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LOWER BOUNDS FOR THE BUCKLING

PRESSURE OF SPHERICAL SHELLS*

By Nicholas J. Hoff and Tsai-Chen Soong

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SUMMARY

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An approximate large-deflection theory is developed for the calculation of the hydrostatic pressure under which a thin-walled spherical shell buckles. The theory is based on the assumptions that the buckle has the shape of a spherical cap and that a yield hinge develops along the circumference of the cap. The effect of the test arrangement on the buckling pressure is taken into account. In spite of the rough approximate nature of the analysis reasonable numerical values are obtained for the buckling pressure.

INTRODUCTION

In a review of the state of knowledge regarding the buckling of thin elastic shells, Fung and Sechler[1] ** noted that "a complete spherical shell buckles in the form of a small inward dimple of small solid angle at some point on the surface of the shell; a fact entirely at variance with the linear theory of buckling". The non-linear theories developed to explain the discrepancies have all dealt with shallow spherical caps rather than with complete spheres and almost all the experimental effort related to spherical shells was made with spherical caps supported along their edge. The only known test with a complete spherical shell was carried out by Sechler and Bollay and was reported by Tsien.[2] In this test a brass shell buckled under approximately one-quarter the critical value of the external pressure of the linear theory. More recently, Thompson[3] published the results of two experiments carried out with polyvinyl chloride shells of a radius to thickness ratio of approximately 20. Buckling occurred after about three-quarters of the classical value of the buckling pressure was reached. The paper also contains a large-deflection strain energy analysis which yielded satisfactory agreement with the results of the experiments.

*The work here described was carried out under NASA Grant NsG-93-60.

**Superscript numbers indicate entries under References at the end of the paper.

In a recent paper[4] the senior author developed a rough approximate theory of the buckling of axially compressed thin-walled circular cylindrical shells on the basis of the assumption that plastic hinges form along the ridges when the shell snaps into the large-deflected shape. Since reasonable agreement was found between available experimental data and the results of the approximate theory, it appears worth while to apply the same method of analysis to the spherical shell. The absence of noticeable permanent deformations after the pressure is released does not prove that yielding had not taken place in the material. If a sufficient amount of elastic energy is stored in the buckled shell, the shell can snap back into a shape that appears to be the same as the initial one. Nevertheless some damage is always done because the buckling experiment cannot be repeated many times with the same specimen.

Since the details of the loading machine were found to influence significantly the buckling load in the case of the cylindrical shell, the loading apparatus to be used with the spherical shells is carefully defined in the present paper. The numerical values obtained for a special case and plotted in two figures appear to be reasonable. Nevertheless no claim is made for the accuracy of the theory because it is based on rather far-fetched assumptions. It obviously cannot yield more than approximate lower bounds on the buckling pressure because it consists of equating energy quantities before and after buckling. The relative success of the present approach may, however, indicate that the yield stress is one of the physical factors to be taken into account in more rigorous analyses of the large-deflection buckling process. It was included in the empirical formula proposed by Donnell[5] twenty-eight years ago.

SYMBOLS

C	correction factor for influence coefficient (see Eq. 23)
d	radial components of displacement
h	wall thickness of specimen
H	wall thickness of container
E	Young's modulus
k	influence coefficient (see Eqs. 21 and 22)
K	Baker-Cline multiplier (see Eq. 23)

M	moment resultant
N	edge moment resultant
p	pressure
P	radial component of force resultant
q	non-dimensional quantity defined in Eq. 24
r	radius of specimen
R	radius of container
v	volume inside specimen
V	volume between spheres
W	work
x	non-dimensional quantity defined in Eq. 25
α	semi-vertex angle of bulge
β	multiplier defined in Eq. 62
γ	ratio of specific heats of air (= 1.4)
Δ	lack of fit defined in Eq. 20
κ	curvature
ν	Poisson's ratio
σ	stress
χ	rotation

In addition, the following subscripts are used:

a	atmospheric
ad	adiabatic
c	inside specimen after loading; also refers to cap
cc	inside specimen after buckling

cr	critical (refers to classical theory)
edge	refers to edge of buckle
el·b	elastic bending
f	final
i	initial after loading (before buckling)
o	initial before loading
out	refers to work done by outside atmosphere
pl	plastic
r	remainder
y	yield

BUCKLING ANALYSIS

Assumptions Regarding the Buckling Process

It is assumed that the thin-walled spherical test specimen is enclosed in a larger spherical container as shown in Fig. 1. The space between the two spheres is filled with water and at the beginning of the test the air outside the container, the water between the spheres and the air inside the test specimen are at the same atmospheric pressure p_a . In the first phase of the test the pressure of the water is increased to $p_a + p_i$ through the use of a pump provided with a no-return valve. The increase in the pressure expands the large sphere and compresses the small sphere; the effective pressure on the test specimen is therefore $p_i - p_c$ if the notation of Fig. 2 is employed. As the pressure in the water should be increased very slowly, it can be assumed that the compression inside the test specimen takes place isothermally. Thus the pressure inside the specimen before and after the compression is related to the volume inside the specimen before and after the compression through the equation

$$p_a v_o = (p_a + p_c) v_i \tag{1}$$

if the notation of Fig. 2b is used.

Under the conditions just described, the specimen is assumed to be in a neutral equilibrium. It suddenly jumps over into a buckled shape characterized by a dimple of a semivertex angle α (see Fig. 1). The dimple is again spherical and it has the same radius as the original spherical specimen would have if it were subjected to the prevailing pressure difference. The deformations during the buckling process consist therefore of a uniform expansion corresponding to the change in the pressure difference and an inextensional reversal of the curvature of the buckled portion of the median surface of the spherical specimen.

After buckling the water pressure is $p_a + p_f$ (see Fig. 2c), the pressure inside the specimen is $p_a + p_{cc}$, and the effective pressure on the spherical specimen is $p_f - p_{cc}$. The volume between the two spheres does not change during buckling as the water is incompressible. The change in the pressure inside the specimen is governed by the **adiabatic law**

$$\frac{p_a + p_{cc}}{p_a + p_c} = \left(\frac{v_i}{v_f}\right)^{\gamma} \tag{2}$$

where for air the value of γ is

$$\gamma = 1.4 \tag{2a}$$

Strain Energy in Unbuckled State

If the letters R and r denote the median surface radii of the outer and inner shells, and the letters H and h the uniform wall thicknesses of the outer and inner shells, the uniform normal stresses in the shell walls are

$$\sigma = p_i R/2H \qquad \text{in the outer shell}$$

$$\sigma = (p_i - p_c)r/2h \qquad \text{in the inner shell} \tag{3}$$

The elastic strain energy stored in the outer shell is

$$U = (\sigma^2/2E)4\pi R^2 H = (2\pi R^2 H/E)(p_i R/2H)^2 = (\pi/2)p_i^2 R^4/EH$$

The total energy stored initially in the two shells is therefore

$$U_i = \frac{\pi}{2}\frac{p_i^2}{E}\frac{R^4}{H} + \frac{\pi}{2}\frac{(p_i - p_c)^2}{E}\frac{r^4}{h} \tag{4}$$

Elastic Membrane Strain Energy after Buckling

After buckling the final pressure $p_a + p_f$ prevails between the two shells. Even though part of the inner shell is now concave, the absolute magnitude of the membrane stress is the same throughout the inner shell as the magnitude of the curvature is the same everywhere. The total membrane strain energy stored in the system is

$$U_f = \frac{\pi}{2} \frac{p_f^2}{E} \frac{R^4}{H} + \frac{\pi}{2} \frac{(p_f - p_{cc})^2}{E} \frac{r^4}{h} \qquad (5)$$

Elastic Energy of Bending of Cap

After buckling the portion of the shell, called the cap, that jumped over into a position of opposite curvature, is under the action of bending moments. The magnitude of the elastic strain energy stored because of bending can be evaluated approximately if the energy necessary to bend a flat circular plate of a radius $r \sin \alpha$ into a spherical cap of radius r is calculated and multiplied by 4. The curvature κ caused by uniform bending moment resultants N distributed along the entire circular edge of the plate and acting perpendicularly to the edge is

$$\kappa = [12(1 - \nu)/Eh^3]N \qquad (6)$$

Since the curvature sought is $1/r$, Eq. 6 can be solved for N and $\kappa = 1/r$ can be substituted in the result obtained:

$$N = \frac{Eh^3/r}{12(1 - \nu)} \qquad (7)$$

Since the edge of the plate must rotate through the angle 2α, and since the work done by the edge moment resultant 2N is $(1/2)2N2\alpha$ per unit length of the circumference, the total work W done is

$$W = 2[12(1 - \nu)]^{-1} (Eh^3/r)\alpha \, 2\pi \, r \sin \alpha$$

As the work done is equal to the energy stored, the following expression is obtained for the energy $U_{el.b}$ of elastic bending:

$$U_{el.b} = \frac{\pi E}{3(1 - \nu)} h^3 \alpha \sin \alpha \qquad (8)$$

Plastic Work Performed

In agreement with the assumptions the edge of the buckled cap deforms as a yield hinge. When a yield hinge forms in a ideally rigid-plastic material, half the thickness of the sheet is subjected to the tensile yield stress, and the other half to the compressive yield stress as indicated in Fig. 3. If the two yield stresses are equal numerically, the bending moment per unit length is

$$M_{pl} = \sigma_y (h/2)^2 \qquad (9)$$

This moment is constant during the deformations because no plastic deformations can take place in a rigid-plastic solid until the bending moment reaches the value given in Eq. 7. Hence the total plastic work done amounts to

$$W_{p\ell} = M_{pl}\ 2\alpha\ 2\pi\ r \sin\alpha$$

where 2α is the total angle of rotation and $2\pi r \sin\alpha$ is the length of the yield hinge. Substitution yields

$$W_{p\ell} = \pi\ \sigma_y\ h^2 r\ \alpha \sin\alpha \qquad (10)$$

Changes in Volume of the Two Shells

The volume between the two shells in their natural state is

$$V_o = (4\pi/3)(R^3 - r^3) \qquad (11)$$

After the water between the two shells is brought to the pressure $p_a + p_i$, the outer shell expands and the inner shell is compressed with the result that the volume increases. Since the strains in the outer and inner shells are

$$\epsilon = (1 - \nu)(\sigma/E) = [(1 - \nu)/2](p_i R/EH) \quad \text{in the outer shell}$$

$$\epsilon = -(1 - \nu)(\sigma/E) = -[(1 - \nu)/2](p_i - p_c)(r/Eh) \quad \text{in the inner shell} \qquad (12)$$

the volume of air just before buckling is

$$V_i = \frac{4\pi}{3}\left\{R^3\left[1 + \frac{1-\nu}{2}\,\frac{p_i R}{EH}\right]^3 - r^3\left[1 - \frac{1-\nu}{2}\,\frac{(p_i - p_c)r}{Eh}\right]^3\right\} \qquad (13)$$

The volume V_f after buckling can be calculated in a similar manner but in addition the volume of the buckle itself must also be taken into account.

The volume of the cap is

$$V_c = (1/3)\pi r^3(\cos^3 \alpha - 3 \cos \alpha + 2) \qquad (14)$$

Hence the volume V_f after buckling is

$$V_f = \frac{4\pi}{3}\left\{R^3\left[1 + \frac{1-\nu}{2}\,\frac{p_f R}{EH}\right]^3 - r^3\left[1 - \frac{1-\nu}{2}\,\frac{(p_f - p_{cc})r}{Eh}\right]^3\right\} + (2\pi/3)r^3(\cos^3 \alpha - 3 \cos \alpha + 2) \qquad (15)$$

The volume of the test sphere immediately before buckling is

$$v_i = \frac{4\pi}{3}\,r^3\left[1 - \frac{1-\nu}{2}\,\frac{(p_i - p_c)r}{Eh}\right]^3 \qquad (16)$$

and after buckling

$$v_f = \frac{4\pi}{3}\,r^3\left[1 - \frac{1-\nu}{2}\,\frac{(p_f - p_{cc})r}{Eh}\right]^3 - (2\pi/3)r^3\,(\cos^3 \alpha - 3 \cos \alpha + 2) \qquad (17)$$

Work Done by the Air

The air outside the container remains at the atmospheric pressure during the buckling process. Hence the work done by the outside air is

$$W_{out} = \frac{4\pi}{3}\,R^3\,p_a\left\{\left[1 + \frac{1-\nu}{2}\,\frac{p_i R}{EH}\right]^3 - \left[1 + \frac{1-\nu}{2}\,\frac{p_f R}{EH}\right]^3\right\} \qquad (18)$$

The work necessary to compress adiabatically the air inside the test specimen is

$$W_{ad} = \frac{(p_a+p_c)v_i}{\gamma - 1}\left[-1 + \left(\frac{p_a + p_{cc}}{p_a + p_c}\right)^{\frac{\gamma-1}{\gamma}}\right] \tag{19}$$

Elastic Strain Energy in Edge Zone

It is easy to see that the deformations of the cap and the remainder in the state of membrane stress existing after the buckle was formed are not compatible. The remainder is in compression and the cap is in tension. The relative displacement Δ between cap and remainder can be easily calculated with the aid of Figs. 1 and 2:

$$\Delta = (1 - \nu)(r^2/h) \sin \alpha[(p_f - p_{cc})/E] \tag{20}$$

In order to reestablish continuity, edge forces P (perpendicular to the axis of the cap) and edge moments M must develop. These can be calculated with the aid of formulas derived in Reference 6. The displacement d perpendicular to the axis of the cap and the rotation χ of the end tangent are given by

$$\begin{aligned} d &= k_{dM} M + k_{dP} P \\ \chi &= k_{\chi M} M + k_{\chi P} P \end{aligned} \tag{21}$$

with the sign convention shown in Fig. 4. For the cap, the influence coefficients can be given as

$$\begin{aligned} k_{dM} &= -k_{\chi P} = 2[3(1-\nu^2)]^{\frac{1}{2}} (1/E)(r/h^2) \sin \alpha\, C_{12} \\ k_{dP} &= -2[3(1-\nu^2)]^{1/4} (1/E)(r/h)^{3/2} \sin^2 \alpha\, C_{11} \\ k_{\chi M} &= 4[3(1-\nu^2)]^{3/4} (1/E)(r^{1/2}/h^{5/2}) C_{22} \end{aligned} \tag{22}$$

where the factors C are defined as

$$C_{11} = (K_{11}/\sqrt{2}) \frac{(1 + R^*)^{1/2}}{1 + 2R^*}$$

$$C_{22} = (K_{22}/\sqrt{2}) \frac{(1 + R^*)^{1/2}}{1 + 2R^*}$$

$$C_{12} = K_{12} \frac{1}{1 + 2R^*} \tag{23}$$

The coeficients K were derived by Baker and Cline[7]; they are tabulated in the reference quoted.[6] It is of interest to note that they vary little with α but change more rapidly with the quantity

$$x = q\,\alpha \tag{24}$$

where

$$q = \sqrt{2}\ [3(1 - \nu^2)]^{1/4}\ (a/b)^{1/2} \tag{25}$$

As the ratio h/a approaches zero, the values of K_{11} , K_{22} , and K_{12} approach the values $\sqrt{2}$, $\sqrt{2}$, and 1 , respectively. Moreover

$$R^* = \sigma/\sigma_{cr} \tag{26}$$

with the classical critical stress

$$\sigma_{cr} = [3(1 - \nu^2)]^{-1/2}\ E(h/a) \tag{27}$$

The influence coefficients for the remainder are given by the same expressions except for the sign: In the expression for k_{dP} the right-hand member is positive and in the expression for k_{XM} the right-hand member is negative. Equations 23 remain unchanged but the values of the Baker-Cline coefficients must be taken from the table established for the remainder, and not for the cap. The value of R^* also changes sign; it is positive for the cap where the stress is tensile and negative for the remainder where the stress is compressive.

With the sign convention used

$$P_c = P_r = P \qquad\qquad M_c = -M_r = -M \tag{28}$$

One condition of compatible deformations is that the lack of fit Δ must be equal in magnitude to the difference in the radial edge displacement d of the remainder and the cap:

$$\Delta = d_r - d_c \tag{29}$$

Finally, the sign convention for the rotation of the edge tangent is such that

$$\chi_c + \chi_r = 0 \tag{30}$$

Solution of these equations yields

$$M = -(1/2)[3(1-\nu^2)]^{-1/2}E(h^2/r)\operatorname{cosec}\alpha \frac{C_{12c} + C_{12r}}{2(C_{22c}+C_{22r})(C_{11c}+C_{11r})-(C_{12c}+C_{12r})^2}\Delta$$

$$P = [3(1-\nu^2)]^{-1/4}\ E(h/r)^{3/2}\operatorname{cosec}^2\alpha \frac{C_{22c} + C_{22r}}{2(C_{22c}+C_{22r})(C_{11c}+C_{11r})-(C_{12c}+C_{12r})^2}\Delta$$

$$(31)$$

Since the work done by the edge moment resultants is zero as the relative rotation between the edges of cap and remainder is zero, work is done only by the edge force resultants P . This work is

$$W_{el} = (1/2)2\pi r \sin\alpha\, P\, \Delta$$

This work is equal to the elastic energy stored in the edge zone. Substitutions yield

$$U_{edge} = \pi \frac{(1-\nu)^2}{[3(1-\nu^2)]^{1/4}} \frac{(p_f-p_{cc})^2}{E} \frac{r^{7/2}}{h^{1/2}} \sin\alpha \frac{C_{22c} + C_{22r}}{2(C_{22c}+C_{22r})(C_{11c}+C_{11r})-(C_{12c}+C_{12r})^2}$$

$$(32)$$

It is of interest to note that the value of the last fraction approaches 1/2 as the final pressure $p_f - p_{cc}$ (and thus R^*) decreases and as the ratio h/r decreases and approaches zero.

Buckling Condition

Obviously buckling cannot occur unless the energy stored in the system before buckling plus the potential of the external loads is at least as large as the energy stored in the system after buckling plus the work done in an irreversible manner and transformed into heat. Hence a lower limit on the buckling pressure p_i can be obtained by solving the energy equation

$$U_i + W_{out} = U_f + U_{el.b} + U_{edge} + W_{pl} + W_{ad} \tag{33}$$

In addition the geometric constraint

$$V_f = V_i \tag{34}$$

must be satisfied and the conditions connecting the changes in volume of the air contained in the test specimen with the changes in pressure (Eqs. 1 and 2) must be observed. The equations contain the nine unknowns p_i, p_f, p_c, p_{cc}, V_i, V_f, v_i, v_f and α. For their solution there are available four equations of geometry (Eqs. 13, 15, 16, and 17), two equations of change of state of the air (Eqs. 1 and 2), one equation of incompressibility of the water (Eq. 34) and one energy equation (Eq. 33). If a value of α is arbitrarily selected, the eight equations suffice to determine all the remaining unknowns. One can plot then p_i as a function of α and consider the minimal value of p_i as the lower bound on the buckling pressure.

Numerical Values

The calculations outlined in the preceding sections were programmed for the Burroughs 220 digital computer in order to obtain some numerical results. For prescribed values of α, the semivertex angle of the buckle, the gauge pressure p_i in the container was calculated and plotted. Figure 5 contains six such curves corresponding to different ratios of the radius R of the container to the radius r of the specimen. The six values are 1.1 , 1.2 , 1.3 , 1.6 , 2.0 , and 2.2. The other parameters of the problem are fixed in the following manner:

$$E = 15 \times 10^6 \text{ psi} \qquad \sigma_y = 30{,}000 \text{ psi}$$

$$h/H = 0.1 \qquad r/h = 500 \qquad \nu = 0.3$$

It is to be expected that buckling will occur at the minimal value of the gauge pressure. Hence the test arrangement corresponding to $R/r = 1.2$ yields a buckling pressure p_i which is about 26.5 percent of the classical value. This is in good agreement with the test results obtained by Sechler and Bollay with the brass shell mentioned in the Introduction. The semivertex angle is about 11 degrees according to Fig. 5.

As the volume enclosed between test specimen and container is increased, the semivertex angle increases slowly and the pressure ratio decreases slowly. For $R/r = 2.2$ the values obtained are $\alpha = 13$ degrees and $p_i/p_{cr} = 0.18$.

The effect of a change in the material of the specimen (and of the container) is illustrated in Fig. 6. When $R/r = 1.2$, the minimal values of the gauge pressure of buckling amount to approximately 25, 26 and 41 percent of the classical values of the critical pressure for carbon steel, brass and an aluminum alloy, respectively. The corresponding semivertex angles are about 11, 11, and 12.5 degrees.

To illustrate details of the buckling process, the following data are given:

Characteristics of brass test specimen and container:

$R/r = 1.2 \qquad h/H = 0.1 \qquad r/h = 500$

$E = 15 \times 10^6$ psi $\qquad \sigma_y = 30{,}000$ psi

Semivertex angle of buckle	$\alpha = 11$ degrees
Gauge pressure before buckling	$p_i = 19.30$ psi
Gauge pressure after buckling	$p_f = 7.39$ psi
Gauge pressure inside specimen after buckling	$p_{cc} = 0.012$ psi

The relative importance of the various energy quantities can be seen from the following data:

$$(1/\pi r^3)(U_i - U_f) = 6.392 \times 10^{-3} \text{ in. lb per in.}^3$$

$$(1/\pi r^3)U_{el.b} = 2.170 \times 10^{-3} \text{ in. lb per in.}^3$$

$$(1/\pi r^3)U_{edge} = 2.361 \times 10^{-6} \text{ in. lb per in.}^3$$

$$(1/\pi r^3)W_{out} = 1.694 \times 10^{-3} \text{ in. lb per in.}^3$$

$$(1/\pi r^3)W_{pl} = 4.320 \times 10^{-3} \text{ in. lb per in.}^3$$

$$(1/\pi r^3)W_{ad} = 1.676 \times 10^{-3} \text{ in. lb per in.}^3$$

Simplified Solution

If use is made of the observation that the changes in volume caused by the elastic strains are small compared to the initial volumes, and that the semivertex angle α of the bulge is likely to be small, and if in addition the container is assumed to be rigid

$$H = \infty \tag{35}$$

the expressions presented simplify considerably. In particular

$$\cos^3 \alpha - 3 \cos \alpha + 2 = (3/4)\, \alpha^4 \tag{36}$$

and the volumes of interest become

$$v_i = \frac{4\pi}{3} r^3 \left[1 - \frac{3(1 - \nu)}{2} \frac{(p_i - p_c)r}{Eh} \right] \tag{37}$$

$$v_f = \frac{4\pi}{3} r^3 \left[1 - \frac{3(1 - \nu)}{2} \frac{(p_f - p_{cc})r}{Eh} \right] - \frac{\pi}{2} r^3 \alpha^4 \tag{38}$$

$$V_i = \frac{4\pi}{3} (R^3 - r^3) + 2\pi(1 - \nu)(p_i - p_c) \frac{r^4}{Eh} \tag{39}$$

$$V_f = \frac{4\pi}{3} (R^3 - r^3) + 2\pi(1 - \nu)(p_f - p_{cc}) \frac{r^4}{Eh} + \frac{\pi}{2} r^3 \alpha^4 \tag{40}$$

From Eq. 1

$$v_i = \frac{p_a}{p_a + p_c} \frac{4\pi}{3} r^3 \tag{41}$$

Equation 2 can be re-written as

$$\frac{p_a + p_c}{p_a + p_{cc}} = \left(\frac{v_f}{v_i} \right)^{\gamma} \tag{42}$$

and the incompressibility condition becomes

$$p_i - p_c = p_f - p_{cc} + \frac{E}{4(1 - \nu)} \frac{h}{r} \alpha^4 \tag{43}$$

These equations suffice to express all the quantities of interest in terms of the geometric and physical constants of the problem and of the initial pressure p_i and the angle α. Consideration of the energy equation yields then p_i as a function of α.

The energy terms become:

$$U_i = \frac{\pi}{2} \frac{(p_i - p_c)^2}{E} \frac{r^4}{h} \tag{44}$$

$$U_f = \frac{\pi}{2} \frac{(p_f - p_{cc})^2}{E} \frac{r^4}{h} \tag{45}$$

$$U_{el.b} = \frac{\pi E}{3(1 - \nu)} h^3 \alpha^2 \tag{46}$$

$$W_{pl} = \pi\sigma_y h^2 r\alpha^2 \tag{47}$$

$$W_{ad} = \frac{(p_a + p_c)v_i}{\gamma - 1} \left[\left(\frac{v_i}{v_f}\right)^{\gamma-1} - 1\right] \tag{48}$$

$$W_{out} = 0 \tag{49}$$

Moreover, in a first approximation, U_{edge} can be disregarded. Two immediate consequences of the incompressibility condition and Eq. 42 are

$$v_i = v_f \qquad\qquad p_c = p_{cc} \tag{50}$$

and it follows from Eq. 48 that

$$W_{ad} = 0 \tag{51}$$

From Eqs. 44 and 45 it follows that

$$U_i - U_f = (\pi/2)(r^4/Eh)(p_i - p_f)(p_i + p_f - 2p_c) \tag{52}$$

But the condition

$$V_i = V_f$$

implies

$$2\pi(r^4/Eh)(1 - \nu)(p_i - p_f) = (\pi/2)r^3\alpha^4 \tag{53}$$

while the first of Eqs. 50 leads to the same requirement. Hence, Eq. 33 becomes simply

$$U_i - U_f = U_{el.b} + W_{pl} \tag{54}$$

or

$$\frac{\pi}{2}\frac{r^4}{Eh}(p_i - p_f)(p_i + p_f - 2p_c) = \frac{\pi E}{3(1 - \nu)} h^3\alpha^2 + \pi\,\sigma_y\, h^2 r\alpha^2 \tag{55}$$

Eq. 55 together with Eq. 53, can be solved for $(p_i - p_c)$:

$$p_i - p_c = \frac{4(1 - \nu)}{\alpha^2}\left[\frac{E}{3(1 - \nu)}\left(\frac{h}{r}\right)^3 + \sigma_y\left(\frac{h}{r}\right)^2\right] + \frac{E}{8(1 - \nu)}\frac{h}{r}\alpha^4 \tag{56}$$

From Eqs. 41 and 37 one obtains

$$\frac{p_a}{p_a + p_c} = 1 - \frac{3}{2}(1 - \nu)\frac{(p_i - p_c)r}{Eh} \tag{57}$$

that is

$$\frac{1}{1 + (p_c/p_a)} = 1 - \frac{3}{2}(1 - \nu)\frac{(p_i - p_c)r}{Eh}$$

or approximately,

$$\frac{p_c}{p_a} = \frac{3}{2}(1 - \nu)\frac{(p_i - p_c)r}{Eh}$$

that is

$$p_c = \frac{(3/2)(1 - \nu)(r/Eh)}{(1/p_a) + (3/2)(1 - \nu)(r/Eh)}\, p_i \tag{58}$$

Substitution in Eq. 56 yields

$$p_i\left[1 - \frac{(3/2)(1 - \nu)(r/Eh)}{(1/p_a)+(3/2)(1-\nu)(r/Eh)}\right] = \frac{4(1 - \nu)}{\alpha^2}\left[\frac{E}{3(1 - \nu)}\left(\frac{h}{r}\right)^3 + \sigma_y\left(\frac{h}{r}\right)^2\right]$$

$$+ \frac{E}{8(1 - \nu)}\frac{h}{r}\alpha^4$$

or

$$p_i\left[\frac{1}{1+(3/2)(1-\nu)(r/Eh)p_a}\right]=\frac{4(1-\nu)}{\alpha^2}\left[\frac{E}{3(1-\nu)}\left(\frac{h}{r}\right)^3+\sigma_y\left(\frac{h}{r}\right)^2\right]+\frac{E}{8(1-\sigma)}\ \frac{h}{r}\ \alpha^4 \tag{59}$$

This can also be written as

$$\beta p_i=\frac{4(1-\nu)}{\alpha^2}\left[\frac{E}{3(1-\nu)}\left(\frac{h}{r}\right)^3+\sigma_y\left(\frac{h}{r}\right)^2\right]+\frac{E}{8(1-\nu)}\ \frac{h}{r}\ \alpha^4 \tag{60}$$

where β is the expression in brackets multiplying p_i in Eq. 59; its value is almost exactly equal to unity. To obtain $p_{i_{min}}$ set $dp_i/d\alpha$ equal to zero and solve for α :

$$\alpha_m^6=\frac{16(1-\nu)^2}{E}\ \frac{h}{r}\left[\frac{E}{3(1-\nu)}\ \frac{h}{r}+\sigma_y\right] \tag{61}$$

Substitution in Eq. 60 yields

$$p_{i_{min}}=\frac{3}{8}\ \frac{Eh}{(1-\nu)r}\ \alpha_{min}^4\ (1/\beta) \tag{62}$$

A numerical example will show the accuracy of the approximate formula. If a brass sphere is characterized by the values

$$E = 15 \times 10^6 \qquad h/r = 1/500$$

$$\sigma_y = 30{,}000 \qquad \nu = 0.3$$

one obtains from Eq. 61

$$\alpha^6 = 46.2 \times 10^{-6} \qquad \alpha = 0.189 = 10^\circ\ 45'$$

and from Eq. 62

$$p_i = 20.2 \text{ psi} \qquad p_i/p_{cr} = (20.2/72.3)100 = 27.9\%$$

SIMPLE BUCKLING FORMULAS

If the assumptions of the last section are maintained, namely that the container is perfectly rigid, and in addition ν is taken as 0.3, the results of the analysis can be presented in the following simple formulas:

$$\alpha_{min}=1.47\left(\frac{\sigma_{cr}}{E}\right)^{1/6}\left(\frac{\sigma_{cr}}{E}+1.27\ \frac{\sigma_y}{E}\right)^{1/6} \tag{63}$$

$$\alpha_{min} = 1.24\left(\frac{h}{r}\right)^{1/6}\left(\frac{h}{r} + 2.1\ \frac{\sigma_y}{E}\right)^{1/6} \tag{64}$$

$$p_{i_{min}} = 4.16\ \sigma_{cr}\left(\frac{\sigma_{cr}}{E}\right)^{2/3}\left(\frac{\sigma_{cr}}{E} + 1.27\ \frac{\sigma_y}{E}\right)^{2/3} \tag{65}$$

$$p_{i_{min}} = 1.26\ E\left(\frac{h}{r}\right)^{5/3}\left(\frac{h}{r} + 2.1\ \frac{\sigma_y}{E}\right)^{2/3} \tag{66}$$

$$\sigma_{i_{min}} = 0.630\ E\left(\frac{h}{r}\right)^{2/3}\left(\frac{h}{r} + 2.1\ \frac{\sigma_y}{E}\right)^{2/3} \tag{67}$$

where σ_{cr} is the critical stress defined in Eq. 27.

The last equation shows that for comparatively thick shells with a low value of the yield stress the buckling stress decreases more rapidly with decreasing h/r ratio than is predicted by the classical small-deflection formula, while for comparatively thin shells with a high value of the yield stress the opposite is true.

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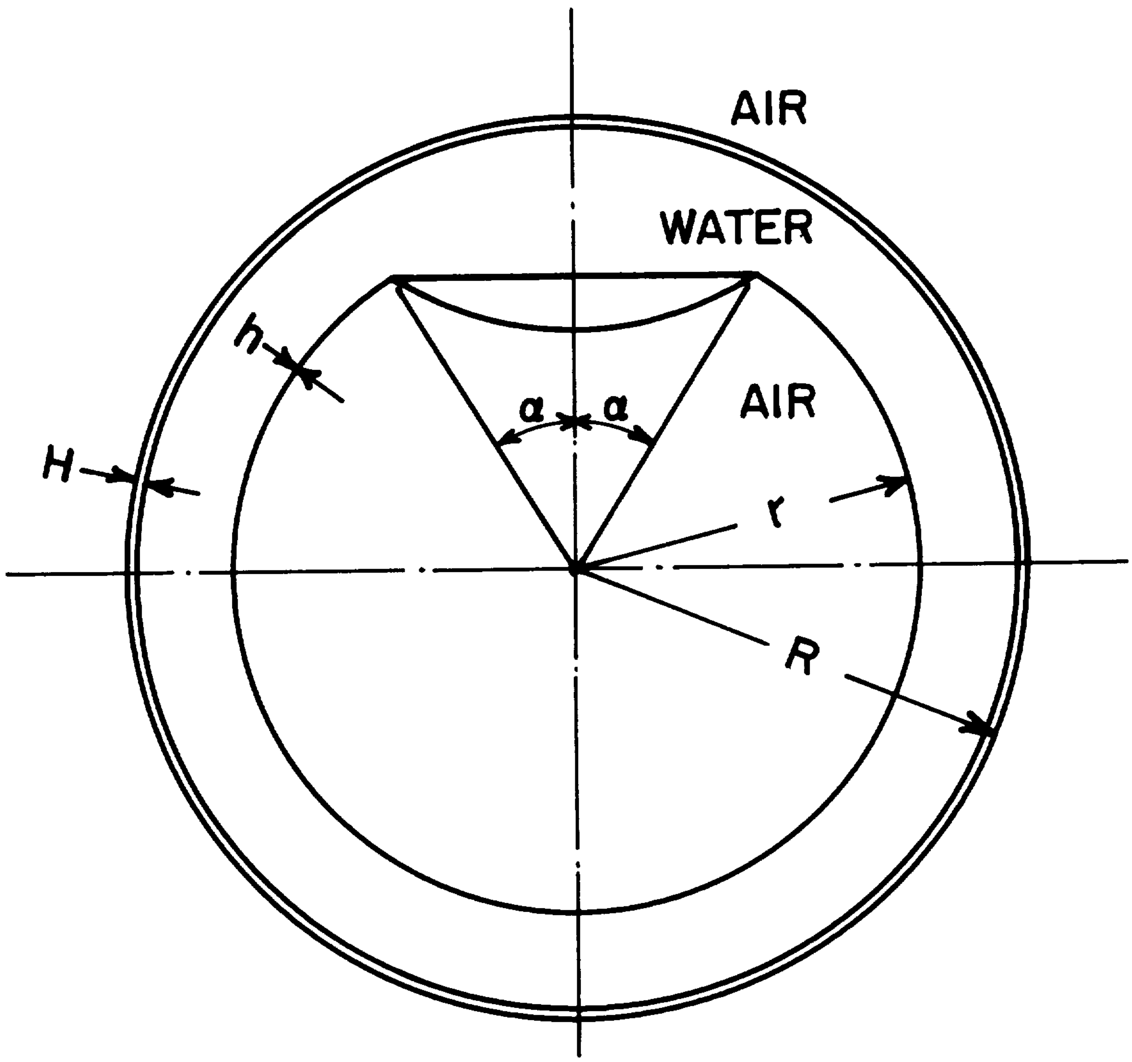
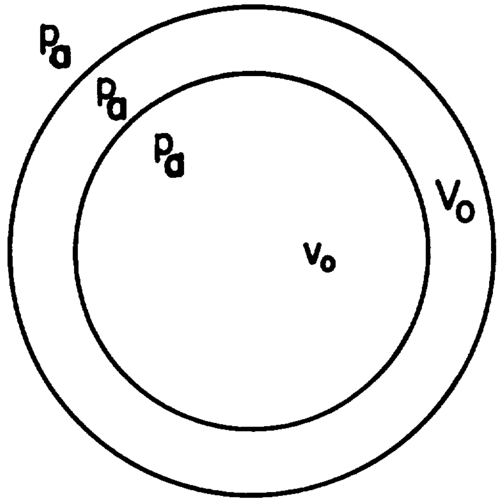
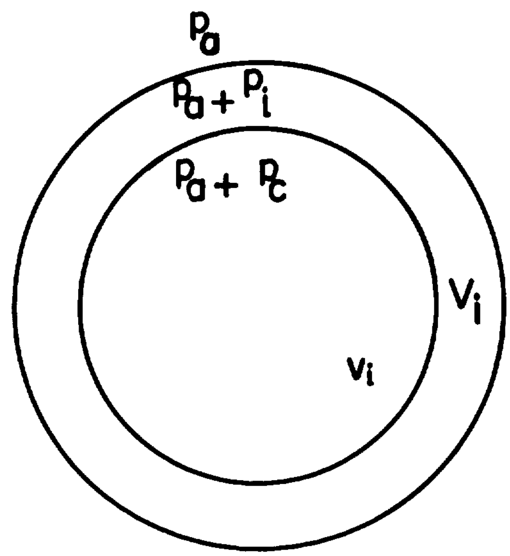


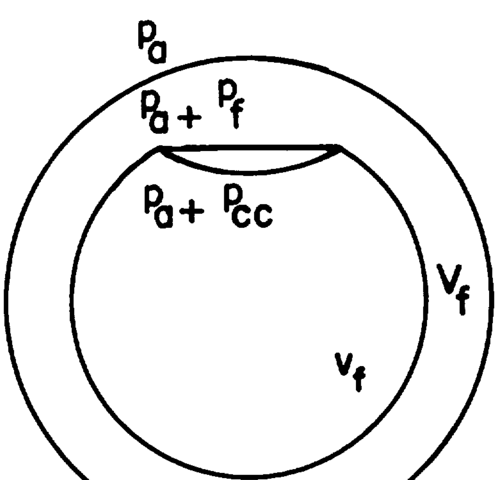
FIG. 1. BUCKLED SPHERICAL TEST SPECIMEN IN SPHERICAL CONTAINER



(a) Phase 0: before application of pressure.

(b) Phase 1: after application of pressure just before buckling





(c) Phase 2: after buckling

FIG. 2 THREE PHASES OF BUCKLING TEST

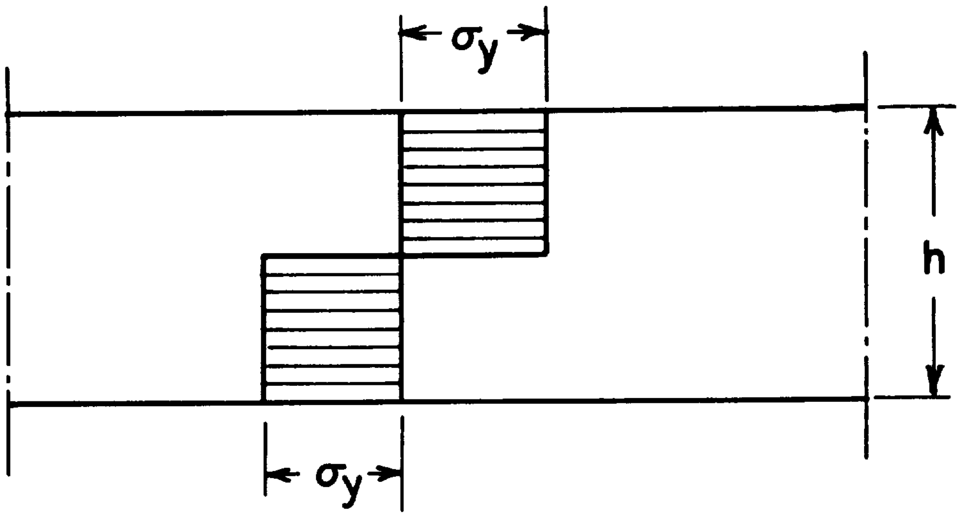


FIG. 3. STRESS DISTRIBUTION IN WALL OF SHELL CORRESPONDING TO PLASTIC HINGE.

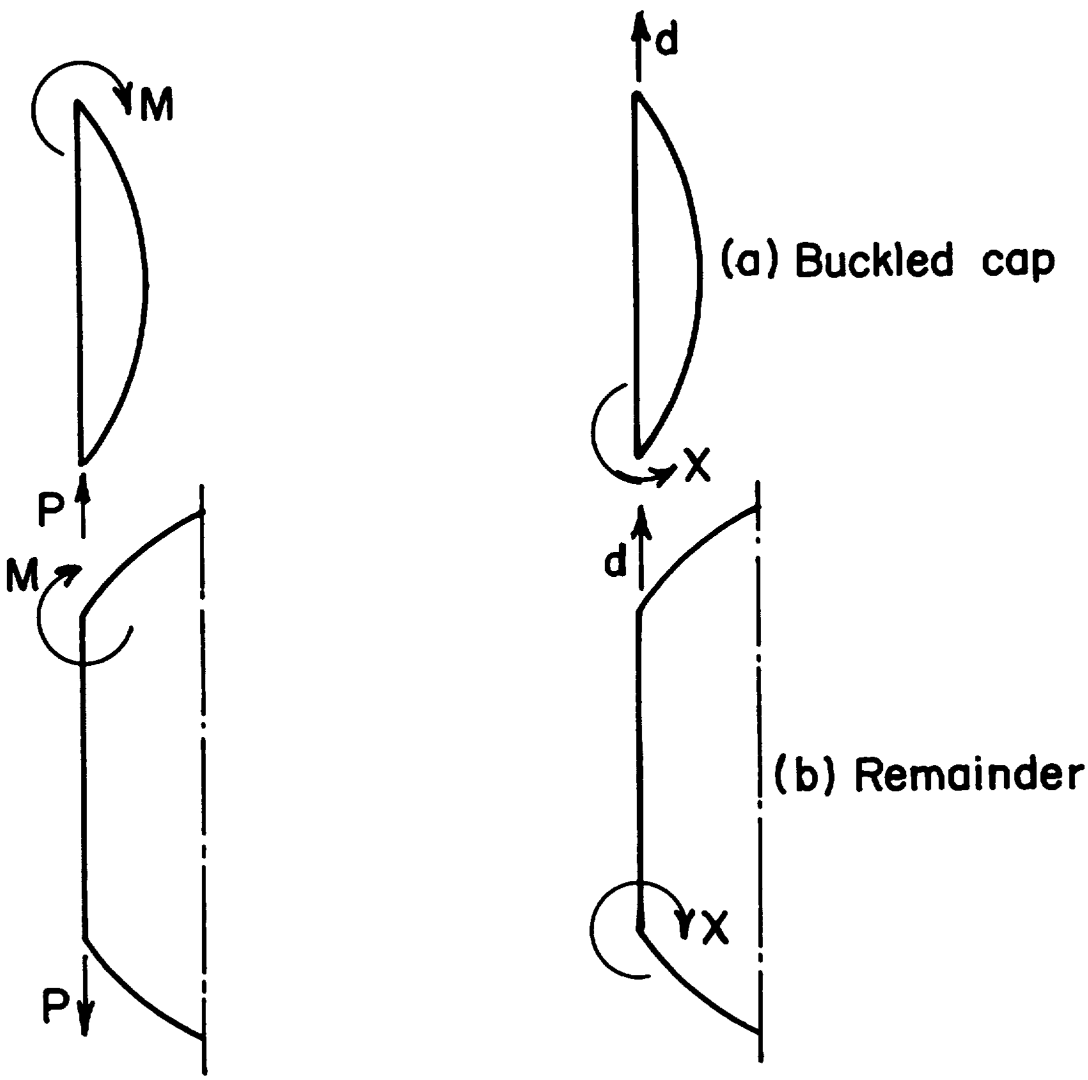


FIG. 4. CONVENTION FOR EDGE DISPLACEMENTS AND STRESS RESULTANTS.

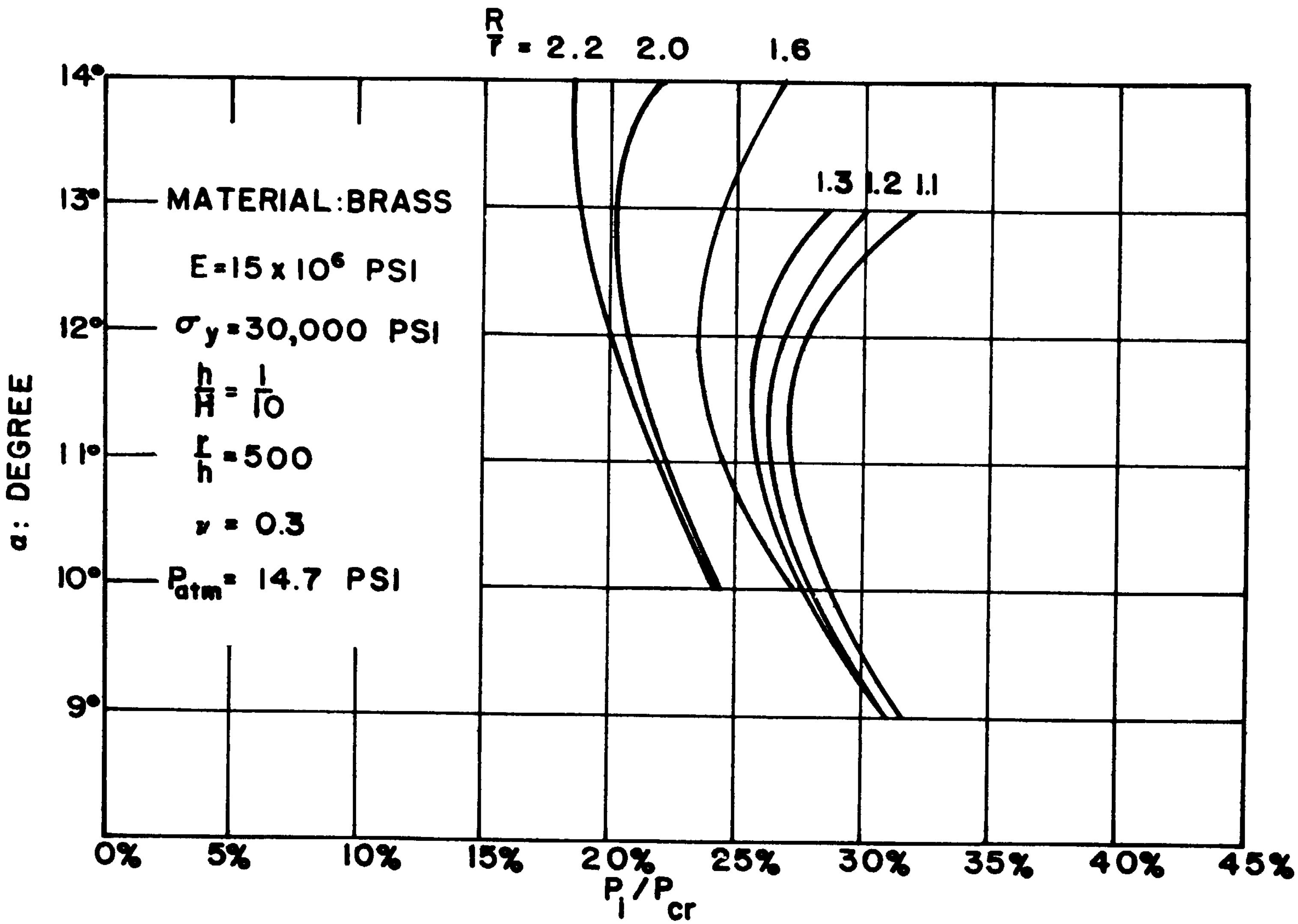


FIG.5 EFFECT OF VOLUME OF CONTAINER ON BUCKLING PRESSURE

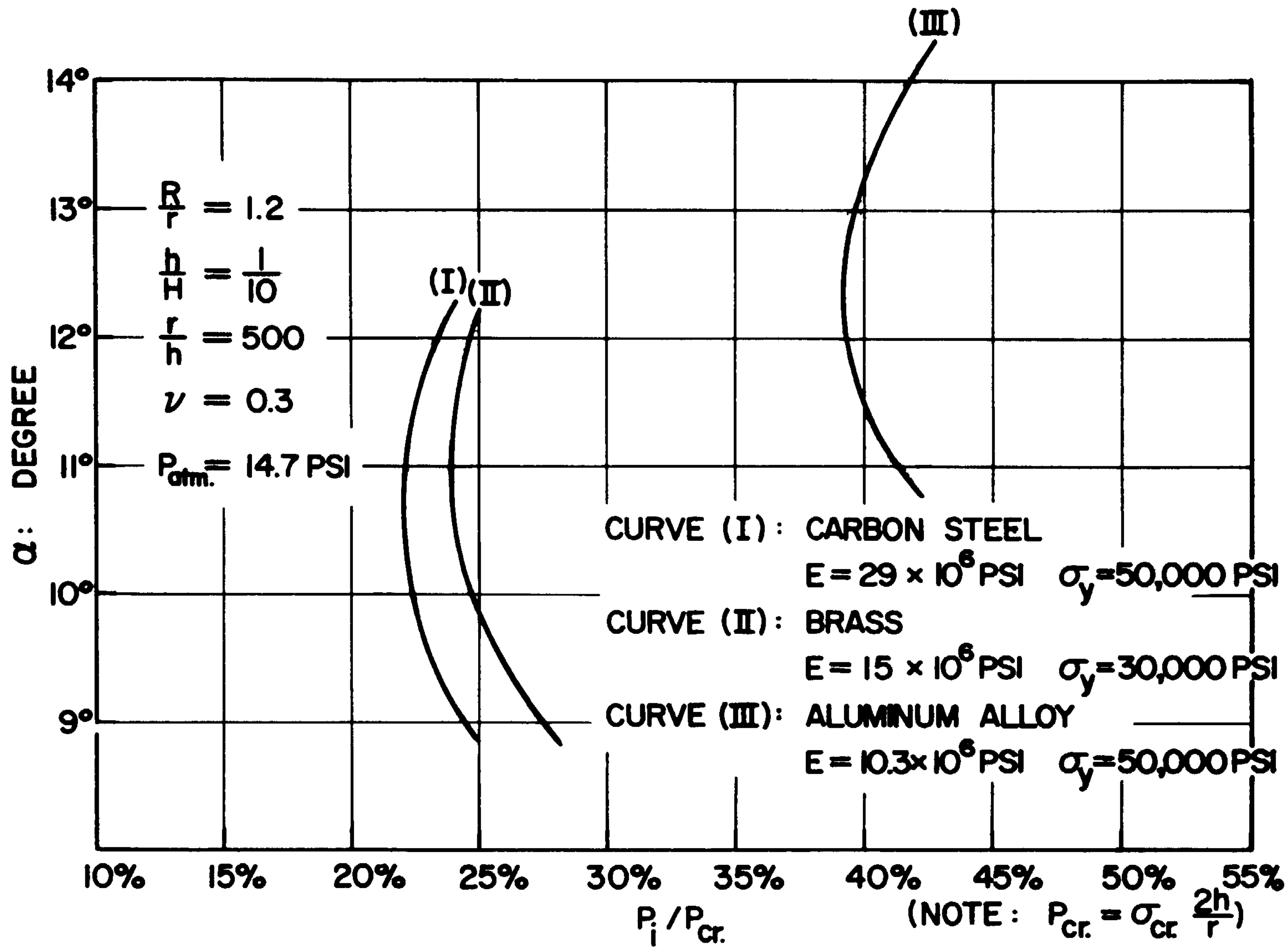


FIG. 6. EFFECT OF MATERIAL OF SPECIMEN ON BUCKLING PRESSURE