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EARTH REFLECTED SOLAR RADIATION INCIDENT UPON AN ARBITRARILY ORIENTED SPINNING FLAT PLATE

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SUMMARY   18131

A general derivation is given for the earth reflected solar radiation input to a flat plate—a solar cell paddle, for example—which is spinning about an axis coincident with the axis of symmetry of the satellite to which it is affixed. The resulting equations are written for the general case so that arbitrary orientations of the spin axis with respect to the earth-satellite line and arbitrary orientations of the normal to the plate with respect to the spin axis can be treated.

No attempt is made to perform the resulting integrations because of the complexity of the equations; nor is there any attempt to delineate the integration limits for the general case. However, the equations governing these limits are given. The appendixes contain: the results, in graphical form, of two representative examples; the general computer program for the calculation is given in Fortran notation; and the results of a calculation of the distribution of albedo energy on the proposed Echo II satellite. The value of the mean solar constant used is 1.395×10^6 erg/cm^2-sec; the mean albedo of the earth is assumed to be 0.34; and the earth is assumed to be a diffuse reflector.

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INTRODUCTION

In a recent series of papers (References 1 through 5) an attempt has been made to derive and present in a clear, useful manner the methods and results by which some of the power inputs to an orbiting earth satellite can be determined (this is necessary before a thermal analysis can be undertaken). The intention has been to treat the problems by assuming models which would provide the most useful and accurate information for the design engineer, and which would be soluble within the scope of the limited mathematics used in this paper and throughout the previous work. The approach used in this work assumes that (1) any latitudinal fluctuations and variations in the surface radiation and reflection properties of the earth, including the effect of the atmosphere, can be overlooked; and (2) that the net effect on the satellite can be predicted, to within useful limits, by assuming an average, uniform distribution of these properties over the surface of the earth. The advantage of this type of treatment is that these mean properties enter into the problem simply as multiplicative factors and, hence, the results already obtained can be brought up to date (as new values of the mean parameters become available) by a simple multiplication. It is apparent that a detailed analysis of the spectral distribution, intensity, variation with altitude, variation with latitude, etc., of earth radiation and earth reflected solar radiation is certainly necessary for any study pertaining to the physics of the atmosphere and the radiation balance of the earth, from which more accurate values of these "mean" parameters can be determined.

The previous papers (Reference 1 through 5) have not considered the problems which allow for easy calculation (for example, the earth radiation to a sphere reduces to a trivial problem once the model for the earth has been chosen) nor do they contain approximations for satellite geometries which are so complicated that even an attempt at a description of the problem would be excessively complicated and tedious. The analyses have been limited to spheres and to flat plates, both stationary and spinning. It is hoped that all these results can be applied usefully to the bulk of satellites which do approximate spherical shapes or which are composed of a group of planes or flat plates. In addition, this analysis lends itself to the solution of the unshaded solar cell paddle problem.

The present paper extends the results of the previous work to the problem of determining the earth reflected solar radiation (the albedo) incident upon a spinning flat plate. This flat plate is, presumably, the best representation of a typical solar cell paddle commonly in use today.

The following assumptions are made:

(1) The earth can be represented by a uniform sphere whose radius is equal to its mean radius, 6367.5 km (Reference 6);

(2) The albedo of the earth is latitude and longitude independent and can best be represented by its mean value, 0.34 (Reference 7);

(3) The earth is a diffuse reflector of solar radiation;

(4) The solar constant can be replaced by its mean value, 1.395×10^6 erg/cm^2-sec (Reference 8);

(5) The axis of spin of the satellite is coincident with its axis of symmetry (Reference 4).

Save for these assumptions, the paper contains no further approximations. In fact, the paper is perfectly general, and in it an attempt has been made to treat every possible orientational configuration regardless of the geometrical complexity. Of course, this means that it is impossible to define all of the possibilities separately and list the limits of integration for each case. However, the rules for determining these limits are clearly listed just as they were written into the computer program.

In Appendixes F and G, there are presented in graphical form the results of two calculations. Appendix F treats what is hoped to be a suitable representative example of the case where the spin axis of the plate (satellite) is coincident with the radius vector from the satellite to the earth's center. Appendix G treats a representative example from the general case where the spin axis makes an arbitrary angle with the radius vector. In Appendix H, there is presented a general Fortran program for the computation. Since an elemental area of any satellite—whatever its configuration—can be considered a flat plate and its normal defined, the method developed in this paper lends itself to the determination of the albedo energy distribution on the satellite, albeit a tedious process for the general case. However, for a spherical satellite the problem is quite simple. Consequently, in Appendix I there are presented curves showing the distribution of albedo energy on the proposed Echo II satellite. This example is included to illustrate the use of the method for determining the energy distribution on unshielded portions of a satellite.

ANALYTICAL TREATMENT

The geometrical situation and the relevant parameters are depicted in Figure 1. The associated definitions are:

$\mathcal{S}$ = the mean solar constant;

$\mathbf{S}$ = the solar vector;

α = the mean albedo of the earth;

$d\Sigma$ = the element of terrestial surface area;

$\mathbf{r}$ = the vector between the plate and the earth's center;

$\boldsymbol{\rho}$ = the vector between the plate and $d\Sigma$;

$\mathbf{A}$ = the normal to the plate;

β = the angle between the negative of the solar vector and $d\Sigma$;

θ_s = the angle between the solar vector and $\mathbf{r}$;

θ = the colatitudinal coordinate of $d\Sigma$;

λ = the angle between $\mathbf{A}$ and $\mathbf{r}$;

η = the angle between $\mathbf{A}$ and $\boldsymbol{\rho}$;

ϕ = the azimuthal coordinate of $d\Sigma$;

r_o = the radial coordinate of $d\Sigma$, r_o is equal to 1 by definition;

ξ = the angle between $d\Sigma$ and $-\boldsymbol{\rho}$; and

σ = the angle between $\mathbf{r}$ and $\boldsymbol{\rho}$.

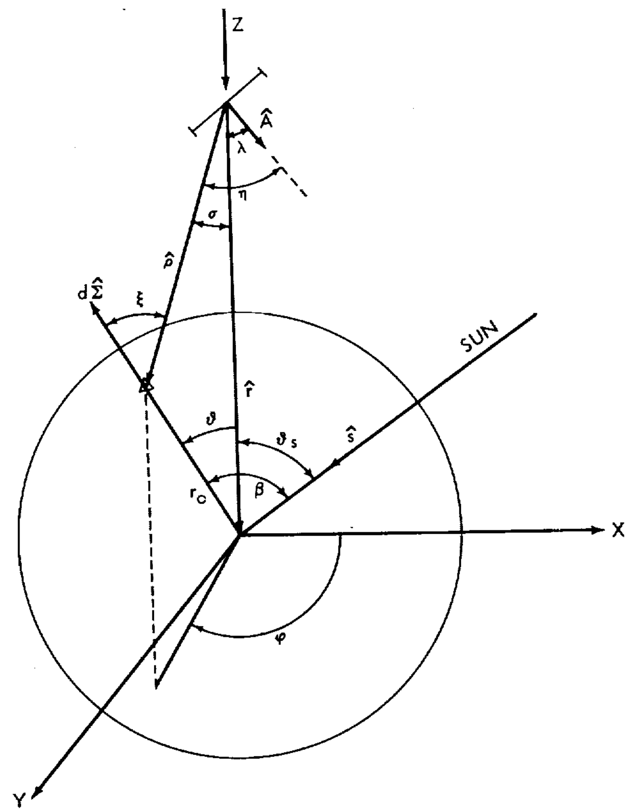

Figure 1—Geometry of the problem.

The general expression for the reflected solar radiation incident upon a flat plate of unit area is

$$P = \int_\Sigma \frac{S\alpha \cos\beta}{\pi} \cos\xi \, \frac{\cos\eta}{\rho^2} \, d\Sigma \, . \tag{1}$$

In this expression it is not necessary to refer to the vector properties of the parameters because the angular dependence is already indicated. In Equation 1, $S\alpha \cos\beta \, d\Sigma$ gives the amount of incident solar energy reflected by $d\Sigma$. This quantity multiplied by $(\cos\xi)/\pi$ gives the amount of energy reflected by $d\Sigma$ in the direction of the plate per unit solid angle. The factor $(\cos\eta)/\rho^2$ gives the solid angle subtended at $d\Sigma$ by the plate of unit area; and $d\Sigma$ is the relevant terrestrial surface area over which the function is integrated.

As is shown in Reference 2 and Appendix A, Equation 1 becomes

$$P = \frac{S\alpha}{\pi} \int_\theta \int_\phi \frac{(r\cos\theta - 1)\left(\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos\phi\right)\cos\eta\sin\theta \, d\theta \, d\phi}{\left(r^2 + 1 - 2r\cos\theta\right)^{3/2}} , \tag{2}$$

where r is in mean earth radii. A method for determining all of the relevant geometrical quantities using simple vector analysis is presented in the Appendixes. In Appendix A the derivation for $\cos\xi$ is given.

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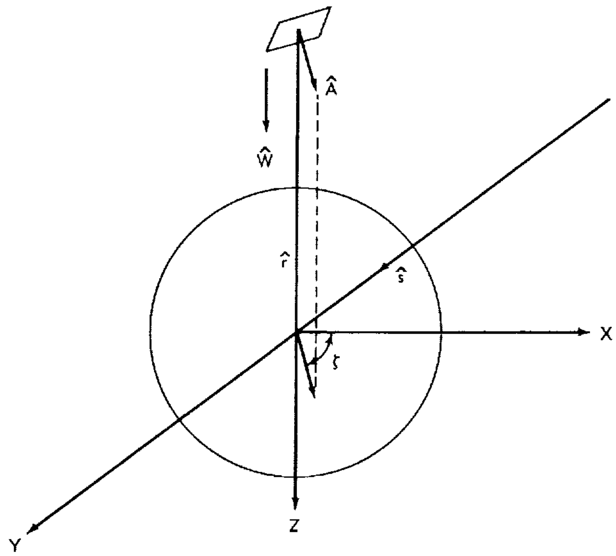

Figure 2—Definition of the angle ζ.

The expression for $\cos\eta$ in Equation 2 has not yet been determined. Before this can be done another angle, ζ, must be introduced, which will help define the position of **A** more precisely (see Figure 2). It is obvious that even though the angle λ remains constant, the value of the angle η is not invariant for rotation of **A** about **r**. The angle ζ introduced above will define this angle of rotation of **A** about **r**. In fact, the case mentioned here is just a special case of a more general rotation, or spin, about an arbitrary axis, which shall be considered. Clearly, we must define a system reference from which all quantities will be measured. The obvious one is the combination of the solar vector S, and the plane of S and **r**, that is, the S**r** plane. The Z axis is chosen to lie along **r** and the X axis lies perpendicular to **r** in the S**r** plane. The azimuthal angle ϕ, as well as ζ, is measured from the X axis in the XY plane. The angle ζ is taken equal to zero when **A** lies in the positive XZ plane, as shown in Figure 1. Following the right-hand rule, the motion is then in the positive Y direction.

The first and simplest case we shall consider, then, is the situation represented by the plate spinning about an axis coincident with the vector **r**. This axis of spin will be denoted henceforth by ω. We shall then extend this to the most general case, where the spin axis makes an arbitrary angle with the vector **r**.

Part I: The Spin Axis ω is Coincident with r

In this section the case for which the spin axis ω is coincident with **r** is considered.

From Appendix B we have

$$\cos\eta = \frac{\sin\lambda\sin\theta\,(\sin\zeta\sin\phi + \cos\zeta\cos\phi)}{\left(r^2+1-2r\cos\theta\right)^{1/2}} + \frac{(r-\cos\theta)\cos\lambda}{\left(r^2+1-2r\cos\theta\right)^{1/2}} \,. \tag{3}$$

Inserting Equation 3 into Equation 2 we have

$$P = \frac{S\alpha}{\pi}\int_\theta\int_\phi \frac{(r\cos\theta-1)\left(\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos\phi\right)}{\left(r^2+1-2r\cos\theta\right)^2}\Big[(r-\cos\theta)\cos\lambda + \sin\lambda\sin\theta\,(\sin\zeta\sin\phi+\cos\zeta\cos\phi)\Big]\sin\theta\,d\theta\,d\phi \,. \tag{4}$$

We now consider Equation 1 from which some qualitative statements can be made concerning the limits of the θ and ϕ integration.

The upper limit θ_m of the θ integration is determined from the value of ξ for which the terrestrial surface element $d\Sigma$ is no longer visible from the plate, that is, $\xi = \pi/2$. Then, $\theta_m = \cos^{-1}(1/r)$. Because the upper limit of the ϕ integration is determined by the fact that the side of the plate in question no longer receives any reflected solar radiation from that particular element $d\Sigma$, one of whose coordinates is ϕ, the upper limit ϕ_m of the ϕ integration can be determined in two ways.

The two quantities which determine ϕ_m are the source function $S a \cos \beta \, d\Sigma$ and the solid angle factor $(\cos \eta)/\rho^2$. In the first instance the source function becomes 0 when $\beta = (\pi/2)$. Then $\cos\beta = 0$, from which

$$\phi_m = \cos^{-1}\left(-\cot\theta \cot\theta_s\right) . \tag{5}$$

For $\beta > (\pi/2)$, the source function, of course, remains 0. For many values of θ and θ_s the argument of the $\cos^{-1}$ in Equation 5 is less than -1, for which the upper limit of the ϕ integration is π (if we take advantage of the symmetry and multiply by 2). For other values of θ and θ_s the argument will be greater than $+1$, for which the source function is always 0.

In the second case where the upper limit of the ϕ integration is determined by the solid angle dependence we see that the incident energy from any element $d\Sigma$ is 0 when $\eta \geq \pi/2$. The geometrical representation for this case is shown very clearly in Reference 1. A symmetry condition cannot be used here because the source function is symmetric only about the **rS** plane and this symmetry condition is applicable only to the case where $\zeta = 0$. Hence, we must consider the situation presented for a given value of ζ as the value of ϕ is varied. For many values of the parameters, the value of $\cos\eta$ will never become 0 for any value of ϕ. However, for other values of the parameters, the side of the plate in question ceases to be visible from the elements of area $d\Sigma$ (denoted by some specific values of the parameter ϕ) and $\eta \geq \pi/2$. Then, for higher values of ϕ the plate will once again become visible from $d\Sigma$. From Reference 1 it is clear that the symmetry plane for this situation is the **rA** plane. Therefore, we consider the biquadrant $\zeta \leq \phi \leq (\zeta + \pi)$. If the value of $\cos\eta$ because 0 at all, it will do so somewhere in this range of ϕ. Because of the symmetry with respect to the **rA** plane, the value of ϕ somewhere in this range for which $\cos\eta = 0$, call it ϕ', is mirrored in the plane. In other words, the angle ϕ'' at which the plate again becomes visible in the range $\zeta + \pi \leq \phi \leq \zeta + 2\pi$ is given by $\zeta + \pi - \phi' = \phi'' - (\zeta + \pi)$ or $\phi'' = 2(\zeta + \pi) - \phi'$. In order to determine ϕ' we equate Equation 3 to 0 from which we obtain

$$-\frac{(r - \cos\theta)\cos\lambda}{\sin\lambda \sin\theta} = \sin\zeta \sin\phi + \cos\zeta \cos\phi .$$

Transferring the second term on the right-hand side to the left-hand side and squaring gives

$$\cos^2\phi + \frac{2(r - \cos\theta)\cos\lambda \cos\zeta}{\sin\lambda \sin\theta}\cos\phi + \frac{(r - \cos\theta)^2 \cos^2\lambda}{\sin^2\lambda \sin^2\theta} - \sin^2\zeta = 0 . \tag{6}$$

The solution of Equation 6 for ϕ will yield, when applicable, two solutions, one each in the ranges $\zeta \leq \phi \leq \zeta + \pi$ and $\zeta + \pi \leq \phi \leq \zeta + 2\pi$ symmetric about $\phi = \zeta + \pi$. The solution of Equation 6 is

$$\cos\phi = \frac{-(r - \cos\theta)\cos\lambda\cos\zeta}{\sin\lambda\sin\theta} \pm \sin\zeta\sqrt{1 - \frac{(r-\cos\theta)^2\cos^2\lambda}{\sin^2\lambda\sin^2\theta}}\,, \tag{7}$$

where the double roots are given by the plus and minus signs. Obviously, when ζ is equal to 0 or π, Equation 7 yields only one result, because of the multivaluedness of the cosine function. This follows quite clearly from the symmetry condition. For these cases, the roots will always lie in the second and third or first and fourth quadrants, with the limiting case $\phi = \pi/2$ and $3\pi/2$; one root will never be found in the first quadrant and the other in third, for example.

If the absolute value of the argument in Equation 7 is greater than 1, then the roots do not exist and either the entire spherical cap of the earth lying within the cone defined by θ is visible from the plate for all ζ or it is never visible. The former case occurs when

$$0 \leq \lambda < \sin^{-1}\frac{(r - \cos\theta)}{\left(r^2 + 1 - 2r\cos\theta\right)^{1/2}}$$

(first quadrant), and the second when

$$\lambda \geq \sin^{-1}\frac{(r - \cos\theta)}{\left(r^2 + 1 - 2r\cos\theta\right)^{1/2}}$$

(second quadrant). An even more useful range of the parameter λ is $0 \leq \lambda \leq \theta_m$ for which the entire spherical cap of the earth (that portion lying within the tangent cone delineated by θ_m) is visible. For this case the problem is greatly simplified.

Until this point we have discussed only the determination of the upper limit of the ϕ integration by using the conditions considered above. However, from the considerations of Equation 7 we have actually obtained both the lower and upper limit. For those cases where $\lambda > \theta_m$, Equation 7 is applicable and the limits are given by the two roots already stated. This does not mean that the region of integration contains the origin necessarily. For example, considering the situation presented by $\zeta = \pi$ and $\lambda = \pi/2$, we see that the roots given by Equation 7 are $\phi = \pi/2$ and $3\pi/2$. However, the geometry of the situation clearly shows that the range of integration is from $\pi/2$ to $3\pi/2$. Therefore, the criterion which determines the way of proceeding from limit to limit is that the range must always contain the angle ζ.

Since Equations 5 and 7 must both be consulted in order to determine the relevant range of the variable ϕ, it is not practical to attempt to write down Equation 4 as a sum of terms each with the proper limits of integration clearly shown. A complicating feature is that the limit on the θ integration is dependent upon the limits of the ϕ integration; that is, if for a particular case the upper limit of the ϕ integration is equal to π for certain values of θ where $0 \leq \theta \leq \theta' < \theta_m$ and is given by Equation 5 for $\theta' < \theta \leq \theta_m$, then the appropriate range of the variable θ must be coupled with corresponding values of the limits of the ϕ integration.

For any given value of the parameter ζ the computer program is written so that Equations 5 and 7 are solved for their respective ranges of the variable ϕ for which the input to the plate is not 0. The overlapping portions of the regions determined in the above manner, then, represent the cogent ranges of ϕ over which the function in Equation 4 must be integrated.

Thus far, all discussions of Equation 4 and the limits of integration of θ and ϕ have been based on the assumption that the value of ζ is fixed. Since this is not in general the case, it is necessary to average the results of Equation 4 over a spin period. This problem was discussed extensively in Reference 4 and the relevant expressions for various ranges of the parameter ζ were written. In Equation 7, the values of the range of the ϕ integration are clearly seen to be dependent upon ζ as is the function itself given by Equation 4. The average over a spin period is given by

$$\langle P \rangle = \frac{1}{2\pi} \int_0^{2\pi} P(\zeta)\, d\zeta \, . \tag{8}$$

Clearly, the integrals of Equation 4 could be given explicitly, after much labor, as a function of the parameters and as a function of the upper limits of ϕ. However, it is doubtful whether Equation 8 could be integrated explicitly except for a few special cases. Consequently, Equation 8 is integrated by the computer and the results thereby obtained are the spin averages. Of course, in order to perform the integration over ζ the integration over θ and ϕ for every ζ ($\Delta\zeta$ apart) must be performed. If in a specific application results should be required for any of these intermediate values of ζ, a very simple modification of the program will suffice. These, however, do not constitute by themselves spin averages.

Thus far, the only case considered has been that in which the spin axis is coincident with **r**. Variation of the angle λ has been sufficient to describe all the degrees of freedom here allowed; that is, the variation of the angle between the normal to the plate and the spin axis.

Part II: The Spin Axis ω Makes an Arbitrary Angle χ with r

In this section the case for which the spin axis ω makes an angle χ with **r** is considered. Figure 3 shows the geometry of this new situation. We introduce a new angle ϵ which defines the angle between the normal to the plate **A** and the axis of symmetry of the satellite ω (the spin axis for our problem). The quantity ϵ is generally fixed for a given satellite. In addition, we must introduce another angle κ, the azimuthal angle of ω from the **rS** plane (an angle similar to ϕ).

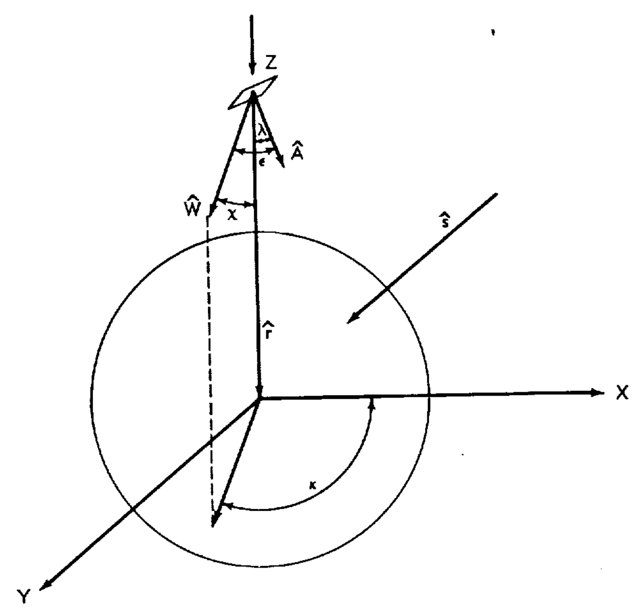

Figure 3—Definitions of the parameters defining the general orientation of the flat plate.

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For this case it becomes necessary to take the zero point of ζ as the point when $\mathbf{A}$ lies in the $r\omega$ plane and the motion is in the direction of the X′ axis (see Appendix C, Figure C5). From Appendix C we have

$$\cos\eta = \frac{(r-\cos\theta)}{\left(r^2+1-2r\cos\theta\right)^{1/2}}\left(\cos\chi\cos\epsilon - \sin\chi\sin\epsilon\cos\zeta\right) + \frac{\sin\theta}{\left(r^2+1-2r\cos\theta\right)^{1/2}}\Big[\cos\phi\cos\kappa\cos\epsilon\sin\chi$$

$$+ \sin\phi\sin\kappa\cos\epsilon\sin\chi + \cos\phi\cos\kappa\sin\epsilon\cos\chi\cos\zeta + \sin\phi\sin\kappa\sin\epsilon\cos\chi\cos\zeta$$

$$+ \sin\phi\cos\kappa\sin\epsilon\sin\zeta - \cos\phi\sin\kappa\sin\epsilon\sin\zeta\Big] . \qquad (9)$$

Equation 9 substituted into Equation 2 now gives the albedo input.

To determine the range of the ϕ integration, equate Equation 9 to 0 making the following substitutions for convenience:

$$F = \frac{(r-\cos\theta)}{\sin\theta}\left(\cos\chi\cos\epsilon - \sin\chi\sin\epsilon\cos\zeta\right) ,$$

$$G = \left(\cos\kappa\cos\epsilon\sin\chi + \cos\kappa\sin\epsilon\cos\chi\cos\zeta - \sin\kappa\sin\epsilon\sin\zeta\right) ,$$

$$H = \left(\sin\kappa\cos\epsilon\sin\chi + \sin\kappa\sin\epsilon\cos\chi\cos\zeta + \cos\kappa\sin\epsilon\sin\zeta\right) .$$

Then,

$$-F = G\cos\phi + H\sin\phi . \qquad (10)$$

Transferring the first term on the right-hand side to the left-hand side and squaring we have

$$F^2 + 2FG\cos\phi + G^2\cos^2\phi = H^2\sin^2\phi = H^2\left(1-\cos^2\phi\right) ,$$

from which

$$\left(G^2+H^2\right)\cos^2\phi + 2FG\cos\phi + F^2 - H^2 = 0 . \qquad (11)$$

Solving Equation 11 for $\cos\phi$ we have

$$\cos\phi = \frac{-FG \pm H\sqrt{G^2+H^2-F^2}}{\left(G^2+H^2\right)} . \qquad (12)$$

When $\kappa = 0$ and $\zeta = 0$, the problem is symmetric in ϕ as before and $H = 0$. Therefore, it is necessary to remember that the solution of Equation 12 for these conditions includes two roots, one each in the first and fourth quadrants or one each in the second and third quadrants. The situation is much more complicated for the general case because there the angle η is symmetric about the $r\mathbf{A}$ plane, which is not, in general, coincident with the rS plane which defines the angle ϕ. If, for example, we consider the case presented by $\kappa = \pi/2$ and $\zeta = 0$, or π (where the **rA** plane makes an angle κ with the **rS** plane), the solution of Equation 12 reduces to

$$\cos\phi = \pm \frac{1}{H}\sqrt{H^2 - F^2} . \tag{13}$$

Now, we know that if $\eta \geq \pi/2$ at all, one root ϕ' say, of Equation 13 must lie in the range $0 \leq (\phi - \pi/2) \leq \pi$ and from this we obtain $\pi/2 \leq \phi \leq 3\pi/2$ (for the general case when $\zeta = 0$, or π and $\kappa \leq \phi \leq \pi + \kappa$); the other corresponding value ϕ'' is given by $(3\pi/2 - \phi') \equiv (\phi'' - 3\pi/2)$.

Equation 13 has two roots, one given by the minus sign and the other given by the plus sign. From the geometry we know that one root must lie in the range $\pi/2 \leq \phi \leq 3\pi/2$, a solution corresponding to the minus sign. However, $\cos\phi < 0$ for both quadrants given in the range $\pi/2 \leq \phi \leq 3\pi/2$, so another criterion is needed by which the computer can select the proper value of ϕ in such cases. Since the only ϕ values of interest here are those in the range for which $\eta \leq \pi/2$, the value of $\cos\eta$ given by Equation 9 must be positive in all such cases. If it is desired to determine the range of ϕ for any particular values of the set of parameters, and if such a case as the above should occur, the only course open is to calculate the value of $\cos\eta$ for both possible values of ϕ and thereby directly determine the one corresponding to $\eta = \pi/2$. If this value of ϕ is the larger value, the smaller ϕ value will automatically meet all criterion. However, a slightly different approach can be used when the computer program is written. We must integrate over the relevant range of ϕ for a given value of the set of parameters, κ, χ, ϵ, and ζ (whether or not we ultimately want the average over all values of ζ). Since the computer does this in arithmetic steps, changing the value of ϕ by $\Delta\phi$ each time, it may as well start at the value of ϕ representing the symmetry of the given condition. The plane of symmetry for this part of the problem is the **rA** plane. If we can determine the value of ϕ for this case, the computer can be programmed to start here and go in the plus and minus directions relative to ϕ, and continue the computation until $\cos\eta$ (which must be calculated at each step anyway) $= 0$. If $\cos\eta$ does not become 0 at all, the integration is terminated at $\pm\pi$ radians from the starting point. If at the starting point $\eta \geq \pi/2$, no further consideration is necessary. Proceeding thus, we need only correlate with the apposite range of ϕ dictated by the source function which is symmetric about the **rS** plane. This restriction is given by Equation 5. The starting point ϕ_0 of ϕ as developed in Appendix D is given by $\phi_0 = (\kappa + \gamma)$, for $0 \leq \zeta \leq \pi$ and $\phi_0 = (\kappa - \gamma)$ for $\pi \leq \zeta \leq 2\pi$, where

$$\cos\gamma = \frac{\cos\epsilon \sin\chi + \sin\epsilon \cos\chi \cos\zeta}{\left[1 - (\cos\chi \cos\epsilon - \sin\epsilon \sin\chi \cos\zeta)^2\right]^{1/2}} , \tag{14}$$

and where γ is the angle between the projection of **A** and ω onto the XY plane. It will be helpful to remember that $\gamma \leq \epsilon$, the equal sign holding only when $\chi = \pi/2$ and $\zeta = 3\pi/2$ or $\pi/2$. The appropriate sign from the square root can then be readily determined. When $\chi = 0$ the equation has no meaning.

RECAPITULATION

The development is now complete for all the equations necessary to determine the albedo input to a spinning flat plate in the most general orientation, under the condition, of course, that the plate is not shielded at any time by other members of the satellite. However, the equation for this input has been developed for a "stationary" satellite, that is, a satellite at a fixed altitude and whose satellite-earth line makes an angle θ_s with the solar vector. Appendix E gives the development for expressions for $\mathbf{r}$ and θ_s in terms of general orbital parameters so that transformations may be made with facility from one set of parameters to the other.

The equation giving the albedo input is

$$P = \frac{S\alpha}{\pi} \int_\theta \int_\phi \frac{(r\cos\theta - 1)\left(\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos\phi\right)\cos\eta\sin\theta\, d\theta\, d\phi}{\left(r^2 + 1 - 2r\cos\theta\right)^{3/2}} . \tag{2}$$

The value of $\cos\eta$ is given by:

$$\cos\eta = \frac{\sin\lambda\sin\theta\,(\sin\zeta\sin\phi + \cos\zeta\cos\phi)}{\left(r^2 + 1 - 2r\cos\theta\right)^{1/2}} + \frac{(r - \cos\theta)\cos\lambda}{\left(r^2 + 1 - 2r\cos\theta\right)^{1/2}} ; \tag{3}$$

when the spin axis $\boldsymbol{\omega}$ is coincident with $\mathbf{r}$ (Part I) and

$$\begin{aligned}\cos\eta = {} & \frac{(r - \cos\theta)}{\left(r^2 + 1 - 2r\cos\theta\right)^{1/2}}\left(\cos\chi\cos\epsilon - \sin\chi\sin\epsilon\cos\zeta\right) + \frac{\sin\theta}{\left(r^2 + 1 - 2r\cos\theta\right)^{1/2}}\big[\cos\phi\cos\kappa\cos\epsilon\sin\chi \\ & + \sin\phi\sin\kappa\cos\epsilon\sin\chi + \cos\phi\cos\kappa\sin\epsilon\cos\chi\cos\zeta + \sin\phi\sin\kappa\sin\epsilon\cos\chi\cos\zeta \\ & + \sin\phi\cos\kappa\sin\epsilon\sin\zeta - \cos\phi\sin\kappa\sin\epsilon\sin\zeta .\end{aligned} \tag{9}$$

when the spin axis $\boldsymbol{\omega}$ makes an arbitrary angle χ with $\mathbf{r}$ (Part II). The range of the θ integration is $0 \le \theta \le \theta_m$ where $\theta_m = \cos^{-1}(1/r)$. The limits of the ϕ integration are given by those values of ϕ which simultaneously satisfy two conditions. For the case considered in Part I ($\boldsymbol{\omega}$ coincident with $\mathbf{r}$) the two conditions given by (a) and (b):

(a) $$0 \le \phi \le \phi_m , \text{ where } \phi_m = \cos^{-1}\left(-\cot\theta\cot\theta_s\right) \tag{5}$$

$$\left(2\pi - \phi_m\right) \le \phi \le 2\pi \qquad \text{note: } \phi_m \le \pi ,$$

and

(b) $\phi_1 \le \phi \le \phi_2$ where ϕ_1 and ϕ_2 are given by the solution of the equation

$$\cos\phi = \frac{-(r - \cos\theta)\cos\lambda\cos\zeta}{\sin\lambda\sin\theta} \pm \sin\zeta\sqrt{1 - \frac{(r - \cos\theta)^2\cos^2\lambda}{\sin^2\lambda\sin^2\theta}} , \tag{7}$$

with the condition that $\phi_1 \le \zeta \le \phi_2$, where we mean only to imply that the angle ζ lies between the limiting values of ϕ, thereby excluding the path from ϕ_1 to ϕ_2 that does not include the angle ζ. For the case considered in Part II (ω makes an angle χ with $\mathbf{r}$) the two conditions given by (c) and (d):

(c) $$0 \leq \phi < \phi_m, \text{ where } \phi_m = \cos^{-1}\left(-\cos\theta \cot\theta_s\right)$$

$$\left(2\pi - \phi_m\right) \leq \phi \leq 2\pi \qquad \text{note: } \phi_m \leq \pi \ ,$$

and

(d) $\phi_1' \leq \phi \leq \phi_2'$ where the roots ϕ_1' and ϕ_2' are given by the solutions to

$$\cos\phi = \frac{-\,FG \pm H\sqrt{G^2 + H^2 - F^2}}{\left(G^2 + H^2\right)}, \tag{12}$$

where

$$F = \frac{r - \cos\theta}{\sin\theta}\left(\cos\chi\cos\epsilon - \sin\chi\sin\epsilon\cos\zeta\right) ,$$

$$G = \left(\cos\kappa\cos\epsilon\sin\chi + \cos\kappa\sin\epsilon\cos\chi\cos\zeta - \sin\kappa\sin\epsilon\sin\zeta\right) ,$$

$$H = \left(\sin\kappa\cos\epsilon\sin\chi + \sin\kappa\sin\epsilon\cos\chi\cos\zeta + \cos\kappa\sin\epsilon\sin\zeta\right) ,$$

with the condition that the path from ϕ_1' to ϕ_2' must contain the angle $(\kappa + \gamma)$, for $0 \leq \zeta \leq \pi$, and $(\kappa - \gamma)$ for $\pi \leq \zeta \leq 2\pi$, where

$$\cos\gamma = \frac{\cos\epsilon\sin\chi + \sin\epsilon\cos\chi\cos\zeta}{\left[1 - \left(\cos\chi\cos\epsilon - \sin\epsilon\sin\chi\cos\zeta\right)^2\right]^{1/2}} . \tag{14}$$

In fact the angles $(\kappa + \gamma)$ or $(\kappa - \gamma)$ —depending upon the value of ζ in question—lie exactly midway between the angles ϕ_1' and ϕ_2'.

This equation has no meaning when $\chi = 0$.

In addition, as is pointed out in Appendix C, we need only consider the following ranges of the parameters κ, χ, and ϵ for the general case: $0 \leq \chi \leq \pi$, $0 \leq \epsilon \leq \pi/2$, and $0 \leq \kappa \leq \pi$. For any value of $\epsilon > (\pi/2)$ the same physical picture is obtained if χ is replaced by $(\pi - \chi)$, ϵ by $(\pi - \epsilon)$, and κ by $(\pi + \kappa)$. Any value of $\kappa > \pi$ can be replaced by its equivalent value $(2\pi - \kappa)$. Therefore, the incident power (averaged over ζ) obtained, for example, for $\chi = 150°$, $\epsilon = 110°$ and $\kappa = 30°$, is identical to the result obtained for $\chi = 30°$, $\epsilon = 70°$ and $\kappa = 150°$.

CONCLUDING REMARKS

It should be quite clear from the foregoing that the equation for the reflected solar power input to the spinning plate—even though it has not been solved explicitly—is an exact expression only for the geometric aspects of the problem. The assumption that the earth is a perfectly spherical diffuse reflector is necessary if the equation is not to be a much more complicated expression than it already is. However, these assumptions are not as serious (the error introduced by them is probably negligible in a first approximation even if enough were known to treat them analytically) as the assumption that the reflectivity is latitude and longitude independent. Undoubtedly there is a variation in this parameter with position on the earth and also a variation with time. If this θ and ϕ dependence were known in some explicit form it could easily be introduced into the equations. However, since information of this kind is not now available, the use of a hemispherical average would seem justified. It is also obvious that the figure of 0.34 used for the magnitude of the albedo, being a yearly hemispherical average* is not correct for many atmospheric conditions, in particular extreme cloud cover. Assuming that these conditions can always be represented reasonably well by an average value it still follows that the albedo might increase or decrease at any given time by a factor of 2 or more. However, by taking the mean value we introduce this parameter as a multiplicative factor. Consequently, the results of this paper can easily be modified at any time.

Appendix F and G contain in graphical form the results of two example calculations; Appendix H contains a Fortran program for the general problem; and Appendix I contains curves showing the distribution of albedo energy on the proposed Echo II satellite, thereby illustrating the utility of a flat plate analysis for the purpose of determining energy distributions.

ACKNOWLEDGMENT

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Appendix A

The Calculation of the Angles β and ξ

From Figure 1 we have the two angles β and ξ to calculate. The angle β is the angle between $d\Sigma$ and $-S$. The coordinates in the X, Y, Z frame of reference are:

$$\left.\begin{aligned} -S_X &= \delta \sin\theta_s & d\Sigma_X &= d\Sigma \sin\theta \cos\phi \,, \\ -S_Y &= 0 & d\Sigma_Y &= d\Sigma \sin\theta \sin\phi \,, \\ -S_Z &= -\delta \cos\theta_s & d\Sigma_Z &= -d\Sigma \cos\theta \,. \end{aligned}\right\} \tag{A1}$$

Therefore,

$$\cos\beta = \cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos\phi \,. \tag{A2}$$

The angle ξ is the angle between $d\Sigma$ and $-\rho$. We need only determine the coordinates of $-\rho$ in the X, Y, Z frame which are,

$$\left.\begin{aligned} -\rho_X &= -\rho\sin\sigma\cos\phi \,, \\ -\rho_Y &= -\rho\sin\sigma\sin\phi \,, \\ -\rho_Z &= -\rho\cos\sigma \,. \end{aligned}\right\} \tag{A3}$$

Therefore,

$$\cos\xi = \cos\sigma\cos\theta - \sin\sigma\sin\theta \,. \tag{A4}$$

In terms of the angle of integration θ the expression for $\cos\xi$ becomes,

$$\cos\xi = \frac{r\cos\theta - 1}{\left(r^2 + 1 - 2r\cos\theta\right)^{1/2}} \,. \tag{A5}$$

Appendix B

The Calculation of cos η when ω is Coincident with r

To determine the value of $\cos\eta$ for the case when $\boldsymbol{\omega}$ is coincident with $\mathbf{r}$ consider Figure B1. The angle σ is the angle between $\mathbf{r}$ and $\boldsymbol{\rho}$. The X axis lies in the Sr plane normal to $\mathbf{r}$, or the Z axis. The spin $\boldsymbol{\omega}$ of the satellite is determined from the right-hand rule. The azimuthal angle of spin ζ delineates the position of $\mathbf{A}$ as it spins about $\boldsymbol{\omega}$. Since the vector $\mathbf{A}$ will lie in the XZ plane for two values of the angle ζ, π radians apart, the zero value of ζ is defined to occur when $\mathbf{A}$ lies in the XZ plane and when the motion at that time is in the positive Y direction. This zero value of ζ is shown in Figure B1. In the general case, the symbol ϵ represents the angle between $\mathbf{A}$ and $\boldsymbol{\omega}$, but in the present case this is synonymous with λ so we shall use the latter notation.

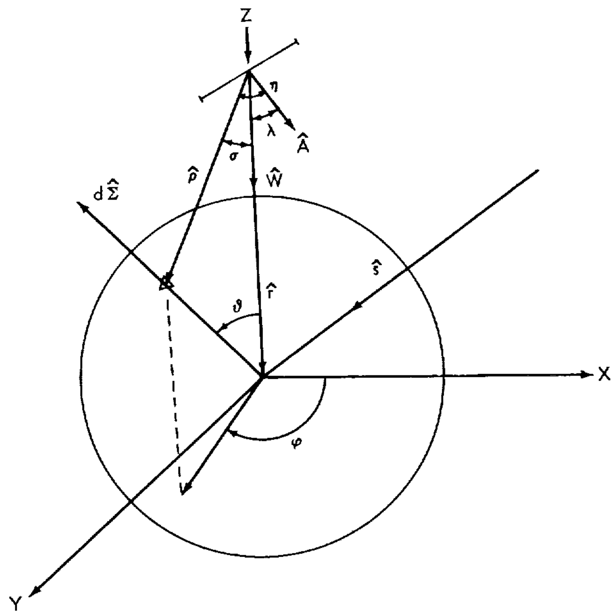


Figure B1—Geometry for determining $\cos\eta$ when ω is coincident with $\mathbf{r}$.

The components of the vectors $\mathbf{A}$ and $\boldsymbol{\rho}$ are given by

$$\left.\begin{aligned} \rho_X &= \rho\sin\sigma\cos\phi & A_X &= A\sin\lambda\cos\zeta\,, \\ \rho_Y &= \rho\sin\sigma\sin\phi & A_Y &= A\sin\lambda\sin\zeta\,, \\ \rho_Z &= \rho\cos\sigma & A_Z &= A\cos\lambda\,. \end{aligned}\right\} \tag{B1}$$

Therefore,

$$\cos\eta = \frac{\mathbf{A}\cdot\boldsymbol{\rho}}{A\rho} = \sin\sigma\cos\phi\sin\lambda\cos\zeta + \cos\sigma\cos\lambda + \sin\sigma\sin\phi\sin\lambda\sin\zeta\,. \tag{B2}$$

In terms of θ, the angle used in the integration, we have,

$$\cos\eta = \frac{(r-\cos\theta)\cos\lambda}{\left(r^2+1-2r\cos\theta\right)^{1/2}} + \frac{\sin\theta\sin\lambda}{\left(r^2+1-2r\cos\theta\right)^{1/2}}(\cos\phi\cos\zeta+\sin\phi\sin\zeta)\,. \tag{B3}$$

Appendix C

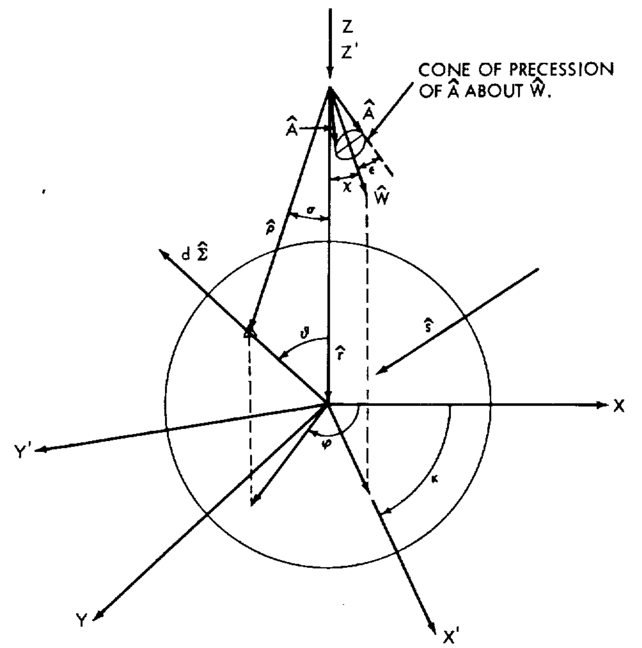
The Calculation of $\cos\eta$ when ω Makes an Angle χ with r

Figure C1 shows the general case. In order to determine $\cos\eta$ we introduce a new coordinate system, the primed system. The projection of ω onto the XY plane is defined as the X′ axis. The X′ axis makes an angle κ with the X axis. Therefore, the primed coordinate system is obtained by a rotation of angle κ about the Z axis. The ωr plane, by definition the X′Z′ plane, is now the plane from which the angle ζ is measured.

As before $\zeta \equiv 0$ when A lies in the X′ Z′ plane and the motion is in the positive Y′ direction. In Figure C1, the top position of A shows the $\zeta = 0$ position.

We now introduce the direction cosines $\cos\mu$, $\cos\nu$, and $\cos\lambda$ where μ, ν, and λ are the angles between A and X′, Y′, and Z′, respectively.

The components of the vectors A and ρ are now:



ϵ = the angle between A and ω.
χ = the angle between ω and r.

Figure C1—Geometry for determining $\cos\eta$ when ω makes an angle χ with r′.

$$\left.\begin{aligned} A_{X}' &= A\cos\mu & \rho_{X}' &= \rho\sin\sigma\cos(\phi-\kappa)\,, \\ A_{Y}' &= A\cos\nu & \rho_{Y}' &= \rho\sin\sigma\sin(\phi-\kappa)\,, \\ A_{Z}' &= A\cos\lambda & \rho_{Z}' &= \rho\cos\sigma\,. \end{aligned}\right\} \tag{C1}$$

However, before calculating $\cos\eta$, we must determine $\cos\mu$, $\cos\nu$, and $\cos\lambda$. In Figure C2 we depict the coordinate system $\underline{X}$, $\underline{Y}$, and $\underline{Z}$, used for determining $\cos\lambda$. The $\underline{Z}$ axis lies along ω; the $\underline{X}$ axis is normal to ω in the Z′ω plane and the $\underline{Y}$ axis lies along Y′. The $\underline{X}$, $\underline{Y}$, $\underline{Z}$ coordinate system is obtained by a simple rotation of the primed system an angle χ about Y′. When $\zeta = 0$, the motion of A is in the

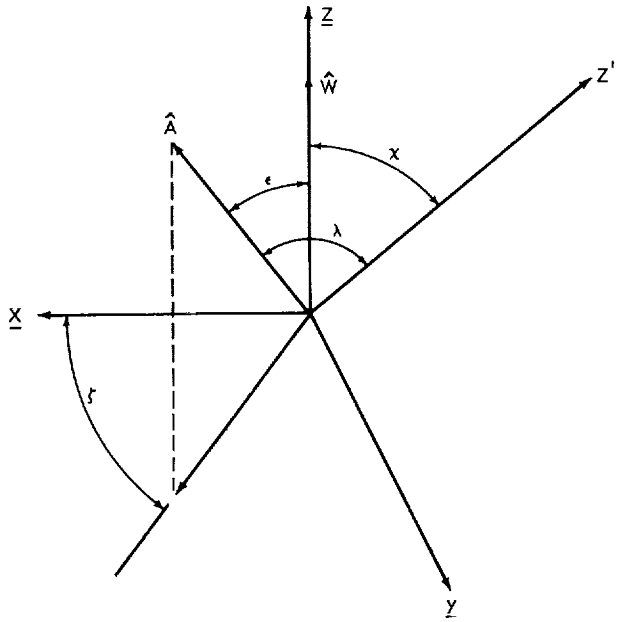


Figure C2—Illustration of the "barred" coordinate system.

positive $\underline{Y}$ direction. This coordinate system meets the specifications as shown in Figure C1: that is,

$$\zeta = 0\,, \qquad \lambda = \chi + \epsilon$$

$$\zeta = \pi\,, \qquad \lambda = \chi - \epsilon \quad \text{for} \quad \epsilon < \chi$$

$$\lambda = \epsilon - \chi \quad \text{for} \quad \epsilon > \chi\,.$$

The coordinates of $\mathbf{A}$ and $\mathbf{Z}'$ are:

$$\left.\begin{aligned} A_{\underline{X}} &= A \sin\epsilon \cos\zeta & Z'_{\underline{X}} &= -Z' \sin\chi\,, \\ A_{\underline{Y}} &= A \sin\epsilon \sin\zeta & Z'_{\underline{Y}} &= 0\,, \\ A_{\underline{Z}} &= A \cos\epsilon & Z'_{\underline{Z}} &= Z' \cos\chi\,. \end{aligned}\right\} \tag{C2}$$

Therefore,

$$\cos\lambda = \frac{\mathbf{A} \cdot \mathbf{Z}'}{AZ'} = \cos\epsilon \cos\chi - \sin\epsilon \sin\chi \cos\zeta\,. \tag{C3}$$

In Figure C3 we depict the geometry from which we can determine the angle μ. The coordinate system used is the $\underline{X}, \underline{Y}, \underline{Z}$ system defined in Figure C2. The coordinates of $\mathbf{A}$ and $\mathbf{X}'$ are:

$$\left.\begin{aligned} A_{\underline{X}} &= A \sin\epsilon \cos\zeta & X'_{\underline{X}} &= X' \sin\left(\frac{\pi}{2} - \chi\right), \\ A_{\underline{Y}} &= A \sin\epsilon \sin\zeta & X'_{\underline{Y}} &= 0\,, \\ A_{\underline{Z}} &= A \cos\epsilon & X'_{\underline{Z}} &= X' \cos\left(\frac{\pi}{2} - \chi\right). \end{aligned}\right\} \tag{C4}$$

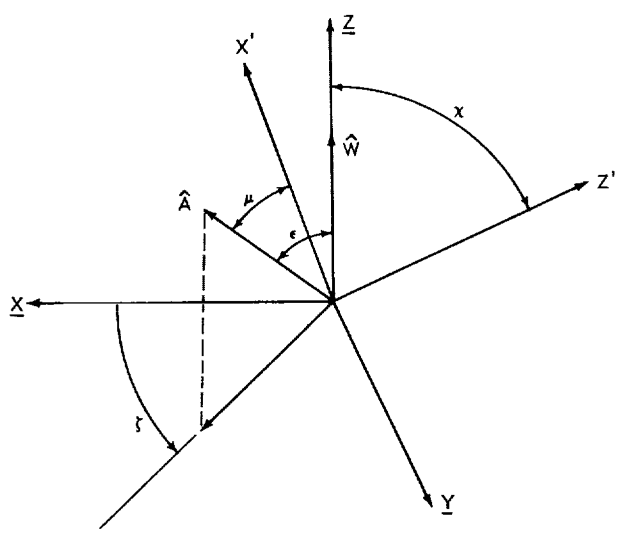


Figure C3—Geometry for determining the angle μ.

Therefore,

$$\cos\mu = \frac{\mathbf{A} \cdot \mathbf{X}'}{AX'} = \cos\epsilon \sin\chi + \sin\epsilon \cos\chi \cos\zeta\,. \tag{C5}$$

Using the condition that the sum of the squares of the direction cosines is equal to unity, we have

$$\cos^2\nu = 1 - \left(\cos\epsilon \cos\chi - \sin\epsilon \sin\chi \cos\zeta\right)^2$$

$$- \left(\cos\epsilon \sin\chi + \sin\epsilon \cos\chi \cos\zeta\right)^2\,. \tag{C6}$$

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Squaring and reducing Equation C6 we have

$$\cos\nu = \sin\epsilon \sin\zeta \ . \tag{C7}$$

Now, from Equations C3, C5, and C7 we have

$$\cos\eta = \sin\sigma\cos(\phi-\kappa)\cos\mu + \sin\sigma\sin(\phi-\kappa)\cos\nu + \cos\sigma\cos\lambda \ . \tag{C8}$$

Expanding Equation C8 we have

$$\begin{aligned}\cos\eta = {} & \sin\sigma(\cos\phi\cos\kappa + \sin\phi\sin\kappa)\left(\cos\epsilon\sin\chi + \sin\epsilon\cos\chi\cos\zeta\right) \\ & + \sin\sigma(\sin\phi\cos\kappa - \cos\phi\sin\kappa)\sin\epsilon\sin\zeta + \cos\sigma\left(\cos\chi\cos\epsilon - \sin\chi\sin\epsilon\cos\zeta\right) \ .\end{aligned} \tag{C9}$$

And in terms of the angle of integration θ we have finally

$$\begin{aligned}\cos\eta = {} & \frac{\sin\theta}{\left(r^2+1-2r\cos\theta\right)^{1/2}}\Big[\cos\phi\cos\kappa\cos\epsilon\sin\chi + \sin\phi\sin\kappa\cos\epsilon\sin\chi + \cos\phi\cos\kappa\sin\epsilon\cos\chi\cos\zeta \\ & + \sin\phi\sin\kappa\sin\epsilon\cos\chi\cos\zeta + \sin\phi\cos\kappa\sin\epsilon\sin\zeta - \cos\phi\sin\kappa\sin\epsilon\sin\zeta \\ & + \frac{(r-\cos\theta)}{\left(r^2+1-2r\cos\theta\right)^{1/2}}\left[\cos\epsilon\cos\chi - \sin\epsilon\sin\chi\cos\zeta\right] \ .\end{aligned} \tag{C10}$$

It might be well to point out here that identical results will be obtained by a reflection of the vectors $\boldsymbol{\omega}$ and $\mathbf{A}$ in a plane normal to $\boldsymbol{\omega}$. It then follows that for any given values of κ, χ, and ϵ an entirely equivalent physical picture is obtained if we replace these values by $(\kappa+\pi)$, $(\pi-\chi)$, and $(\pi-\epsilon)$, respectively. However, this reflection causes the rotation of $\mathbf{A}$ about $\boldsymbol{\omega}$ to be in the opposite sense from what it was before. The zero point and positive direction of ζ remain the same, but the coordinate system is rotated, so the consequences of the reflection of $\boldsymbol{\omega}$ is that the geometrical configuration that was presented at $\zeta = \pi/2$ before reflection is presented at $\zeta = 3\pi/2$ after reflection. Consequently, the total number of calculations required to cover the entire range of the parameters χ and ϵ is reduced by a factor of 2. In addition, geometrical changes given by the parameter κ are symmetric about the **Sr** plane so that we need not consider any values of κ other than $0 \le \kappa \le \pi$.

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Appendix D

The Calculation of the Angle γ

To determine the angle γ between the **Ar** plane and the **rω** plane consider Figure D1. From Appendixes B and C we can write

$$A \sin\lambda \cos\gamma \;=\; A \cos\mu \;. \qquad \text{(D1)}$$

Here, γ is the angle between the projection of A onto the X′ Y′ plane and the X′ axis (the angle between the **Ar** and **rω** planes. Obviously, $\varphi = (\kappa + \gamma)$ for $0 < \zeta < \pi$, and $\varphi = (\kappa - \gamma)$ for $\pi < \zeta < 2\pi$. Then

$$\cos\gamma \;=\; \frac{\cos\epsilon \sin\chi + \sin\epsilon \cos\chi \cos\zeta}{\left[1 - \left(\cos\chi \cos\epsilon - \sin\chi \sin\epsilon \cos\zeta\right)^2\right]^{\frac{1}{2}}} \;. \qquad \text{(D2)}$$

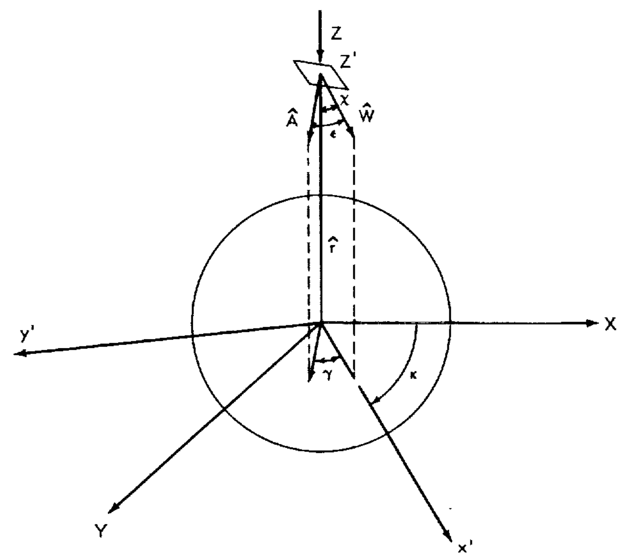

Figure D1—Geometry for determining the angle γ.

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Appendix E

Determination of the Parameters Used in This Report in Terms of the General Orbital Parameters

If we assume the gravitational field to be spherical (that is, to have a $1/r$ potential), then the orbit on an artificial satellite will be an ellipse with the earth at a focus. Such an orbit represented in polar coordinates, is

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos(\psi - B)}, \tag{E1}$$

where

- r = the radius vector from the earth's center to the satellite,
- a = the semimajor axis of the orbit,
- e = the eccentricity of the orbit,
- ψ = the azimuthal position of the satellite in orbit measured from some fixed direction in space,
- B = the angle between the projection of the solar vector (taken this time to be positive from the earth to the sun) onto the orbital plane and r_p where r_p is the radius vector at perigee.

The angle B, then, serves as one of the angles which determines the orientation of the satellite orbit. The remaining angle δ, which determines the orientation, is defined as the angle between the orbital plane and the solar vector. The zero point of the angle ψ is measured from the projection of the solar vector onto the orbital plane and the positive direction is taken as the direction of motion of the satellite. Hence, the value of r at perigee, r_p, is given by Equation E1 when $(\psi - B) = 0$; and likewise the value of r at apogee r_a is given when $(\psi - B) = \pi$.

Since for many purposes the satellite orbit is specified in normal apogee and perigee values we have

$$a = \frac{r_a + r_p}{2}$$

and

$$e = \frac{r_a - r_p}{r_a + r_p}.$$

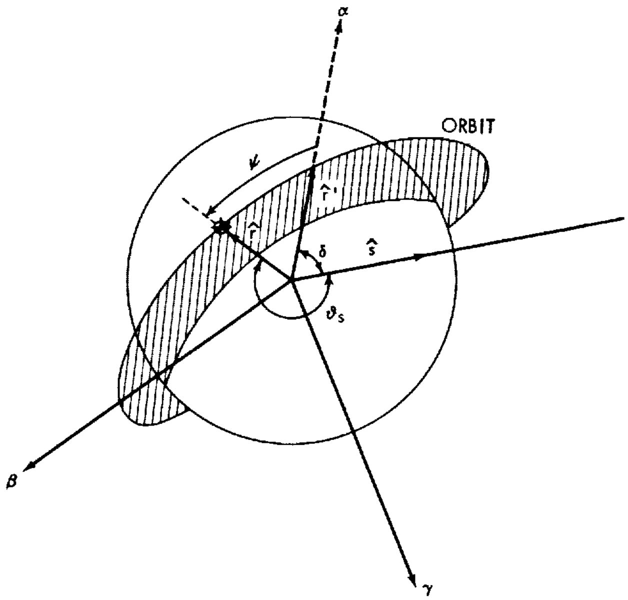

Figure E1—A pictorial representation of the α, β, γ coordinate system and the satellite orbital parameters.

Figure E1 shows the geometry of the general case. In it the azimuthal angle ψ and the angle δ are clearly shown. The vector $\mathbf{r}$ corresponding to $\psi = 0$ is called $\mathbf{r}'$. The α axis lies in the plane of the orbit along $\mathbf{r}'$. The β axis lies in the plane of the orbit normal to $\mathbf{r}'$, and the γ axis lies in the $S\mathbf{r}'$ plane normal to $\mathbf{r}'$. Then

$$r_\alpha = r \cos \psi \qquad S_\alpha = S \cos \delta ,$$

$$r_\beta = r \sin \psi \qquad S_\beta = 0 ,$$

$$r_\gamma = 0 \qquad S_\gamma = S \sin \delta ,$$

from which we have

$$\cos \theta_s = \cos \psi \cos \delta . \tag{E2}$$

Throughout the body of this report we have made use of a set of parameters defining the orientation of the satellite and its spin axis with respect to a coordinate system whose origin lies in the earth. And, the results of sample calculations for specific values of these parameters are presented in Appendixes F and G. In the following discussions the type of problem presented in Appendix G is used as it represents the most general case.

The values of the parameters there chosen represent only a possible set of the instantaneous values an actual satellite will encounter. A satellite in orbit, whether it is "free" so that its angular momentum vector is constant—apart from external perturbations—or whether its angular momentum is fixed with respect to the earth or fixed with respect to the sun (the latter two conditions being accomplished by internal orientational correction mechanisms) will experience instantaneously changing values of the parameters θ_s, χ, and κ. In order to determine these values for any given time, it is necessary to know not only the orientation of the orbit and its parameters but also the angle between the solar vector (now defined to be positive from the satellite to the sun) and the spin axis Γ, and the variation of this angle with time.

Of course, there is nothing sacred about this particular angle insofar as the contents of this paper are concerned. However, the satellite is, after all, only a means of placing scientific instruments in the space environment: for any meaningful interpretation of data it is necessary that the orientation of the satellite be known, and the most readily available and usable reference system is the sun. Consequently, this angle Γ will always be continuously made available as part of the data from the satellite itself. This is not so for some of the parameters used in this work. Since no unique set of parameters exists with which to treat the problem it can always be argued that a rather unfortunate choice was made. However, since this is predominately a matter of taste the author deems it essential to retain parameters pertaining to the satellite-earth system—the only problem under consideration—in order to minimize the difficulty of the required integrations. And since a solution in closed form may not exist, at least in any form which could be termed reasonable, the usefulness of retaining a parameter (the angle Γ) the functional dependence of which is required but cannot be determined explicitly, is doubtful. It can also be argued that the methods derived herein will be useful not for the purpose of determining the incident energy at every instant (which would scarcely be of interest) but will be useful for determining the upper and lower limits that any particular satellite component can be expected to experience.

As alluded to above, we have three situations with which to deal. The simplest of these from the present point of view is the case of fixed orientation of the spin axis with respect to the satellite-earth line. If $\chi = 0$, which for the case under consideration will probably be most often the situation, the problem reduces to the least complicated form. A knowledge of Γ here (though necessary to insure the constancy of χ) is not essential for the solution of our problem. If, however, $\chi \neq 0$ but is constant, the problem is only slightly more complicated because now κ is free to change unless the spin axis is also fixed with respect to the $\mathbf{Sr}$ plane (see Figure C1). This complication is rather trivial, however, and if κ is not fixed, all values will probably be experienced and the maximum and minimum power inputs can be readily determined. The consideration of this point will be left to the reader (a knowledge of Γ may be helpful here).

A more complicated problem presented by an equally likely number of satellites is that for which the spin axis orientation is constant with respect to the sun. Here we need only a functional relation between Γ and χ for various values of the position of the satellite in orbit.

The third situation is the one for which the angular momentum vector is fixed in space (a quite common state of affairs apart from perturbations) and it requires, in addition to the functional relation just mentioned, a functional relation giving the variation of Γ with time of year; and this in turn requires a knowledge of the value of Γ at injection.

We will now derive the most important of these functional relations.

Figure E2 shows the satellite orbit geometry and defines a new coordinate system, the X'', Y'', Z'' system. The X'' axis is coincident with the radius vector $\mathbf{r}$ and the Y'' axis lies normal to $\mathbf{r}$ in the plane of the orbit. The Z'' axis is normal to the orbital plane and its sense is determined by the direction of motion of the satellite in orbit (normal to the page and towards the reader in Figure E2). The positive or negative direction of δ is determined by whether S lies above or below the orbital plane (that is, its projection along Z'' is positive if above the orbital plane, or negative if below). In addition, the absolute value of δ need never be taken greater than $\pi/2$.

To determine the relation between Γ and χ we need only determine the components of S and $\boldsymbol{\omega}$ (not shown) in the X'', Y'', Z'' system. Using the angle δ as previously defined, we have the components

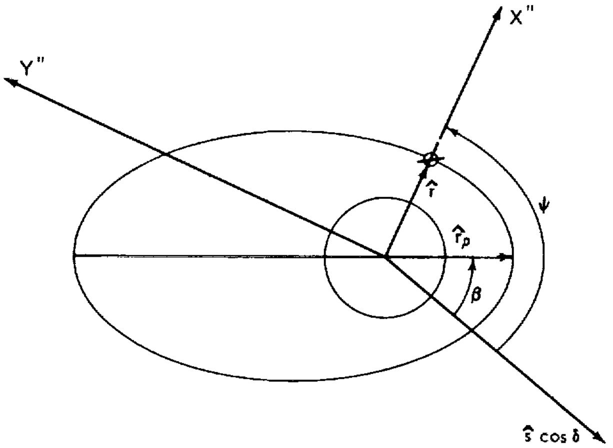

Figure E2—A pictorial representation of satellite orbit geometry and definition of the double primed (") coordinate system.

of S:

$$
\begin{aligned}
S_{X''} &= S\cos\delta\cos\psi\,,\\
S_{Y''} &= -S\cos\delta\sin\psi\,, \qquad\qquad (E3)\\
S_{Z''} &= S\sin\delta\,.
\end{aligned}
$$

To determine the components of ω we need to know the angle Δ between ω and the orbital plane and the angle Ω between $\mathbf{r}$ and the projection of ω onto the orbital plane. For the case of constant angular momentum, the angle Δ will be constant and equal to its value at injection Δ_0. This does not take into consideration its change due to environmental perturbing effects, but such changes are made available from satellite data. Its value is also taken plus or minus in the same sense as the angle δ. For the possible case where the spin axis is constantly oriented to coincide with the solar vector—either in a plus or minus sense—the angle Δ is equal to $\pm\delta$. In the former instance (when ω is constant in space) the angle Ω is given by

$$\Omega = \Omega_0 - (\psi - B')\,, \qquad (E4)$$

where B' is the value of ψ at injection. If the satellite is injected at perigee $B' \equiv B$. The value Ω_0 of the angle between $\omega\cos\Delta$ and $\mathbf{r}_0$ (where $\mathbf{r}_0$ is the radius vector at injection) is taken positive in the direction of ψ, and ψ is positive in the direction of motion of the satellite: that is, if $\omega\cos\Delta$ is normal to $\mathbf{r}_0$ at injection (tangent to the orbital plane) and in the direction of motion, then $\Omega_0 = \pi/2$; and if $\omega\cos\Delta$ is tangent to $\mathbf{r}_0$ but in the direction opposite to the motion, then $\Omega_0 = 3\pi/2$. In Equation E4, the resulting value of Ω will sometimes be negative but will always correspond to the proper value of Ω measured positive as defined for Ω_0. The angles Δ_0 and Ω_0 defined here are commonly referred to as the angles of yaw and pitch at injection, respectively. Therefore, the components of ω are:

$$
\left.
\begin{aligned}
\omega_{X''} &= \omega\cos\Delta\cos\Omega\,,\\
\omega_{Y''} &= \omega\cos\Delta\sin\Omega\,,\\
\omega_{Z''} &= \omega\sin\Delta\,.
\end{aligned}
\right\} \qquad (E5)
$$

We can now write

$$\cos\Gamma = \cos\delta\cos\psi\cos\Delta\cos\Omega - \cos\delta\sin\psi\cos\Delta\sin\Omega + \sin\Delta\sin\delta\,. \qquad (E6)$$

As yet we have not derived an explicit functional relation between Γ and χ. In fact, it should be obvious that since we can determine the X″ component of **ω** we automatically have the angle between **ω** and **r**. From Equation E5 we have

$$\cos \chi' = \cos \Lambda \cos \Omega , \tag{E7}$$

where χ' is the angle between **ω** and **r**. Remembering that the **r** used in this appendix is in the opposite sense from the **r** used in the actual problem (see Figure E2) we have $\chi = (\pi - \chi')$. The solution of χ' from Equation E7 lies in the range $0 \leq \chi' \leq \pi$. The expression given in Equation E6, then, is not necessary to determine the value of χ. However, in the event that the value of Δ or Ω has changed so that one or the other cannot readily be determined, Equation E6 provides a method for determining the unknown one from a knowledge of Γ which is normally made available.

The value of the angle κ (the angle between the projection of both **ω** and S onto the plane normal to **r**, the Y″ Z″ plane) is now given by

$$\cos \kappa = \frac{(S \sin \theta_s) \cdot (\boldsymbol{\omega} \sin \chi)}{S\omega \sin \theta_s \sin \chi} ,$$

where $S \sin \theta_s$ and $\boldsymbol{\omega} \sin \chi$ are for the present considered as vectors.

Clearly the Y″ and Z″ components of these vectors are identical to the Y″ and Z″ components of S and **ω** determined in a different manner above, that is, Equations E3 and E5,

$$(S \sin \theta_s)_{Y''} = - S \cos \delta \sin \psi ,$$

$$(S \sin \theta_s)_{Z''} = S \sin \delta ,$$

and

$$(\boldsymbol{\omega} \sin \chi)_{Y''} = \omega \cos \Delta \sin \Omega ,$$

$$(\boldsymbol{\omega} \sin \chi)_{Z''} = \omega \sin \Delta ,$$

so that

$$\cos \kappa = \frac{\sin \delta \sin \Delta - \cos \delta \cos \Delta \sin \Omega \sin \psi}{\sin \theta_s \sin \chi} . \tag{E8}$$

Because of the great length required to discuss the variation of δ, Δ, and Ω as a function of time of year, these considerations will not be included here.

Appendix F

Graphic Results Giving the Incident Power as a Function of the Parameters $\mathbf{r}, \theta_s$, and λ for a Representative Sample of the Case for which ω is Coincident with $\mathbf{r}$

The graphs that follow present the incident energy to a spinning flat plate for the case in which the spin axis is coincident with the satellite-earth line. The results have been calculated to an altitude of 200 km. Each of the four graphs has been calculated for a different value of the parameter θ_s: These values are: θ_s = 0, 30, 60, and 90 degrees. On each graph, various curves generated by the parameter λ are given.

Thus far, we have only spoken about the case for which χ, the angle between ω and $\mathbf{r}$, is zero. We need not go to the more complicated general case to treat the case given by $\chi = \pi$. The results presented herein are applicable for this latter situation. However, before the values can be found from the graphs, it is necessary to first determine the appropriate values of the parameter λ. To determine the value of λ to which any of the given curves corresponds when $\chi = \pi$, merely subtract the value of λ given on the curve from 180 degrees. Hence, the curve drawn for χ = 0 degrees, λ = 40 degrees corresponds to the situation given by χ = 180 degrees, λ = 140 degrees. And conversely, to determine the given curve to use for a chosen value of λ and for $\chi = \pi$, merely subtract the desired value of λ from 180 degrees. Hence, the curve to use for χ = 180 degrees, λ = 80 degrees is the given curve for χ = 0 degrees, λ = 100 degrees.

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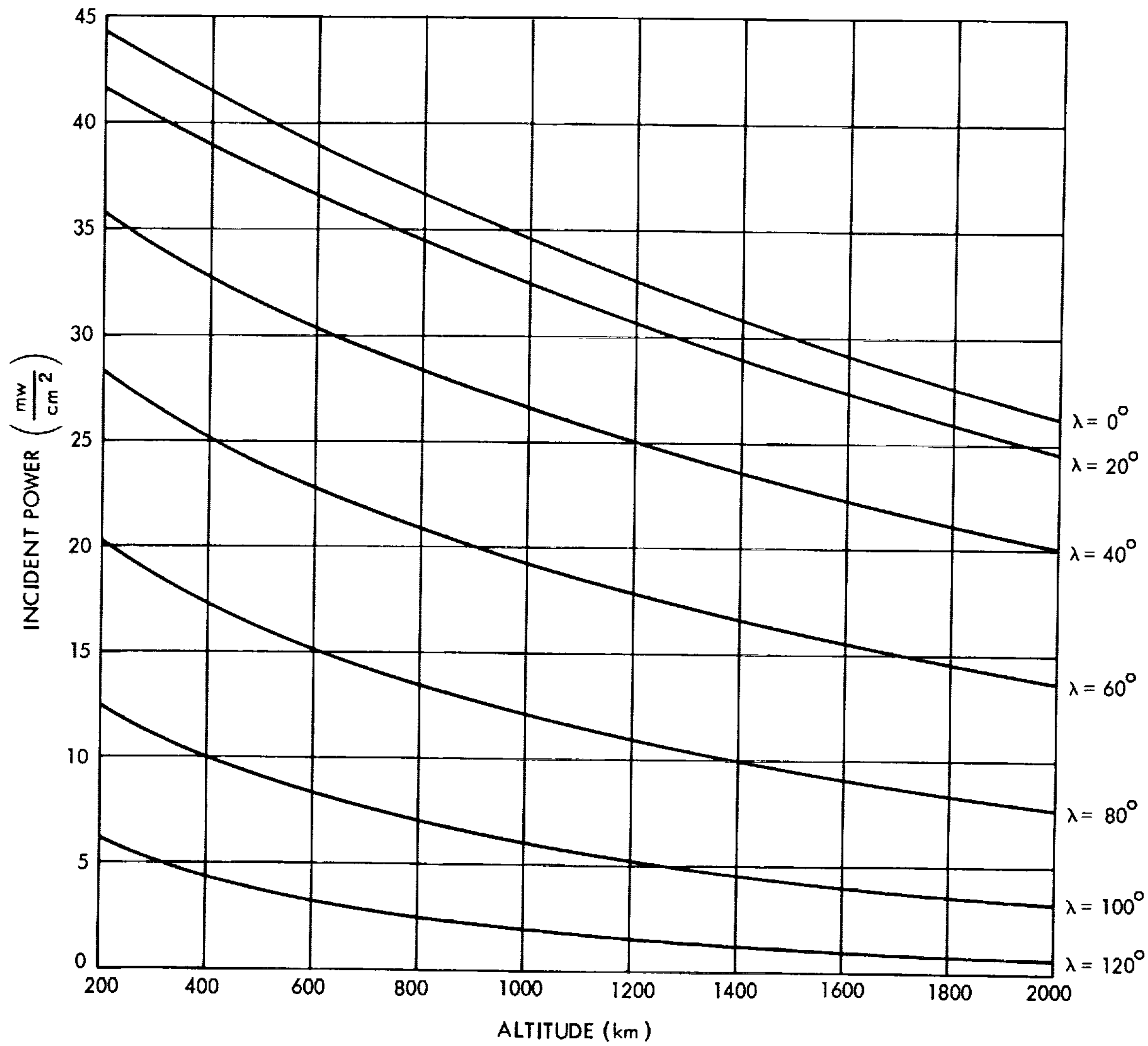

Figure F1—Incident power versus altitude for $\theta_s = 0°$.

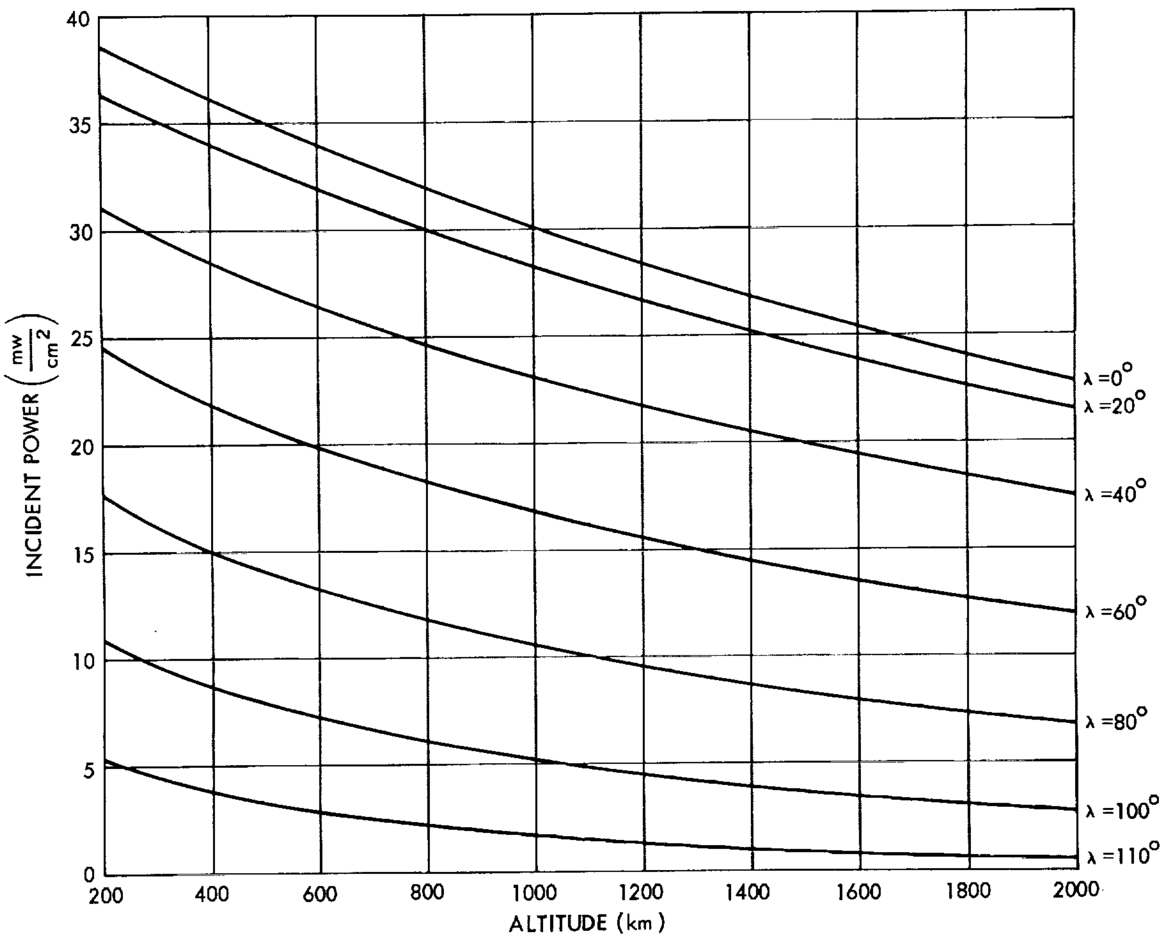

Figure F2—Incident power versus altitude for θ_s = 30°.

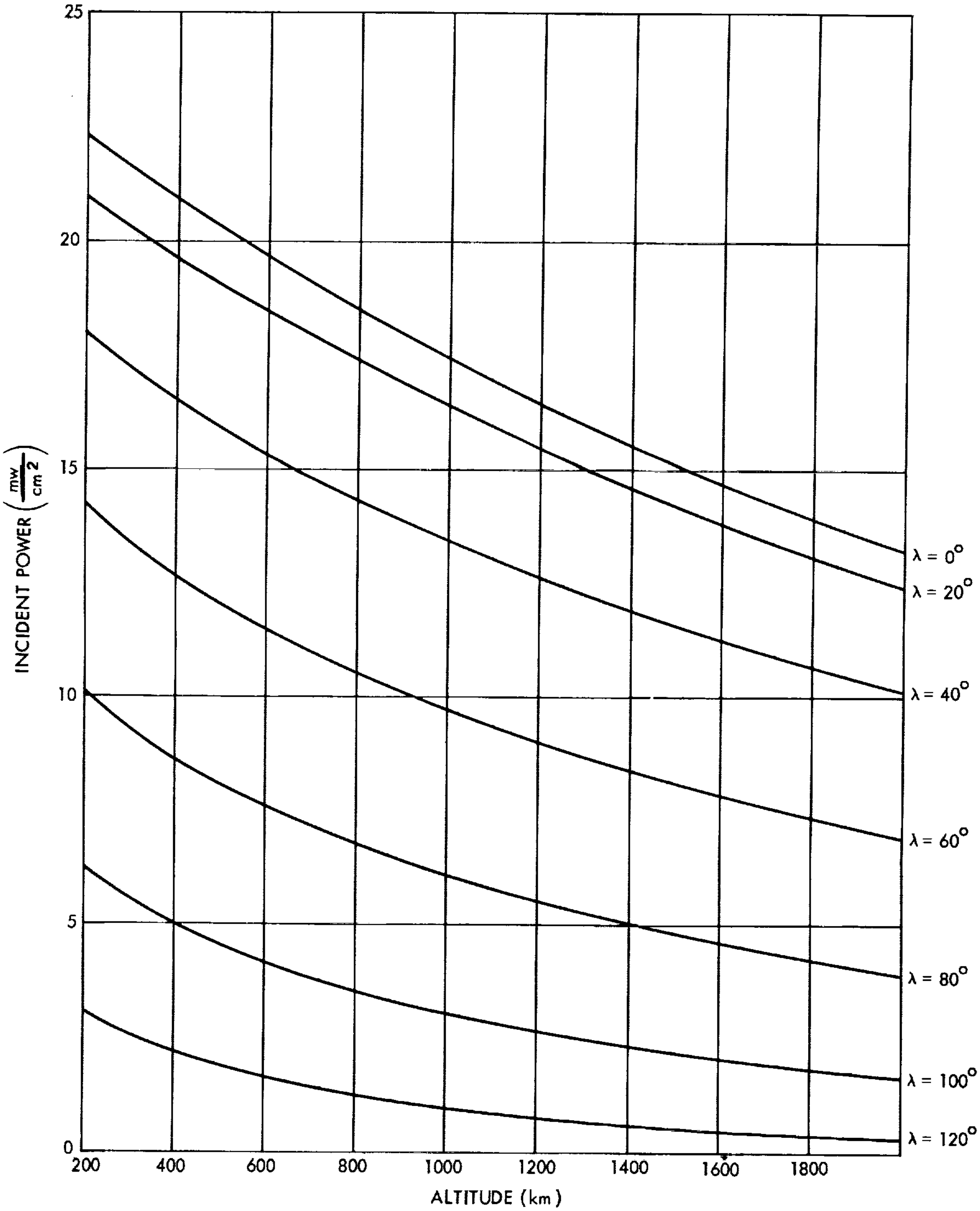

Figure F3—Incident power versus altitude for $\theta_s = 60°$.

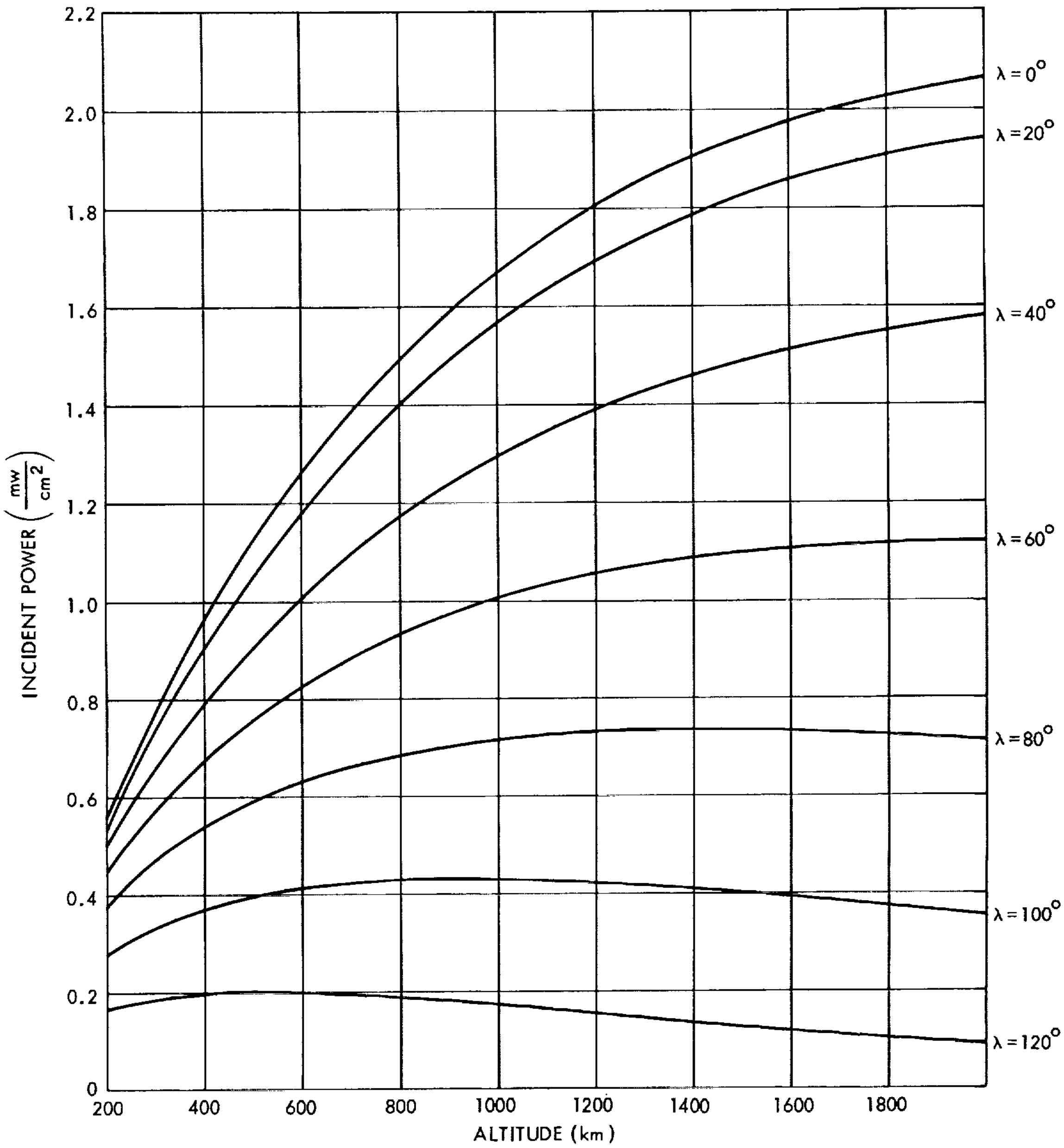

Figure F4—Incident power versus altitude for $\theta_s = 90°$.

Appendix G

Graphic Results Giving the Incident Power as a Function of the Parameters r, θ_s, κ, χ, and ϵ for a Representative Sample for the General Case

The graph (Figure G1) contained in this appendix presents the results of an example calculation. The values of the parameters used are: χ = 60 degrees, θ_s = 30 degrees; κ = 0 degrees and 180 degrees; and ϵ = 30 and 60 degrees. The average incident power is plotted as a function of altitude out to 2000 km. This example is included only as an illustration of the method presented in this paper.

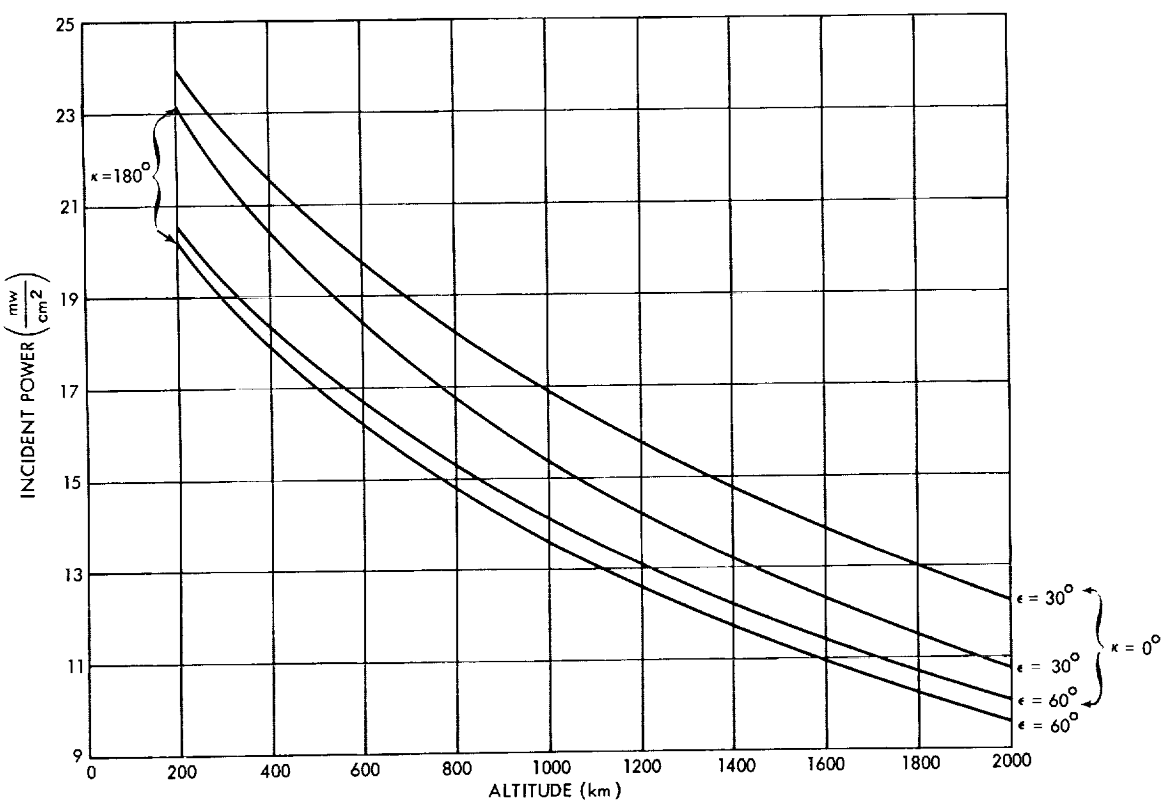

Figure G1—Incident power versus altitude for θ_s = 30°, χ = 60°.

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Appendix H

Fortran Program for the General Case

This appendix gives the program (in Fortran notation) for the general case. In addition, the results of a sample problem are given so that the readout format can be seen.

For the evaluation of the double integral, Equation 4 (which give P), and to obtain the average (Equation 8), the following data must be put into the program. All angles are expressed in degrees.

Table H1

Correspondence of Computer Notation to Analytical Notation.

Computer Notation	Analytic Notation
GL1	κ
THS1	θ_s that is, the first θ_s
THMX	last θ_s
DTHS	the increment by which the θ_s's are augmented
THS1	$\theta_s, \theta_s + \Delta\theta_s, \ldots\ldots\ \theta_s + j\theta_{s'} =$ THMX
S2	δ
ALF	α
DX1	the increment of ζ
DPHI	the increment by which ϕ is measured
CHI1	χ
EI1	ϵ

The Fortran listings of the program are given first; the subroutine used by the program is given next. Sample data cards are also listed.

FORTRAN IBM 7090 PROGRAM FOR CALCULATIONS OF THE GENERAL CASE

```
 EARTH REFLECTED SOLAR RADIATION FALLING ON SPINNING FLAT PLATE

      DIMENSION P(1000),T(1000),XS(1000),SX(1000),CX(1000),CP(1000),
     1 SP(1000),CTH(1000),STH(1000)
    3 FORMAT (0PF7.4,1P7E15.4)
    4 FORMAT (7E10.5)
    5 FORMAT (1P6E19.7)
 1300 FORMAT (1H0,13X,5HKAPPA,16X,3HCHI,12X,7HEPSILON,18X,1HR,12X,7HTHET
     1A S,12X,7HAVERAGE)
 1301 FORMAT (1H1,39X,20HKAPPA...............,1PE15.7/40X,20HINITIAL THE
     1TA-S.....,1PE15.7/40X,20HMAX. THETA-S........,1PE15.7/40X,20HTHETA
     2-S INCREMENT...,1PE15.7/40X,20HS...................,1PE15.7/40X,20
     3HALPHA...............,1PE15.7/40X,20HDZETA INCREMENT.....,1PE15.7/
     440X,20HPHI INCREMENT.......,1PE15.7/40X,20HCHI.................,1P
     5E15.7/40X,20HEPSILON.............,1PE15.7//)
 1302 FORMAT (1H0,3X,I4,43H VALUES OF DOUBLE INTEGRAL FOR GIVEN DZETAS)
      DTR= 1.74532925E-02
      RTD= 1.0/DTR
      PI=3.1415926
      TUPI=PI+PI
      PI2=0.5*PI
      PI4=0.25*PI
    6 READ INPUT TAPE 2,4,GL1,THS1,THMX,DTHS,SZ,ALF,DX1,DPHI,CHI1,EI1
      WRITEOUTPUTTAPE3,1301,GL1,THS1,THMX,DTHS,SZ,ALF,DX1,DPHI,CHI1,EI1
      SAP=ALF*SZ/PI
      THS1=DTR*THS1
      THMX=DTR*THMX
      DTHS=DTR*DTHS
      PGL=GL1
      PCHIL=CHI1
      PEIL=EI1
      CHI1=DTR*CHI1
      EI1=DTR*EI1
      GL1=DTR*GL1
      GLMX=DTR*GLMX
      DGL=DTR*DGL
      DPHI=DTR*DPHI
      DX1=DTR*DX1
      I=0
      X=-DX1
      TUPE=TUPI - DX1
   11 X= X+DX1
      IF (X-TUPE) 12,12,13
   12 I=I+1
      CX(I)=COSF(X)
      SX(I)=SINF(X)
      GO TO 11
   13 IM=I
      FIM=FLOATF(IM)
      I=0
      PHI=-DPHI
   14 PHI=PHI+DPHI
      IF(PHI-TUPI)15,15,16
   15 I=I+1
      CP(I)=COSF(PHI)
      SP(I)=SINF(PHI)
      GO TO 14
```

EARTH REFLECTED SOLAR RADIATION FALLING ON SPINNING FLAT PLATE

```
 16 KM=I
 10 READ INPUT TAPE 2,4,R
    IF (R) 6,6,111
111 GRA= 1.0/R
    R21=R**2+1.0
    THM=ATANF(SQRTF(1.0-GRA**2)/GRA)
    IF (THM) 112,100,100
112 THM=THM + PI
100 IF (THM-PI4) 50,50,51
 50 DTH=THM/26.0
    GO TO 52
 51 DTH=THM/50.0
 52 GL=GL1
    CHIL=CHI1
    EIL=EI1
    I=0
    TH=-DTH
 17 TH=TH+DTH
    IF(TH-THM)18,18,20
 18 I=I+1
    CTH(I)=COSF(TH)
    STH(I)=SINF(TH)
    GO TO 17
 20 JM=I
 31 CL=COSF(GL)
    SL=SINF(GL)
    CCHIL=COSF(CHIL)
    SCHIL=SINF(CHIL)
    CEIL=COSF(EIL)
    SEIL=SINF(EIL)
    THS=THS1
    GO TO 36
 35 THS=THS+DTHS
    IF(THS-THMX)36,36,10
 36 CSTS=COSF(THS)
    SNTS=SINF(THS)
    PTHS=RTD*THS
    DO 1250 I=1,IM
    DO 1100 J=1,JM
    RCTH=R*CTH(J)
    RCTHM1=RCTH-1.0
    R21R=R21-2.0*RCTH
    FAC=STH(J)*RCTHM1/(R21R*R21R)
    CTTHS=CTH(J)*CSTS
    STTHS=STH(J)*SNTS
    SLST=SL*STH(J)
    RMCTHL=R-CTH(J)
    DO 1007 K=1,KM
    CB=CTTHS+STTHS*CP(K)
    IF(CB)1003,1003,1004
1003 P(K)=0.0
    GO TO 1007
1004 CETA=RMCTHL*(CCHIL*CEIL-SCHIL*SEIL*CX(I))+STH(J)*(CP(K)*CL*CEIL*SC
   1HIL + SP(K)*SL*CEIL*SCHIL + CP(K)*CL*SEIL*CCHIL*CX(I) + SP(K)*SL*S
   2EIL*CCHIL*CX(I) + SP(K)*CL*SEIL*SX(I) - CP(K)*SL*SEIL*SX(I))
    IF(CETA)1005,1005,1006
```

EARTH REFLECTED SOLAR RADIATION FALLING ON SPINNING FLAT PLATE

```
1CC5 P(K)=C.C
     GC TO 1C07
1CC6 P(K)=CB*CETA
1CC7 CCNTINUE
1C10 CALL SIMP(P,KM,ANS)
11CC T(J)=ANS*FAC
12CC CALL SIMP(T,JM,ANS)
1250 XS(I)=SAP*ANS*DPHI*DTH/9.0
     TCUT=0.0
     DC 1251 IX=1,IM
1251 TCUT=TOUT+XS(IX)
     AVE=TOUT/FIM
     WRITE CUTPUT TAPE 3,1300
     WRITE OUTPUT TAPE 3,5, PGL,PCHIL,PEIL,R,PTHS,AVE
     WRITE CUTPUT TAPE 3,1302,IM
     WRITE CUTPUT TAPE 3,5, (XS(I2),I2=1,IM)
     GC TC 35
     END(0,0,0,1,0,0,1,0,0,0,0,0,0,0,0)
```

```
     SUBRCUTINE SIMP(P,K,ANS)

     SUBRCUTINE SIMP(P,K,ANS)
     DIMENSICN P(1C00)
1C10 IEND=K-1
     SUM=C.0
     DC 1C11 IX=2,IEND,2
1C11 SUM=SUM+P(IX)
     IEND=IEND-1
     SUM2=0.C
     DC 1C12 IX=3,IEND,2
1012 SUM2=SUM2+P(IX)
     ANS=P(1)+P(K)+4.0*SUM+2.0*SUM2
     RETURN
     END(0,0,0,1,0,0,1,0,0,0,0,0,0,0,0)
```

```
KAPPA................  1.8000000E 02
INITIAL THETA-S......  3.0000000E 01
MAX. THETA-S.........  3.0010000E 01
THETA-S INCREMENT....  3.0000000E 01
S....................  1.3950000E 02
ALPHA................  3.4000000E-01
DZETA INCREMENT......  1.0000000E 01
PHI INCREMENT........  1.0000000E 00
CHI..................  5.9999999E 01
EPSILCN..............  3.0000000E 01

          KAPPA            CHI        EPSILON              R        THETA S        AVERAGE
 1.8000000E 02  5.9999999E 01  3.0000000E 01  1.0314000E 00  2.9999999E 01  2.3202087E 01

36 VALUES OF DOUBLE INTEGRAL FOR GIVEN DZETAS
 1.3761604E 01  1.3888321E 01  1.4266056E 01  1.4886069E 01  1.5734338E 01  1.6792933E 01
 1.8043202E 01  1.9480528E 01  2.1106825E 01  2.2781268E 01  2.4476020E 01  2.6184151E 01
 2.7816340E 01  2.9400020E 01  3.0794582E 01  3.1967988E 01  3.2874174E 01  3.3444360E 01
 3.3639385E 01  3.3444357E 01  3.2874168E 01  3.1967979E 01  3.0794570E 01  2.9400007E 01
 2.7816328E 01  2.6184139E 01  2.4476009E 01  2.2781258E 01  2.1106816E 01  1.9480520E 01
 1.8043196E 01  1.6792928E 01  1.5734336E 01  1.4886067E 01  1.4266054E 01  1.3888320E 01
```

SAMPLE PROBLEM AND RESULT FORMAT

Appendix I

Energy Distribution Predicted for the Proposed Echo II Satellite

This appendix presents a graph (Figure I1) showing the distribution of albedo energy on the proposed Echo II satellite. The satellite is assumed stationary at an altitude of 700 nautical miles. In this case, the angle λ designates the angle between the normal to the surface area of the sphere and the satellite-earth line. The azimuthal average input to a zone lying between λ and $(\lambda + d\lambda)$, is identical to the average input to a flat plate of elemental area spinning about the satellite-earth line at an angle λ. The average incident power is plotted as a function of λ for various values of the parameter θ_s.

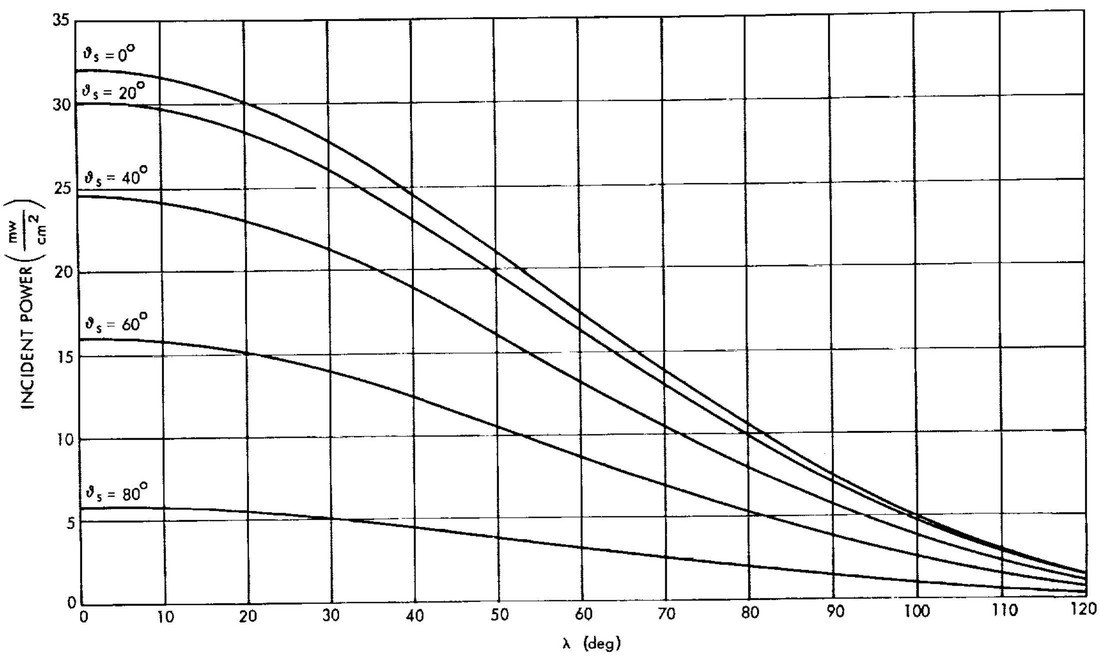

Figure I1—Prediction of the distribution of incident albedo energy on the Echo II (R = 700 nautical miles) satellite for various values of θ_s.

NASA-Langley, 1963 G-336