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II. INTEGRATION OF THE HEAT FLOW EQUATION

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ABSTRACT

The temperature $T(r)$ in a stellar corona is computed under the circumstances that energy is supplied outward from the base of the corona only by thermal conduction. The heat flow equation is solved analytically under a variety of circumstances. In a corona of very low density the energy consumed by expansion of the corona can be neglected and $T(r) \propto r^{-2/7}$, as in Chapman's original static coronal model. The result is a supersonic stellar wind with a velocity $v(\infty)$ of the same order as the gravitational escape velocity $2^{1/2} w$. In a corona with medium density and sufficiently low temperature that $v(\infty)$ is small compared to w, a near region, in which $T(r) \propto r^{-4/7}$ extends for some distance outward from the star before the far region, $T(r) \propto r^{-2/7}$, takes over. The result is a supersonic stellar wind velocity $v(\infty)$ of the same order as the characteristic thermal velocity c_0 at the base of the corona. In a corona which is exceedingly dense, an intermediate region in which $T(r) \propto r^{-1}$ appears between the near and the far regions, which has the result of extending to large distance the point at which the coronal expansion becomes supersonic. In a corona

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which is exceedingly hot $(c_0 \approx w)$ the expansion becomes so violent that thermal conduction becomes negligible and the behavior of the corona is approximately adiabatic.

It is shown that any effect which tends to reduce the thermal conductivity of the coronal gases at large distance from the star has the effect of enhancing the velocity of the stellar wind.

Comparison with Chamberlain's earlier discussion of the solution of the momentum and heat flow equations in his "solar breeze" model shows that he made two self-consistent errors in his assumption that the energy flux in the solar wind is identically zero and that the gas motion is adiabatic at large radial distances from the Sun. It is shown that neither assumption is correct in a corona of finite density. It is shown, however, that the analytical form $T(r) \propto 1/r$ suggested by Chamberlain is obtained in the limit as the density of the corona is made large without limit, in which case all motion in the corona approaches zero.

Application of the solutions of the heat flow equation to the sun — assuming that the solar corona is heated solely by thermal conduction—show that at least under present conditions the solar corona and wind would lie in the middle ground between high and low density and temperature. Assuming that they have coronas heated solely by conduction it is suggested that some of the giant stars with the low gravitational escape velocities, may fall into the high density case, and certain dwarfs into the low density case. Some of the very active stars may fall into the high temperature quasi-adiabatic case.

I. INTRODUCTION

In a previous paper (Parker, 1963a hereafter referred to as Paper I) the mass and momentum conservation equations were solved for a stellar corona in which the temperature was taken to be a given function $T(r)$ of radial distance from the star. The present paper goes on from there to consider the form of $T(r)$ in a corona in which energy transport is limited to thermal conduction. The paper is written from a purely academic point of view for the purpose of exploring the presently unknown dynamical properties of a conductive stellar corona. It is an open question at the present time to what extent the outer solar corona is supplied by thermal conduction from the low corona, as opposed to direct heating of the outer corona by the dissipation of wave motion originating beneath the corona.

In this connection it is easy to show that thermal conduction is probably important, but it has not yet been possible to establish whether wave dissipation is, or is not, important, too. To show that thermal conduction is at least important, note that the observed solar wind strength at the orbit of Earth (Shklovskii, 1960; Gringauz, et al, 1960; Bridge et al, 1962; Bonetti, et al, 1962; Neugebauer and Snyder, 1962) is of the order of 500 km/sec and, say, 5 or more ions/cm^3. The solar corona is observed to be at least as active at low solar latitudes, whence comes the solar wind observed near the plane of the ecliptic, as it is at high latitudes. Hence an upper limit to the energy carried away by expansion of the corona into the solar wind is obtained if it is assumed that the solar wind has the observed strength everywhere around the sun that it is observed to have near the ecliptic. The result is an estimated efflux from the sun of 7×10^{35} protons/sec in the solar wind. The energy (gravitational energy plus kinetic energy) consumed by each proton in the wind is 5×10^{-9} ergs, so that the efflux of 7×10^{35} protons/sec means an energy consumption of 3.5×10^{27} ergs/sec by the expanding corona. Radiation losses above the base of the corona are somewhat less than this, so 3.5×10^{27} ergs/sec is a rough order of magnitude figure for the energy that is supplied to the outer corona. Now thermal conduction is expected to supply $-4\pi a^2 \kappa(T_0)(\nabla T)_0$ across $r = a$ where the temperature is T_0 and the temperature gradient is $(\nabla T)_0$. Suppose that $a = 1\ R_\odot$. Billings and Lilliequist (1963) suggest that $(\nabla T)_0$ may be of the order of 3° per km. Using (2) this leads to an energy flux by thermal conduction of 1.1×10^{27} ergs/sec if $T_0 = 1 \times 10^6\ ^\circ K$ and 6.2×10^{27} ergs/sec if $T_0 = 2 \times 10^6\ ^\circ K$. Thus thermal conduction could be the sole source of energy to the expanding outer corona (see discussion in Parker, 1963b). But the question is clearly open. Contemporary theory of coronal heating by wave dissipation (see for instance Osterbrock, 1961; Whitaker, 1963) is not sufficiently quantitative to be able to help in the decision. The one remaining approach to the problem is to solve the momentum and heat flow equations with simple assumptions concerning the configuration and structure of the corona to see if the observed solar wind velocity and density at the orbit of Earth can be accounted for in a quantitative way by the coronal temperature and density observed at the sun. A rough numerical investigation was begun by de Jager (1962) using Chapman's temperature distribution (Chapman, 1957, 1959) for a static corona. Noble and Scarf have recently begun an investigation of the solar corona and solar wind by numerical methods, giving a proper simultaneous solution of the momentum and heat flow equations. Their first paper (Noble and Scarf, 1963) on the problem shows that no energy source other than thermal conduction seems to be required beyond a couple of solar radii, within the present uncertainties in the observations. However they point out that further investigation into some of the evident complications should be carried out before any final conclusion is reached.

Altogether, then, it is clear that thermal conduction is an important process in determining the dynamical behavior of the solar corona. It may be presumed, therefore, that thermal conduction plays an important role in the coronas of many kinds of stars other than the sun. In consequence of the evident widespread importance of thermal conductivity in stellar coronas the present paper undertakes a general study of the dynamical properties of the hypothetical stellar corona in which thermal conduction is the only form of energy transport beyond some given radial distance $r = a$. The purpose will be to examine the various qualitative dynamical features of the conduction corona under different circumstances of temperature and density. We shall be concerned more with limiting cases, to illustrate the various features, rather than extensive numerical results for any single model. Thus the present study will add little or no quantitative information to the question of the dominance of thermal conduction in the solar corona, discussed above. Rather it is aimed at illustrating the various asymptotic classes of coronal behavior so that the position occupied by the solar corona may be seen in its proper perspective. The aim is also to illustrate the dynamical possibilities available to stars other than the sun, many of which must have coronas with rather different values of density and temperature.

Consider a stellar corona in which $T(r)$ is assumed to be determined by the stationary heat flow equation*

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* See discussion and derivation in Appendix I.

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\kappa(T)\frac{dT}{dr}\right] = Nv\left[3k\frac{dT}{dr} - \frac{2kT}{N}\frac{dN}{dr}\right] \quad (1)$$

where N is the number of ions per cm^3 and v is the velocity of expansion, determined from $T(r)$ by the methods outlined in I. Radiation losses are neglected (Chapman, 1957, 1959). It is assumed throughout that $T(\infty) = 0$. There are obviously certain special cases one can imagine for which $T(\infty)$ may not be essentially zero, but they will have to be taken up elsewhere. The thermal conductivity is denoted by $\kappa(T)$ and the numerical coefficients on the right hand side of (1) are appropriate to fully ionized hydrogen, for which (Chapman, 1954; Spitzer, 1956)

$$\kappa(T) \cong 6 \times 10^{-7}\, T^{5/2} \text{ ergs/cm sec °K}. \quad (2)$$

As in the previous paper the discussion will be limited to coronas in which the thermal velocity is small compared to the gravitational escape velocity, so that the corona is tightly bound to the star by the gravitational field.

Now the heat flow equation has been discussed previously by Chamberlain (1961) in connection with his own ideas of the dynamics of the solar corona. Unfortunately he based his numerical solutions on the assumption that the total energy flow from the corona to $r = \infty$ is exactly zero. That is to say, Chamberlain postulated that the expanding corona forms the perfect thermal insulator, so that with 10^6 °K in the corona and 0 °K in interstellar space there is no heat transfer between. He also postulated that the flow was exactly adiabatic at large r, rather than solving the heat flow and momentum equations there, so he did not discover the error of his basic assumption. The analytical solutions given in the present paper show that the energy flow to $r = \infty$ is not zero, with the result that the stellar corona heated by thermal conduction expands into space with supersonic velocity, rather than with an evaporative velocity going to zero like $1/r^{1/2}$ as suggested by Chamberlain. It is interesting to note that Chamberlain's velocity dependence turns up in the present paper as the limiting form of the supersonic solution of the momentum and heat flow equations when the ratio of density to thermal conductivity, N/κ, becomes large without limit. The way in which this limit comes about is as follows: For any finite N/κ the expansion becomes supersonic at some critical distance r_c (see discussion in I) and remains supersonic everywhere beyond; the distance r_c increases without limit as $N/\kappa \rightarrow \infty$, so that the supersonic portion of the solution moves out of the picture, leaving behind the evaporative solution $v \propto 1/r^{1/2}$ discussed by Chamberlain. Of course, the velocity of expansion goes to zero at the same time that its form approaches $1/r^{1/2}$, so it is probably just as meaningful to say that in the limit as $N/\kappa \rightarrow \infty$ the corona approaches stasis.

II. BASIC CONSIDERATIONS IN THE SOLUTION OF THE CORONAL HEAT FLOW EQUATION

In order to carry out an analytical solution of the heat flow equation (1) it is necessary to consider the limiting conditions under which its solution may be desired. The study of the momentum equation in Paper I for arbitrary $T(r)$

showed two opposite extremes for expansion of a stellar corona. The first extreme is that as the temperature at the base of a corona is increased, a point is reached at which the effective enthalpy of the gas becomes equal in magnitude to the negative gravitational energy. When this occurs the corona is no longer bound to the star by the gravitational field. The expansion starts with the velocity of sound at the base of the corona, and the equivalent de Laval nozzle (see Paper I) looses its throat. The important point is that the energy loss to the star becomes so great with increasing coronal temperature that the energy transported by thermal conduction becomes negligible. The heat flow eqn. (1) becomes more or less irrelevant. The expansion of the corona becomes approximately adiabatic, and this case has been dealt with at some length elsewhere (Parker, 1960, 1963b). Adiabatic coronal expansion has the general property, already described, of starting with supersonic velocity at the base of the corona and decelerating outward to $r = \infty$. Any enthalpy excess over the gravitational energy is converted to kinetic energy. This circumstance of an overheated corona may perhaps have some application to the coronal outburst from the sun following a solar flare. This has been dealt with elsewhere, too (Parker, 1961). Altogether, then, the limit of high coronal temperature does not involve the heat flow equation in any fundamental way and it has been dealt with previously. It will be discussed no further here.

The second and opposite extreme is in the limit of low temperature, in which the corona becomes to tightly bound gravitationally that the energy consumed by its expansion is negligible compared to the energy transported by thermal conduction. In this case thermal conduction is the dominant factor in determining $T(r)$ and the effects of the heat flow equation are most readily illustrated. Thus it is with the lower coronal temperatures that we shall be most concerned here, and consequently it is advisable at this point to consider a little more closely how the energy $O\left[N_0 v_0 a^2 (G M_* M / a)\right]$ consumed by coronal expansion compares with the energy flow $O\left[\kappa(T_0)\, a\, T_0\right]$ by thermal conduction. Let M_* represent the mass of the star, a the radius of the star, and M the mass of a hydrogen atom. It may be seen from either eqn. (25) or from eqn. (42) of paper I that very roughly

$$\ln v_0 = O\left(-\frac{G M_* M}{a k T_0}\right)$$

as T_0 becomes small. In general $G M_* M / a k T_0$ is of the order of 10 or more, so that decreasing T_0 to one half means a reduction in v_0 by a factor of 10^{+4}. On the other hand the thermal flux is proportional to $T_0^{7/2}$, which means a reduction only by about a factor of 10. At lower temperatures v_0, and hence $N_0 v_0 a^2 (G M_* M / a)$ decreases even more rapidly in comparison to $\kappa(T_0)\, a\, T_0$. Altogether, then, it is evident that in the limit of low coronal temperature T_0, the energy consumed by coronal expansion may be made arbitrarily small compared to the energy flowing outward through the corona by the mechanism of thermal conduction. In the limit of small coronal temperature and density, the coronal temperature approaches the static temperature distribution

$$T(r) = T_0 \left(\frac{a}{r}\right)^{2/7}$$

originally proposed by Chapman (1957, 1959) for a static model of the solar corona.

To facilitate solution of the heat flow equation (1) it is convenient to express it in terms of the same reduced variables as employed in Paper I in the discussion of the momentum and mass conservation equations. Thus let a designate the radial distance at which the heat flow equation is assumed to become valid and put $\xi = r/a$. Let $c^2 = 2kT/M$, $w^2 = GM_*/a$, $n = N/N_0$. The subscript zero is used to denote the value at $\xi = 1$. Then write $\kappa_0 \equiv \kappa(T_0)$ and

$$\kappa(T) = \kappa_0 f(c^2/c_0^2)$$

so that $f(c^2/c_0^2) = (c^2/c_0^2)^{5/2}$ for fully ionized hydrogen. The heat flow equation (1) may be written

$$\frac{d}{d\xi}\left[\xi^2 f\left(\frac{c^2}{c_0^2}\right)\frac{dc^2}{d\xi}\right] = \frac{N_0 v_0 a k}{\kappa_0}\left(3\frac{dc^2}{d\xi} - \frac{2c^2}{n}\frac{dn}{d\xi}\right) \quad (3)$$

upon using the condition for mass conservation, eqn. (6) of Paper I. Solving the momentum equation (5) of Paper I for $(c^2/n)\,dn/d\xi$, substituting into (3), and integrating yields the usual energy equation

$$\xi^2 f\left(\frac{c^2}{c_0^2}\right)\frac{d}{d\xi}\left(\frac{c^2}{c_0^2}\right) = \frac{N_0 v_0 a^2 M}{\kappa_0 a T_0}\left(\frac{1}{2}v^2 + \frac{5}{2}c^2 - \frac{w^2}{\xi}\right) - \frac{F_\infty}{\kappa_0 a T_0} \quad (4)$$

where F_∞ is the energy flow per steradian at $\xi = 0$. The quantity $\kappa_0 a T_0$ represents the order of magnitude of the energy flux transported by thermal conduction outward from $\xi = 1$. It is assumed that the temperature falls to zero at $\xi = \infty$, i.e. $c^2(\infty) = 0$.

III. LOW DENSITY AND LOW TEMPERATURE APPROXIMATIONS

The simplest asymptotic class of solutions of the energy equation (4) is in the low density approximation. For any given value of T_0, the density N_0 may be made sufficiently small that any quantity multiplied by $N_0 v_0 a^2 / \kappa_0 a T_0$ may be neglected. With $f(c^2/c_0^2) = (c^2/c_0^2)^{5/2}$, appropriate for ionized hydrogen, and with $c^2(1) = c_0^2$, integration of (4) yields

$$c^2(\xi) = \frac{c_0^2}{\xi^{2/7}}. \tag{5}$$

The energy flux is

$$F_\infty = \frac{2}{7} \kappa_0 a T_0 . \tag{6}$$

The expansion $v(\xi)$ of the corona follows now from eqns (39) and (40) of Paper I with $\beta = 2/7$. The velocities v_0 and $v(\infty)$ follow from eqns. (42), (43) and (45), (46), and are illustrated in Figure 4 of Paper I. As already noted, this low density approximation was applied by de Jager (1962) by numerical methods to the expansion of the solar corona. It is based on the assumption that the expansion energy is small compared to the conduction energy

$$N_0 v_0 a^2 M \left[\frac{1}{2} v^2(\infty) + w^2\right] << F_\infty , \qquad (7)$$

which can be achieved for any T_0 by making N_0 sufficiently small. Hence we refer to it as the low density approximation.

The condition (7) can also be satisfied for any N_0 by making T_0 sufficiently small, but if this involves making T_0 so small that $w^2/c_0^2 >> 10$, then $v^2(\infty) << w^2$ and the much better low temperature approximation is available. With $w^2/c_0^2 >> 10$ there is a region $(1, \xi_1)$ near the star where $5c^2(\xi)/2$ and $v^2(\xi)$ are both small compared to w^2/ξ. In this region the energy equation (4) may be approximated as

$$\xi^2 \left(\frac{c^2}{c_0^2}\right)^{5/2} \frac{d}{d\xi}\left(\frac{c^2}{c_0^2}\right) \cong -\frac{N_0 v_0 a^2 M w^2}{\kappa_0 a T_0} \frac{1}{\xi} - \frac{F_\infty}{\kappa_0 a T_0} . \qquad (8)$$

This equation is valid out to ξ_1, where $c^2(\xi)$ becomes comparable to w^2/ξ. We recall from eqn. (8) of Paper I that $c^2(\xi)$, w^2/ξ, and $v^2(\xi)$ are all of the same order of magnitude at the critical point. Hence the distance ξ_c to the critical point must be of the same general order of magnitude as ξ_1.

Now the total energy flux F_∞ at $\xi = \infty$ is made up of the convection of kinetic energy and of thermal conduction. Denoting the latter by F_c we have

$$F_\infty = N_0 v_0 a^2 \frac{1}{2} M v^2(\infty) + F_c . \tag{9}$$

Let it be assumed that the temperature is sufficiently low that

$$N_0 v_0 a^2 \frac{1}{2} M v^2(\infty) << F_\infty \tag{10}$$

This is a weaker restriction on N_0 and v_0 than (7). Since $v^2(\infty)$ is presumably comparable to $v^2(\xi)$ at the critical point near ξ_1, it follows that all the terms on the right hand side of (4) must be small compared to F_∞ by the time ξ_1 is reached. Under these conditions (8) is valid for all $\xi \geq 1$ The term in w^2/ξ on the right hand side of (8) is not correct beyond ξ_1, but it is negligible there so incorrectness causes no error.

Integration of (8) subject to the boundary conditions that $c^2(1) = c_0^2$ and $c^2(\infty) = 0$ yields

$$c^2(\xi) = c_0^2 \left(\frac{Q}{\xi^2} + \frac{1 - Q}{\xi} \right)^{2/7}, \tag{11}$$

where

$$Q \equiv \frac{7 N_0 v_0 a^2 M w^2}{4 K_0 a T_0} \tag{12}$$

We require that $0 \leq Q \leq 1$ in order that $c^2(\xi)$ be real everywhere in $(1, \infty)$. The energy flux transported by thermal conduction to $\xi = \infty$ is readily shown to be

$$F_c = \frac{2}{7} \kappa_0 a T_0 (1 - Q) \text{ ergs/sec steradian}. \tag{13}$$

The conduction flux across $\xi = 1$ is readily shown to be

$$F_0 = \frac{2}{7} \kappa_0 a T_0 (1 + Q) \tag{14}$$

which is larger than F_c by just the amount of energy $N_0 v_0 a^2 M w^2$ consumed in lifting the gas out of the gravitational field of the star. In terms of the definition of Q the basic energy relation

$$N_0 v_0 a^2 M w^2 = \frac{4}{7} Q \kappa_0 a T_0 \tag{15}$$

serves to determine v_0 as a function of N_0 and T_0, or if v_0 is known from the momentum equation, it serves to determine Q. Now as we have already noted, the low temperature approximation is based upon the inequality (10) which is a weaker condition at low coronal temperatures, $w^2/c_0^2 >> 10$, than (7). If the coronal temperature is not low, then of course $v^2(\infty) = O(w^2)$ and the two are equivalent, requiring small N_0. The condition (10) may be expressed in terms of Q as

$$\frac{w^2}{v^2(\infty)} \gg \frac{Q}{1-Q} \quad . \tag{16}$$

This inequality establishes an upper limit on Q for the validity of the low temperature approximation. Note that only as $w^2/c_0^2 \longrightarrow \infty$ can $Q \longrightarrow 1$.

There are several implications of (10) and (16) that should be noted. The inequality (16) requires that $Q < 1$. It is obvious from (15) that $Q > 0$, so that altogether $0 < Q < 1$. The non-vanishing of Q follows from the fact that with $T_0 > 0$ we must have v_0 nonvanishing. For if $v_0 = 0$, then $c^2(\xi) \propto \xi^{-2/7}$, and the momentum equations predict a nonvanishing v_0, in contradiction to the assumption that

Now it follows from (11) that the outward decline of $c^2(\xi)$ must have an effective exponent β somewhere between 2/7 and 4/7, with $\beta \cong$ 4/7 close to the star and $\beta \cong$ 2/7 far away. The relative importance of the two regions depends upon the value of Q. It was shown in Paper I that neither v_0 nor $v(\infty)$ is very sensitive to the value of β in this range. Thus for a given T_0, both v_0 and $v(\infty)$ are very approximately independent of Q. Thus for a given Q, (16) is principally a restriction on T_0, placing an upper limit on T_0 for the validity of the low temperature approximation. Eqn. (15) then gives N_0 once T_0 has been chosen (so as to satisfy (16) of course). The maximum value of N_0 occurs for the maximum Q permitted by (16) for the chosen T_0.

The lower is T_0 the closer may Q approach to 1.0. This may be stated a little differently by considering that as $Q \rightarrow 1$, (16) requires that T_0 must become small. To satisfy (15) it is then necessary that N_0 become large at a very rapid rate. For intermediate values of Q, say Q = 0.5, making N_0 large, requires through (15), that T_0 must become small, which automatically satisfies (16). Of course, a moderate N_0, requiring a moderate T_0 through (15), may satisfy (16) just as well in this case. Altogether, the maximum value of N_0 for any particular temperature T_0 occurs when Q is as large as (16) will permit. For moderate or low T_0 this means that Q is near 1.0 and $c^2(\xi)$ declines like $\xi^{-4/7}$ for some distance out from the star.

Consider now the relative magnitudes of the distance ξ_c to the critical point, the distance ξ_1 to the point where $c^2(\xi)$ becomes comparable to w^2/ξ, and the distance ξ_2 to the point where $c^2(\xi)$, given by (11) flattens out from the $\xi^{-4/7}$ near the star to the $\xi^{-2/7}$ at large radial distance. Then ξ_c is given by eqn. (8) of Paper I, and ξ_1 and ξ_2 may be defined to be

$$\frac{5}{2} c^2(\xi_1) = \frac{w^2}{\xi_1}, \tag{17}$$

$$\xi_2 = \frac{Q}{1-Q}. \tag{18}$$

It was already noted that ξ_c is comparable to ξ_1 because $c^2(\xi)$ and w^2/ξ are comparable at ξ_c. More precisely, if $c^2(\xi) = c_0^2/\xi^\beta$, then solution of eqn. (8) of Paper I yields

$$\frac{\xi_c}{\xi_1} = \left[\frac{5}{2(2+\beta)}\right]^{1/(1-\beta)} = O(1). \tag{19}$$

To demonstrate the relative magnitude of ξ_2, note from the definition of ξ_2 that

$$\frac{N_0 v_0 a^2 M w^2}{\xi_2} = O(F_c).$$

Since $v^2(\infty)$ is of the same order as w^2/ξ_c, we have

$$N_0 v_0 a^2 \tfrac{1}{2} M v^2(\infty) = O\left(N_0 v_0 a^2 M \frac{w^2}{\xi_c}\right)$$

$$= \frac{\xi_2}{\xi_c} O\left(N_0 v_0 a^2 M \frac{w^2}{\xi_2}\right)$$

$$= \frac{\xi_2}{\xi_c} O(F_c).$$

Then if (10) is to be satisfied, it follows that $\xi_2 << \xi_c, \xi_1$. It follows from the definition of ξ_2 that if $\xi_c >> \xi_2$, then $(1-Q)\xi_c >> Q$, which is evidently equivalent to the statement (16). Physically this means that the region $(1, \xi_2)$ near the star in which $c^2(\xi) \propto \xi^{-2/7}$ terminates well before the critical point. In the vicinity of the critical point and beyond, then

$$c^2(\xi) \cong c_0^2 \left(\frac{1-Q}{\xi}\right)^{2/7}. \tag{20}$$

From eqn. (8) of Paper I it follows that

$$\xi_c \cong \left(\frac{7w^2}{16c_0^2}\right)^{7/5} \frac{1}{(1-Q)^{2/5}} . \tag{21}$$

This distance becomes very large as $Q \longrightarrow 1$. The expansion velocity at the critical point is, from eqn. (9) of Paper I

$$v^2(\xi_c) = c^2(\xi_c)$$

$$= c_0^2 \left[\frac{16 c_0^2 (1-Q)}{7w^2}\right]^{2/5} . \tag{22}$$

Noting that $v^2(\infty)$ is of the same order as $v^2(\xi_c)$, it is a simple matter to show that (16) and $(1-Q)\xi_c >> Q$ are equivalent, which in turn is equivalent to

$$\left(\frac{7w^2}{16c_0^2}\right)^{7/5} (1-Q)^{3/5} >> Q . \tag{23}$$

Consider now the expansion velocity $v(\xi)$ that results from the temperature distribution (11) of the low temperature limit. The first approximation, given by eqns. (13) and (22) of Paper I, will be sufficient for the present purposes. Define $I(\xi)$ as

$$I(\xi) \equiv \int_1^{\xi} \frac{du \; w^2}{u^2 \; c^2(u)} \tag{24}$$

with $c^2(\xi)$ given by (11). Then from eqn. (13) of Paper I, the expansion velocity in $\xi \leq \xi_c$ is

$$v_1^2(\xi) = v_0^2 \frac{Q^{4/7}}{\xi^{34/7}} \left[1 + \frac{1-Q}{Q}\xi\right]^{4/7} \exp\left[2I(\xi)\right]. \tag{25}$$

The function $I(\xi)$ is related to the incomplete beta function and is expressed in terms of hypergeometric functions in Appendix I. The density in $\xi < \xi_c$ is given as

$$n(\xi) = \exp\left[-2I(\xi)\right].$$

Putting $\xi = \xi_c$ in (24) and using (22) for $v(\xi_c)$ it follows that

$$\left(\frac{v_0^2}{c_0^2}\right)_1 = \frac{c_0^2}{(1-Q)^2} \left(\frac{7w^2}{16 c_0^2}\right)^6 \exp\left[-2I(\xi_c)\right] \tag{26}$$

with $I(\xi_c)$ given in Appendix I. This serves to determine the mass flow $N_0 v_0 a^2$ and, upon substitution into (15), it relates Q to N_0 and T_0.

In the region beyond ξ_c the expansion velocity follows from eqn. (22) of Paper I as

$$v_1^2(\xi) = 3c^2(\xi_c) - 2c_0^2\left(\frac{Q}{\xi^2} + \frac{1-Q}{\xi}\right)^{2/7} - \frac{2w^2}{\xi_c}\left(1 - \frac{\xi_c}{\xi}\right) + 4J(\xi) \tag{27}$$

where

$$J(\xi) \equiv \int_{\xi_c}^{\xi} \frac{du}{u}\, c^2(u) \tag{28}$$

with $c^2(\xi)$ given by (11). The function $J(\xi)$ can be expressed in terms of hypergeometric functions and is given in Appendix I. Noting that (11) approximates to (20) everywhere in $\xi > \xi_c$, it follows that

$$v_1^2(\infty) = \frac{87}{7}\, c_0^2\left[\frac{16c_0^2(1-Q)}{7w^2}\right]^{2/5}. \tag{29}$$

To illustrate the results of the formal calculations under the low density $(Q \cong 0)$ and low temperature $(0 < Q < 1)$ approximations the position ξ_c of the critical point and the velocities $(v_0^2/c_0^2)_1^{1/2}$, $v(\xi_c)/c_0$, $v_1(\infty)/c_0$ are computed from the temperature distribution given by (11) and plotted in Fig. 1 as a function of w^2/c_0^2. The asymptotic density $n(\xi)\,\xi^2$ is plotted in Fig. 2. The values φ = 0, 4/7, 1.0 were chosen to illustrate the full range of variation of φ. Since (23) must be satisfied, it is obvious that φ = 1.0 has no physical significance. It is included only to exhibit the mathematical boundary of the domain of ξ_c etc. φ = 4/7 is valid only for $w^2/c_0^2 \gtrsim 5$, and of course φ = 0 is valid for all w^2/c_0^2. For the intermediate case φ = 4/7, about half of the thermal energy is consumed by coronal expansion, the other half flowing out to $\xi = \infty$. There is the interesting matter of the cross-over of the density curves in Fig. 2 in the vicinity of $w^2/c_0^2 \cong$ 5 or 6. At large w^2/c_0^2 the curve for φ = 1 lies below the curve for φ = 0 for the reason that $c^2(\xi)$ declines outward more rapidly for φ = 1. At smaller w^2/c_0^2 the curves are reversed for the two reasons that with $\varphi = 1$ $(\beta = 4/7)$ the minimum value of w^2/c_0^2 for $\xi_c = 1$ (given as $2+\beta$ by eqn. (41) of Paper I) is approached more quickly and the density at the minimum value of w^2/c_0^2 (given as $[\beta/(4-\beta-2\beta^2)]^{1/2}$ by eqn. (47) of Paper I) is larger.

To illustrate the range of validity of the approximations used in this section for a star with one solar mass we have plotted the thermal energy flux $2K_0 a T_0/7$ ergs/sec steradian in Fig. 3 in comparison with the energy $N_0 v_0 a^2 M w^2$ ergs/sec steradian consumed in lifting the expanding corona in the gravitational field of the star. The numerical values of v_0 (shown by the broken line in Fig. 3(a)) were computed for an isothermal corona $(\beta = 0)$ because such values are typical for β anywhere between zero and 4/7 (see Fig. 4, Paper I). Fig. 3(a) was computed for $a = 1\ R_\odot$, for which the maximum temperature is $5.62 \times 10^6\ {}^\circ K$, and Fig. 3(b) for $a = 4\ R_\odot$, for which the maximum temperature is $1.4 \times 10^6\ {}^\circ K$. A typical density in the lower solar corona, $r \cong 1 R_\odot$ is 10^8 atoms/cm^3. It is readily seen from Fig. 3(a) that the energy consumed by expansion is larger than that transported by thermal conductivity for $T_\circ \gtrsim 0.9 \times 10^6\ {}^\circ K$, i.e. (7) is satisfied only for $T_\circ$ less than about $0.7 \times 10^6\ {}^\circ K$. On the other hand, a star with a coronal density of $10^6/cm^3$ will fall within the low density approximation for all $T_\circ$.

If it is assumed that the corona is heated by thermal conduction only beyond $4 R_\odot$, then Fig. 3(b) is applicable. The density of the solar corona at $4 R_\odot$ is observed to be of the order of 10^5 atoms/cm^3 (Van de Hulst, 1953), which leads to the conclusion that the energy consumed by expansion may be neglected if $T_\circ \lesssim 0.3 \times 10^6\ {}^\circ K$.

It is evident from all this that the energy consumed by expansion of the solar corona is of the same order rather than much smaller, than the thermal energy flux transported by conduction. The solar corona seems to be too dense to fall into the low density limit, and somewhat too hot to fall into the low temperature limit. On the other hand, we have already pointed out that the solar corona is not so hot that it falls into the high temperature adiabatic limit. In the next section we take up the high density approximation.

IV. HIGH DENSITY APPROXIMATION

In this section we undertake the integration of the heat flow equation (4) under the circumstances that $N_\circ$ is taken to be extremely large while $T_\circ$ is maintained at some moderate value. The effect of making $N_\circ$

large is to make $v(\xi)$ small, so that the high density approximation consists of integrating (4) neglecting only the term $v^2(\xi)$ on the right hand side. It constitutes the next order of analytical complexity after the low temperature approximation. As we shall see, the solar corona is not sufficiently dense as to allow application of the high density approximation, but other stars with somewhat denser coronas may well fall into the high density category.* The principal interest in the high density approximation is that it illustrates the behavior of the stellar corona in which the heat supply for a given temperature T_0 becomes small. The expansion velocity then exhibits a maximum at some intermediate distance and declines for a time before becoming supersonic at large distance. It is shown that in the limit as $N_0/\kappa_0 \longrightarrow \infty$ the expansion goes over into the algebraic form $v(\xi) = v_1 \xi^{-1/2}$ proposed by Chamberlain (1961) for the expansion of the solar corona, though of course $v_1 \longrightarrow 0$ at the same time.

1. Semi-Quantitative Discussion

As a consequence of the greater complexity of the high density approximation, there are a number of qualitative and semi-quantitative points that are best cleared up before presenting the formal solution of the energy equation. They are taken up under the paragraph headings which follow.

(a) The Nonvanishing of F_∞ :

A fundamental point to be kept in mind throughout the discussion of the high density approximation is that F_∞ is non zero for all finite N_0.

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*It should be noted that radiation losses become a serious concern when the density becomes, say, a factor of ten greater than the solar corona. This complication will not be included in the present formal solution of (4).

The nonvanishing of F_∞ is readily established by noting that for any N_0, no matter how large, the energy equation (4) may be reduced to

$$-\xi^2\left(\frac{c^2}{c_0^2}\right)^{5/2}\frac{d}{d\xi}\left(\frac{c^2}{c_0^2}\right) = +\frac{F_\infty}{\kappa_0 \lambda T_0}$$

by choosing T_0 small enough.* The reduction follows from the rapid decline of v_0, discussed earlier. The boundary condition $c^2(\infty) = 0$ means in this case that $F_\infty > 0$. We assume that increasing T_0 can only increase the total energy flow in the corona to $\xi = \infty$. Hence

$$F_\infty > 0$$

for all higher T_0. QED. As was pointed out earlier, Chamberlain (1961) made the error of supposing that F_∞ was identically zero for all finite N_0 and T_0. He made the supporting assumption that the coronal expansion becomes exactly adiabatic at large ξ. The low density and low temperature approximations have already demonstrated that the flow at large ξ is not at all adiabatic . The same will again be true in the high density approximation.

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* This is a purely formal argument on the mathematical properties of eqn. (4), so the fact (4) may not be physically correct for a corona of arbitrarily low temperature is irrelevant.

(b) The Near Region:

It has been shown that the energy flux F_∞ is nonzero for all finite N_0 and vanishes only in the limit as $N_0 \to \infty$. On the other hand, the energy flux transported across $\xi = 1$ by thermal conduction is of the order of $\kappa_0 a T_0$, which is essentially independent of N_0. With large N_0 only the small portion F_∞ of $\kappa_0 a T_0$ is transported to $\xi = \infty$ by conduction and convection. Hence, it is evident that most of $\kappa_0 a T_0$ must go into lifting the gas in the gravitational field of the star, which is $N_0 v_0 a^2 M w^2$. Hence, in order of magnitude

$$N_0 v_0 a^2 M w^2 = O(\kappa_0 a T_0). \tag{30}$$

It follows that for a fixed T_0, v_0 varies inversely with N_0 and becomes arbitrarily small as N_0 becomes large. The expansion velocity $v(\xi)$ for $\xi > 1$ is proportional to v_0, so that the velocity $v(\xi)$ is generally inversely proportional to N_0 for fixed T_0. It follows for sufficiently large N_0, that in the region near the star both the term F_∞ and the term in $v^2(\xi)$ may be dropped from the right hand side of (4). The problem may be further simplified with the assumption that the temperature of the corona is fixed at some moderate value, say $w^2/c_0^2 \cong 10$. Then for some distance out from the base of the corona the term $c^2(\xi)$ may be neglected compared to w^2/ξ and (4) may be integrated to give

$$c^2(\xi) \cong c_0^2\left[1 - Q\left(1 - \frac{1}{\xi^2}\right)\right]^{2/7} \qquad (31)$$

Now in this expression for $c^2(\xi)$ it must be required that Q is neither much greater than 1.0, nor much less than 1.0. For if, on the one hand Q were much greater than 1.0, we would have $c^2(\xi)$ vanishing near the star, which would require an enormous heat sink at the point of vanishing. No such sink exists. If on the other hand Q were much less than 1.0, $c^2(\xi)$ would quickly level off at $c_0^2(1-Q)^{2/7} \cong c_0^2$, and a nearly isothermal coronal would be the result. But an isothermal corona yields a v_0 which is much too large to satisfy (30) as $N_0 \longrightarrow \infty$. Thus, we conclude that, as was shown for the low temperature approximation, we must have $Q \longrightarrow 1$ as N_0 becomes large. It follows that

$$c^2(\xi) \cong \frac{c_0^2}{\xi^{4/7}} \qquad (32)$$

in the near region. The thermal flux from the base of the corona is then approximately $4\kappa_0 a T_0/7$, and in place of (29) we may now write

$$N_0 v_0 a^2 M w^2 \cong \frac{4}{7}\kappa_0 a T_0 . \qquad (33)$$

This expression determines v_0 as a function of N_0 and T_0.

With the expression (32) for $c^2(\xi)$ in the near region it is readily shown that $5c^2(\xi)/2$ becomes equal to w^2/ξ on the right hand side of (4) at ξ_1 where

$$\xi_1 = \left(\frac{2w^2}{5c_0^2}\right)^{7/3}. \tag{34}$$

The near region is defined to be $(1, \xi_1)$. In it the velocity is given by the equation for conservation of mass

$$v(\xi) = \frac{v_0}{n(\xi)\,\xi^2} \tag{35}$$

where $n(\xi)$ is determined by the hydrostatic barometric equation (see eqn.(12), Paper I) to be

$$n(\xi) = \xi^{4/7} \exp\left[-\frac{7w^2}{3c_0^2}\left(1 - \frac{1}{\xi^{3/7}}\right)\right]. \tag{36}$$

It follows at once that the coronal density and expansion velocity at the outer end of the near region are

$$n(\xi_1) = \left(\frac{2w^2}{5c_0^2}\right)^{4/3} \exp\left(\frac{35}{6} - \frac{7w^2}{3c_0^2}\right) \tag{37}$$

and

$$\xi_1 = \left(\frac{2w^2}{5c_o^2}\right)^{7/3}. \tag{38}$$

For fixed w_o^2/c_o^2, $v(\xi_1)$ is proportional to v_o, and hence inversely proportional to N_o in the near region.

(c) The Intermediate Region:

The energy relation (33) declares that v_o is inversely proportional to N_o as N_o becomes large while T_o is fixed. We know from Paper I that v_o can be made small for fixed T_o only if the outward temperature decline in the corona becomes as steep as $1/\xi$. This is the role played by the intermediate region, which lies beyond the outer boundary $\xi = \xi_1$ of the near region. The intermediate region, in which $c^2(\xi) \propto 1/\xi$, is the distinguishing feature of the high density approximation. We will have more to say on this later.

In the vicinity of ξ_1, and beyond, the term $5c^2(\xi)/2$ cannot be neglected on the right hand side of (4). We obtain

$$\xi^2\left(\frac{c^2}{c_o^2}\right)^{5/2}\frac{d}{d\xi}\left(\frac{c^2}{c_o^2}\right) = \frac{N_o v_o a^2 M}{\kappa_o a T_o}\left[\frac{5}{2}c^2(\xi) - \frac{w^2}{\xi}\right].$$

It is obvious from this differential equation that $c^2(\xi)$ must decrease like $1/\xi$ in the intermediate region. For if $c^2(\xi)$ decreased less rapidly, the right hand side would become positive and $c^2(\xi)$ would then increase outward from the star. If $c^2(\xi)$ decreased more rapidly, it would become negligible again, leading to $c^2(\xi) \propto \xi^{-4/7}$ which would be contrary to the assumption that $c^2(\xi)$ decreased more rapidly than w^2/ξ. Now if $c^2(\xi)$ is proportional to $1/\xi$, then the left hand side of the equation decreases like $\xi^{-5/2}$, which rapidly becomes negligible compared to either term on the right hand side, both of which decline only as fast as ξ^{-1}. Equating the right hand side to zero, then, yields

$$c^2(\xi) \cong \frac{2w^2}{5\xi} \tag{39}$$

throughout the intermediate region. It is this $1/\xi$ temperature dependence through the intermediate region that leads to the small value of v_0 required by (33) for large N_0. The gas flow through the intermediate region is approximately adiabatic, since thermal conduction, represented by the left hand side of (3), rapidly becomes negligible with increasing ξ. If $v^2(\xi)$ is included on the right hand side of (4), the result is the wellknown adiabatic flow relation

$$\frac{1}{2} v^2(\xi) + \frac{5}{2} c^2(\xi) - \frac{w^2}{\xi} \cong 0 \tag{40}$$

If (39) were exact, rather than only approximate, the hydrostatic barometric equation would give

$$n(\xi) = n(\xi_1)\left(\frac{\xi_1}{\xi}\right)^{3/2} \tag{41}$$

in the intermediate region. The equation for conservation of mass would then give

$$v(\xi) = v(\xi_1)\left(\frac{\xi_1}{\xi}\right)^{1/2}. \tag{42}$$

The values of $n(\xi_1)$ and $v(\xi_1)$ are given by (37) and (38) respectively. As a matter of fact, $c^2(\xi)$ must be a little less than (39) because of the neglect of the left hand side of (4) and because of the neglect of $v^2(\xi)$. Thus $n(\xi)$ must decrease a little more rapidly than (41), and the velocity is accordingly somewhat greater than given by (42). This is discussed further in paragraph (d).

The intermediate region terminates where the individual terms on the right hand side of (4) become comparable to F_∞. If this occurs in the vicinity of $\xi = \xi_3$, then in order of magnitude ξ_3 is defined by

$$N_0 v_0 a^2 M \frac{w^2}{\xi_3} = O(F_\infty).$$

But the numerator of the left hand side of this equation is approximately equal to the energy flux F_0 transported across the base of the corona. Hence

$$\xi_3 = O\left(\frac{F_0}{F_\infty}\right). \tag{43}$$

(d) The Far Region:

Beyond ξ_3, where F_∞ can no longer be neglected on the right hand side of (4), the terms which become negligible are $c^2(\xi)$ and w^2/ξ, both of which vanish as $\xi \longrightarrow \infty$. Then if F_∞ is written in the form (9), the energy equation becomes

$$-\kappa_0 a T_0 \, \xi^2 \left(\frac{c^2}{c_0^2}\right)^{5/2} \frac{d}{d\xi}\left(\frac{c^2}{c_0^2}\right) = F_\infty - N_0 v_0 a^2 \tfrac{1}{2} M v^2(\xi)$$

$$= F_c + N_0 v_0 a^2 \tfrac{1}{2} M \left[v^2(\infty) - v^2(\xi)\right].$$

Since $F_\infty > 0$, it follows that F_c and $v(\infty)$ cannot both be identically equal to zero. Let it be supposed that $F_c \neq 0$. It is an easy matter to show from this that $v(\infty) \neq 0$, for neglecting v^2, the energy equation yields

$$c^2(\xi) \cong c_0^2 \left(\frac{7 F_c}{2\kappa_0 a T_0 \xi}\right)^{2/7}$$

$$\cong c_0^2 \left(\frac{2 F_c}{F_0 \xi}\right)^{2/7}.$$

It was shown in Paper I that for any outward temperature decline slower than $1/\xi$ the expansion velocity approaches a constant $v(\infty) > 0$ as ξ becomes large. The temperature here declines only as fast as $\xi^{-2/7}$. Hence $v^2(\infty)$ is nonvanishing. The only other possibility is now that $F_c = 0$, but since $F_\infty \neq 0$, this automatically gives $v(\infty) \neq 0$. QED. Since $c^2(\infty) = 0$, it follows that the expansion becomes supersonic in the far region.

There must exist a critical point at which $v^2(\xi)$ crosses over $c^2(\xi)$. The temperature, given by (39) in the intermediate region is so high that the total enthalpy plus the gravitational energy of the gas is

$$\frac{5}{2} c^2(\xi) - \frac{w^2}{\xi} \cong 0. \tag{44}$$

We know from previous discussion that when this is the case, the expansion goes supersonic very soon after $c^2(\xi)$ declines less rapidly than $1/\xi$. Thus, we expect that in general order of magnitude

$$\xi_c = O(\xi_3) \tag{45}$$

and

$$v^2(\xi_3) = O\left[c^2(\xi_3)\right]. \tag{46}$$

Presumably, then

$$v^2(\infty) = O\left[v^2(\xi_3)\right] \tag{47}$$

$$= O\left(\frac{w^2}{\xi_3}\right). \tag{48}$$

It is immediately evident that the coefficient 2/5 on the right hand side of (39) may not be entirely correct toward the outer side of the intermediate region. It is readily seen from (40) that $c^2(\xi)$ is probably somewhat less than given by (39), with the consequence, mentioned earlier, that $n(\xi)$ tends to be less than given by (41). The velocity $v^2(\xi)$ tends to be larger than given by (42), so that it reaches a value $O\left[c^2(\xi_3)\right]$ at ξ_3. The important point is that the small deviation from strictly adiabatic flow is an essential qualitative feature of the coronal expansion.

(e) Discussion:

The arguments presented in the foregoing paragraphs show that in the limit of large N_0 the velocity v_0 varies inversely with N_0, as given by (33). With $w^2 >> 5c_0^2/2$ most of the energy is transported outward from $\xi = 1$ by thermal conduction. The convection of enthalpy may be neglected. The velocity increases throughout the near region $(1, \xi_1)$ to a maximum $v(\xi_1)$ given by (38) at the transition from the near to the intermediate region. Throughout the intermediate region the velocity declines. The conduction flux decreases outward through the near region like $1/\xi$ whereas the convection of enthalpy decreases only as fast as $1/\xi^{4/7}$. The near region ends where thermal conduction falls below convection. In the intermediate region thermal conduction may be neglected and the gas motion is approximately (but not exactly) adiabatic. The expansion velocity declines outward through the intermediate region almost as rapidly as given by (42). The intermediate region ends at ξ_3 where the convection of energy becomes comparable to F_∞. The energy flux F_∞ is nonvanishing for all finite N_0 and guarantees that the expansion velocity will be supersonic at infinity. The smallness of the thermal conduction flux at ξ_3 suggests that the expansion proceeds to $\xi = \infty$ with only the total enthalpy that it has at ξ_3. Consequently we suggest that the critical point lies not too far beyond ξ_3 and that $v^2(\infty)$ is of the order of magnitude* of w^2/ξ_3.

It is of some interest to consider the form of $v^2(\xi)$ as $N_0 \longrightarrow \infty$. Eqn. (33) requires then $v_0 \longrightarrow 0$ like $1/N_0$. This reduction can only be brought about by the length ξ_3 of the intermediate region becoming large without limit. Then the flux of kinetic energy at $\xi = 0$ is

$$N_0 v_0 a^2 \frac{1}{2} M v^2(\infty) = N_0 v_0 a^2 M \, O\left(\frac{w^2}{\xi_3}\right)$$

$$= O\left(\frac{F_0}{\xi_3}\right). \qquad (49)$$

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* It should be understood that the term "order of magnitude" as used through this paper is meant in the formal analytical sense. It is not meant necessarily to imply approximate arithmetic equality.

The thermal conduction flux at $\xi = \infty$ cannot be more than the conduction flux at ξ_3, which is smaller than the above expression by $1/\xi_3^{3/2}$

$$F_c \leq O\left(\frac{F_o}{\xi_3^{5/2}}\right) \tag{50}$$

The energy flux at $\xi = \infty$ is predominantly kinetic energy, and this kinetic energy goes to zero as $N_o \to \infty$. The result of $N_o \to \infty$ is then that $\xi_3 \to \infty$ so that the supersonic region beyond ξ_3 moves out of the picture. The approximate conditions given by (42) and (43) become exact and extend all the way to $\xi = \infty$. We recognize this as the adiabatic subsonic coronal expansion proposed by Chamberlain (1961) as applying for all finite N_o. We have shown here that it arises only in the limit as $N_o \to \infty$. We note also, in this limit, that $v(\xi) \to 0$ for all ξ, indicating that the corona approaches stasis.

It is of interest to see under what circumstances the intermediate region, which is characteristic of the high density approximation, may be expected to occur in the corona of a star. There are a number of requirements for the existence of the intermediate region. In the first place the corona must be tightly bound to the star, or else the whole corona will be moving outward with supersonic velocity. This requires that the enthalpy plus gravitational energy in the low corona must be rather less than zero,

$$\frac{5}{2} c_o^2 << w^2. \tag{51}$$

It follows from (34) that this is just the requirement for the existence of the near region, i.e. $\xi_1 > 1$. In addition to this, the intermediate region is subsonic, requiring that

$$v(\xi_1) \ll c(\xi_1). \tag{52}$$

The velocity $v(\xi_1)$ follows from (42), $c(\xi_1)$ follows from (34), and is given by (39). Thus (52) becomes

$$N_0 \gg \frac{4K_0}{35\,a k c_0}\left(\frac{5c_0^2}{2w^2}\right)^{13/3} \exp\left(\frac{7w^2}{3c_0^2} - \frac{35}{6}\right) \tag{53}$$

for the existence of the intermediate region.

To see where the solar corona falls, put $a = 7 \times 10^{10}$ cm, $w^2 = 2.19 \times 10^{15}$ cm^2/sec^2, and put $M = 2.06 \times 10^{-24}$ gm for one atom in ten being helium. Then for a moderate temperature $w^2/c_0^2 = 10$ ($T_0 = 1.6 \times 10^6$ °K) we have $\xi_1 = 25.4$, $c_0 = 1.48 \times 10^7$ cm/sec, $c(\xi_1) = 0.58 \times 10^7$ cm/sec. The requirement (53) for the existence of the intermediate region becomes

$$N_0 \gg 1 \times 10^{11}/cm^3.$$

Observation suggests that N_0 is somewhere in the vicinity of 10^8/cm^3. Raising T_0 relaxes the requirement on N_0 but we cannot go very far toward increasing T_0 because of the requirement (51). For the marginal case, that

w^2/c_0^2 is as small as 5 ($T_0 \cong 3.3 \times 10^6$ °K), reached only under transient conditions at the peak of solar activity, we have $\xi_1 = 5$, $c_0 = 2.1 \times 10^7$ cm/sec, $c(\xi_1) = 1.32 \times 10^7$ cm/sec and

$$N_0 >> 0.3 \times 10^8/cm^3.$$

It is observed that N_0 exceeds this value, but whether it exceeds $0.3 \times 10^8/cm^3$ by a large enough amount for a long enough period of time to produce an intermediate region, we cannot say. It would be extremely interesting to see the phenomenon of an intermediate region in the solar corona sometime, somewhere beyond a distance of a few solar radii, but the numbers do not make it look very hopeful. Certainly any temporary intermediate region in the solar corona would be of limited radial extent, i.e. ξ_3 only a little larger than ξ_1.

It is not difficult to imagine stars with weaker gravitational fields and larger radii, so that κ_0/c_0^2 is smaller for a given value of w^2/c_0^2, in which case the inequality (53) is easily satisfied. Perhaps some giant stars qualify for this condition. Then an intermediate region with a velocity maximum at ξ_1 would result. In some extreme case it is conceivable that N_0 is so large that $v(\infty)$ might be only a few km/sec with the adiabatic solutions (42) and (43) extending far into space.

2. Formal Discussion

With the substitutions

$$X = \left(\frac{2w^2}{5c_0^2}\right)^{7/3} \left(\frac{\kappa_0 a T_0}{N_0 v_0 a^2 M w^2}\right)^{2/3} \times \left[\frac{1}{\xi} + \frac{F_\infty}{N_0 v_0 a^2 M w^2}\right], \tag{54}$$

$$Y = \left(\frac{2w^2}{5c_0^2}\right)^{4/3} \left(\frac{\kappa_0 a T_0}{N_0 v_0 a^2 M w^2}\right)^{2/3} \frac{c^2(\xi)}{c_0^2} \tag{55}$$

$$U = \frac{1}{5}\left(\frac{2w^2}{5c_0^2}\right)^{4/3} \left(\frac{\kappa_0 a T_0}{N_0 v_0 a^2 M w^2}\right)^{2/3} \frac{v^2(\xi)}{c_0^2} \tag{56}$$

the energy equation (4) may be written in the reduced form

$$Y^{5/2} \frac{dY}{dX} = X - Y - U. \tag{57}$$

Consider the integration of (57) neglecting the velocity U. As $\xi \longrightarrow \infty$ it is readily seen that X approaches X_∞ where

$$X_\infty \equiv \left(\frac{2w^2}{5c_0^2}\right)^{7/3} \left(\frac{K_0 a T_0}{N_0 v_0 a^2 M w^2}\right)^{2/3} \frac{F_\infty}{N_0 v_0 a^2 M w^2} \tag{58}$$

Then since $c^2(\infty) = 0$, it follows that $Y(X_\infty) = 0$. Since F_∞ approaches zero as N_0 becomes large, it is evident that $X_\infty \longrightarrow 0$. Thus the limiting solution, as $N_0 \longrightarrow \infty$, passes through the origin $Y(0) = 0$. It is readily shown by repeated iteration of (57), taking advantage of the smallness of the left hand side, that the limiting solution is

$$Y(X) \cong X - \left[X - (X - X^{5/2})^{5/2}\left(1 - \frac{5}{2}X^{3/2}\right)\right]^{5/2}$$

$$\times \left[1 - \frac{5}{2}(X - X^{5/2})^{3/2}\left(1 - \frac{5}{2}X^{3/2}\right) + \frac{15}{4}X^{1/2}(X - X^{5/2})^{5/2}\right] \cdots$$

$$\cong X\left[1 - X^{3/2} - 5X^3 - \frac{325}{8}X^{9/2} + O(X^6)\right] \tag{59}$$

in the neighborhood of the origin.

When N_o is large but finite, $Y(X)$ has the form

$$Y(X) \cong \left(\frac{7}{4}\right)^{2/7} \left(X^2 - X_\infty^2\right)^{2/7} \tag{60}$$

as $X \longrightarrow X_\infty$.

An important point to note before discussing $Y(X)$ at large X is that the entire family of solutions $Y(X)$ tend to converge rapidly toward the limiting solution $(N_o \longrightarrow \infty)$ as X increases from X_∞ . To demonstrate this convergence for small X_∞ , consider a solution $Y(X)$ which lies a very small distance ϵ below the limitng solution $Y_1(X)$ at $X = X_\epsilon$ i.e.

$$Y(X_\epsilon) = Y_1(X_\epsilon) - \epsilon \ .$$

Let $\epsilon << X_\epsilon Y(X_\epsilon)$. In the neighborhood of $[X_\epsilon, Y(X_\epsilon)]$ let the solution be represented by

$$Y(X) = Y_1(X) - h(X)$$

so that $\epsilon = h(X_\epsilon)$. Make X_ϵ so small that (59) reduces to

$Y_1(X) \cong X$. Then (57) reduces to

$$\frac{dh}{dX} = 1 - \left(\frac{5}{2X} + \frac{1}{X^{5/2}}\right) h ,$$

upon neglecting all terms second order in h . The desired solution of this equation is

$$h(X) = \frac{1}{X^{5/2}} \exp\left(\frac{2}{3X^{3/2}}\right)\left[\epsilon X_\epsilon^{5/2} \exp\left(-\frac{2}{3X_\epsilon^{3/2}}\right)\right.$$

$$\left. + \int_{X_\epsilon}^{X} ds\ s^{5/2} \exp\left(-\frac{2}{3s^{5/2}}\right)\right]$$

$$\cong \epsilon \left(\frac{X_\epsilon}{X}\right)^{5/2} \exp \frac{2}{3}\left(\frac{1}{X^{3/2}} - \frac{1}{X_\epsilon^{3/2}}\right) \tag{61}$$

Since $X_\epsilon << 1$, it is obvious that $h(X)$ decreases extremely rapidly with increasing X , demonstrating the convergence of the solution toward the limiting solution $Y_1(X)$ for $N_0 \longrightarrow \infty$.

Now consider the solution of (57) as X becomes large. This asymptotic limit is of physical interest because for moderate or low coronal temperature, $w^2/c_0^2 >> 1$ and X becomes large toward the base of the corona, $\xi = 1$.

The asymptotic form of $Y(X)$ for large X is readily obtained from (57) by noting that $Y << X$. The result is

$$Y(X) \sim \left(\frac{7}{4}\right)^{2/7} X^{4/7} \left[1 - \left(\frac{4}{7}\right)^{5/7} \frac{4}{11} \frac{1}{X^{3/7}} \right. \tag{62}$$

$$\left. + O\left(\frac{1}{X^{6/7}}\right)\right].$$

This asymptotic form is valid for all large N_0, as a consequence of the convergence of the solutions toward the limiting solution.

The limiting solution was computed from (57) by numerical methods and is shown in Fig. 4. The other solutions of (57), which do not pass through the origin, are indicated in Fig. 4 by the short arrows which represent the slope dY/dX at various points in the (X, Y) plane. The convergence toward the limiting solution is clearly evident.

The formal asymptotic form (62) reduces directly to (32) upon neglecting F_∞ in (54). This gives formal proof of (33) for large N_0 and w^2/c_0^2. Note that (54) - (56) can be considerably reduced through application of (33). The near region, then, is described by (62) and extends from $X_1 = (7/4)^{2/3}$, corresponding to $\xi = \xi_1$ as defined in (34), to the base of the corona $X_b \cong \xi_1 (7/4)^{2/3}$.

When X becomes small compared to X_1, the solution lies close to the expansion of the limiting form given by (59). The leading term of this expansion gives simply $Y \cong X$, which reduces immediately to (39). The solution follows

$Y \cong X$ until X nears X_∞ . The intermediate region, then, lies between $O(X_\infty)$ and X_1 .

As X begins to approach X_∞ the solution veers away from the limiting solution and follows the asymptotic form (60). With the aid of (33) we write

$$X = X_\infty \left[1 + \frac{F_0}{F_\infty \xi} \right] \tag{63}$$

where, it will be remembered, F_0 is the total energy flux $4\kappa_0 a T_0 / 7$ at $\xi = 1$. Substituting this into (60) and neglecting terms second order in $1/\xi$, it is readily shown that $c^2(\xi) \propto \xi^{-2/7}$. This constitutes the far region.

Now consider the effect of including U in (57). The effect of U is to decrease dY/dX slightly. The decrease in dY/dX is negligible in the near region and throughout much of the intermediate region. The decrease becomes significant only in the outer portions of the intermediate region and in the far region. The result is that the solution $Y(X)$ which includes U starts out at large X essentially coincident with the solution neglecting U . As the far region is approached, it declines less steeply than the solutions which neglect U . At small X the kinetic energy U approaches a constant value, which when subtracted from F_∞ leaves only the conduction flux F_c .

Thus X_∞ must be modified by replacing F_∞ on the right hand side of (48) with F_c. It was noted earlier that $F_c << F_\infty$, with the result that X_∞, at which Y vanishes, is moved much closer to the origin. The solution of (57) including U is sketched in Fig. 5. Noting that $U \longrightarrow 0$ as $N_0 \longrightarrow \infty$, it is evident in this limit that the solution including U converges to the solution neglecting U, and both converge to the solution (59) through the origin.

V. THE ROLE OF THE VARIATION OF $\kappa(T)$

The discussion has thus far been confined to the hypothetical case of a stellar corona in which energy is transported beyond the base of the corona only by thermal conduction for which $\kappa(T)$ is proportional to $T^{5/2}$. There are obviously many other possible forms for the conductivity $\kappa(T)$ in the circumstances encountered in stellar coronas. For instance transverse and disordered magnetic fields may cause $\kappa(T)$ to diminish outward from the star more rapidly than $T^{5/2}$. Or if we admit the possibility of un-ionized gas in, say, the outer atmosphere of a red giant, then $\kappa(T)$ may decline more slowly than $T^{5/2}$ being proportional only to $T^{1/2}$. It is not possible at the present time to state precisely just what effects in $\kappa(T)$ should occur in the solar corona, etc., except that there are several effects suggesting that $\kappa(T)$ falls below the value given by (2) beyond some distance of the order of 0.1 a.u. or more. (See section VI.) Thus at the present time the best approach would seem to be an inquiry into the general effects of deviation of $\kappa(T)$ from (2). Then when we understand the consequences of variation in $\kappa(T)$ in a general way it will be possible to state the results of specific variations in $\kappa(T)$ when they become known.

Several effects turn up in the general inquiry, the principal of which is that reduction in $\kappa(T)$ beyond distances of several stellar radii may have the effect of enhancing coronal expansion. In fact it will be shown that a steady outward expansion of a stellar corona follows from the hydrodynamic equations if, and only if, $\kappa(T)$ declines outward from the star at a suitably rapid rate.

1. Effects of the Form of $\kappa(T)$

The simplest variation in $\kappa(T)$ is a variation in the proportionality constant κ_0. It is readily seen from (4) that κ_0 appears only in the ratio N_0/κ_0, so a reduction in κ_0 is equivalent to an increase in N_0, which was discussed at some length in the preceding section.

The next simplest variations in $\kappa(T)$ are either to introduce a sharp cutoff in $\kappa(T)$ at some fixed radial distance from the star or to assume that the functional dependence of $\kappa(T)$ on T is something other than $T^{5/2}$. The effects of a cutoff in $\kappa(T)$ are considered under the next paragraph heading. The present discussion centers on the functional importance of $\kappa(T)$.

Let it be assumed that

$$\kappa(T) = \kappa_0\left(\frac{c^2}{c_0^2}\right)^{\gamma} \qquad (64)$$

where γ is a numerical constant, with the values $\gamma = 0$ for uniform conductivity, $\gamma = \frac{1}{2}$ for an un-ionized gas, and $\gamma = 5/2$ for a fully ionized gas. To demonstrate the necessary conditions on γ for coronal expansion consider the case that the temperature c_0^2 or the density N_0 at the base of the corona is small enough that the terms involving $N_0 v_0$ on the right hand side of (4) may be neglected. Then subject to the boundary conditions that $c^2(1) = c_0^2$ and $c^2(\infty)$, integration yields

$$c^2(\xi) = \frac{c_0^2}{\xi^{1/(1+\gamma)}} \tag{65}$$

and

$$F_\infty = \frac{\kappa_0 a T_0}{1+\gamma} \tag{66}$$

The discussion of the hydrodynamic momentum equation in Paper I shows that a corona will expand to supersonic velocity if, and only if, the temperature decreases outward less rapidly than $1/\xi$. A more rapid decline gives a static corona. Thus it is necessary and sufficient that $\gamma > 0$ in order for the corona to expand into a supersonic stellar wind. The corona is static if the conductivity is independent of the temperature or if the conductivity increases with decreasing temperature.

There are some circumstances when $\kappa(T)$ may be more a function of position than of the local temperature. Then putting $\kappa(T) = \kappa_0 \xi^\delta$ we have

$$c^2(\xi) = \frac{c_0^2}{\xi^{1+\delta}}$$

and

$$F_\infty = (1+\delta)\,\kappa_0\, a\, T_0$$

for $\delta > -1$.* The corona will expand to yield a supersonic stellar wind only if $\delta < 0$, i.e. only if κ declines outward from the star.

The physical explanation for the outward declining thermal conductivity to favor coronal expansion is simple and straightforward. The coronal expansion leading to the stellar wind occurs in the region between the energy source at the base of the corona and the energy sink at $\xi = \infty$ where the temperature vanishes. If the conductivity declines at large radial distance, the effect is to sever the connection with the cold $\xi = \infty$. The result is a relatively slow outward temperature decline, leading to rapid coronal expansion. On the other hand, if the conductivity is large at large radial distance, the coronal thermal energy drains rapidly to $\xi = \infty$. The result is a rapid outward temperature decline, which discourages coronal expansion. Of course, the heat flow from the base of the corona is larger when the conductivity is large at large radial distance, but the flow passes to $\xi = \infty$ and does not serve to enhance coronal expansion.

It would be a straightforward matter to repeat the calculations of the preceding sections for values of γ other than the $5/2$ already considered, but the complete calculation is probably not of sufficient interest at the

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* If $\delta < -1$, the temperature falls to zero at finite ξ , at which point a heat sink is implied. This is physically unrealistic for the present discussion.

present time to justify presenting it here in its entirety. The low density approximation has already been given. For the low temperature and high density approximations it is interesting to note that

$$c^2(\xi) = \frac{c_0^2}{\xi^{2/(1+\gamma)}} \tag{67}$$

in the near region. The efflux of coronal gas is given by

$$N_0 v_0 a^2 M w^2 = \frac{2}{1+\gamma} K_0 a T_0$$

in analogy to (33). In the far region $c^2(\xi)$ is proportional to $\xi^{1/(1+\gamma)}$, which is just half as fast a decline as in the near region. Of course $c^2(\xi)$ is still proportional to $1/\xi$ in the intermediate region of the high density approximation. It is interesting to note that when $\gamma \leq 1$, the intermediate region disappears for the reason that $c^2(\xi)$ declines at least as rapidly as w^2/ξ. Then with $5c_0^2 < 2w^2$ at the base of the corona, $5c^2(\xi)/2$ remains less than w^2/ξ until the far region is reached where F_∞ becomes non-negligible.

The case $\gamma = 1$ is of formal mathematical interest because the energy equation in the high density approximation (omitting $v^2(\xi)$) can be integrated in closed form (see Appendix II). When $\gamma \leq 0$ the energy equation can be integrated because v_0 is identically zero for the moderate and low temperatures $2w^2/5c_0^2 >> 1$ considered here.

2. Effects of a Cutoff of $\kappa(T)$

There are a number of physical reasons, discussed a little later, why beyond some suitably large distance from the star $\kappa(T)$ may decline rather considerably below the value given by (2). If the decline should be abrupt, its effects can be approximated by introducing a sharp cutoff at some suitable radial distance ξ_κ up to which $\kappa(T)$ has its normal value and beyond which $\kappa(T)$ is identically zero. Such an idealized form for $\kappa(T)$ lends itself to simple presentation of the physical consequences of a rapid decline in $\kappa(T)$.

It is evident from the previous discussion of the effects of the analytical form of $\kappa(T)$ that cutting off $\kappa(T)$ at some distance ξ_κ may enhance coronal expansion. To illustrate the effects note that the motion of the coronal gas beyond ξ_κ is completely adiabatic, because we are considering the hypothetical case that no energy source or sink besides thermal conduction is available above the base of the corona. The properties of an adiabatic corona have been discussed elsewhere (Parker, 1960). Briefly, the atmosphere will be static for moderate and low coronal temperatures unless $\xi_\kappa > 2w^2/5c_0^2$: This may be shown by noting that the temperature in a static corona will be uniform out to ξ_κ ; the adiabatic corona beyond ξ_κ can expand only if c^2 at the base of the adiabatic corona exceeds $2w^2/5\xi_\kappa$.

The really interesting case is when $\xi_\kappa >> 2w^2/5c_0^2$ in a corona of low density. Then in the low density approximation (dropping all terms except F_∞ on the right hand side of (4)) integration of eqn. (4) yields the temperature distribution

$$c^2(\xi) = c_0^2 \left\{ 1 - \left[1 - \left(\frac{c^2(\xi_K)}{c_0^2} \right)^{7/2} \right] \left[\frac{1 - 1/\xi}{1 - 1/\xi_K} \right] \right\}^{2/7} \tag{69}$$

and the energy flux

$$F_0 = \frac{2}{7} K_0 a T_0 \left\{ \frac{1 - \left[c^2(\xi_K)/c_0^2 \right]^{7/2}}{1 - 1/\xi_K} \right\} \tag{70}$$

at $\xi = 1$. The energy is transported entirely by thermal conduction in the low corona and entirely by convection beyond ξ_K, where $c^2(\xi)$ is obtained from the adiabatic condition $v^2(\xi) + 5c^2(\xi) - 2w^2/\xi = v^2(\infty)$. At $\xi = \infty$ the energy flux has all been converted into kinetic energy. Thus

$$F_\infty = F_0 - N_0 v_0 a^2 M w^2 = N_0 v_0 a^2 \frac{1}{2} M v^2(\infty). \tag{71}$$

Now the velocity $v(\infty)$ can be computed from $c^2(\xi)$ by the methods outlined in Paper I. The temperature $c^2(\xi_K)$ will adjust itself so that $v(\infty)$ in (71) will lead to an F_0 which is equal to the value given by (70). In the limit as N_0 becomes small it is evident that F_0

must become small and $c^2(\xi_f) \longrightarrow c_o^2$, i.e. the corona becomes isothermal in $(1, \xi_f)$, with a temperature c_o^2. The dynamics of an isothermal-adiabatic atmosphere have been treated elsewhere (Parker, 1960). A more interesting limit is obtained however if N_o is fixed at some small value and ξ_f is permitted to become large. Then $c^2(\xi)$, as given by (69), declines rapidly from c_o^2 to $c^2(\xi_f)$ close to the star and remains nearly isothermal with a temperature near $c^2(\xi_f)$ from there all the way out to ξ_f. For an approximately isothermal atmosphere it is readily shown from the momentum equation that

$$v(\infty) \sim 2\, c(\xi_f)\, \ln^{1/2}\left[\frac{2c^2(\xi_f)\,\xi_f}{w^2}\right] \tag{72}$$

for large ξ_f.* We note that for a given value of $c(\xi_f)$, the velocity $v(\infty)$ increases without bound as ξ_f becomes large. The velocity v_o, on the other hand is essentially independent of ξ_f when ξ_f is large. Hence, it may be seen that with small N_o (71) gives $F_\infty \cong F_o$. Combining (70) and (71) yields

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* The approximations involved are (a) that $c(\xi)$ is not precisely $c(\xi_f)$ near the star, where $c(\xi) \longrightarrow c_o$ and (b) $c(\xi)$ is not precisely $c(\xi_f)$ as ξ becomes comparable to ξ_f because the convection term $v^2 + 5c^2 - 2w^2/\xi$ on the right hand side of (4) is not negligible there. Neither of these approximations is essential to the present argument, however.

$$N_o v_o a^2 \frac{1}{2} M v^2(\infty)$$

$$\cong \frac{2}{7} \kappa_o a T_o \left\{ 1 - \left[\frac{c^2(\xi_\kappa)}{c_o^2} \right]^{7/2} \right\}. \qquad (73)$$

For any given $c^2(\xi_\kappa)$ the left hand side of this equation can be made as large as desired by making ξ_κ sufficiently large. Thus, no matter how small may be N_o, the left hand side of (73) can be made significantly greater than zero, with the result that $c^2(\xi_\kappa)$ on the right hand side is required by (73) to be significantly less than c_o. Note however that $c^2(\xi_\kappa)/c_o^2$ appears on the right hand side to such a high power that if $c(\xi_\kappa)$ is even ten percent less than c_o, the term in $c^2(\xi_\kappa)/c_o^2$ may be neglected and

$$N_o v_o a^2 \frac{1}{2} M v^2(\infty) \cong \frac{2}{7} \kappa_o a T_o \qquad (74)$$

This equation tells us that for small N_o, almost the entire energy flux $2\kappa_o a T_o / 7$ is converted into kinetic energy as ξ_κ becomes large. Consequently, if N_o is small, the resulting $v(\infty)$ may be enormous. Further, the velocity v_o is determined principally by $c^2(\xi_\kappa)$ and diminishes rapidly with declining $c^2(\xi_f)$, as noted earlier,

$$\frac{v_0}{c(\xi_f)} \sim \left[\frac{w^2}{2c^2(\xi_f)}\right]^2 \exp\left[\frac{3}{2} - \frac{w^2}{c^2(\xi_f)}\right]. \tag{75}$$

Then if ξ_f is caused to become large without limit, $v(\infty)$ in (72) becomes large without limit and $c(\xi_f)$ diminishes in order to reduce v_0 sufficiently to keep (74) satisfied. The point of this is that, within the framework of the formal heat flow equation (4), the cutoff of $\kappa(T)$ at some extremely large radial distance ξ_f can lead to an arbitrarily large stellar wind velocity $v(\infty)$. This is true independently of the density N_0. The only effect of N_0 is that for a larger N_0, v_0 must be correspondingly smaller so that $c(\xi_f)$ is smaller and ξ_f larger in order to achieve the same $v(\infty)$

VI. SUMMARY AND DISCUSSION

The present paper has concerned itself with the dynamics of the stellar corona in which heat is supplied above the base of the corona solely by thermal conduction. The corona of the sun may, or may not, be an example of pure conductive heating. One of the first points that should be taken up in the discussion of the formal mathematical examples presented above is the question of the general validity of eqn. (2) for the thermal conductivity. Eqn. (2) was derived for an infinitesimal heat flow in the direction parallel to magnetic field (Chapman and Cowling, 1958). Taking these two assumptions one at a time, we note that an absolute upper limit to the heat flux is obtained by assuming that all of the electrons are moving in the direction of the heat flow with the rms thermal velocity. The resulting heat flux would then be $(3NkT/2)(3kT/m)^{1/2}$ ergs/cm^2 sec. This maximum energy transport may be rather small if the coronal density is very low, and eqn.(2) is not valid if the actual flux begins to approach the maximum. The heat flux in a stellar corona is of the general order of $\kappa(T)\, r\, T(r)$ ergs/sec steradian, at a radial distance r provided that the corona is dense enough to transport it. We must require then that

$$\kappa(T)\, r\, T(r) \ll \frac{3}{2} N(r)\, k\, T(r) \left(\frac{3kT}{m}\right)^{1/2} r^2. \qquad (76)$$

If it be assumed that $\kappa(T)$ is given by (2), the inequality may be written in the numerical form

$$N(r) \gg 4 \times 10^3 \, \frac{T^2(r)}{r} \quad /cm^3 \qquad (77)$$

This inequality must be satisfied or else the effective value of the thermal conductivity will be depressed below eqn. (2); the effective conductivity $\kappa(T)$ will be enough so that (76) is satisfied. Now the numerical values indicate that except in an extremely tenuous corona (77) will be satisfied near the star. On the other hand unless $T(r)$ drops off as fast as $1/r^{1/2}$ at large r, (77) will not be satisfied as $r \to \infty$ because $N(r)$ is asymptotically proportional to $1/r^2$. These circumstances may be illustrated by the numerical values for the solar corona. In the low corona where $r = 10^6$ km and $T \cong 10^6\ {}^\circ K$ the inequality is $N \gg 6 \times 10^4/cm^3$, which is satisfied by a large margin. On the other hand, at the orbit of Earth where $r = 1.5 \times 10^{13}$ cm and $T(r)$ seems to be of the general order of $10^5\ {}^\circ K$ (Bonetti, et al, 1962; Neugebauer and Snyder, 1962) the inequality becomes $N \gg 3/cm^3$ which is just barely satisfied, if at all. Observations (see for instance Bonetti, et al, 1962) suggest that $N \cong 2 - 10/cm^3$. A slightly higher temperature of $3 \times 10^5\ {}^\circ K$ yields $N \gg 24/cm^3$, which is not satisfied. The conclusion is that, in the absence of any other effects, the low density of the solar wind must lead to a depression of the thermal conductivity below eqn. (2) at least at some distance beyond the orbit of Earth. The conductivity may often be depressed even inside the orbit of Earth. More complete and quantitative temperature and density measurements of the interplanetary plasma will have to be carried out before more can be said.

Whatever may be the circumstances for the sun, it is evident that some caution must be exercised in application of the formal examples worked out in the text. The low density approximation and the cutoff with ξ_5 arbitrarily large are particularly suspect when it comes to actual models of existing coronas. As was stated in the beginning, the purpose of the formal models has been to illustrate the properties of the formal heat flow and momentum equations in a hypothetical stellar corona, concerning which there have been so many mistaken ideas.

The second effect on the thermal conductivity is the well known channeling by magnetic fields. The reduction of the effective thermal conductivity in the direction perpendicular to the local magnetic field is by the factor $\left(1+\omega_{ce}^2 t_{De}^2\right)$ (Chapman and Cowling, 1958) when the electrons are responsible for most of the thermal conduction. Here ω_{ce} is the electron cyclotron frequency and t_{De} is the deflection time for Coulomb collisions (Spitzer, 1956). When this reduction is by more than a factor of 10^2 it can be shown (Rosenbluth and Kaufmann, 1958; Vaughn-Williams and Haas, 1961) that the ions take over from the electrons and the reduction is not as large as the factor $\left(1+\omega_{ce}^2 t_{De}^2\right)$. In the low solar corona where $N \sim 10^8/cm^3$ the time between electron Coulomb collisions is of the order of 10^{-1} sec, whereas a magnetic field of 1 gauss yields a cyclotron frequency of the order of 10^7 radians/sec. The reduction of the thermal conductivity perpendicular to the field is clearly enormous. The same is true at the orbit of Earth and beyond. Altogether then, we may conclude that in stellar coronas and stellar winds the flow of energy by thermal conduction is channelled almost entirely along the magnetic fields. It follows at once that if the lines of force should become sinuous and generally non-radial, as they are observed to be in interplanetary space (McCracken, 1962; Smith et al, 1963), the effective path length will be increased and the cross section decreased, with a corresponding reduction in heat transport. It follows that if the field should become completely disorganized, as it appears to be at some distance beyond the orbit of Earth (Meyer, et al, 1956), the flow of heat might be cut off altogether. The result of reducing or cutting off thermal conduction in interplanetary space was shown to enhance the expansion of the corona.

Now to summarize the results of the formal calculations. Assuming that the thermal conductivity is of the general form given by (2) it was shown that the temperature in a corona of extremely low density declines outward like $r^{-2/7}$. At higher densities but for low temperature, a near region where the temperature declines like $r^{-4/7}$ appears at the base of the corona and extends for some distance outward before the decline goes over to $r^{-2/7}$. If the density is high and the temperature is not low, an intermediate region where the temperature declines like r^{-1} appears between the near $r^{-4/7}$ and far $r^{-2/7}$ regions. It was shown, from the fact that a sufficiently low coronal temperature reduces all these cases to the first, that the energy flow to $r = \infty$ is non-vanishing. Hence in all cases the coronal expansion becomes supersonic at large r to form a stellar wind. It was shown that the tendency of thermal conductivity to decline with decreasing temperature plays an essential role in bringing about supersonic coronal expansion. Coronal expansion is generally enhanced by any mechanism which tends to decrease the effective thermal conductivity at large distances from the star. It can be asserted that so long as thermal conductivity is present the temperature declines outward from a star enough less rapidly than $1/r$ that a supersonic stellar wind is the result. Only in the limit of large $N_o/K(T_o)$ is there a possibility of limiting coronal expansion to subsonic velocities. This limiting case may perhaps prove to be of interest in the expansion of the coronas of some red giants where the low coronal temperature leads to very small $K(T_o)$

Numerical estimates for the solar corona suggest that it is neither so tenuous, nor so cool, now so dense that it can be approximated by any of the three cases cited above. Nor is it so hot as to approach the adiabatic case. Rather the sun seems to lie solidly in the middle ground where much, but not all, of the energy transported by thermal conduction is consumed in the expansion. The corona is hot enough that the ultimate expansion velocity is not only supersonic but it is comparable to the gravitational escape velocity from the base of the corona. Thus, simple models, such as the isothermal corona, or the isothermal-adiabatic corona (Parker, 1958, 1960) may be used to fit empirical data, but numerical methods, such as employed by Noble and Scarf (1963), are required to deduce quantitative conduction models of the solar corona and solar wind from the mass, momentum, and energy conservation equations.

Several of the qualitative conclusions resulting from the high and low density approximations may prove to be of interest for understanding the changes in the solar corona and solar wind over the 11 or 22-year cycle of solar activity. For instance, for a fixed coronal temperature T_0 the stellar wind velocity $v(\infty)$ declines to zero in the limit as $N_0 \longrightarrow \infty$. On the other hand $v(\infty)$ becomes comparable to the gravitational escape velocity $2^{1/2} w$ in the limit as N_0 becomes small. Note also that for fixed $N_0 / K(T_0)$ the stellar wind velocity varies approximately as $T_0^{1/2}$. Now it is observed that both the temperature and the density in the low solar corona tend to decline during the years of minimum solar activity, particularly at high solar latitudes. One expects a simultaneous decline in the density of the solar wind and a decline in the solar wind flux $N_0 v_0 a^2$. But it is not at all clear to what extent the solar wind velocity will decline, because the decline of N_0 tends to <u>increase</u> the velocity and the decline of T_0 tends to <u>decrease</u> the velocity. There is the additional possibility that with the declining density the thermal conductivity may cut off more completely at, say, one or two a.u., which would further enhance $v(\infty)$. It is not inconceivable that the net effect could be an <u>increase</u> in the solar wind velocity sometime during the years of low solar activity and a decrease of velocity sometime during the years of high solar activity. Such an effect might seem surprising, but at the present moment we cannot rule it out. Observations over the next 22-year solar cycle would settle the matter. The fundamental theoretical question concerns the relation between N_0 and T_0 in a stellar corona. Presumably N_0 and T_0 are determined by the wave dissipation which heats the corona, and while considerable progress has been made in this field (see for instance Osterbrock, 1961 and Whitaker, 1963 and references therein) there is not yet any quantitative result of which we can be sure.

When we come to consider the coronas of stars which are different from the sun, only the most general speculations can be made (see for instance Parker, 1960, 1963b). Let it be assumed that there are stars in which the energy transport outward from the base of the corona is principally thermal conduction. From the general restriction that $w^2/c_0^2 > 5$ it follows that c_0^2 must be very small for giant stars, and probably very large for dwarfs. The total heat flow is proportional to $\kappa(T_0)\, a\, T_0$, so that if $c_0^2 \propto 1/a$ we have that the heat flow is proportional to $a^{-5/2}$. This may be very small for giants and large for dwarfs, suggesting that coronal expansion in giants may tend to fall into the high density category, and dwarfs into the low density and/or the low temperature category. It would be expected that some of the particularly active stars might fall into the high temperature category with a violent and nearly adiabatic expansion.

APPENDIX I

In order to evaluate the integral defined in eqn. (24) for the temperature $c^2(\xi)$ given by (11), put

$$I(\xi) \equiv \frac{w^2}{c_o^2} \int_1^{\xi} \frac{du}{u^2 \left[(1-Q)/u + Q/u^2\right]^{2/7}}$$

Then with $z \equiv u(1-Q)/Q$,

$$I(\xi) = \frac{w^2 (1-Q)^{3/7}}{c_o^2 \, Q^{5/7}} \int_{(1-Q)/Q}^{\xi(1-Q)/Q} \frac{dz}{z^{10/7} (1+z)^{2/7}} .$$

Note that by expanding the integrand in ascending powers of z it can be shown that

$$\int_x^y \frac{dz}{z^{1-m} (1+z)^{m+n}} =$$

$$\frac{1}{m} \left\{ y^m F(m, m+n; m+1; -y) - x^m F(m, m+n; m+1; -x) \right\}$$

when $x, y < 1$. Here F represents the hypergeometric function. When y exceeds 1.0, it is convenient to use the well known relation

$$\frac{y^m}{m} F(m, m+n; m+1; -y) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$- \frac{1}{ny^n} F\left(n, m+n; n+1; -\frac{1}{y}\right).$$

A similar expression may be used when x exceeds 1.0. With $m = -3/7$ and $n = 5/7$, it follows that

$$I(\xi) = \frac{7w^2}{3c_0^2 Q^{2/7}}\left[F\left(-\frac{3}{7}, \frac{2}{7}; \frac{4}{7}; -\frac{1-Q}{Q}\right)\right.$$

$$\left. - \frac{1}{\xi^{3/7}} F\left(-\frac{3}{7}, \frac{2}{7}; \frac{4}{7}; -\frac{(1-Q)\xi}{Q}\right)\right]$$

when $(1-Q)\xi/Q < 1$. If $(1-Q)/Q < 1$ but $(1-Q)\xi/Q > 1$, then

$$I(\xi) = \frac{7w^2}{3c_0^2 Q^{2/7}}\left[F\left(-\frac{3}{7}, \frac{2}{7}; \frac{4}{7}; -\frac{1-Q}{Q}\right)\right.$$

$$- \frac{3}{5}\left(\frac{Q}{1-Q}\right)^{2/7} \frac{1}{\xi^{5/7}} F\left(\frac{5}{7}, \frac{2}{7}; \frac{12}{7}; -\frac{Q}{(1-Q)\xi}\right)$$

$$\left. - \frac{7}{10}\left(\frac{1-Q}{Q}\right)^{3/7} \frac{\Gamma(11/7)\Gamma(12/7)}{\Gamma(9/7)}\right].$$

Finally, when $(1-Q)/Q > 1$,

$$I(\xi) = \frac{7w^2}{5c_0^2(1-Q)^{2/7}}\left[F\left(\frac{5}{7}, \frac{2}{7}; \frac{12}{7}; -\frac{Q}{1-Q}\right)\right.$$

$$\left. - \frac{1}{\xi^{5/7}} F\left(\frac{5}{7}, \frac{2}{7}; \frac{12}{7}; -\frac{Q}{(1-Q)\xi}\right)\right].$$

For purposes of numerical computation note that

$$F\left(-\frac{3}{7}, \frac{2}{7}; \frac{4}{7}; -x\right) = 1 + \frac{3}{14}x - \frac{27}{539}x^2 + \frac{8}{343}x^3$$

$$- \frac{628}{60025}x^4 + \frac{621}{67228}x^5 + \cdots$$

$$F\left(\frac{5}{7}, \frac{2}{7}; \frac{12}{7}; -x\right) = 1 - \frac{5}{42}x + \frac{45}{931}x^2 - \frac{120}{4459}x^3 + \cdots$$

Since according to (16) we have $(1-Q)\xi_c/Q >> 1$, it follows that with $\epsilon \equiv Q/(1-Q)\xi_c$,

$$I(\xi_c) \cong \frac{7w^2}{5c_0^2(1-Q)^{2/7}}\left\{F\left(\frac{5}{7}, \frac{2}{7}; \frac{12}{7}; -\frac{Q}{1-Q}\right)\right.$$

$$\left. - \frac{1}{\xi_c^{5/7}}\left[1 + O(\epsilon)\right]\right\}$$

for $Q < (1-Q)$ and

$$I(\xi_c) \cong \frac{7w^2}{3c_0^2 Q^{2/7}} \left\{ F\left(-\frac{3}{7}, \frac{2}{7}; \frac{4}{7}; -\frac{1-Q}{Q}\right) \right.$$

$$- \frac{7}{10}\left(\frac{1-Q}{Q}\right)^{3/7} \frac{\Gamma(11/7)\,\Gamma(12/7)}{\Gamma(9/7)}$$

$$\left. - \frac{3}{5}\left(\frac{Q}{1-Q}\right)^{2/7} \frac{1}{\xi_c^{5/7}} \left[1 + O(\epsilon)\right] \right\}$$

for $(1-Q) < Q$.

To evaluate the integral defined in eqn.(24) for the temperature given by (11),

$$J(\xi) = c_0^2 \int_{\xi_c}^{\xi} \frac{du}{u} \left[\frac{1-Q}{u} + \frac{Q^2}{u^2}\right]^{2/7},$$

put

$$w = \frac{u\,\xi_c(1-Q)}{Q}.$$

Then

$$J(\xi) = \frac{c_0^2(1-Q)^{4/7}}{Q^{2/7}} \int_1^{\xi/\xi_c} \frac{dw}{w^{11/7}} (1+w)^{2/7}.$$

It follows at once that

$$J(\xi) = 7c_0^{\,2}(1-Q^2)^{2/7}\left\{\frac{1}{\xi_c^{\,2/7}}F\left[\frac{2}{7}, -\frac{2}{7}; \frac{9}{7}; -\frac{Q}{(1-Q)\xi_c}\right]\right.$$

$$\left. - \frac{1}{\xi^{2/7}}F\left[\frac{2}{7}, -\frac{2}{7}; \frac{9}{7}; -\frac{Q}{(1-Q)\xi}\right]\right\}$$

$$\cong 7c_0^{\,2}(1-Q)^{2/7}\left(\frac{1}{\xi_c^{\,2/7}} - \frac{1}{\xi^{2/7}}\right).$$

APPENDIX II

When the thermal conductivity $\kappa(T)$ is proportional to the first power of the temperature, the energy equation may be written

$$y \frac{dy}{dx} = x - \lambda y \tag{A1}$$

where

$$y = Q \frac{c^2}{c_0^2} \tag{A2}$$

$$x = \frac{Q^2}{\xi} + \frac{F_\infty}{\kappa_0 a T_0} \tag{A3}$$

and

$$\lambda = \frac{5 c_0^2}{2 w^2} Q \tag{A4}$$

with

$$Q \equiv \frac{N_0 v_0 a^2 M w^2}{\kappa_0 a T_0} \,. \tag{A5}$$

To integrate this differential equation let $y = v x$ and $z = \ln x$. The result may be written

$$(v_2 x - y)^{v_2} (v_1 x + y)^{v_1} = C \tag{A6}$$

where C is the integration constant and

$$\left. \begin{aligned} v_1 &\equiv \tfrac{1}{2}\left[(4+\lambda^2)^{1/2} + \lambda\right], \\ v_2 &= \tfrac{1}{2}\left[(4+\lambda^2)^{1/2} - \lambda\right]. \end{aligned} \right\} \tag{A7}$$

The solutions through the origin are $C = 0$ and $y = v_2 x$, $y = -v_1 x$. The solution of physical interest is the one for which $c^2(\infty) = 0$. This requires that y vanish at $x = x_\infty$ where

$$x_\infty \equiv \frac{F_0}{\kappa_0 a T_0}, \tag{A8}$$

in which case

$$C = v_1^{v_1} v_2^{v_2} x_\infty^{v_1 + v_2}. \tag{A9}$$

For a dense corona, $x_\infty << 1$ as a consequence of the smallness of F_∞. Hence $C << 1$ and the solution rapidly approaches the limiting solution

$y = \nu_2 x$ as x increases.

$$y \cong \nu_2 x \left\{ 1 - \left[\left(\frac{\nu_1}{\nu_1 + \nu_2} \right)^{\nu_1} \left(\frac{x_\infty}{x} \right)^{\nu_1 + \nu_2} \right]^{1/\nu_2} + \cdots \right\}. \tag{A10}$$

In the neighborhood of x_∞, the solution approximates to

$$y \cong \left[2 x_\infty (x - x_\infty) \right]^{1/2} \tag{A11}$$

which is to say that $c^2 \propto \xi^{-1/2}$ in the far region.

The boundary condition $c^2(1) = c_0^2$ serves to determine Q for a given w^2/c_0^2 and x_∞. The quantity x_∞ can be determined only by simultaneous solution with the momentum approximation. In the low density approximation this is a simple matter, as was pointed out in the text. In the high density approximation it may be effected for moderate or low coronal temperatures, for in that case x_∞, which is nonvanishing for all finite N_0, goes to zero in the limit as $N_0 \longrightarrow \infty$ and (A6) reduces to

$$y = \nu_2 x \tag{A12}$$

In the limit as $N_0 \to \infty$ we have $x = Q^2$ and $y = Q$ where $c^2 = c_0^2$, $\xi = 1$. Then with $\lambda = Q(5c_0^2/2w^2)$ it is readily shown with the aid of (A7) that

$$Q = \frac{1}{(1 - 5c_0^2/2w^2)^{1/2}}, \tag{13}$$

which reduces to

$$\frac{N_0 v_0 a^2 M w^2}{K_0 a T_0} \cong 1 \tag{14}$$

for moderate or low coronal temperatures. This is in agreement with (68) in the text since $\gamma = 1$ in the present case. Combining (A4) and (A13) there results the useful relation

$$\frac{w^2}{c_0^2} = \frac{5}{4}\left[1 + \left(1 + \frac{4}{\lambda^2}\right)^{1/2}\right]$$

in the limit of large N_0

The solution of the energy equation for $\gamma = 1$ has the interesting property that the temperature is proportional to $1/\xi$ in both the near region and the intermediate region. Thus

$$c^2(\xi) \cong \frac{c_0^2}{\xi} \tag{A15}$$

all the way to the neighborhood of x_∞ where the far region begins. This may be seen from (67) or from (A12). It is readily shown from eqn. (12) of Paper I that throughout this entire region

$$n(\xi) \cong \frac{1}{\xi^{w^2/c_0^2 - 1}} \tag{A16}$$

and

$$v(\xi) \cong v_0 \, \xi^{w^2/c_0^2 - 3} . \tag{A17}$$

The velocity becomes comparable to $c(\xi)$ in the general vicinity of

$$\xi_2 \cong \left(\frac{c_0}{v_0}\right)^s \tag{A18}$$

where

$$S = \frac{1}{w^2/c_0^2 - 5/2} ,$$

and must be included in the energy equation from ξ_2 on and out to $\xi = \infty$. The critical point lies at some radial distance of the same order as ξ_2 and the expansion velocity is supersonic beyond.

The solutions of (A5), with C given by (A9) are plotted in Fig. 6 for the moderate coronal temperature λ = 0.25, for which ν_1 = 1.133, ν_2 = 0.883. The limiting case of very high coronal density corresponds to the solution through the origin, $x_\infty = 0$. The value of w^2/c_0^2 for this case is 11.3. Solutions for $x_\infty > 0$ (finite N_0) are plotted in Fig. 6 to show their rapid convergence toward the limiting solution (A12) with increasing x.

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FIGURE CAPTIONS

Fig. 1. A plot of ξ_c, $v_1(\infty)/c_0$, $v_1(\xi_c)/c_0$, and $(v_0^2/c_0^2)_1^{1/2}$ in the low temperature approximation as a function of w^2/c_0^2 for the three cases Q = 0, 4/7, and 1.0. The case Q = 0 represents diversion of only a vanishing portion of the energy transported by thermal conduction into lifting the expanding corona in the gravitational field of the star; $Q = 1$ represents diversion of all but a vanishing portion of the energy transported by thermal conduction; Q = 4/7 represents an intermediate case where about half the conduction energy is diverted into coronal expansion. The break in the curve of $(v_0^2/c_0^2)_1^{1/2}$ for Q = 4/7 results from the use of the asymptotic expression for $w^2/c_0^2 > 10$ for the purpose of illustrating the order of the approximation involved.

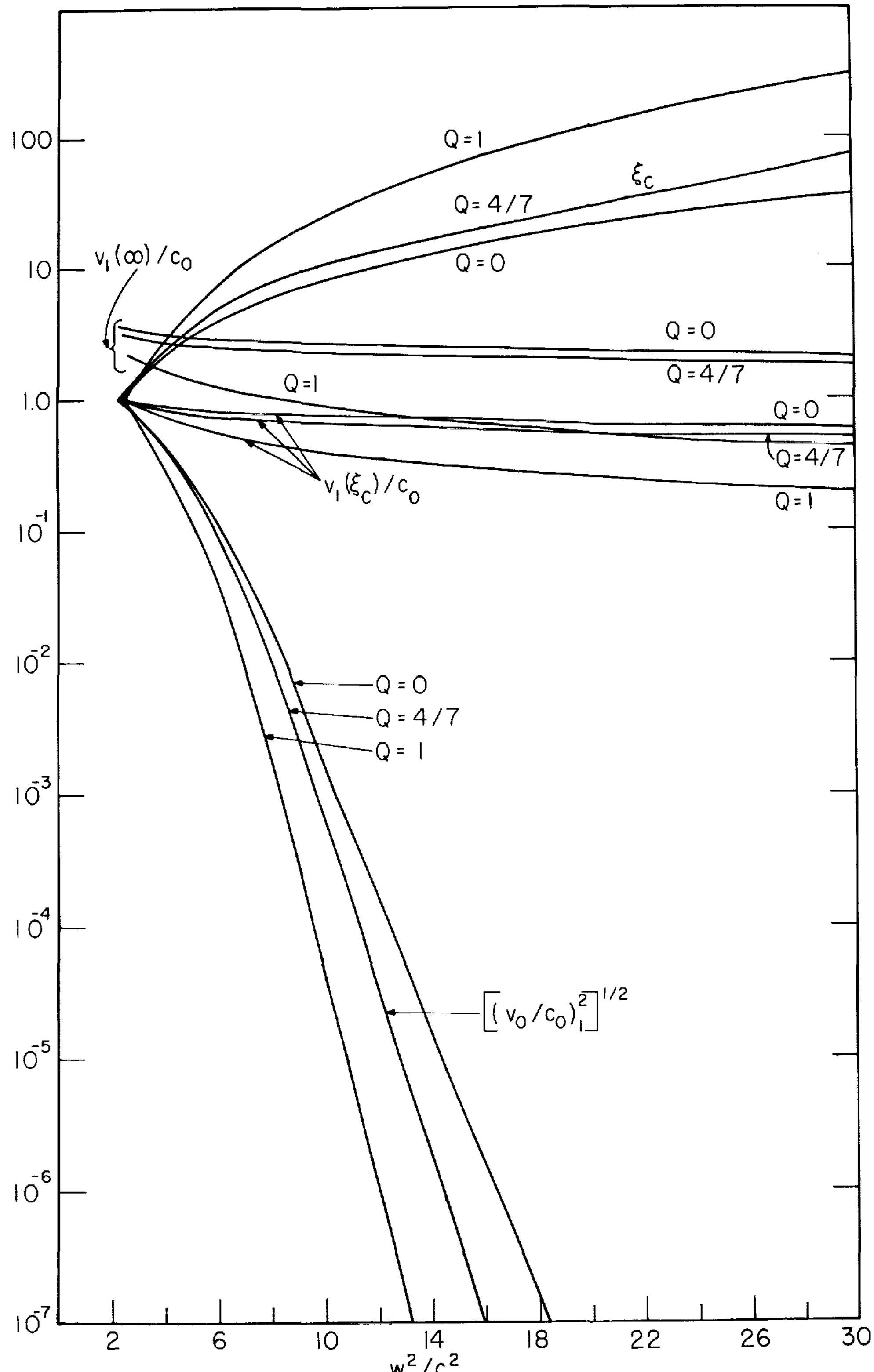
Fig. 2. A plot of the asymptotic density $n(\xi)\,\xi^2$ for large ξ in the low temperature approximation as a function of w^2/c_0^2.

Fig. 3. A plot of the energy flux $2K_0 a T_0/7$ from the base of the corona in the low density approximation as a function of temperature T_0 in comparison with the energy $N_0 v_0 a^2 M w^2$ for various values of N_0 consumed in lifting an isothermal expanding corona of the same temperature T_0 in the gravitational field of the star. The numerical values apply to the sun with (a) $a = 1 R_\odot = 7 \times 10^5$ km and (b) $a = 4 R_\odot$.

Fig. 4. A plot of the limiting solution $Y_1(X)$ as $N_0 \rightarrow \infty$. The short lines indicate the tangents to the family of solutions of (57) (neglecting U) through the positions of the lines.

Fig. 5. The light lines represent a sketch of the solutions of (57) neglecting U, as approximated by (59) and (61). The heavy line illustrates how the solution of (57) including U cuts across those solutions and reaches the X-axis at a small value of X.

Fig. 6. A plot of $y(x)$ from (A5) and (A9) for the intermediate value $\lambda = 0.25$ and for $x_\infty = 0, 0.01, 0.02$, etc.



Q=1
ξ_c
Q=4/7
Q=0
$v_1(\infty)/c_0$
Q=0
Q=4/7
Q=1
Q=0
Q=4/7
Q=1
$v_1(\xi_c)/c_0$
Q = 0
Q = 4/7
Q = 1
$\left[(v_0/c_0)_1^2\right]^{1/2}$
100
10
1.0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
2
6
10
14
18
22
26
30
w^2/c_0^2

Fig. 1

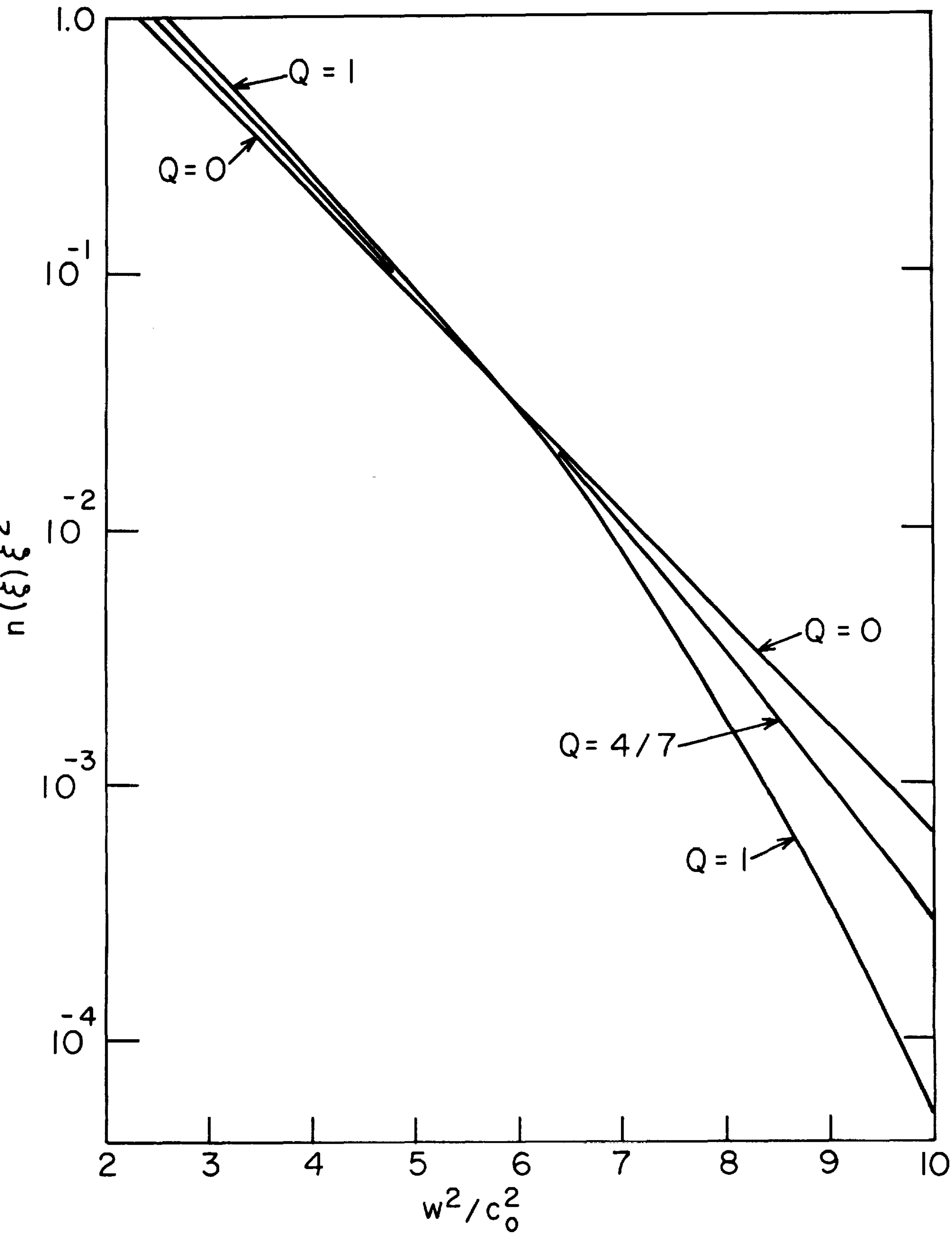
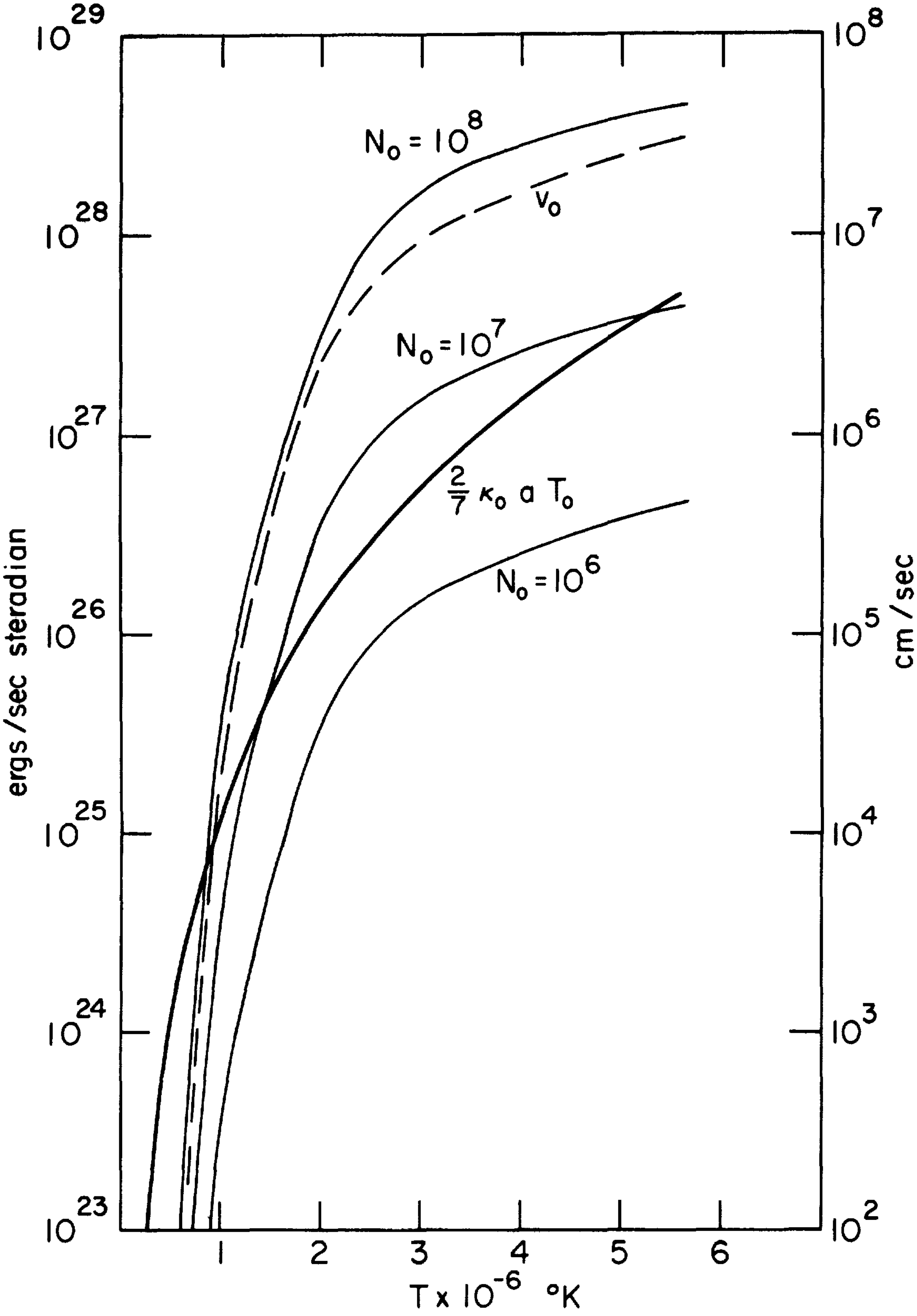
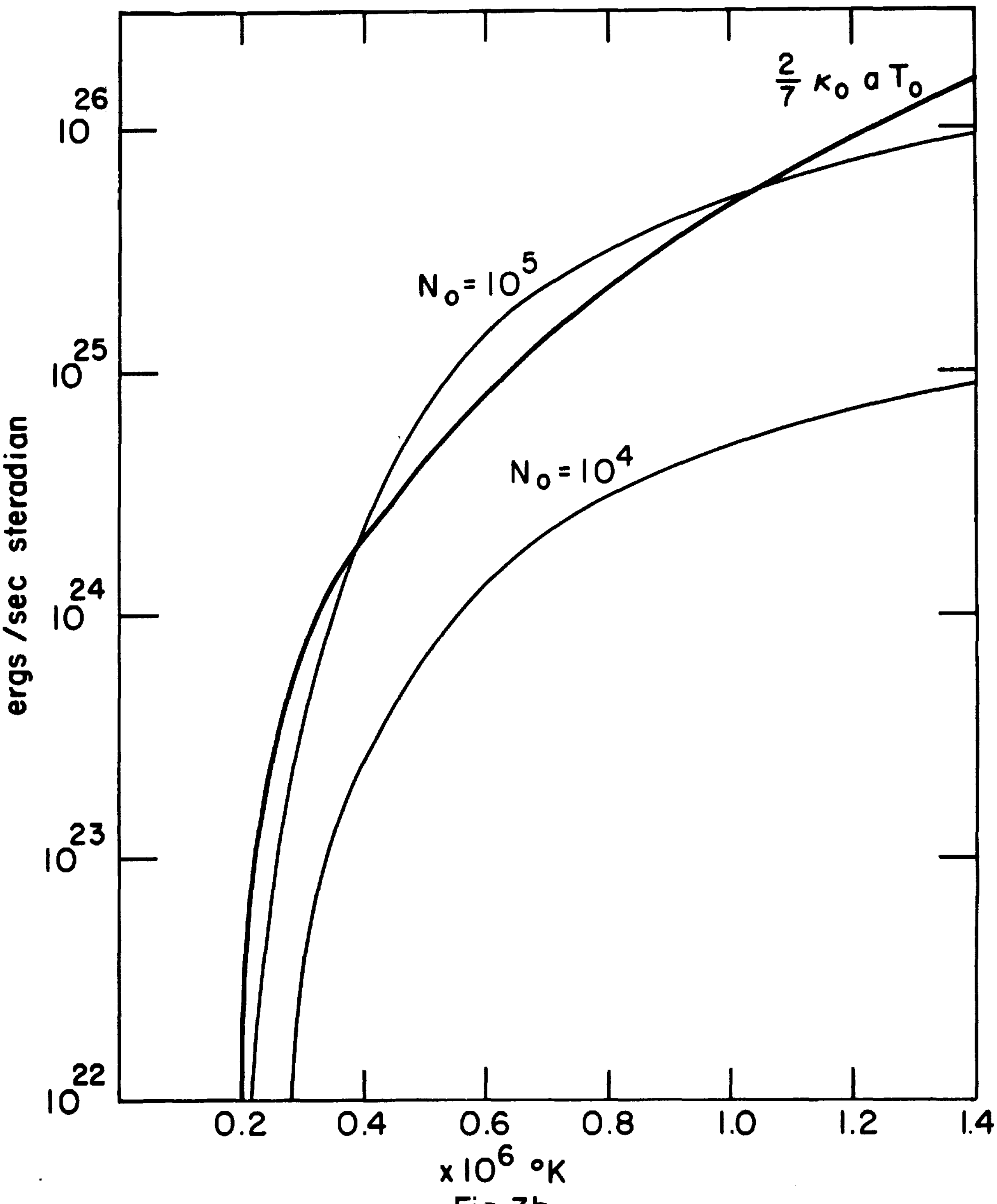


Fig. 2



10^{29}
10^{28}
10^{27}
10^{26}
10^{25}
10^{24}
10^{23}
ergs/sec steradian
10^8
10^7
10^6
10^5
10^4
10^3
10^2
cm/sec
$N_o = 10^8$
v_o
$N_o = 10^7$
$\frac{2}{7}\kappa_o\ a\ T_o$
$N_o = 10^6$
1
2
3
4
5
6
$T \times 10^{-6}$ °K

Fig. 3a



$\frac{2}{7}\kappa_0 a T_0$
$N_o = 10^5$
$N_o = 10^4$
ergs /sec steradian
10^{26}
10^{25}
10^{24}
10^{23}
10^{22}
0.2
0.4
0.6
0.8
1.0
1.2
1.4
$\times 10^6$ °K

Fig. 3b

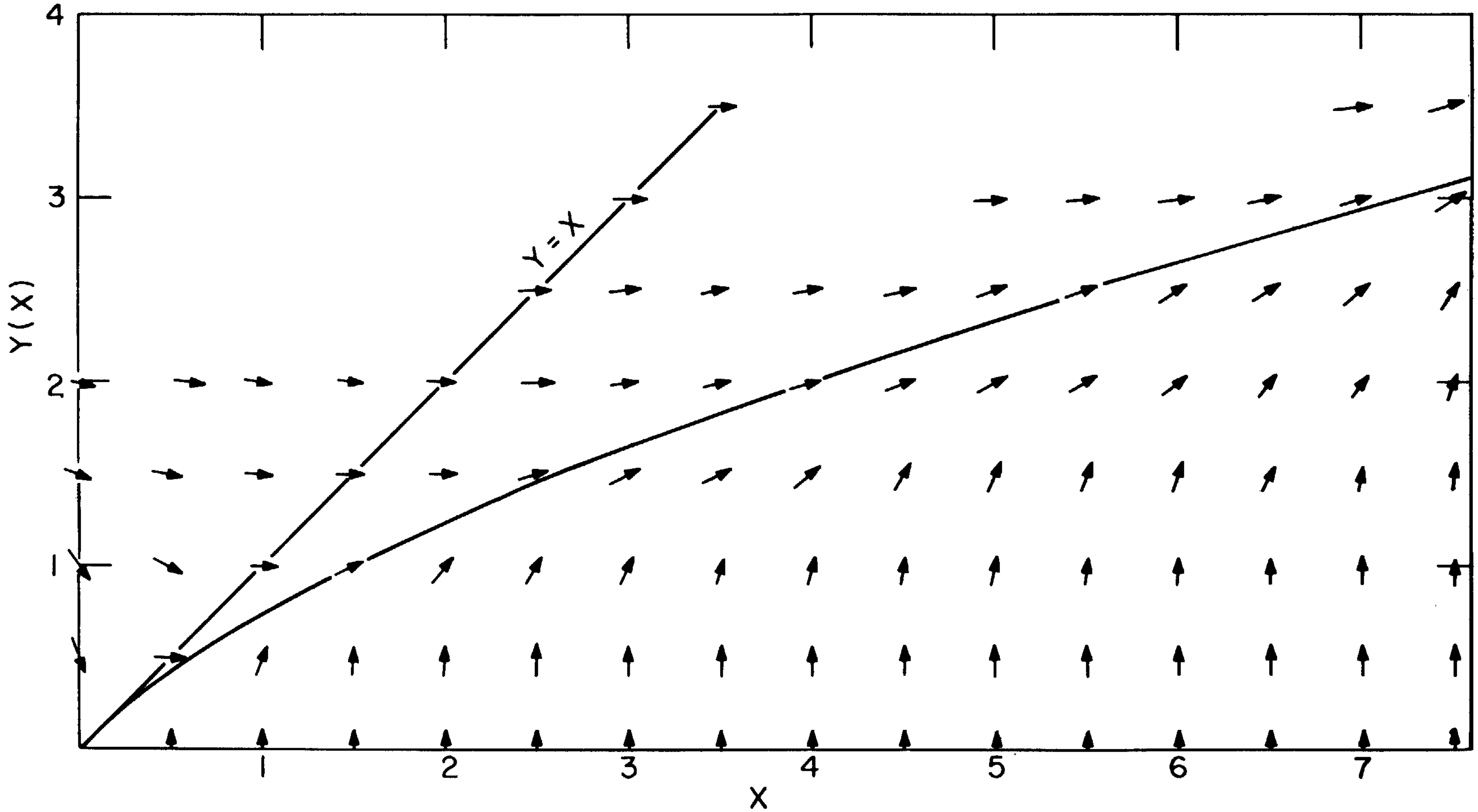


Fig.4

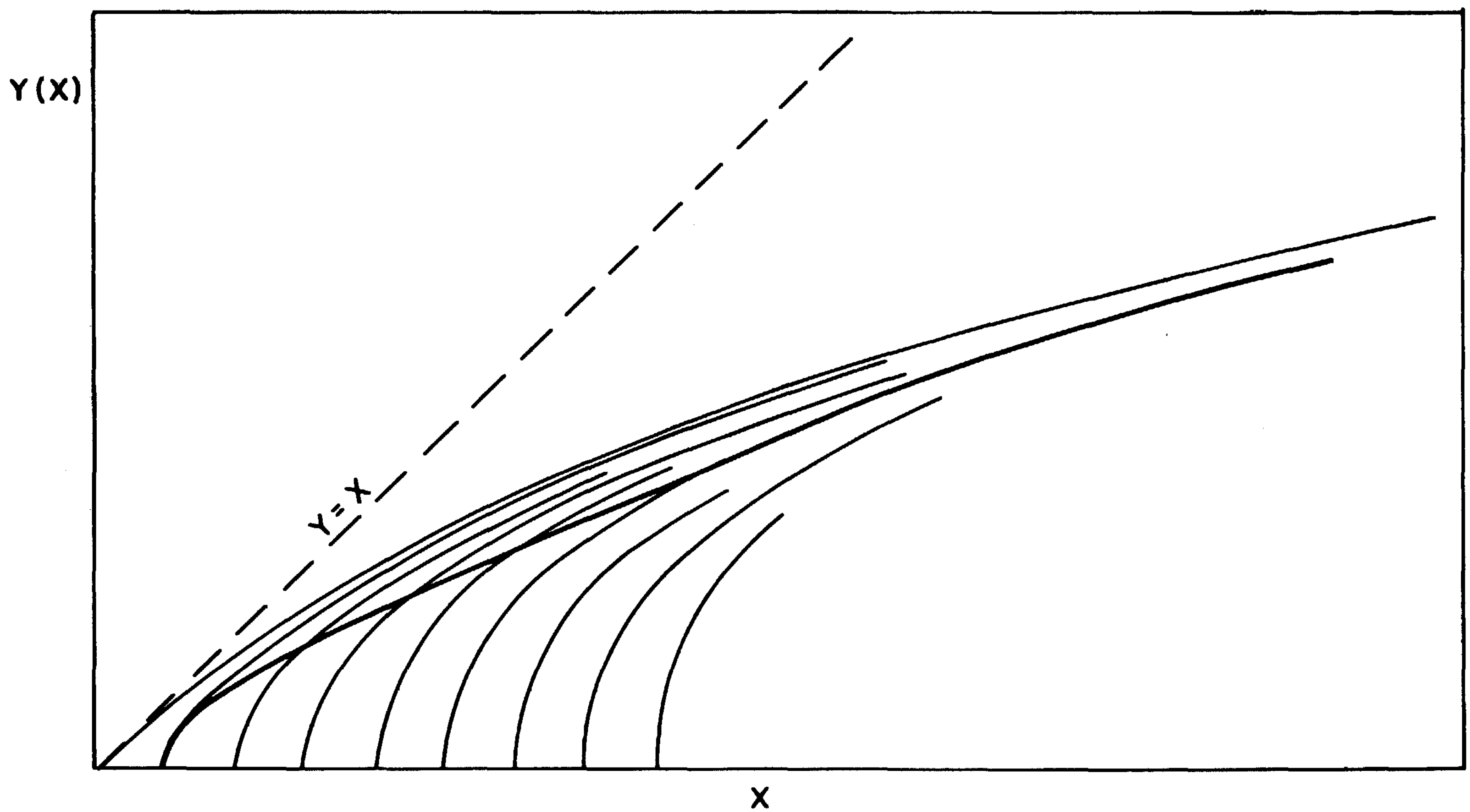


Fig. 5

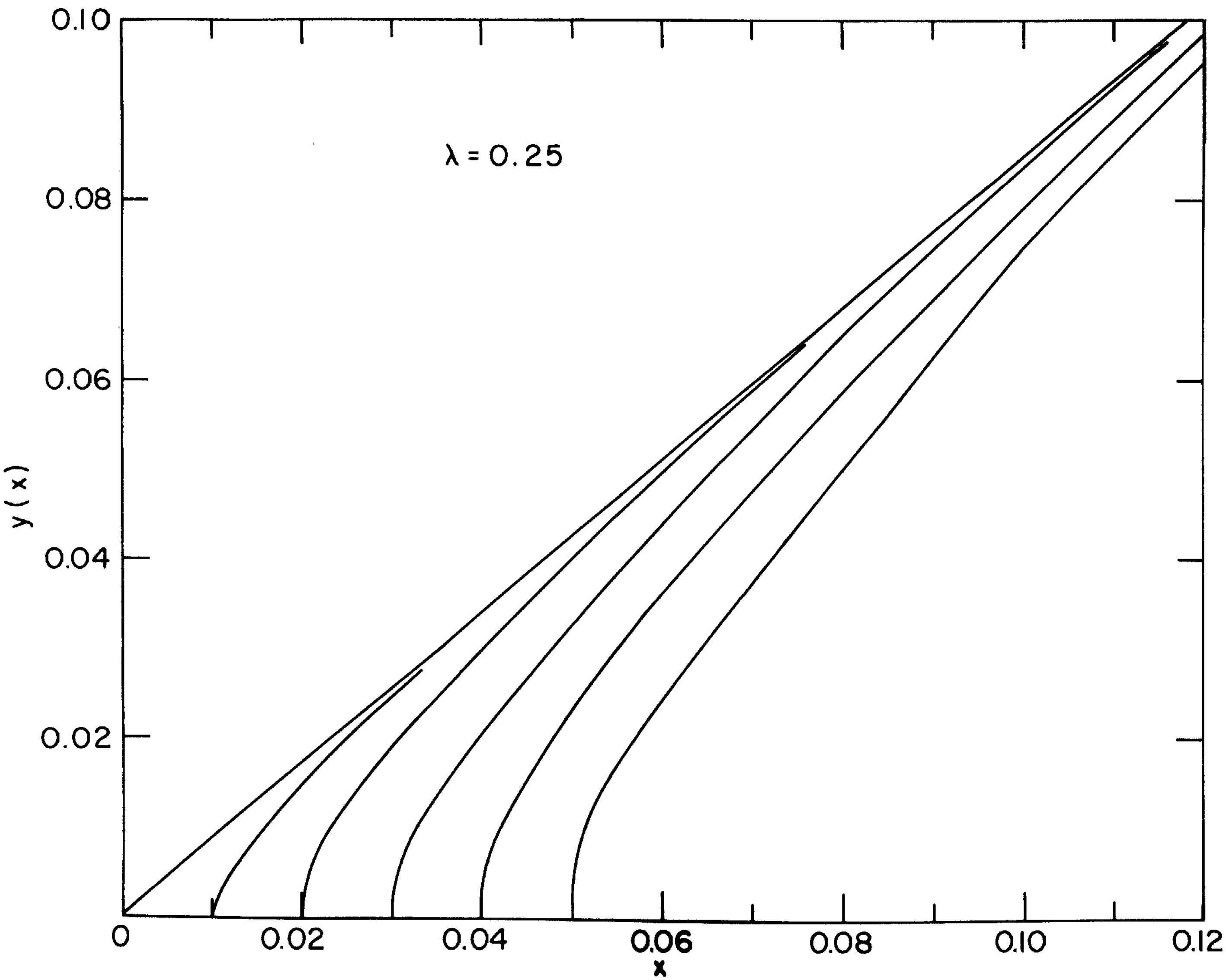


Fig.6