NASr-21(02)

MEMORANDUM
RM-3861-NASA
SEPTEMBER 1963

PERTURBATIONS OF EQUATORIAL SATELLITES DUE TO EQUATORIAL ELLIPTICITY

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This research is sponsored by the National Aeronautics and Space Administration under Contract No. NASr-21. This report does not necessarily represent the views of the National Aeronautics and Space Administration.

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PREFACE

This Memorandum is part of a series of continuing studies on orbital perturbations under NASA contract NASr-21 (02). It represents an extension to a previous RAND report, R-399-NASA, "Perturbations of a Synchronous Satellite," by R. H. Frick and T. B. Garber. The results should be of interest to any agency concerned with long term satellite station-keeping.

SUMMARY

This Memorandum analyzes the perturbations of circular equatorial orbits due to the ellipticity of the earth's equatorial section. The results indicate that with the exception of orbits with periods of 12, 24, and 36 hours, the perturbations due to equatorial ellipticity are negligible and completely dominated by the perturbations resulting from initial condition errors. In particular, initial errors in orbital radius and orbital velocity cause steady state drifts in angular position of the satellite which are not influenced by the earth's equatorial ellipticity.

The three special periods include the 24-hour synchronous orbit which, as shown previously,[1] undergoes long period oscillations in longitude about the position of the minor axis of the earth's equatorial section. In addition, the orbits with 12 and 36 hour periods are resonant orbits in which the driving function resulting from equatorial ellipticity has a frequency equal to orbital frequency. This results in a divergent oscillatory perturbation at orbital frequency in both orbital radius and orbital angle. The rates of buildup of the amplitudes of these oscillations are 30.4 n mi/yr in radius and .243 deg/yr in angle for the 12-hour orbit. The corresponding figures for the 36-hour orbit are 34.1 n mi/yr and .131 deg/yr.* These divergent oscillations are the dominant oscillatory effect for these two orbits, but they are still subject to the steady state drift in angle due to initial condition errors.

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*The exact values of these rates are dependent on the assumed ellipticity of the earth's equatorial section.

CONTENTS

SYMBOLS

g_o = gravitational acceleration at the earth's surface

J_2 = earth oblateness coefficient

$J_2^{(2)}$ = equatorial ellipticity coefficient

k_2 = modified ellipticity coefficient

n = ratio of orbital rate to earth rate

R_E = mean earth radius

r = radial distance from earth's center to satellite

r_c = circular orbital radius corrected for oblateness

r_s = synchronous orbital radius

r_o = circular orbital radius without oblateness

Δr = perturbation in r

Δr_m = amplitude of divergent oscillation in Δr

Δr_o = initial value of Δr

$\Delta \dot{r}_o$ = initial value of rate of change of Δr

$\delta(\Delta r_o)$ = error in Δr_o

$\delta(\Delta \dot{r}_o)$ = error in $\Delta \dot{r}_o$

t = time

U = earth's gravitational potential

δV_o = error in initial orbital velocity

γ = angle between r and minor axis of earth's equatorial section

γ_o = initial value of γ

$\dot{\gamma}$ = rate of change of γ

θ = satellite central angle measured from inertial reference

$\dot{\theta}$ = rate of change of θ

θ_E = earth rotation angle measured from inertial reference

$\dot{\theta}_E$ = earth's angular rate

$\dot{\theta}_o$ = unperturbed orbital angular rate

$\Delta\dot{\theta}$ = perturbation in $\dot{\theta}$

$\Delta\theta_m$ = amplitude of divergent oscillation in $\Delta\theta$

$\Delta\dot{\theta}_o$ = initial value of $\Delta\dot{\theta}$

$\Delta\dot{\theta}_{ss}$ = steady state value of $\Delta\dot{\theta}$

$\delta(\Delta\dot{\theta}_o)$ = error in $\Delta\dot{\theta}_o$

I. INTRODUCTION

It has been shown in Refs. 1 and 2 that a synchronous equatorial satellite under the influence of the earth's equatorial ellipticity will undergo long period oscillations in longitude about the position of the minor axis of the earth's equatorial section.

This Memorandum considers the effect of this same ellipticity on circular equatorial orbits of any period. The analysis as presented in Section II is similar to the perturbation method used in Ref. 1 with the generalization that the orbital angular rate no longer equals the earth's rate. The solution of the perturbation equations shows the sensitivity of the resulting drift rate to initial condition errors. Three special cases of the general solution are also considered: the synchronous orbit and the two resonant orbits with angular rates of twice and two-thirds of earth rate.

II. ANALYSIS

STATEMENT OF THE PROBLEM

If a satellite is established in a circular equatorial orbit with an angular rate of $\dot{\theta}_o$, what are the perturbations in angular rate and orbital radius resulting from the earth's equatorial ellipticity?

REFERENCE SYSTEM

In Fig. 1 the XY coordinate system is a geocentric inertial reference system in the earth's equatorial plane. The line AA$'$ is the minor axis of the earth's equatorial section which rotates at earth rate, $\dot{\theta}_E$. The instantaneous position of the satellite S is specified by its radial distance, r, and its central angle θ, measured from the X axis. The angle γ is the instantaneous central angle between the satellite radius, r, and the minor axis AA$'$.

EQUATIONS OF MOTION

In Ref. 1, the general equations of motion are developed. The potential function used is of the following form for the equatorial plane:

$$U = \frac{g_o R_E^2}{r}\left[1 + J_2 \frac{R_E^2}{2r^2} + 3J_2^{(2)} \frac{R_E^2}{r^2} \cos 2\gamma\right] \tag{1}$$

The resulting equations of motion in the equatorial plane are of the form

$$\frac{d^2r}{dt^2} - r\dot{\theta}^2 = - \frac{g_o R_E^2}{r^2} - \frac{3J_2 g_o R_E^4}{2r^4} - \frac{9J_2^{(2)} g_o R_E^4}{r^4} \cos 2\gamma \tag{2}$$

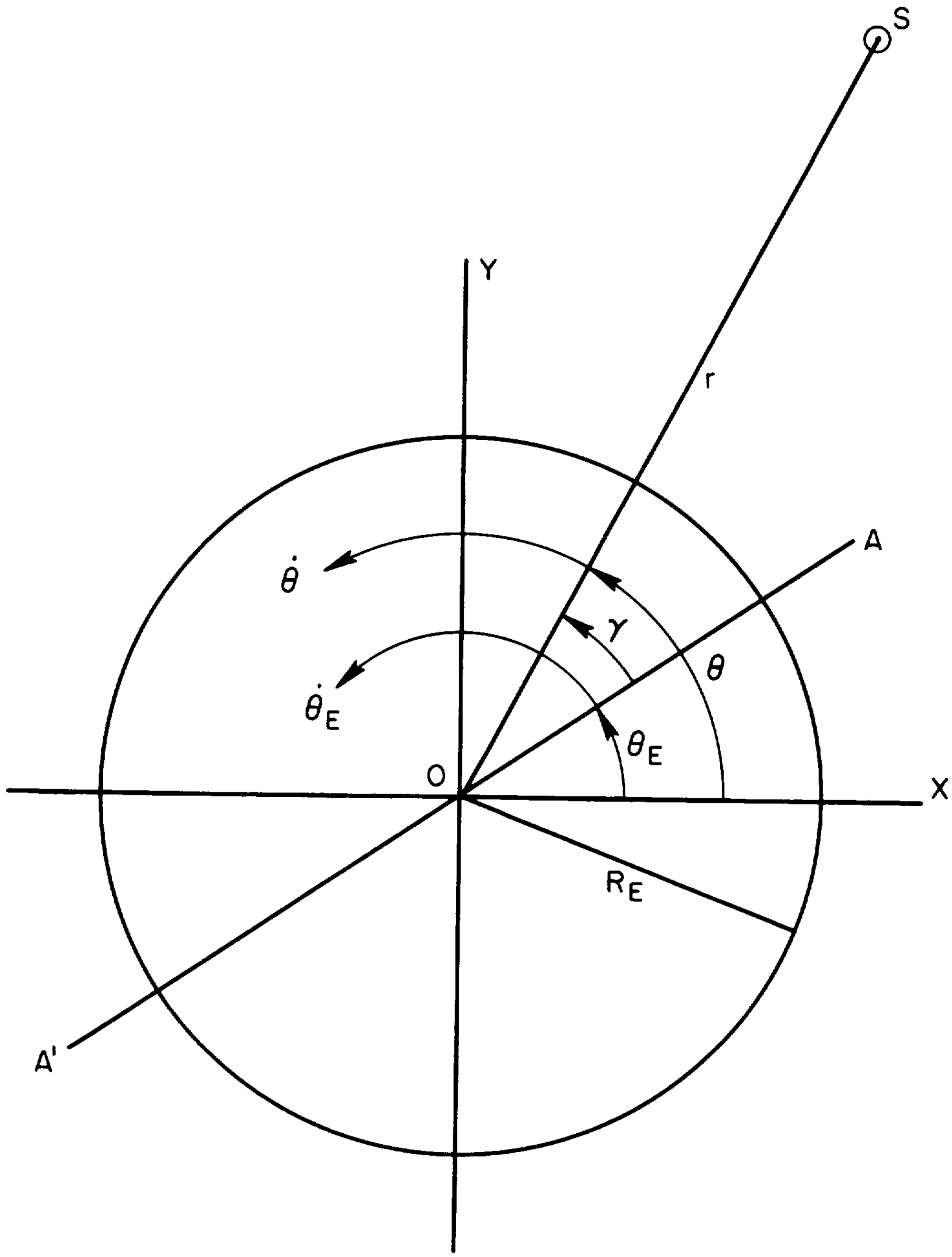

Fig. 1— Reference system

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{6J_2^{(2)}g_oR_E^4}{r^4}\sin 2\gamma \qquad (3)$$

PERTURBATION EQUATIONS

If the desired orbital angular rate is $\dot{\theta}_o$, a set of perturbation variables can be defined as follows:

$$r = r_c + \Delta r \qquad (4)$$

$$\dot{\theta} = \dot{\theta}_o + \Delta\dot{\theta} \qquad (5)$$

where r_c is the circular orbital radius corresponding to $\dot{\theta}_o$ (r_c is corrected for the effect of earth oblateness).

Also the angle γ can be expressed in the form

$$\gamma = \dot{\gamma}t + \gamma_o \qquad (6)$$

where

$$\dot{\gamma} \doteq \dot{\theta}_o - \dot{\theta}_E \qquad (7)$$

Substitution of Eqs. (4), (5), and (6) into Eqs. (2) and (3) gives the desired perturbation equations as follows:*

$$\frac{d^2}{dt^2}\left(\frac{\Delta r}{r_c}\right) - 3\dot{\theta}_o^2\left(\frac{\Delta r}{r_c}\right) - 2\dot{\theta}_o^2\left(\frac{\Delta\dot{\theta}}{\dot{\theta}_o}\right) = 9k_2\,\dot{\theta}_o^2\cos 2(\dot{\gamma}t + \gamma_o) \qquad (8)$$

$$2\frac{d}{dt}\left(\frac{\Delta r}{r_c}\right) + \frac{d}{dt}\left(\frac{\Delta\dot{\theta}}{\dot{\theta}_o}\right) = 6k_2\,\dot{\theta}_o\sin 2(\dot{\gamma}t + \gamma_o) \qquad (9)$$

where

*For details see pp 6-8 in Ref. 1.

$$k_2 = -J_2^{(2)} \frac{R_E^2}{r_c^2} \tag{10}$$

and the value of r_c is obtained from the equation

$$r_c \dot{\theta}_o^2 = \frac{g_o R_E^2}{r_c^{\,2}} + \frac{3J_2\, g_o R_E^4}{2r_c^{\,4}} \tag{11}$$

The approximate solution of Eq. (11) is

$$r_c = r_o \left(1 + \frac{J_2\, R_E^2}{2r_o^{\,2}} \right) \tag{12}$$

where r_o is the orbital radius corresponding to $\dot{\theta}_o$ if there were no oblateness, as given by

$$r_o^3 = \frac{g_o R_E^2}{\dot{\theta}_o^{\,2}} \tag{13}$$

PERTURBATION SOLUTION

Equations (8) and (9) can be solved simultaneously to give

$$\begin{aligned} \frac{\Delta r}{r_c} = 3k_2 \Bigg[& \frac{2n \cos 2\gamma_o}{n-1} - \frac{n(5n-8) \cos 2\gamma_o \cos n\, \dot{\theta}_E t}{(n-2)\,(3n-2)} \\ & - \frac{2n\,(n-3) \sin 2\gamma_o \sin n\, \dot{\theta}_E t}{(n-2)\,(3n-2)} - \frac{n^2\,(n-3) \cos 2\left[(n-1)\, \dot{\theta}_E t + \gamma_o\right]}{(n-1)\,(n-2)\,(3n-2)} \Bigg] \\ & + (4 - 3 \cos n\, \dot{\theta}_E t) \frac{\Delta r_o}{r_c} + \frac{\sin n\, \dot{\theta}_E t}{n\dot{\theta}_E} \frac{\Delta \dot{r}_o}{r_c} + 2(1 - \cos n\, \dot{\theta}_E t) \frac{\Delta \dot{\theta}_o}{n\dot{\theta}_E} \end{aligned} \tag{14}$$

$$\frac{\Delta\dot{\theta}}{\dot{\theta}_E} = 6k_2 n \left[- \frac{3n \cos 2\gamma_o}{2(n-1)} + \frac{n(5n-8) \cos 2\gamma_o \cos n \dot{\theta}_E t}{(n-2)(3n-2)} \right.$$

$$+ \frac{2n(n-3) \sin 2\gamma_o \sin n \dot{\theta}_E t}{(n-2)(3n-2)}$$

$$\left. - \frac{n(n^2-2n+4) \cos 2\left[(n-1)\dot{\theta}_E t + \gamma_o\right]}{2(n-1)(n-2)(3n-2)} \right]$$

$$- 6n(1-\cos n \dot{\theta}_E t) \frac{\Delta r_o}{r_c} - \frac{2 \sin n \dot{\theta}_E t}{\dot{\theta}_E} \frac{\Delta \dot{r}_o}{r_c}$$

$$+ (4 \cos n \dot{\theta}_E t - 3) \frac{\Delta\dot{\theta}_o}{\dot{\theta}_E} \tag{15}$$

where Δr_o, $\Delta\dot{r}_o$ and $\Delta\dot{\theta}_o$ are the initial values of Δr, $\Delta\dot{r}$, and $\Delta\dot{\theta}$ respectively, and n is given by

$$n = \frac{\dot{\theta}_o}{\dot{\theta}_E} \tag{16}$$

An examination of Eqs. (14) and (15) shows that they are indeterminate for n = 2/3, 1, and 2, corresponding to orbital periods of 36, 24, and 12 hours. These cases are evaluated in the next section of the report.

For any other value of n than the three specified above, Eqs. (14) and (15) are valid solutions. By a suitable selection of initial conditions, the coefficients of $\cos n\,\dot{\theta}_E t$, $\sin n\,\dot{\theta}_E t$ and the constant term in Eq. (15) can be made zero. The required initial conditions are as follows:

$$\frac{\Delta r_o}{r_c} = -\frac{3k_2\, n^2\, (n-3)\cos 2\gamma_o}{(n-1)(n-2)(3n-2)} + \frac{\delta(\Delta r_o)}{r_c} \tag{17}$$

$$\frac{\Delta \dot{r}_o}{r_c} = \frac{6k_2\, \dot{\theta}_E n^2\, (n-3)\sin 2\gamma_o}{(n-2)(3n-2)} + \frac{\delta(\Delta \dot{r}_o)}{r_c} \tag{18}$$

$$\frac{\Delta \dot{\theta}_o}{\dot{\theta}_E} = -\frac{3k_2\, n^2\, (n^2-2n+4)\cos 2\gamma_o}{(n-1)(n-2)(3n-2)} + \frac{\delta(\Delta \dot{\theta}_o)}{\dot{\theta}_E} \tag{19}$$

In each of these expressions the first term is the required initial condition while $\delta(\Delta r_o)$, $\delta(\Delta \dot{r}_o)$ and $\delta(\Delta \dot{\theta}_o)$ are the residual errors in achieving the desired value. Substitution of Eqs. (17), (18), and (19) into Eqs. (14) and (15) reduces the solution to

$$\begin{aligned}\frac{\Delta r}{r_c} = &-\frac{3k_2\, n^2\, (n-3)\cos 2\left[(n-1)\,\dot{\theta}_E t + \gamma_o\right]}{(n-1)(n-2)(3n-2)} \\ &+ (4-3\cos n\,\dot{\theta}_E t)\,\frac{\delta(\Delta r_o)}{r_c} + \frac{\sin n\,\dot{\theta}_E t}{n\dot{\theta}_E}\,\frac{\delta(\Delta \dot{r}_o)}{r_c} \\ &+ 2\,(1-\cos n\,\dot{\theta}_E t)\,\frac{\delta(\Delta \dot{\theta}_o)}{n\dot{\theta}_E}\end{aligned} \tag{20}$$

$$\frac{\Delta\dot{\theta}}{\dot{\theta}_E} = - \frac{3k_2 n^2 (n^2 - 2n + 4) \cos 2\left[(n-1)\dot{\theta}_E t + \gamma_o\right]}{(n-1)(n-2)(3n-2)}$$

$$- 6n (1-\cos n\,\dot{\theta}_E t) \frac{\delta(\Delta r_o)}{r_c} - \frac{2 \sin n\,\dot{\theta}_E t}{\dot{\theta}_E} \frac{\delta(\Delta\dot{r}_o)}{r_c}$$

$$+ (4 \cos n\,\dot{\theta}_E t - 3) \frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E} \qquad (21)$$

An examination of Eqs. (20) and (21) shows that the earth's equatorial ellipticity causes a bounded oscillation with a frequency of $2(n-1)\,\dot{\theta}_E$ in both Δr and $\Delta\dot{\theta}$. In addition the errors in the initial conditions introduce oscillatory terms at orbital frequency, $n\,\dot{\theta}_E$ as well as constant bias terms. In particular, the steady state bias term in Eq. (21) would result in a steady drift of the satellite relative to the desired orbital rate, $n\,\dot{\theta}_E$.

SPECIAL CASES

It was indicated in the previous section that Eqs. (14) and (15) become indeterminate for n = 2/3, 1 and 2. These cases are considered below.

Synchronous Satellite (n = 1)

If the limits of Eqs. (14) and (15) are taken as n approaches 1 the following expressions are obtained

$$\frac{\Delta r}{r_c} = 3k_2 \Big[3 \cos 2\gamma_o - 3 \cos 2\gamma_o \cos \dot{\theta}_E t$$

$$- 4 \sin 2\gamma_o \sin \dot{\theta}_E t + 4\dot{\theta}_E t \sin 2\gamma_o\Big]$$

$$+ (4-3 \cos \dot{\theta}_E t) \frac{\Delta r_o}{r_c} + \frac{\sin \dot{\theta}_E t}{\dot{\theta}_E} \frac{\Delta \dot{r}_o}{r_c}$$

$$+ 2 (1 - \cos \dot{\theta}_E t) \frac{\Delta \dot{\theta}_o}{\dot{\theta}_E} \tag{22}$$

$$\frac{\Delta \dot{\theta}}{\dot{\theta}_E} = 6k_2 \Big[- 3 \cos 2\gamma_o + 3 \cos 2\gamma_o \cos \dot{\theta}_E t$$

$$+ 4 \sin 2\gamma_o \sin \dot{\theta}_E t - 3\dot{\theta}_E t \sin 2\gamma_o\Big]$$

$$- 6 (1 - \cos \dot{\theta}_E t) \frac{\Delta r_o}{r_c} - \frac{2 \sin \dot{\theta}_E t}{\dot{\theta}_E} \frac{\Delta \dot{r}_o}{r_c}$$

$$+ (4 \cos \dot{\theta}_E t - 3) \frac{\Delta \dot{\theta}_o}{\dot{\theta}_E} \tag{23}$$

As before, the constant term and the coefficients of $\cos \dot{\theta}_E t$ and $\sin \dot{\theta}_E t$ in Eq. (23) can be set equal to zero, so that the desired initial conditions are

$$\frac{\Delta r_o}{r_c} = - 3k_2 \cos 2\gamma_o + \frac{\delta(\Delta r_o)}{r_c} \tag{24}$$

$$\frac{\Delta\dot{r}_o}{r_c} = 12k_2\,\dot{\theta}_E \sin 2\gamma_o + \frac{\delta(\Delta\dot{r}_o)}{r_c} \tag{25}$$

$$\frac{\Delta\dot{\theta}_o}{\dot{\theta}_E} = 0 + \frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E} \tag{26}$$

These initial conditions reduce Eqs. (22) and (23) to

$$\frac{\Delta r}{r_c} = -\,3k_2 \cos 2\gamma_o + 12k_2\,\dot{\theta}_E t \sin 2\gamma_o$$

$$+ (4 - 3\cos\dot{\theta}_E t)\,\frac{\delta(\Delta r_o)}{r_c} + \frac{\sin\dot{\theta}_E t}{\dot{\theta}_E}\,\frac{\delta(\Delta\dot{r}_o)}{r_c}$$

$$+ 2\,(1 - \cos\dot{\theta}_E t)\,\frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E} \tag{27}$$

$$\frac{\Delta\dot{\theta}}{\dot{\theta}_E} = -\,18k_2\,\dot{\theta}_E t \sin 2\gamma_o$$

$$-\,6\,(1 - \cos\dot{\theta}_E t)\,\frac{\delta(\Delta r_o)}{r_c} - \frac{2\sin\dot{\theta}_E t}{\dot{\theta}_E}\,\frac{\delta(\Delta\dot{r}_o)}{r_c}$$

$$+ (4\cos\dot{\theta}_E t - 3)\,\frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E} \tag{28}$$

The residual bias term in Eq. (27) can be regarded as an additional correction to the steady state radius r_c. Then it is seen that if the residual initial condition errors are reduced to zero, Eqs. (27) and (28) become

$$\frac{\Delta r}{r_c} = 12k_2\, \dot{\theta}_E t \sin 2\gamma_o \tag{29}$$

$$\frac{\Delta\dot{\theta}}{\dot{\theta}_E} = -\, 18k_2\, \dot{\theta}_E t \sin 2\gamma_o \tag{30}$$

These relations are equivalent to Eqs. (53) and (54) of Ref. 1.

Twelve Hour Period Satellite (n = 2)

If the limits of Eqs. (14) and (15) are taken as n approaches 2, the following expressions are obtained

$$\begin{aligned}\frac{\Delta r}{r_c} = 3k_2 \Big[& 4 \cos 2\gamma_o - 4 \cos 2\gamma_o \cos 2\dot{\theta}_E t \\ & + \frac{1}{2} \sin 2\gamma_o \sin 2\dot{\theta}_E t - \dot{\theta}_E t \sin 2(\dot{\theta}_E t + \gamma_o)\Big] \\ & + (4 - 3 \cos 2\dot{\theta}_E t)\, \frac{\Delta r_o}{r_c} + \frac{\sin 2\dot{\theta}_E t}{2\dot{\theta}_E}\, \frac{\Delta \dot{r}_o}{r_c} \\ & + (1 - \cos 2\dot{\theta}_E t)\, \frac{\Delta\dot{\theta}_o}{\dot{\theta}_E}\end{aligned} \tag{31}$$

$$\frac{\Delta\dot{\theta}}{\dot{\theta}_E} = 12k_2 \left[- 3 \cos 2\gamma_o + 3 \cos 2\gamma_o \cos 2\dot{\theta}_E t + \frac{1}{2} \sin 2\gamma_o \sin 2\dot{\theta}_E t + \dot{\theta}_E t \sin 2(\dot{\theta}_E t + \gamma_o)\right]$$

$$- 12(1 - \cos 2\dot{\theta}_E t) \frac{\Delta r_o}{r_c} - \frac{2 \sin 2\dot{\theta}_E t}{\dot{\theta}_E} \frac{\Delta \dot{r}_o}{r_c}$$

$$+ (4 \cos 2\dot{\theta}_E t - 3) \frac{\Delta\dot{\theta}_o}{\dot{\theta}_E} \tag{32}$$

As before the constant term and the coefficients of $\cos 2\dot{\theta}_E t$ and $\sin 2\dot{\theta}_E t$ are set equal to zero. The required initial conditions are obtained as follows:

$$\frac{\Delta r_o}{r_c} = - 3k_2 \cos 2\gamma_o + \frac{\delta(\Delta r_o)}{r_c} \tag{33}$$

$$\frac{\Delta \dot{r}_o}{r_c} = 3k_2 \dot{\theta}_E \sin 2\gamma_o + \frac{\delta(\Delta \dot{r}_o)}{r_c} \tag{34}$$

$$\frac{\Delta\dot{\theta}_o}{\dot{\theta}_E} = 0 + \frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E} \tag{35}$$

Substitution for the initial conditions in Eqs. (31) and (32) gives

$$\frac{\Delta r}{r_c} = - 3k_2 \cos 2(\dot{\theta}_E t + \gamma_o) - 3k_2 \dot{\theta}_E t \sin 2(\dot{\theta}_E t + \gamma_o)$$

$$+ (4-3 \cos 2\dot{\theta}_E t) \frac{\delta(\Delta r_o)}{r_c} + \frac{\sin 2\dot{\theta}_E t}{2\dot{\theta}_E} \frac{\delta(\Delta \dot{r}_o)}{r_c}$$

$$+ (1 - \cos 2\dot{\theta}_E t) \frac{\delta(\Delta \dot{\theta}_o)}{\dot{\theta}_E} \tag{36}$$

$$\frac{\Delta \dot{\theta}}{\dot{\theta}_E} = 12k_2 \dot{\theta}_E t \sin 2(\dot{\theta}_E t + \gamma_o)$$

$$- 12 (1 - \cos 2\dot{\theta}_E t) \frac{\delta(\Delta r_o)}{r_c} - \frac{2\sin 2\dot{\theta}_E t}{\dot{\theta}_E} \frac{\delta(\Delta \dot{r}_o)}{r_c}$$

$$+ (4 \cos 2\dot{\theta}_E t - 3) \frac{\delta(\Delta \dot{\theta}_o)}{\dot{\theta}_E} \tag{37}$$

From Eqs. (36) and (37) it is seen that both Δr and $\Delta\dot{\theta}$ contain oscillatory terms with amplitudes which increase with time.

<u>36 Hour Period Satellite (n = 2/3)</u>

If the limits of Eqs. (14) and (15) are taken as n approaches 2/3, the following expressions are obtained

$$\frac{\Delta r}{r_c} = 3k_2 \left[- 4 \cos 2\gamma_o + 4 \cos 2\gamma_o \cos \frac{2\dot{\theta}_E t}{3} \right.$$

$$\left. + \frac{7}{2} \sin 2\gamma_o \sin \frac{2\dot{\theta}_E t}{3} + \frac{7}{3} \dot{\theta}_E t \sin 2\left(\frac{\dot{\theta}_E t}{3} - \gamma_o\right) \right]$$

$$+ \left(4 - 3 \cos \frac{2\dot{\theta}_E t}{3}\right) \frac{\Delta r_o}{r_c} + \frac{3 \sin \frac{2\dot{\theta}_E t}{3}}{2\dot{\theta}_E} \frac{\Delta \dot{r}_o}{r_c}$$

$$+ 3\left(1 - \cos \frac{2\dot{\theta}_E t}{3}\right) \frac{\Delta \dot{\theta}_o}{\dot{\theta}_E} \tag{38}$$

$$\frac{\Delta \dot{\theta}}{\dot{\theta}_E} = 4k_2 \left[3 \cos 2\gamma_o - 3 \cos 2\gamma_o \cos \frac{2\dot{\theta}_E t}{3} \right.$$

$$\left. - \frac{5}{2} \sin 2\gamma_o \sin \frac{2\dot{\theta}_E t}{3} - \frac{7}{3} \dot{\theta}_E t \sin 2\left(\frac{\dot{\theta}_E t}{3} - \gamma_o\right) \right]$$

$$- 4\left(1 - \cos \frac{2\dot{\theta}_E t}{3}\right) \frac{\Delta r_o}{r_c} - \frac{2 \sin \frac{2}{3} \dot{\theta}_E t}{\dot{\theta}_E} \frac{\Delta \dot{r}_o}{r_c}$$

$$+ \left(4 \cos \frac{2\dot{\theta}_E t}{3} - 3\right) \frac{\Delta \dot{\theta}_o}{\dot{\theta}_E} \tag{39}$$

If as before the constant term and the coefficients of $\cos \frac{2\dot{\theta}_E t}{3}$ and $\sin \frac{2\dot{\theta}_E t}{3}$ in Eq. (39) are equated to zero, the following initial condition relations are obtained

$$\frac{\Delta r_o}{r_c} + 3k_2 \cos 2\gamma_o + \frac{\delta(\Delta r_o)}{r_c} \tag{40}$$

$$\frac{\Delta \dot{r}_o}{r_c} = - 5k_2 \dot{\theta}_E \sin 2\gamma_o + \frac{\delta(\Delta \dot{r}_o)}{r_c} \tag{41}$$

$$\frac{\Delta\dot{\theta}_o}{\dot{\theta}_E} = 0 + \frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E} \tag{42}$$

Substitution of these initial conditions in Eqs. (38) and (39) gives

$$\begin{aligned}\frac{\Delta r}{r_c} = {} & 3k_2 \cos 2\left(\frac{\dot{\theta}_E t}{3} - \gamma_o\right) + 7k_2\, \dot{\theta}_E t \sin 2\left(\frac{\dot{\theta}_E t}{3} - \gamma_o\right) \\ & + \left(4 - 3 \cos \frac{2\dot{\theta}_E t}{3}\right) \frac{\delta(\Delta r_o)}{r_c} + \frac{3 \sin \frac{2\dot{\theta}_E t}{3}}{2\dot{\theta}_E} \frac{\delta(\Delta \dot{r}_o)}{r_c} \\ & + 3\left(1 - \cos \frac{2\dot{\theta}_E t}{3}\right) \frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E}\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\Delta\dot{\theta}}{\dot{\theta}_E} = {} & - \frac{28}{3} k_2\, \dot{\theta}_E t \sin 2\left(\frac{\dot{\theta}_E t}{3} - \gamma_o\right) \\ & - 4\left(1 - \cos \frac{2\dot{\theta}_E t}{3}\right) \frac{\delta(\Delta r_o)}{r_c} \\ & - \frac{2 \sin \frac{2\dot{\theta}_E t}{3}}{\dot{\theta}_E} \frac{\delta(\Delta \dot{r}_o)}{r_c} \\ & + \left(4\cos \frac{2\dot{\theta}_E t}{3} - 3\right) \frac{\delta(\Delta\dot{\theta}_o)}{\dot{\theta}_E}\end{aligned} \tag{44}$$

As in the case of the 12-hour period satellite, a divergent oscillation appears in both Δr and $\Delta\dot{\theta}$.

III. RESULTS AND DISCUSSION

GENERAL SOLUTION

If n is not equal to 2/3, 1 or 2, an examination of Eqs. (20) and (21) shows that the earth's equatorial ellipticity produces bounded oscillations in both Δr and $\Delta\dot{\theta}$ at a frequency of $2(n-1)\,\dot{\theta}_E$ or twice the orbital rate relative to the earth. These oscillations result in displacements of the order of a few hundred feet relative to an unperturbed satellite with an orbital rate $n\dot{\theta}_E$.

If in addition any residual initial condition errors exist, oscillatory terms will be introduced in both Δr and $\Delta\dot{\theta}$ with a frequency of $n\dot{\theta}_E$, the orbital rate relative to inertial space. In addition, the initial condition errors will introduce constant bias errors in Δr and $\Delta\dot{\theta}$ which will cause a steady state drift relative to the unperturbed satellite. The magnitude of this drift is given by the relation

$$\Delta\dot{\theta}_{ss} = -\,6n\dot{\theta}_E\,\frac{\delta(\Delta r_o)}{r_c} - 3\delta(\Delta\dot{\theta}_o) \tag{45}$$

where

$$r_c = \frac{r_s}{n^{2/3}} \tag{46}$$

$$\delta(\Delta\dot{\theta}_o) = \frac{\delta V_o}{r_c} = \frac{n^{2/3}\,\delta V_o}{r_s} \tag{47}$$

r_s = orbital radius for a synchronous satellite (22748.4 n mi)

δV_o = orbital velocity error

Substitution of Eqs. (46) and (47) into (45) gives

$$\Delta\dot{\theta}_{ss} = -\frac{6\dot{\theta}_E}{r_s} n^{5/3}\,\delta(\Delta r_o) - \frac{3}{r_s} n^{2/3}\,\delta V_o \tag{48}$$

Since the errors $\delta(\Delta r_o)$ and δV_o are independent, the resulting contributions to the steady state drift can be treated separately so that

$$\Delta\dot{\theta}_{ss} = -\frac{6\dot{\theta}_E}{r_s} n^{5/3}\,\delta(\Delta r_o)$$

$$= 34.66\ n^{5/3}\,\delta(\Delta r_o)\ \text{deg/yr/n mi} \tag{49}$$

where $\delta(\Delta r_o)$ is expressed in nautical miles.

$$\Delta\dot{\theta}_{ss} = -\frac{3n^{2/3}\,\delta V_o}{r_s}$$

$$= -\ 39.19\ n^{2/3}\,\delta V_o\ \frac{\text{deg/yr}}{\text{ft/s}} \tag{50}$$

where δV_o is in ft/s.

Figure 2 is a plot of Eqs. (49) and (50) as a function of n plotted from n = 0 to n = 17.067 which corresponds to a surface orbit with a period of 84.4 min.

As an example, for n = 5, corresponding to an orbital period of 4 hrs and 48 min and a nominal orbital radius equal to 7779.9 n mi, an error of .1 n mi in radius would result in a drift rate of 50.7 deg/yr.

Similarly an error of 1 ft/sec in orbital velocity would result in a drift rate of 114.6 deg/yr.

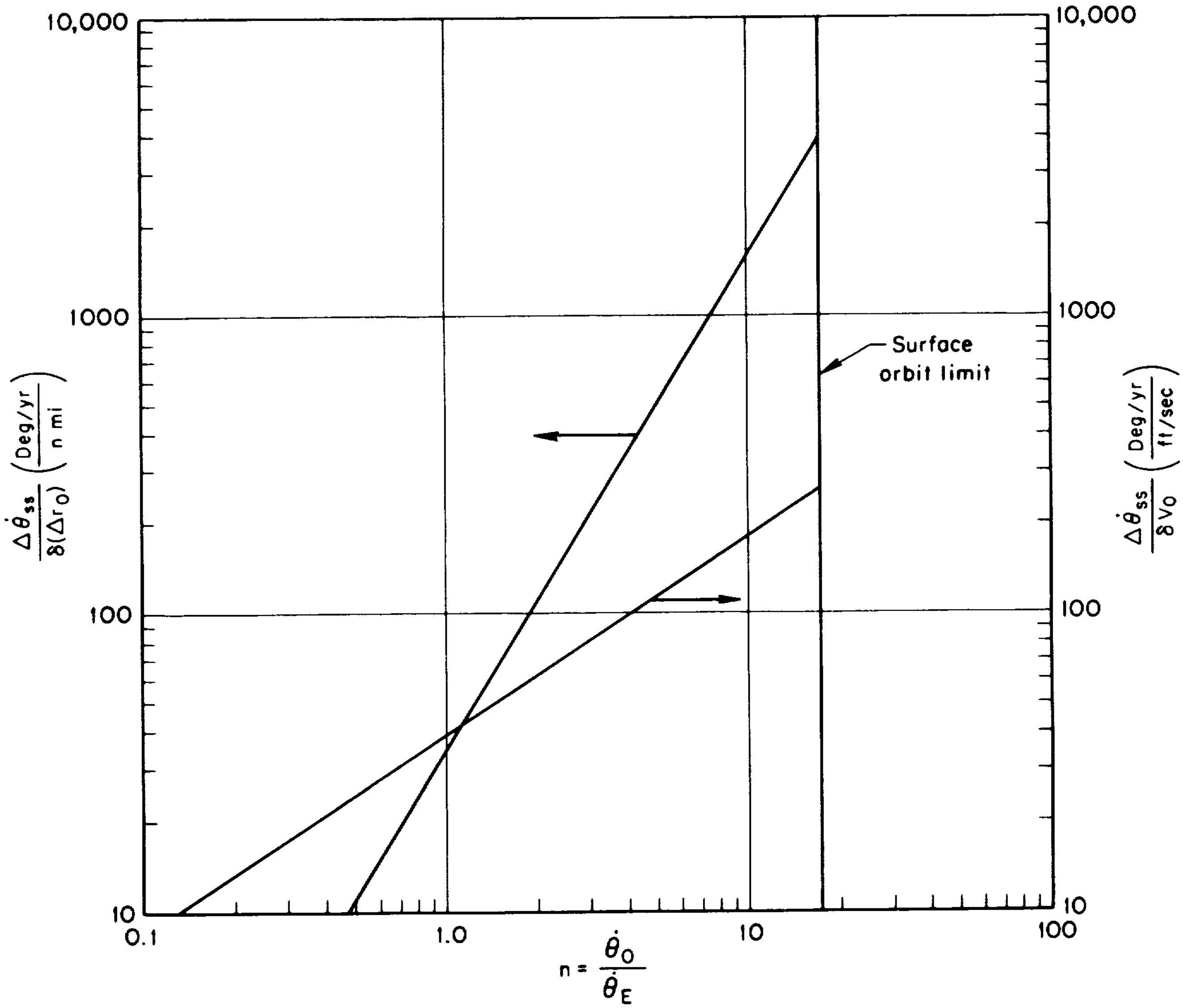

Fig. 2—Steady state drift rate

This sensitivity of drift rate to initial condition errors is identical with that shown in Fig. 51 of Ref. 3. Thus, it is seen that the resulting drift rate is determined by the initial condition errors. The presence of equatorial ellipticity merely changes the desired values of the initial conditions as specified in Eqs. (17), (18), and (19).

SPECIAL CASES

In the special cases when n = 2/3, 1, or 2 it is seen that the solutions for Δr and $\Delta\dot{\theta}$ include terms with amplitudes which increase with time. The physical significance of these time dependent solutions is discussed in the following paragraphs.

Synchronous Satellite (n = 1)

From Eqs. (29) and (30), it is seen that both Δr and $\Delta\dot{\theta}$ increase linearly with time. However, in Eq. (7) which defines $\dot{\gamma}$ the additive term of $\Delta\dot{\theta}$ has been neglected as being small compared to $\dot{\theta}_o - \dot{\theta}_E$. In the case of a synchronous satellite, this is no longer valid, since $\dot{\theta}_o$ is equal to $\dot{\theta}_E$. If the $\Delta\dot{\theta}$ term is included in $\dot{\gamma}$, it effectively causes a slow change in γ_o in Eqs. (29) and (30). Thus, these expressions for Δr and $\Delta\dot{\theta}$ are only valid for small values of $\Delta\theta$ (of the order of 10^o). However, in Ref. 1 a more general treatment of this problem is presented in which the limit on the size of $\Delta\theta$ is removed. This more general treatment shows that Eqs. (29) and (30) represent the initial part of a long period oscillation in satellite longitude about the position of the minor axis of the earth's equatorial section.

Twelve-Hour Period Satellite (n = 2)

This case and the one following for n = 2/3 are resonant orbits in which the frequency of the perturbing function, $2\dot{\gamma}$, is equal in magnitude to the orbital angular rate $\dot{\theta}_o$. This condition is given by the relation

$$2\dot{\gamma} = 2(n-1)\ \dot{\theta}_E = \pm\ n\dot{\theta}_E = \pm\ \dot{\theta}_o \qquad (51)$$

which is satisfied for n = 2 or 2/3.

For n = 2, Eqs. (36) and (37) show that both Δr and $\Delta\dot{\theta}$ contain terms of the form t sin $2(\dot{\theta}_E t + \gamma_o)$ which is the typical buildup of an undamped resonant system excited at its natural frequency. Thus, the amplitudes of Δr and $\Delta\theta$ apparently grow without bound and the solution is only valid during the time that $\frac{\Delta r}{r_c}$ and $\frac{\Delta\dot{\theta}}{\dot{\theta}_E}$ remain small.

It should be noted that the $\Delta\dot{\theta}$ contribution to $\dot{\gamma}$ would detune the system slightly so that a beat frequency response would result between the frequencies $2\dot{\gamma}$ and $\dot{\theta}_o$. However, the initial buildup of this beat is represented quite well by Eqs. (36) and (37).

If the divergent term in Eq. (37) is integrated, the following expression for the divergent term in $\Delta\theta$ is obtained

$$\Delta\theta = -\ 6k_2\ \dot{\theta}_E t\ \cos\ 2(\dot{\theta}_E t + \gamma_o) \qquad (52)$$

Thus for this orbit $\Delta\theta$ has an oscillatory buildup at orbital frequency, $2\dot{\theta}_E$, and with an amplitude $\Delta\theta_m$ which increases at the rate

$$\frac{\Delta\theta_m}{t} = 6k_2\ \dot{\theta}_E$$

$$= .243\ \text{deg/yr} \qquad (53)$$

where

$$k_2 = - J_2^{(2)} \frac{R_E^2}{r_c^2} \quad (54)$$

$$r_c = \frac{r_s}{(2)^{2/3}} = .6335 \, r_s \quad (55)$$

and

$$J_2^{(2)} = - 5.35 \times 10^{-6} \text{ (Ref. 4)} \quad (56)$$

Similarly, it is seen from Eq. (36) that Δr also has an oscillatory buildup at orbital frequency with an amplitude Δr_m which increases at a rate given by

$$\frac{\Delta r_m}{t} = - 3k_2 \, r_c \, \dot{\theta}_E$$

$$= 30.4 \text{ n mi/yr} \quad (57)$$

If it is assumed that the initial errors in the radial and tangential velocities are both 1 ft/sec and that the initial error in orbital radius is 1 n mi, it can be shown that at the end of one year of operation, the divergent terms described above are the dominant <u>oscillatory</u> terms in both Δr and $\Delta\dot{\theta}$ as expressed in Eqs. (36) and (37)

However, the steady state drift rate in $\Delta\dot{\theta}$ is still present, and its magnitude can be determined from Fig. 2 as

$$\Delta\dot{\theta}_{ss} = 110.0 \text{ deg/yr due to } \delta r_o$$

and

$$\Delta\dot{\theta}_{ss} = 62.2 \text{ deg/yr due to } \delta V_o$$

Thus a satellite with a 12-hour period would deviate from its unperturbed position with an oscillatory divergence superposed on a steady state drift rate in angular position.

Thirty-Six Hour Period Satellite ($n = 2/3$)

The behavior of the 36-hour period satellite is similar to that described above. For the same initial position and velocity errors, it can be shown from Eqs. (43) and (44) that the dominant oscillatory terms in both Δr and $\Delta\dot{\theta}$ are the divergent oscillations at orbital rate, $2\dot{\theta}_E/3$, with amplitudes which increase at rates given by

$$\frac{\Delta\theta_m}{t} = 14k_2\,\dot{\theta}_E$$

$$= 0.131 \text{ deg/yr} \tag{58}$$

and

$$\frac{\Delta r_m}{t} = 7k_2\,r_c\dot{\theta}_E$$

$$= 34.1 \text{ n mi/yr} \tag{59}$$

As before, the steady state drift rate in $\Delta\dot{\theta}$ can be evaluated for the assumed initial conditions from Fig. 2 as

$$\Delta\dot{\theta}_{ss} = 17.6 \text{ deg/yr due to } \delta r_o$$

and

$$\Delta\dot{\theta}_{ss} = 29.9 \text{ deg/yr due to } \delta V_o$$

Thus, as in the case of the 12-hour period satellite, the deviation from the unperturbed position is an oscillatory divergence at orbital frequency superposed on a steady state drift rate in angular position.

DISCUSSION

The analysis presented in this Memorandum is similar to that developed by Blitzer in Ref. 5, in which he also investigates the perturbations of equatorial orbits, and in particular, the resonant 12-hour and 36-hour orbits. The rate of increase in the divergent amplitudes of Δr and $\Delta\theta$, determined in Eqs. (53), (57), (58), and (59), differ from those obtained by Blitzer, since the ellipticity which he assumed corresponds to a value of 1.67×10^{-6} for $J_2^{(2)}$ based on Ref. 6. If this value of $J_2^{(2)}$ is used in Eqs. (53), (57), (58), and (59), the resulting numerical values are identical with those obtained by Blitzer.

As a result of the analysis presented here and in Ref. 5, it is seen that for most equatorial orbits the perturbations due to equatorial ellipticity are negligible, the principal perturbations affecting station keeping are the steady state drifts in angular position due to initial condition errors. These steady state drifts are not influenced by the earth's equatorial ellipticity.

The only cases in which equatorial ellipticity has any significant effect are the synchronous orbit and the two resonant orbits with 12- and 36-hour periods. The perturbation of the synchronous orbit is in the nature of a long period oscillation in longitude about the position of the minor axis of the earth's equatorial section and a small amplitude long period variation in orbital radius.

The perturbations of the two resonant orbits in both angular position, $\Delta\theta$, and radial distance, Δr, are also oscillatory, but at orbital frequency and with an amplitude which increases slowly with time. Superposed on these divergent solutions are the usual perturbations resulting from initial condition errors.

It should be noted that the orbital rates have been specified relative to inertial space. Thus, the orbital rate as seen from the rotating earth is given by (n-1) $\dot{\theta}_E$. In the case of the 12-hour orbit (n = 2), the relative orbital rate is + $\dot{\theta}_E$ and the satellite moves toward the east and is over the same equatorial position every 24 hours. Similarly for the 36-hour orbit, n = 2/3, the relative orbital rate is - $\dot{\theta}_E/3$ and the satellite apparently moves to the west and is over the same equatorial position every 72 hours.

IV. CONCLUSIONS

As a result of this analysis, the following specific conclusions can be stated:

- o The only significant effect of the earth's equatorial ellipticity on equatorial orbits is for orbital periods of 12, 24, and 36 hours.
- o The 24-hour or synchronous orbit has been discussed in detail in Ref. 1.
- o The 12- and 36-hour orbits are resonant orbits which display a divergent oscillatory perturbation at orbital frequency.
- o The amplitude of the divergent oscillatory perturbation in angular position grows at the rate of .243 deg/yr for the 12-hour orbit and .131 deg/yr for the 36-hour orbit.
- o The amplitude of the divergent oscillatory perturbation in orbital radius grows at the rate of 30.4 n mi/yr for the 12-hour orbit and 34.1 n mi/yr for the 36-hour orbit.
- o For any orbital period other than 12, 24, or 36 hours, the perturbations due to equatorial ellipticity are negligible.
- o From the point of view of long term station-keeping, the drift rates due to initial errors in orbital radius and orbital velocity are the dominant effect and may result in large displacements in angle from the unperturbed position. These steady state drifts are not influenced by the earth's equatorial ellipticity.

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