} = \frac{1+\cos i}{2}\sin(f+\omega+\Omega-n't) - \frac{1-\cos i}{2}\sin(f+\omega-\Omega+n't) ,$$

$$\frac{z}{r} = \sin(f+\omega)\sin i$$

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into Equation 1, we obtain

$$F_1 = \frac{3}{8}\frac{\mu A_{22}}{a^3}(1+\cos i)^2\left(\frac{a}{r}\right)^3\cos(2f+2\omega+2\Omega-2n't)$$

$$+\frac{3}{4}\frac{\mu A_{22}}{a^3}\sin^2 i\left(\frac{a}{r}\right)^3\cos(2\Omega-2n't)$$

$$+\frac{3}{8}\frac{\mu A_{22}}{a^3}(1-\cos i)^2\left(\frac{a}{r}\right)^3\cos(2f+2\omega-2\Omega+2n't)\,. \tag{2}$$

Developing Equation 2 into a series in terms of the mean anomaly l with coefficients developed in powers of the eccentricity, and retaining the long period terms only, we deduce

$$F_1 = Q_0\cos(2l+2\omega+2\Omega-2n't)$$

$$+\,Q_1\cos(2l+2\Omega-2n't) + Q_2\cos(2l-2\omega+2\Omega-2n't)\,, \tag{3}$$

where

$$Q_0 = \frac{3}{8}\frac{\mu A_{22}}{a^3}(1+\cos i)^2\left(1-\frac{5}{2}e^2+\frac{13}{16}e^4-\frac{35}{288}e^6\right),$$

$$Q_1 = \frac{3}{4}\frac{\mu A_{22}}{a^3}\sin^2 i\left(\frac{9}{4}e^2+\frac{7}{4}e^4+\frac{141}{64}e^6\right),$$

$$Q_2 = \frac{3}{8}\frac{\mu A_{22}}{a^3}(1-\cos i)^2\left(\frac{1}{24}e^4+\frac{7}{240}e^6\right).$$

If the mean motion of the satellite is such that it causes the satellite to remain above a particular longitude of the earth for some time, the first term in Equation 3 is the most significant, and will be treated in accordance with the theory of resonance. The last two terms will produce only small, long period terms in the osculating elements.

The coefficient A_{22} is approximately of the same order as the coefficient of the fourth zonal harmonic, and consequently, the secular part F_0 of the disturbing function need not be developed beyond the results established by Brouwer (Reference 1). Adding the term $n'\sqrt{\mu a(1-e^2)}\cos i$, produced by the rotation of the earth to Brouwer's development, we have

$$F_0 = \frac{\mu^2}{2L^2} + n'H + \frac{\mu^4 k_2}{L^3 G^3}\left(-\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2}\right)$$

$$+ \frac{\mu^6 k_4}{L^{10}}\left(\frac{15}{16}\frac{L^7}{G^7} - \frac{9}{16}\frac{L^5}{G^5}\right)\left(1 - 10\frac{H^2}{G^2} + \frac{35}{3}\frac{H^4}{G^4}\right)$$

$$+ \frac{\mu^6 k_2^2}{L^{10}}\left[+ \frac{15}{32}\frac{L^5}{G^5}\left(1 - \frac{18}{5}\frac{H^2}{G^2} + \frac{H^4}{G^4}\right)\right.$$

$$\left. + \frac{3}{8}\frac{L^6}{G^6}\left(1 - 6\frac{H^2}{G^2} + 9\frac{H^4}{G^4}\right) - \frac{15}{32}\frac{L^7}{G^7}\left(1 - 2\frac{H^2}{G^2} - 7\frac{H^4}{G^4}\right)\right],$$

where L, G, and H are Delaunay variables

$$L = \sqrt{\mu a}\ , \quad G = \sqrt{\mu a\left(1 - e^2\right)}\ , \quad H = \sqrt{\mu a\ 1 - e^2)}\cos i\ ,$$

and k_2 and k_4 are the coefficients of the second and fourth zonal harmonics, respectively.

Considering the form of the main argument, it is more convenient to use the canonical set of Poincare rather than that of Delaunay.

Thus

$$\left.\begin{aligned} x_1 &= \sqrt{\mu a} = L, & y_1 &= l + \omega + \Omega - n't\ , \\ x_2 &= \sqrt{\mu a}\left(1 - \sqrt{1 - e^2}\right) = L - G, \text{ and} & y_2 &= -\omega, \text{ and} \\ x_3 &= \sqrt{\mu a}\left(1 - \sqrt{1 - e^2}\cos i\right) = L - H, & y_3 &= -\Omega + n't. \end{aligned}\right\} \tag{4}$$

The complete Hamiltonian is

$$F = F_0 + F_1\ , \tag{5}$$

but the disturbing function can be expressed more conveniently by means of the auxiliary quantities

x_1, ϵ, and γ, where $\epsilon = x_2/x_1$, $\gamma = 1 - x_3/x_1$. And the following relations exist:

$$L = x_1 ,$$

$$\frac{G}{L} = 1 - \epsilon ,$$

$$\frac{H}{G} = \cos i = \frac{\gamma}{1 - \epsilon} ,$$

and

$$e^2 = 2\epsilon - \epsilon^2 .$$

The following expressions were used in the actual computations:

$$\begin{aligned}
F_0 = {} & n'\gamma x_1 + \frac{\mu^2}{2x_1^2} \\
& + \frac{\mu^4 k_2}{2x_1^6} \left[- \left(1 + 3\epsilon + 6\epsilon + 10\epsilon^3\right) + \gamma^2 \left(3 + 15\epsilon + 45\epsilon^2 + 105\epsilon^3\right)\right] \\
& + \frac{\mu^6 k_2^2}{x_1^{10}} \quad \frac{3}{32} \left(4 + 14\epsilon + 19\epsilon^2 - 21\epsilon^3\right) - \frac{3}{16} \gamma^2 \left(16 + 114\epsilon + 459\epsilon^2 + 1371\epsilon^3\right) \\
& \qquad + \frac{3}{32} \gamma^4 \left(76 + 790\epsilon + 4515\epsilon^2 + 18755\epsilon^3\right)\Big] \\
& + \frac{\mu^6 k_4}{x_1^{10}} \left[\frac{3}{16} \left(2 + 20\epsilon + 95\epsilon^2 + 315\epsilon^3\right) - \frac{15}{8} \gamma^2 \left(2 + 24\epsilon + 141\epsilon^2 + 573\epsilon^3\right)\right. \\
& \qquad \left. + \frac{5}{16} \gamma^4 \left(14 + 196\epsilon + 1365\epsilon^2 + 6545\epsilon^3\right) \right] ,
\end{aligned}$$

$$\begin{aligned}
Q_0 = {} & \frac{\mu^4 A_{22}}{x_1^6} \left[\frac{1}{96} \left(36 - 180\epsilon + 207\epsilon^2 - 152\epsilon^3\right) + \frac{1}{48} \gamma \left(36 - 144\epsilon + 63\epsilon^2 - 89\epsilon^3\right)\right. \\
& \qquad \left. + \frac{1}{96} \gamma^2 \left(36 - 108\epsilon - 45\epsilon^2 - 134\epsilon^3\right)\right] ,
\end{aligned}$$

$$Q_1 = \frac{\mu^4 A_{22}}{x_1^6} \left[\frac{3}{32} \left(36\epsilon + 38\epsilon^2 + 85\epsilon^3\right) - \frac{3}{32} \gamma^2 \left(36\epsilon + 110\epsilon^2 + 269\epsilon^3\right) \right] ,$$

$$Q_2 = \frac{\mu^4 A_{22}}{x_1^6} \left[\frac{1}{80} \left(5\epsilon^2 + 2\epsilon^3\right) - \frac{1}{40} \gamma \left(5\epsilon^2 + 7\epsilon^3\right) + \frac{1}{80} \gamma^2 \left(5\epsilon^2 + 12\epsilon^3\right)\right] .$$

The first and second derivatives of F_0 and Q_0 with respect to x_1 can be obtained by using the expression

$$F' = \frac{\partial F}{\partial x_1} + \frac{1-\gamma}{x_1}\frac{\partial F}{\partial \gamma} - \frac{\epsilon}{x_1}\frac{\partial F}{\partial \epsilon} ,$$

from which we obtain:

$$\begin{aligned}
F_0' = \; & n' - \frac{\mu^2}{x_1^3} + \frac{3\mu^4 k_2}{2x_1^7}\left[\left(2+7\epsilon+16\epsilon^2+30\epsilon^3\right) + 2\gamma\left(1+5\epsilon+15\epsilon^2+35\epsilon^3\right) - \gamma^2\left(8+45\epsilon+150\epsilon^2+385\epsilon^3\right)\right] \\
& - \frac{\mu^6 k_2^2}{x_1^{11}}\left[\frac{3}{32}\left(40+154\epsilon+228\epsilon^2-273\epsilon^3\right) + \frac{3}{8}\gamma\left(16+114\epsilon+459\epsilon^2+1371\epsilon^3\right)\right. \\
& \qquad - \frac{9}{16}\gamma^2\left(64+494\epsilon+2142\epsilon^2+6855\epsilon^3\right) - \frac{3}{8}\gamma^3\left(76+790\epsilon+4515\epsilon^2+18755\epsilon^3\right) \\
& \qquad \left. + \frac{3}{32}\gamma^4\left(1064+11850\epsilon+72240\epsilon^2+318835\epsilon^3\right)\right] \\
& - \frac{\mu^6 k_4}{x_1^{11}}\left[\frac{15}{16}\left(4+44\epsilon+228\epsilon^2+819\epsilon^3\right) + \frac{15}{4}\gamma\left(2+24\epsilon+141\epsilon^2+573\epsilon^3\right)\right. \\
& \qquad - \frac{45}{8}\gamma^2\left(8+104\epsilon+658\epsilon^2+2865\epsilon^3\right) - \frac{35}{4}\gamma^3\left(2+28\epsilon+195\epsilon^2+935\epsilon^3\right) \\
& \qquad \left. + \frac{35}{16}\gamma^4\left(28+420\epsilon+3120\epsilon^2+15895\epsilon^3\right)\right] ;
\end{aligned}$$

$$\begin{aligned}
F_0'' = \; & + \frac{3\mu^2}{x_1^4} - \frac{\mu^4 k_2}{x_1^8}\left[3\left(6+23\epsilon+57\epsilon^2+115\epsilon^3\right) + 6\gamma\left(8+45\epsilon+150\epsilon^2+385\epsilon^3\right) - 9\gamma^2\left(12+75\epsilon+275\epsilon^2+770\epsilon^3\right)\right] \\
& + \frac{\mu^6 k_2^2}{x_1^{12}}\left[\frac{3}{16}\left(188+696\epsilon+564\epsilon^2-4653\epsilon^3\right) + \frac{9}{8}\gamma\left(128+988\epsilon+4284\epsilon^2+13710\epsilon^3\right)\right. \\
& \qquad - \frac{9}{8}\gamma^2\left(340+2668\epsilon+11550\epsilon^2+36085\epsilon^3\right) - \frac{3}{4}\gamma^3\left(1064+11850\epsilon+72240\epsilon^2+318835\epsilon^3\right) \\
& \qquad \left. + \frac{45}{16}\gamma^4\left(532+6320\epsilon+40936\epsilon^2+191301\epsilon^3\right)\right] \\
& + \frac{\mu^6 k_4}{x_1^{12}}\left[\frac{45}{8}\left(6+72\epsilon+400\epsilon^2+1529\epsilon^3\right) + \frac{45}{2}\gamma\left(8+104\epsilon+658\epsilon^2+2865\epsilon^3\right)\right. \\
& \qquad - \frac{15}{4}\gamma^2\left(142+1988\epsilon+13440\epsilon^2+62215\epsilon^3\right) - \frac{35}{2}\gamma^3\left(28+420\epsilon+3120\epsilon^2+15895\epsilon^3\right) \\
& \qquad \left. + \frac{525}{8}\gamma^4\left(14+224\epsilon+1768\epsilon^2+9537\epsilon^3\right)\right] ;
\end{aligned}$$

$$Q_0' = -\frac{\mu^4 A_{22}}{x_1^7}\left[\frac{1}{48}\left(72 - 486\epsilon + 765\epsilon^2 - 595\epsilon^3\right) + \frac{3}{4}\gamma\left(6 - 29\epsilon + 17\epsilon^2 - 21\epsilon^3\right) + \frac{1}{48}\gamma^2\left(144 - 486\epsilon - 225\epsilon^2 - 737\epsilon^3\right)\right]$$

$$Q_0'' = +\frac{\mu^4 A_{22}}{x_1^8}\left[\frac{1}{48}\left(288 - 2844\epsilon + 6273\epsilon^2 - 5194\epsilon^3\right) + \frac{1}{24}\gamma\left(720 - 4212\epsilon + 3285\epsilon^2 - 3421\epsilon^3\right) + \frac{1}{16}\gamma^2\left(432 - 1620\epsilon - 825\epsilon^2 - 2948\epsilon^3\right)\right]$$

If only the secular and the first, most important, periodic terms of the disturbing function are retained, we have:

$$F = F_0 + Q_0 \cos 2y_1 . \tag{6}$$

The libration points and the points lying on the intersection of two branches of the separatrix are determined from the equations

$$\frac{\partial F}{\partial x_1} = \frac{\partial F_0}{\partial x_1} + \frac{\partial Q_0}{\partial x_1}\cos 2y_1$$

and

$$\frac{\partial F}{\partial y_1} = -2\,Q_0 \sin 2y_1$$

if the values of x_2 and x_3 are fixed. From Equation 6 we deduce that

$$\frac{\partial F_0}{\partial x_1} + \frac{\partial Q_0}{\partial x_1} = 0 \tag{7}$$

$$\text{for } y_1 = 0,\ \pi, \tag{8}$$

$$\frac{\partial F_0}{\partial x_1} - \frac{\partial Q_0}{\partial x_1} = 0 \tag{9}$$

$$\text{for } y_1 = \frac{\pi}{2},\ \frac{3\pi}{2}. \tag{10}$$

The values $y = \pi/2$, $3\pi/2$ reduce the disturbing function (Equation 5) to a minimum. Consequently, they correspond to the stability position and determine the libration points. The set $y_1 = 0, \pi$ gives the points on the separatrix and corresponds to unstable positions. For satellites moving in the equatorial plane the stable positions are on the equator's minor axis, and the unstable positions are on the major axis (Reference 2). It is convenient in the complete problem, as defined by Equation 5, to retain the development around the characteristic points defined by Equations 7 through 10, and to write the disturbing function in the form

$$F = R_0 + R_1 ,$$

where

$$R_0 = F_0 - Q_0 ,$$

$$R_1 = 2Q_0 \cos^2 y_1 + Q_1 \cos(2y_1 + 2y_2) + Q_2 \cos(2y_1 + 4y_2)$$

for the stable case, and

$$R_0 = F_0 + Q_0 , \tag{11}$$

$$R_1 = -2Q_0 \sin^2 y_1 + Q_1 \cos(2y_1 + 2y_2) + Q_2 \cos(2y_1 + 4y_2) \tag{12}$$

for the unstable case.

The canonical transformation (Equation 4) removes the time and the argument y_3 from the disturbing function. Consequently, this problem contains the energy integral

$$R_0 + R_1 = -C , \tag{13}$$

and the integral

$$x_3 = \sqrt{\mu a}\left(1 - \sqrt{1 - e^2}\cos i\right) = \alpha_3$$

is constant.

THE STABLE CASE

By substituting $x_1 = \partial S/\partial y_1$, and $x_2 = \partial S/\partial y_2$ into Equation 13, we transform it into a Hamilton-Jacobi partial differential equation. Letting

$$S = S_0 + S_1(y_1) + S_2(y_1 , y_2) + S_3(y_1 , y_2) \cdots ,$$

$$C = C_0 + C_1 + C_2 + C_3 + \cdots ,$$

and

$$S_0 = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 ,$$

where α_1 and α_2 can be considered as two constants of integration. We replace the partial differential Equation 13 by the system of Bohlin's equations (Reference 3). These may be deduced by developing the left-hand side of Equation 13 into a Taylor's series about α_1, α_2, α_3:

$$R_0 \left(\alpha_1, \alpha_2, \alpha_3\right) = - C_0 , \tag{14}$$

$$w_{01} \frac{\partial S_2}{\partial y_2} + \frac{1}{2} w_{20} \left(\frac{\partial S_1}{\partial y_1}\right)^2 + w_{10} \frac{\partial S_1}{\partial y_1} + 2Q_0 \cos^2 y_1 + Q_1 \cos\left(2y_1 + 2y_2\right) + Q_2 \cos\left(2y_1 + 4y_2\right) = 0 , \tag{15}$$

$$\begin{aligned} w_{01} \frac{\partial S_3}{\partial y_2} + \left(w_{10} + w_{20} \frac{\partial S_1}{\partial y_1}\right) \frac{\partial S_2}{\partial y_1} + \frac{\partial S_1}{\partial y_1} \frac{\partial S_2}{\partial y_2} w_{11} + \frac{1}{6} w_{30} \left(\frac{\partial S_1}{\partial y_1}\right)^3 & \\ + \left[2Q_0' \cos^2 y_1 + Q_1' \cos\left(2y_1 + 2y_2\right) + Q_2' \cos\left(2y_1 + 4y_2\right)\right] \frac{\partial S_1}{\partial y_1} &= - C_1 , \end{aligned} \tag{16}$$

where

$$w_{ij} = \frac{\partial^{i+j} R_0 \left(\alpha_1, \alpha_2, \alpha_3\right)}{\partial \alpha_1^{\,i} \quad \partial \alpha_2^{\,j}}$$

and

$$Q_i' = \frac{\partial Q_i \left(\alpha_1, \alpha_2, \alpha_3\right)}{\partial \alpha_1}$$

Equation 14 can be considered as a defining equation for C_0.

Imposing on our solution an additional condition – that no secular term with respect to y_2 is contained in S_1, S_2, S_3, ..., but only in S_0 – we deduce from Equations 14, 15, and 16 that

$$S_1 = \phi_1 \left(y_1\right) ,$$

$$S_2 = \phi_2 \left(y_1\right) - \frac{Q_1}{2w_{01}} \sin\left(2y_1 + 2y_2\right) - \frac{Q_2}{4w_{01}} \sin\left(2y_1 + 4y_2\right) ,$$

and also

$$\frac{1}{2} w_{20} \phi_1'^2 + w_{10} \phi_1' + 2Q_0 \cos^2 y_1 = 0 , \tag{17}$$

$$\phi_2' \left(w_{10} + w_{20} \phi_1'\right) + \frac{1}{6} w_{30} \phi_1'^3 + 2\phi_1' Q_0' \cos^2 y_1 = - C_1 . \tag{18}$$

From Equation 17, we have the standard first approximation

$$\phi_1' = - w + A , \tag{19}$$

where

$$w = \frac{w_{10}}{w_{20}} ,$$

and

$$A = \pm \sqrt{w^2 - \frac{4Q_0}{w_{20}} \cos^2 y_1} . \tag{20}$$

If

$$\frac{4Q_0}{w_{20} w^2} > 1 , \tag{21}$$

then $\cos y_1$ oscillates between the limits $-(w/2)\left(\sqrt{w_{20}/Q_0}\right), + (w/2)\left(\sqrt{w_{20}/Q_0}\right)$. This condition (Equation 21) can be written in the form:

$$(F_0' - Q_0')^2 - 4Q_0 (F_0'' - Q_0'') < 0 , \tag{22}$$

(where, in this case, the primes represent differentiation with respect to α_1), together with the condition

$$\cos^2 y_1 < \frac{(F_0' - Q_0')^2}{4Q_0 (F_0'' - Q_0'')}$$

which, for a certain moment of time, must be fulfilled for the motion to be stable. By eliminating ϕ_1' and $\cos^2 y_1$ from Equation 18 by means of Equations 19 and 20, we can write an expression for ϕ_2' :

$$\phi_2' = - \left(\frac{1}{6} \frac{w_{30}}{w_{20}} - \frac{1}{2} \frac{Q_0'}{Q_0}\right) A^2 + \frac{1}{2} \left(\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right) Aw - \frac{1}{2} \left(\frac{w_{30}}{w_{20}} + \frac{Q_0'}{Q_0}\right) w^2 - \frac{1}{A} \left[C_1 - \left(\frac{1}{6} \frac{w_{30}}{w_{20}} + \frac{1}{2} \frac{Q_0'}{Q_0}\right) w^3\right] .$$

The divisor A in the last term may become zero. Therefore, to remove a source of possible discontinuity, we set

$$C_1 = \left(\frac{1}{6}\frac{w_{30}}{w_{20}} + \frac{1}{2}\frac{Q_0'}{Q_0}\right)w^3 , \tag{23}$$

and ϕ_2' becomes a polynomial in A which, taking Equation 20 into account, can also be written as

$$\phi_2' = -\frac{2}{3}w^2\frac{w_{30}}{w_{20}} + \left(\frac{1}{3}\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right)\frac{Q_0}{w_{20}} + \left(\frac{1}{3}\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right)\frac{Q_0}{w_{20}}\cos 2y_1 + \frac{1}{2}\left(\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right)wA .$$

Setting

$$A^{(1)} = -w - \frac{2}{3}w\frac{w_{30}}{w_{20}} + \left(\frac{1}{3}\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right)\frac{Q_0}{w_{20}} , \tag{24}$$

$$A^{(2)} = \left[w + \frac{1}{2}\left(\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right)w^2\right]^2 , \tag{25}$$

$$A^{(3)} = \frac{4Q_0}{w_{20}}\left[1 + \frac{1}{2}\left(\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right)w\right]^2 > 0 , \tag{26}$$

$$A^{(4)} = \frac{Q_0}{w_{20}}\left(\frac{1}{6}\frac{w_{30}}{w_{20}} - \frac{1}{2}\frac{Q_0'}{Q_0}\right), \tag{27}$$

$$A^{(5)} = -\frac{Q_1}{2w_{01}} , \tag{28}$$

$$A^{(6)} = -\frac{Q_2}{4w_{01}} , \tag{29}$$

$$\frac{\partial A^{(i)}}{\partial \alpha_j} = A_j^{(i)} ,$$

$$\frac{\partial C}{\partial \alpha_j} = n_j ,$$

$$\frac{1}{2}\left(A_j^{(2)} - A^{(2)}\frac{A_j^{(3)}}{A^{(3)}}\right) = M_j , \tag{30}$$

we obtain the Hamiltonian function S with its integrals in the form:

$$S = \left(\alpha_1 + A^{(1)}\right) y_1 + \alpha_2 y_2 + \alpha_3 y_3 \pm \int \sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}\, dy_1$$

$$+ A^{(4)} \sin 2y_1 + A^{(5)} \sin\left(2y_1 + 2y_2\right) + A^{(6)} \sin\left(2y_1 + 4y_2\right) ;$$

and

$$x_1 = \frac{\partial S}{\partial y_1} = \left(\alpha_1 + A^{(1)}\right) \pm \sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}$$

$$+ 2A^{(4)} \cos 2y_1 + 2A^{(5)} \cos\left(2y_1 + 2y_2\right) + 2A^{(6)} \cos\left(2y_1 + 4y_2\right) , \tag{31}$$

$$x_2 = \frac{\partial S}{\partial y_2} = \alpha_2 + 2A^{(5)} \cos\left(2y_1 + 2y_2\right) + 4A^{(6)} \cos\left(2y_1 + 4y_2\right) ,$$

$$x_3 = \frac{\partial S}{\partial y_3} = \alpha_3 ; \tag{32}$$

$$\frac{\partial S}{\partial \alpha_1} = n_1 t + \beta_1 = \left(1 + A_1^{(1)}\right) y_1 \pm M_1 \int \frac{dy_1}{\sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}}$$

$$\pm \frac{1}{2} \frac{A_1^{(3)}}{A^{(3)}} \int \sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}\, dy_1$$

$$+ A_1^{(4)} \sin 2y_1 + A_1^{(5)} \sin\left(2y_1 + 2y_2\right) + A_1^{(6)} \sin\left(2y_1 + 4y_2\right) , \tag{33}$$

$$\frac{\partial S}{\partial \alpha_2} \quad n_2 t + \beta_2 = A_2^{(1)} y_1 + y_2 \pm M_2 \int \frac{dy_1}{\sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}}$$

$$\pm \frac{1}{2} \frac{A_2^{(3)}}{A^{(3)}} \int \sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}\, dy_1$$

$$+ A_2^{(4)} \sin 2y_1 + A_2^{(5)} \sin\left(2y_1 + 2y_2\right) + A_2^{(6)} \sin\left(2y_1 + 4y_2\right) , \tag{34}$$

$$\frac{\partial S}{\partial \alpha_3} = n_3 t + \beta_3 = A_3^{(1)} y_1 + y_3 \pm M_3 \int \frac{dy_1}{\sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}}$$

$$\pm \frac{1}{2} \frac{A_3^{(3)}}{A^{(3)}} \int \sqrt{A^{(2)} - A^{(3)} \cos^2 y_1}\, dy_1$$

$$+ A_3^{(4)} \sin 2y_1 + A_3^{(5)} \sin\left(2y_1 + 2y_2\right) + A_3^{(6)} \sin\left(2y_1 + 4y_2\right) . \tag{35}$$

In this representation the terms containing y_2 in the argument are of the first order with respect to the paramater k_2. In addition, these terms contain even powers of the eccentricity as a factor and consequently, will be small from the start; the computation of the second order term in k_2 with the argument y_2 shows that it can be neglected. All quantities can be considered as functions of $n_i\, t + \beta_i$, $(i = 1,2,3)$ but, of course, only y_2 and y_3 will have a secular term, since y_1 does not possess any such term.

It is not difficult to continue the process of computing S, if necessary. However, taking present day knowledge of the numerical values of geodetic parameters into consideration, it was found that even the development of Equations 31 through 35 proved to be accurate – from the practical point of view – overly accurate. In this solution R_1 was originally considered to be of the second order with respect to w_{10}. It must be pointed out that this classification is purely formalistic, and loses its significance after the development is completed. The important characteristic of the solution is that it can be developed into a series in w, with the coefficients depending upon α_1, α_2, α_3. The development is not made in powers of $\sqrt{k_2}$ as might be expected. This feature was observed initially by Izsak (Reference 3) in his solution of the critical inclination problem.

The method presented here does not introduce the small divisor A in the determination of ϕ_n which appears in the expression for S_n :

$$S_n = \phi_n(y_1) + \text{trigonometric terms in } y_1 \text{ and } y_2 .$$

Every ϕ_n' will be a polynomial in A, if the constant of energy C is decomposed properly, to remove the poles with respect to A. This can be easily shown by applying the "from n to n + 1 proof," since the equation for the determination of ϕ_n' has the form:

$$\phi_n' w_{20} A + P_n \left(\phi_1', \phi_2', \cdots, \phi_{n-1}'; \cos^2 y_1\right) = C_n - 1 ,$$

where P_n is a polynomial in $\phi_1', \phi_2', \cdots, \phi_{n-1}'$ $\cos^2 y_1$. The elimination of $\cos^2 y_1$, by means of Equation 17 and the proper determination of C_{n-1}, will lead to the representation of ϕ_n' in polynomial form, providing it has been shown that $\phi_1', \phi_2', \cdots, \phi_{n-1}'$ are polynomials in A. Eliminating higher powers of A in favor of $\cos^2 y_1$, we deduce that

$$\phi_n' = \alpha_n \left(\cos^2 y_1\right) + A\beta_n \left(\cos^2 y_1\right) ,$$

where α_n and β_n are polynomials in $\cos^2 y_1$ with polynomial coefficients in w. This result is similar to that obtained by Izsak (Reference 4) for the critical inclination problem. For the partial derivative of S_n with respect to y_2 we have

$$\frac{\partial S_n}{\partial y_2} = T_n^{(0)}(y_1, y_2) + A T_n^{(1)}(y_1, y_2) ,$$

where $T_n^{(0)}$ and $T_n^{(1)}$ are trigonometric polynomials in $2y_1$ and $2y_2$ with polynomial coefficients in w. When integrated, this has been found to be purely trigonometrical with respect to y_2.

In the stable case y_1 will have a long period oscillation about $\pi/2$ or $3\pi/2$. It is also of interest to know the approximate period of this deviation in longitude. From Equations 23 through 29, we have

$$A_1^{(1)} = -1 - \frac{1}{3}\frac{w_{30}}{w_{20}} w + O\left(w^2\right) , \tag{36}$$

$$A_1^{(2)} = 2w + \left[\frac{w_{30}}{w_{20}} - 3\frac{Q_0'}{Q_0}\right] w^2 + O\left(w^3\right) , \tag{37}$$

$$\frac{A_1^{(3)}}{A^{(3)}} = O\,(w) , \tag{38}$$

$$M_1 = w + \frac{1}{2}\left[\frac{w_{30}}{w_{20}} - 3\frac{Q_0'}{Q_0}\right] w^2 + O\left(w^3\right) , \tag{39}$$

$$\frac{M_1}{\sqrt{A^{(2)}}} = 1 - \frac{Q_0'}{Q_0} w . \tag{40}$$

Neglecting small long-period terms, we can write, from Equation 32, the expression for the period of libration T:

$$\begin{aligned} n_1 T = {} & 2\left(1 + A_1^{(1)}\right)\left[\operatorname{arc\,cos}\left(\frac{1}{k}\right) - \operatorname{arc\,cos}\left(-\frac{1}{k}\right)\right] \\ & + 2\frac{M_1}{\sqrt{A^{(2)}}}\int_{\cos^{-1}(-1/k)}^{\cos^{-1}(1/k)} \frac{dy_1}{\sqrt{1 + k^2\cos^2 y_1}} \\ & + \frac{A_1^{(3)}}{A^{(3)}}\sqrt{A^{(2)}}\int_{\cos^{-1}(-1/k)}^{\cos^{-1}(1/k)} \sqrt{1 - k^2\cos^2 y_1}\, dy_1 , \end{aligned}$$

where

$$k^2 = \frac{A^{(3)}}{A^{(2)}} = \frac{4Q_0}{w^2 w_{20}} .$$

Setting

$$k \cos y_1 = \sin u ,$$

we have

$$\int_{\cos^{-1}(-1/k)}^{\cos^{-1}(1/k)} \frac{dy_1}{\sqrt{1 - k^2 \cos^2 y_1}} = -\frac{2}{k} \int_0^{\pi/2} \frac{du}{\sqrt{1 - \frac{1}{k^2} \sin^2 u}} = -\frac{2}{k} K\left(\frac{1}{k}\right),$$

and

$$\int_{\cos^{-1}(-1/k)}^{\cos^{-1}(1/k)} \sqrt{1 - k^2 \cos^2 y_1}\, dy_1 = -\frac{2}{k} \int_0^{\pi/2} \frac{\cos^2 u\, du}{\sqrt{1 - \frac{1}{k^2} \sin^2 u}} = -\frac{2}{k} K\left(\frac{1}{k}\right) - 2k\left[E\left(\frac{1}{k}\right) - K\left(\frac{1}{k}\right)\right],$$

where $K(1/k)$ and $E(1/k)$ are the standard elliptic integrals of the first and second kinds having the modulus $1/k$. In terms of the new variables,

$$n_1 T = 2\left(1 + A_1^{(1)}\right)\left[2 \arccos\left(\frac{1}{k}\right) - \pi\right] - \frac{4}{k}\left[\frac{M_1}{\sqrt{A^{(2)}}} + \frac{1}{2}\frac{A_1^{(3)}}{A^{(3)}}\sqrt{A^{(2)}}\right] K\left(\frac{1}{k}\right)$$

$$- 2k\sqrt{A^{(2)}}\,\frac{A_1^{(3)}}{A^{(3)}}\left[E\left(\frac{1}{k}\right) - K\left(\frac{1}{k}\right)\right]. \qquad (41)$$

Substituting Equations 36 through 40 into Equation 41 yields:

$$n_1 T = -\frac{2}{3}\frac{w_{30}}{w_{20}} w\left[2\cos^{-1}\left(\frac{1}{k}\right) - \pi\right] - \frac{4}{k}\left(1 - \frac{Q_0'}{Q_0} w\right) K\left(\frac{1}{k}\right) + O\left(\frac{w^2}{k}\right).$$

Equations 14 and 23 give the expression for $C = C_0 + C_1$, from which

$$n_1 = \frac{\partial C}{\partial \alpha_1} = -w_{20} w\left[1 - \frac{1}{2w_{20}}\left(\frac{w_{30}}{w_{20}} + 3\frac{Q_0'}{Q_0}\right) w\right]$$

and

$$T = +\frac{2K\left(\frac{1}{k}\right)}{\sqrt{w_{20} Q_0}} + \frac{2}{3}\frac{w_{30}}{w_{20}^2}\left[2\cos^{-1}\left(\frac{1}{k}\right) - \pi\right] - \frac{4K\left(\frac{1}{k}\right)}{k\, w_{20}}\left[\left(1 - \frac{3}{2}\frac{1}{w_{20}}\right)\frac{Q_0'}{Q_0} - \frac{1}{2}\frac{w_{30}}{w_{20}^2}\right] + O\left(\frac{w}{k}\right). \qquad (42)$$

In order to find the period in the vicinity of the libration point, we expand Equation 42 in powers of $1/k$. Since w approaches zero as the libration becomes smaller,

$$\frac{1}{k} = +\frac{w}{2}\sqrt{\frac{w_{20}}{Q_0}} \to 0;$$

and for small values of $1/k$,

$$\cos^{-1}\left(\frac{1}{k}\right) = \frac{\pi}{2} - \frac{1}{k} - \frac{1}{6}\frac{1}{k^3} + \cdots ,$$

and

$$K\left(\frac{1}{k}\right) = \frac{\pi}{2}\left(1 + \frac{1}{4}\frac{1}{k^2} + \cdots\right) .$$

Consequently we deduce that

$$T = \frac{\pi}{\sqrt{Q_0 w_{20}}}\left\{1 - w\left[\frac{2}{3\pi}\frac{w_{30}}{w_{20}} + \frac{Q_0'}{Q_0} - \frac{1}{2w_{20}}\left(\frac{w_{30}}{w_{20}} + 3\frac{Q_0'}{Q_0}\right)\right]\right\} + O\left(\frac{1}{k^2}\right) .$$

UNSTABLE MOTION

If the condition (Equation 22) is not satisfied, the motion is unstable and the Hamiltonian function takes the form

$$F = R_0 + R_1 ,$$

with R_0 and R_1 defined by Equations 11 and 12. Performing operations similar to the previous ones, we find that

$$S = \left(\alpha_1 + B^{(1)}\right) y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \int \sqrt{B^{(2)} + B^{(3)} \sin^2 y_1}\, dy_1$$
$$+ B^{(4)} \sin 2y_1 + B^{(5)} \sin\left(2y_1 + 2y_2\right) + B^{(6)} \sin\left(2y_1 + 4y_2\right) ,$$

where

$$B^{(1)} = -w - \frac{2}{3} w^2 \frac{w_{30}}{w_{20}} - \left(\frac{1}{3}\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right)\frac{Q_0}{w_{20}} , \tag{43}$$

$$B^{(2)} = w^2\left[1 + \frac{1}{2}\left(\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right) w\right]^2 , \tag{44}$$

$$B^{(3)} = \frac{4Q_0}{w_{20}}\left[1 + \frac{1}{2}\left(\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0}\right) w\right]^2 > 0 , \tag{45}$$

$$B^{(4)} = \left(\frac{1}{6}\frac{w_{30}}{w_{20}} - \frac{1}{2}\frac{Q_0'}{Q_0}\right)\frac{Q_0}{w_{20}} ,$$

$$B^{(5)} = -\frac{Q_1}{2w_{01}},$$

and

$$B^{(6)} = -\frac{Q_2}{4w_{01}}.$$

Despite the similarity between some A's and B's, they are not identical since different values of R_0 are used in each case. The constant of energy has the same analytical form as in the stable case:

$$C = -R_0\left(\alpha_1, \alpha_2\right) + \left(\frac{1}{6}\frac{w_{30}}{w_{20}} + \frac{1}{2}\frac{Q_0'}{Q_0}\right)w^3. \tag{46}$$

Thus we have, as in the previous case,

$$x_1 = \alpha_1 + B^{(1)} + \sqrt{B^{(2)} + B^{(3)}\sin^2 y_1} + 2B^{(4)}\cos 2y_1 + 2B^{(5)}\cos\left(2y_1 + 2y_2\right) + 2B^{(6)}\cos\left(2y_1 + 4y_2\right),$$

$$x_2 = \alpha_2 + 2B^{(5)}\cos\left(2y_1 + 2y_2\right) + 4B^{(6)}\cos\left(2y_1 + 4y_2\right),$$

$$x_3 = \alpha_3,$$

and, putting

$$B_j^{(i)} = \frac{\partial B^{(i)}}{\partial \alpha_j},$$

$$\frac{\partial C}{\partial \alpha_i} = n_i,$$

$$N_j = \frac{1}{2}\left(B_j^{(2)} - B^{(2)}\frac{B_j^{(3)}}{B^{(3)}}\right),$$

we have, designating by the additive constants of integration β_1, β_2, β_3:

$$n_1 t + \beta_1 = \left(1 + B_1^{(1)}\right)y_1 + N_1\int\frac{dy_1}{\sqrt{B^{(2)} + B^{(3)}\sin^2 y_1}} + \frac{1}{2}\frac{B_1^{(3)}}{B^{(3)}}\int\sqrt{B^{(2)} + B^{(3)}\sin^2 y_1}\,dy_1$$

$$+ B_1^{(4)}\sin 2y_1 + B_1^{(5)}\sin\left(2y_1 + 2y_2\right) + B_1^{(6)}\sin\left(2y_1 + 4y_2\right); \tag{47}$$

$$n_2 t + \beta_2 = B_2^{(1)} y_1 + y_2 + N_2 \int \frac{dy_1}{\sqrt{B^{(2)} + B^{(3)} \sin^2 y_1}} + \frac{1}{2} \frac{B_2^{(3)}}{B^{(3)}} \int \sqrt{B^{(2)} + B^{(3)} \sin^2 y_1}\, dy_1$$

$$+ B_2^{(4)} \sin 2y_1 + B_2^{(5)} \sin (2y_1 + 2y_2) + B_2^{(6)} \sin (2y_1 + 4y_2) ;$$

$$n_3 t + \beta_3 = B_3^{(1)} y_1 + y_3 + N_3 \int \frac{dy_1}{\sqrt{B^{(2)} + B^{(3)} \sin^2 y_1}} + \frac{1}{2} \frac{B_3^{(3)}}{B^{(3)}} \int \sqrt{B^{(2)} + B^{(3)} \sin^2 y_1}\, dy_1$$

$$+ B_3^{(4)} \sin 2y_1 + B_3^{(5)} \sin (2y_1 + 2y_2) + B_3^{(6)} \sin (2y_1 + 4y_2) .$$

In the unstable case

$$B^{(2)} + B^{(3)} \sin^2 y_1 > 0 \tag{48}$$

and the square root of Equation 47 (and its reciprocal) may be developed into a Fourier series in $2y_1$; the argument y_1 will possess a real secular term, which is absent in the stable case. The coefficient of y_1 in the right side of Equation 46 is

$$P = 1 + B_1^{(1)} + \frac{2N_1}{\pi} \int_0^{\pi/2} \frac{dy_1}{\sqrt{B^{(2)} + B^{(3)} \sin^2 y_1}} + \frac{1}{\pi} \frac{B_1^{(3)}}{B^{(3)}} \int_0^{\pi/2} \sqrt{B^{(2)} + B^{(3)} \sin^2 y_1}\, dy_1 \tag{49}$$

Substituting in the integrand

$$y_1 = \frac{\pi}{2} - \phi ,$$

and

$$k^2 = \frac{B^{(3)}}{B^{(2)} + B^{(3)}} ,$$

we reduce the integrals to the normal form and Equation 49 becomes:

$$P = 1 + B_1^{(1)} + \frac{2KN_1}{\pi \sqrt{B^{(2)} + B^{(3)}}} + \frac{E}{\pi} \cdot \frac{B_1^{(3)}}{B^{(3)}} \sqrt{B^{(2)} + B^{(3)}} , \tag{50}$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

and

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi \cdot$$

Taking Equations 43, 44, and 45 into account, we further deduce that

$$B_1^{(1)} = -1 - \frac{1}{3} w \frac{w_{30}}{w_{20}} + O(w^2) , \tag{51}$$

$$B_1^{(2)} = 2w + \left(\frac{w_{30}}{w_{20}} - 3 \frac{Q_0'}{Q_0} \right) w^2 + O(w^3) , \tag{52}$$

$$\frac{B_1^{(3)}}{B^{(3)}} = O(w) , \tag{53}$$

and

$$N_1 = w + \frac{1}{2} \left(\frac{w_{30}}{w_{20}} - 3 \frac{Q_0'}{Q_0} \right) w^2 + O(w^3) \tag{54}$$

It follows from Equations 50 through 54, neglecting the terms of the second order in w, that:

$$P = -\frac{1}{3} w \frac{w_{30}}{w_{20}} + \frac{2k' K}{\pi} \left(1 - \frac{Q_0'}{Q_0} \right), \tag{55}$$

where k' is the complementary modulus, $k^2 + k'^2 = 1$, and

$$k' = \frac{w}{\sqrt{B^{(2)} + B^{(3)}}} \left[1 + \frac{1}{2} \left(\frac{w_{30}}{w_{20}} - \frac{Q_0'}{Q_0} \right) w \right] .$$

From Equation 46, neglecting terms of the third order, we have

$$n_1 = \frac{\partial C}{\partial \alpha_1} = -w w_{20} + \frac{1}{2} \left(\frac{w_{30}}{w_{20}} + 3 \frac{Q_0'}{Q_0} \right) w^2 . \tag{56}$$

Using Equations 54 and 55, we can compute the mean motion $\nu_1 = n_1/P$ of the argument y_1 with an accuracy up to the terms of the second order in w. The value of ν_1 denotes the speed with which the satellite will depart from its original position over the earth's surface during the course of time.

CONCLUSION

The theory of motion of a 24-hour satellite under the influence of the ellipticity of earth's equator has been developed using a resonance theory. The expressions for the elements of motion can be represented in the form of a series with respect to the parameter w, which would be closely associated with the mean motion of the main critical argument $(n - n')t + \Omega + \omega$ in a nonresonance case. Canonical elements and the Hamilton-Jacobi partial differential equations were used to solve the problem. This method of solution was chosen because of its flexibility with respect to the form of the integration constants which are adjusted so as to remove small divisors from the solution.

The conditions are established for stable and unstable types of motion; no severe restrictions are imposed on the values of inclination or eccentricity. The formulas are developed to the point where a numerical development can be easily accomplished for any particular case. Using this method, an extension of Hori's critical inclination theory (Reference 5) can be easily obtained. The method described herein can also be applied to a more general case. If the ratio of mean motions is $n/n' = 1/p$, and not $n/n' = 1$ as in our case, the theory given here can be easily extended by using the set of canonical variables:

$$
\begin{aligned}
x_1 &= \frac{\sqrt{\mu a}}{p}, & y_1 &= pl + \Omega + \omega - n't, \\
x_2 &= \sqrt{\mu a}\left(\frac{1}{p} - \sqrt{1-e^2}\right), & y_2 &= -\omega, \\
x_3 &= \sqrt{\mu a}\left(\frac{1}{p} - \sqrt{1-e^2}\cos i\right), & y_3 &= -\Omega + n't.
\end{aligned}
$$

Comparing this theory with observations can help to better determine the coefficient of ellipticity of the earth's equator.

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