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A Table of Integrals Involving Powers, Exponentials, Logarithms, and the Exponential Integral

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PASADENA, CALIFORNIA

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PREFACE

This project was conducted under a research grant from the National Science Foundation at the Department of Chemistry, Johns Hopkins University, Baltimore, Maryland.

ABSTRACT

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The general integrals $\int_a^b x^p e^{-\alpha x} (\ln x)^n dx$ and $\int_a^b x^p e^{-\alpha x} (\ln x)^n [-E_i(-\beta x)] dx$ are investigated, where n is an integer, α and β are real positive numbers, and p is a number greater than -1. Many special cases are obtained, and the results are tabulated in a logical order. Where possible the integrals are expressed in closed form, and several cases are expressed in series expansions.

INTRODUCTION

The following table is a collection of some frequently occurring integrals in quantum mechanics among other applications involving powers, exponentials, logarithms and exponential integrals. Where possible the integrals are expressed in closed form. Also included are several integrals which are expressed in series expansions. It is hoped that these expansions may be useful for many purposes.

The first four sections of the tables deal with powers, exponentials, and logarithms to the zero power, first power, second power, and third power respectively. The fifth section involves some generalizations of the logarithm to the nth power and some miscellaneous results. The same arrangement applies to the sixth through tenth sections except that the exponential integral is included. Within each section, the first subsection is the integral from 0 to ∞, the second from 0 to ρ, and the third from ρ to ∞. Lastly, within each subsection, the general result is obtained for x^p, then examples are given for $p=0,1,2,3$ and for integer n and finally miscellaneous interesting examples, notes and definitions. Throughout the tables p represents any number greater than -1 and n,i,k,ℓ represent integers.

The notation followed consistently is that of Erdelyi, Magnus, Oberhettinger, and Tricomi (Ref. 1). Several of the integrals presented here can be found in the tables of Bierens de Haan (Ref. 2), Hofrieter and Grobner (Ref. 3), Ryshik and Gradstein (Ref. 4), and LeCaine (Ref. 5) and the books of Erdelyi *et al.* (Ref. 1), and Nielsen (Ref. 6). Other references containing pertinent integrals are Levenson (Ref. 7), who discusses the integral denoted $F(n,1,0)$ in Sec. V, Eq. 1, and Busbridge (Ref. 8) and Kourganoff (Ref. 9), who discuss the generalization of $G(p,\alpha,\beta)$ in Sec. VI, Eq. 1.

I. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x} dx$

A.

$$A(p,\alpha) = \int_0^\infty x^p e^{-\alpha x} dx = \frac{\Gamma(p+1)}{\alpha^{p+1}} \tag{1}$$

$$\alpha A(p,\alpha) = pA(p-1,\alpha) \tag{1.1}$$

$$\int_0^\infty e^{-\alpha x} dx = \frac{1}{\alpha} \tag{1.2}$$

$$\int_0^\infty x e^{-\alpha x} dx = \frac{1}{\alpha^2} \tag{1.3}$$

$$\int_0^\infty x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3} \tag{1.4}$$

$$\int_0^\infty x^3 e^{-\alpha x} dx = \frac{6}{\alpha^4} \tag{1.5}$$

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}} \tag{1.6}$$

B.

$$A_1(p,\alpha,\rho) = \int_0^\rho x^p e^{-\alpha x} dx = \rho^{p+1} \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(p+k+1)^1} = \frac{1}{\alpha^{p+1}} \gamma(p+1,\alpha\rho) \tag{2}$$

$$\alpha A_1(p,\alpha,\rho) = pA_1(p-1,\alpha,\rho) - \rho^p e^{-\alpha\rho} \tag{2.1}$$

$$\int_0^\rho e^{-\alpha x} dx = \rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+1)} = \frac{1}{\alpha}\gamma(1,\alpha\rho) = \frac{1}{\alpha}(1-e^{-\alpha\rho}) \tag{2.2}$$

$$\int_0^\rho x e^{-\alpha x} dx = \frac{1}{\alpha^2}\left[1-e^{-\alpha\rho}(1+\alpha\rho)\right] \tag{2.3}$$

$$\int_0^\rho x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3}\left\{1-e^{-\alpha\rho}\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}\right]\right\} \tag{2.4}$$

$$\int_0^\rho x^3 e^{-\alpha x} dx = \frac{6}{\alpha^4}\left\{1-e^{-\alpha\rho}\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}+\frac{(\alpha\rho)^3}{3!}\right]\right\} \tag{2.5}$$

$$\int_0^\rho x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}\left[1-e^{-\alpha\rho}\, e_n(\alpha\rho)\right] \tag{2.6}$$

$$A_1(p,0,\rho) = \int_0^\rho x^p dx = \frac{\rho^{p+1}}{(p+1)} \tag{2.7}$$

C.

$$A_2(p,\alpha,\rho) = \int_\rho^\infty x^p e^{-\alpha x} dx = A(p,\alpha) - A_1(p,\alpha,\rho) = \frac{1}{\alpha^{p+1}}\Gamma(p+1,\alpha\rho) \tag{3}$$

$$\alpha A_2(p,\alpha,\rho) = pA_2(p-1,\alpha,\rho) + \rho^p e^{-\alpha\rho} \tag{3.1}$$

$$\int_\rho^\infty e^{-\alpha x}dx = \frac{1}{\alpha}e^{-\alpha\rho} \tag{3.2}$$

$$\int_\rho^\infty xe^{-\alpha x}dx = \frac{1}{\alpha^2}e^{-\alpha\rho}(1+\alpha\rho) \tag{3.3}$$

$$\int_\rho^\infty x^2e^{-\alpha x}dx = \frac{2}{\alpha^3}e^{-\alpha\rho}\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}\right] \tag{3.4}$$

$$\int_\rho^\infty x^3e^{-\alpha x}dx = \frac{6}{\alpha^4}e^{-\alpha\rho}\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}+\frac{(\alpha\rho)^3}{3!}\right] \tag{3.5}$$

$$\int_\rho^\infty x^ne^{-\alpha x}dx = \frac{n!}{\alpha^{n+1}}e^{-\alpha\rho}e_n(\alpha\rho) \tag{3.6}$$

II. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x} \ln x \, dx$

A.

$$B(p,\alpha) = \int_0^\infty x^pe^{-\alpha x}\ln x\,dx = \frac{\Gamma(p+1)}{\alpha^{p+1}}[\psi(p+1)-\ln\alpha] \tag{1}$$

$$\alpha B(p,\alpha) = pB(p-1,\alpha) + A(p-1,\alpha) \tag{1.1}$$

$$\int_0^\infty e^{-\alpha x}\ln x\,dx = -\frac{1}{\alpha}(\gamma+\ln\alpha) \tag{1.2}$$

$$\int_0^\infty xe^{-\alpha x}\ln x\,dx = -\frac{1}{\alpha^2}(\gamma+\ln\alpha-1) \tag{1.3}$$

$$\int_0^\infty x^2e^{-\alpha x}\ln x\,dx = -\frac{2}{\alpha^3}\left(\gamma+\ln\alpha-1-\frac{1}{2}\right) \tag{1.4}$$

$$\int_0^\infty x^3e^{-\alpha x}\ln x\,dx = -\frac{6}{\alpha^4}\left(\gamma+\ln\alpha-1-\frac{1}{2}-\frac{1}{3}\right) \tag{1.5}$$

$$\int_0^\infty x^ne^{-\alpha x}\ln x\,dx = -\frac{n!}{\alpha^{n+1}}\left(\gamma+\ln\alpha-1-\frac{1}{2}-\cdots-\frac{1}{n}\right) = -\frac{n!}{\alpha^{n+1}}\left(\gamma+\ln\alpha-\sum_{i=1}^n\frac{1}{i}\right) \tag{1.6}$$

$$\int_0^\infty e^{-x}\ln x\,dx = -\gamma \tag{1.7}$$

$$\int_0^\infty x^{\frac{1}{2}} e^{-\alpha x} \ln x \, dx = -\frac{\sqrt{\pi}}{\sqrt{4\alpha^3}} (\gamma + \ln\alpha + 2\ln 2 - 2) \tag{1.8}$$

B.

$$B_1(p,\alpha,\rho) = \int_0^\rho x^p e^{-\alpha x} \ln x \, dx = -\rho^{p+1} \left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(p+k+1)^2} - \ln\rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(p+k+1)^1} \right] \tag{2}$$

$$\alpha B_1(p,\alpha,\rho) = pB_1(p-1,\alpha,\rho) + A_1(p-1,\alpha,\rho) - \rho^p \ln\rho \, e^{-\alpha\rho} \tag{2.1}$$

$$\int_0^\rho e^{-\alpha x} \ln x \, dx = -\rho \left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+1)^2} - \ln\rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+1)} \right] = -\frac{1}{\alpha} \left\{ \gamma + \ln\alpha + \ln\rho \, e^{-\alpha\rho} + [-E_i(-\alpha\rho)] \right\} \tag{2.2}$$

$$\int_0^\rho x e^{-\alpha x} \ln x \, dx = -\frac{1}{\alpha^2} \left\{ \gamma + \ln\alpha + (1 + \alpha\rho) \ln\rho \, e^{-\alpha\rho} + [-E_i(-\alpha\rho)] - 1 + e^{-\alpha\rho} \right\} \tag{2.3}$$

$$\begin{aligned} \int_0^\rho x^2 e^{-\alpha x} \ln x \, dx = -\frac{2}{\alpha^3} \Bigg\{ & \gamma + \ln\alpha + \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} \right] \ln\rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] \\ & - 1 - \frac{1}{2} + e^{-\alpha\rho} \left[1 + \frac{1}{2} + \frac{1}{2}(\alpha\rho) \right] \Bigg\} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \int_0^\rho x^3 e^{-\alpha x} \ln x \, dx = -\frac{6}{\alpha^4} \Bigg\{ & \gamma + \ln\alpha + \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} + \frac{(\alpha\rho)^3}{3!} \right] \ln\rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] - 1 - \frac{1}{2} \\ & - \frac{1}{3} + e^{-\alpha\rho} \left[1 + \frac{1}{2} + \frac{1}{3} + (\alpha\rho)\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{(\alpha\rho)^2}{2!}\left(\frac{1}{3}\right) \right] \Bigg\} \end{aligned} \tag{2.5}$$

$$\int_0^\rho x^n e^{-\alpha x} \ln x \, dx = -\frac{n!}{\alpha^{n+1}} \left\{ \gamma + \ln\alpha + \ln\rho e^{-\alpha\rho} e_n(\alpha\rho) + [-E_i(-\alpha\rho)] + \sum_{i=0}^{n-1} \frac{1}{i+1} [e_i(\alpha\rho) e^{-\alpha\rho} - 1] \right\} \tag{2.6}$$

$$B_1(p,0,\rho) = \int_0^\rho x^p \ln x \, dx = \frac{\rho^{p+1}}{(p+1)} \left(\ln\rho - \frac{1}{p+1} \right) \tag{2.7}$$

$$\sum_{k=0}^\infty \frac{(-x)^k}{k!\,(k+p+1)^1} = \frac{\gamma(p+1, x)}{x^{p+1}} \tag{2.8}$$

$$\sum_{k=0}^\infty \frac{(-x)^k}{k!\,(k+n+1)^1} = \frac{\gamma(n+1, x)}{x^{n+1}} = \frac{n!}{x^{n+1}} [1 - e^{-x} e_n(x)] \tag{2.9}$$

$$\sum_{k=0}^\infty \frac{(-x)^k}{k!\,(n+k+1)^2} = \frac{n!}{x^{n+1}} \left\{ \ln x + [-E_i(-x)] - \psi(n+1) + e^{-x} \sum_{k=0}^{n-1} \frac{e_k(x)}{k+1} \right\} \tag{2.10}$$

C.

$$B_2(p,\alpha,\rho) = \int_\rho^\infty x^p e^{-\alpha x} \ln x \, dx = B(p,\alpha) - B_1(p,\alpha,\rho) \tag{3}$$

$$\alpha B_2(p,\alpha,\rho) = pB_2(p-1,\alpha,\rho) + A_1(p-1,\alpha,\rho) + \rho^p \ln\rho \, e^{-\alpha\rho} \tag{3.1}$$

$$\int_\rho^\infty e^{-\alpha x} \ln x \, dx = \frac{1}{\alpha} \left\{ \ln\rho \, e^{-\alpha\rho} + [-E_i(-\alpha\rho)] \right\} \tag{3.2}$$

$$\int_\rho^\infty x e^{-\alpha x} \ln x \, dx = \frac{1}{\alpha^2} \left\{ (1 + \alpha\rho) \ln\rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] + e^{-\alpha\rho} \right\} \tag{3.3}$$

$$\int_\rho^\infty x^2 e^{-\alpha x}\ln x\,dx = \frac{2}{\alpha^3}\left\{\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}\right]\ln\rho e^{-\alpha\rho} + \left[-E_i(-\alpha\rho)\right] + e^{-\alpha\rho}\left[1+\frac{1}{2}+\frac{1}{2}(\alpha\rho)\right]\right\} \tag{3.4}$$

$$\int_\rho^\infty x^3 e^{-\alpha x}\ln x\,dx = \frac{6}{\alpha^4}\left\{\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}+\frac{(\alpha\rho)^3}{3!}\right]\ln\rho e^{-\alpha\rho} + \left[-E_i(-\alpha\rho)\right]\right.$$

$$\left. + e^{-\alpha\rho}\left[1+\frac{1}{2}+\frac{1}{3}+(\alpha\rho)\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{(\alpha\rho)^2}{2!}\left(\frac{1}{3}\right)\right]\right\} \tag{3.5}$$

$$\int_\rho^\infty x^n e^{-\alpha x}\ln x\,dx = \frac{n!}{\alpha^{n+1}}\left\{\ln\rho e^{-\alpha\rho}e_n(\alpha\rho) + \left[-E_i(-\alpha\rho)\right] + \sum_{i=0}^{n-1}\frac{e_i(\alpha\rho)}{(i+1)}e^{-\alpha\rho}\right\} \tag{3.6}$$

III. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x}\ln^2 x\,dx$

A.

$$C(p,\alpha) = \int_0^\infty x^p e^{-\alpha x}\ln^2 x\,dx = \frac{\Gamma(p+1)}{\alpha^{p+1}}\left\{\left[\psi(p+1)-\ln\alpha\right]^2+\zeta(2,p+1)\right\} \tag{1}$$

$$\alpha C(p,\alpha) = pC(p-1,\alpha) + 2B(p-1,\alpha) \tag{1.1}$$

$$\int_0^\infty e^{-\alpha x}\ln^2 x\,dx = \frac{1}{\alpha}\left[(\gamma+\ln\alpha)^2+\zeta(2)\right] \tag{1.2}$$

$$\int_0^\infty x e^{-\alpha x}\ln^2 x\,dx = \frac{1}{\alpha^2}\left[(\gamma+\ln\alpha-1)^2+\zeta(2)-1\right] \tag{1.3}$$

$$\int_0^\infty x^2 e^{-\alpha x}\ln^2 x\,dx = \frac{2}{\alpha^3}\left[\left(\gamma+\ln\alpha-1-\frac{1}{2}\right)^2+\zeta(2)-1-\frac{1}{2^2}\right] \tag{1.4}$$

$$\int_0^\infty x^3 e^{-\alpha x}\ln^2 x\,dx = \frac{6}{\alpha^4}\left[\left(\gamma+\ln\alpha-1-\frac{1}{2}-\frac{1}{3}\right)^2+\zeta(2)-1-\frac{1}{2^2}-\frac{1}{3^2}\right] \tag{1.5}$$

$$\int_0^\infty x^n e^{-\alpha x}\ln^2 x\,dx = \frac{n!}{\alpha^{n+1}}\left[\left(\gamma+\ln\alpha-\sum_{i=0}^{n-1}\frac{1}{i+1}\right)^2+\zeta(2)-\sum_{i=0}^{n-1}\frac{1}{(i+1)^2}\right] \tag{1.6}$$

$$\zeta(2,n+1) = \zeta(2) - \sum_{i=0}^{n-1}\frac{1}{(i+1)^2} \tag{1.7}$$

$$\zeta(2) = \frac{\pi^2}{6} = 1.644\,934\,0668\cdots \tag{1.8}$$

$$\psi(n+1) = -\gamma+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \tag{1.9}$$

B.

$$C_1(p,\alpha,\rho) = \int_0^{\rho} x^p e^{-\alpha x}\ln^2 x\,dx = 2\rho^{p+1}\left[\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)^3} - \ln\rho\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)^2} + \frac{1}{2}\ln^2\rho\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)}\right] \quad (2)$$

$$\alpha C_1(p,\alpha,\rho) = pC_1(p-1,\alpha,\rho) + 2B_1(p-1,\alpha,\rho) - \rho^p\ln^2\rho\, e^{-\alpha\rho} \quad (2.1)$$

$$\int_0^{\rho} e^{-\alpha x}\ln^2 x\,dx = 2\rho\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+1)^3} - \frac{2\ln\rho}{\alpha}\left\{\gamma + \ln\alpha + \frac{1}{2}\ln\rho\,(1+e^{-\alpha\rho}) + \left[-E_i(-\alpha\rho)\right]\right\} \quad (2.2)$$

$$\int_0^{\rho} xe^{-\alpha x}\ln^2 x\,dx = 2\rho^2\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+2)^3} - \frac{2\ln\rho}{\alpha^2}\left\{\gamma + \ln\alpha + \frac{1}{2}\ln\rho\left[1+(1+\alpha\rho)e^{-\alpha\rho}\right] + \left[-E_i(-\alpha\rho)\right] + e^{-\alpha\rho} - 1\right\} \quad (2.3)$$

$$\int_0^{\rho} x^2e^{-\alpha x}\ln^2 x\,dx = 2\rho^3\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+3)^3} - \frac{4\ln\rho}{\alpha^3}\left(\gamma + \ln\alpha + \frac{1}{2}\ln\rho\left\{1+\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}\right]e^{-\alpha\rho}\right\} + \left[-E_i(-\alpha\rho)\right]\right.$$
$$\left. + e^{-\alpha\rho}\left[1+\frac{1}{2}+(\alpha\rho)\frac{1}{2}\right] - 1 - \frac{1}{2}\right) \quad (2.4)$$

$$\int_0^{\rho} x^3e^{-\alpha x}\ln^2 x\,dx = 2\rho^4\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+4)^3} - \frac{12\ln\rho}{\alpha^4}\left(\gamma + \ln\alpha + \frac{1}{2}\ln\rho\left\{1+\left[1+\alpha\rho+\frac{(\alpha\rho)^2}{2!}+\frac{(\alpha\rho)^3}{3!}\right]e^{-\alpha\rho}\right\}\right.$$
$$\left. + \left[-E_i(-\alpha\rho)\right] + e^{-\alpha\rho}\left[1+\frac{1}{2}+\frac{1}{3}+(\alpha\rho)\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{(\alpha\rho)^2}{2!}\left(\frac{1}{3}\right)\right] - 1 - \frac{1}{2} - \frac{1}{3}\right) \quad (2.5)$$

$$\int_0^{\rho} x^ne^{-\alpha x}\ln^2 x\,dx = 2\rho^{n+1}\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(n+k+1)^3} - \frac{2n!\ln\rho}{\alpha^{n+1}}\left\{\gamma + \ln\alpha + \frac{1}{2}\ln\rho\left[1+e_n(\alpha\rho)e^{-\alpha\rho}\right] + \left[-E_i(-\alpha\rho)\right]\right.$$
$$\left. + \sum_{i=0}^{n-1}\frac{1}{(i+1)}\left[e_i(\alpha\rho)e^{-\alpha\rho} - 1\right]\right\} \quad (2.6)$$

$$C_1(p,0,\rho) = \int_0^{\rho} x^p\ln^2 x\,dx = \frac{\rho^{p+1}}{p+1}\left[\ln^2\rho - \frac{2}{(p+1)}\ln\rho + \frac{2}{(p+1)^2}\right] \quad (2.7)$$

$$S(p,x) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!\,(p+k+1)^3} \quad (2.8)$$

$$xS(n+1,x) = (n+1)\,S(n,x) - \frac{n!}{x^{n+1}}\left\{\gamma + \ln x + \left[-E_i(-x)\right] + \sum_{i=0}^{n-1}\frac{1}{(i+1)}\left[e_i(x)e^{-x} - 1\right]\right\} \quad (2.9)$$

C.

$$C_2(p,\alpha,\rho) = \int_{\rho}^{\infty} x^pe^{-\alpha x}\ln^2 x\,dx = C(p,\alpha) - C_1(p,\alpha,\rho) \quad (3)$$

$$\alpha C_2(p,\alpha,\rho) = pC_2(p-1,\alpha,\rho) + 2B_2(p-1,\alpha,\rho) + \rho^pe^{-\alpha\rho}\ln^2\rho \quad (3.1)$$

$$\int_{\rho}^{\infty} e^{-\alpha x}\ln^2 x\,dx = \frac{1}{\alpha}\left\{(\gamma+\ln\alpha\rho)^2 + \zeta(2) + \ln^2\rho\, e^{-\alpha\rho} + 2\ln\rho\left[-E_i(-\alpha\rho)\right]\right\} - 2\rho\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+1)^3} \quad (3.2)$$

$$\int_\rho^\infty xe^{-\alpha x}\ln^2 x\,dx = \frac{1}{\alpha^2}\Big\{(\gamma + \ln\alpha\rho - 1)^2 + \zeta(2) - 1 + \ln^2\rho\ (1+\alpha\rho)e^{-\alpha\rho} + 2\ln\rho\ e^{-\alpha\rho} + 2\ln\rho\left[-E_i(-\alpha\rho)\right]\Big\}$$
$$- 2\rho^2\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+2)^3} \tag{3.3}$$

$$\int_\rho^\infty x^2e^{-\alpha x}\ln^2 x\,dx = \frac{2}{\alpha^3}\Big\{\left(\gamma + \ln\alpha\rho - 1 - \frac{1}{2}\right)^2 + \zeta(2) - 1 - \frac{1}{2^2} + \ln^2\rho\ e_2(\alpha\rho)e^{-\alpha\rho} + 2\ln\rho\ e^{-\alpha\rho}\left[1 + \frac{1}{2} + (\alpha\rho)\frac{1}{2}\right]$$
$$+ 2\ln\rho\left[-E_i(-\alpha\rho)\right]\Big\} - 2\rho^3\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+3)^3} \tag{3.4}$$

$$\int_\rho^\infty x^3e^{-\alpha x}\ln^2 x\,dx = \frac{6}{\alpha^4}\Big\{\left(\gamma + \ln\alpha\rho - 1 - \frac{1}{2} - \frac{1}{3}\right)^2 + \zeta(2) - 1 - \frac{1}{2^2} - \frac{1}{3^2} + \ln^2\rho\ e_3(\alpha\rho)e^{-\alpha\rho} + 2\ln\rho\ e^{-\alpha\rho}$$
$$\times\left[1 + \frac{1}{2} + \frac{1}{3} + (\alpha\rho)\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{(\alpha\rho)^2}{2!}\left(\frac{1}{3}\right)\right] + 2\ln\rho\left[-E_i(-\alpha\rho)\right]\Big\} - 2\rho^4\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+4)^3} \tag{3.5}$$

$$\int_\rho^\infty x^ne^{-\alpha x}\ln^2 x\,dx = \frac{n!}{\alpha^{n+1}}\Big\{\left(\gamma + \ln\alpha\rho - \sum_{i=0}^{n-1}\frac{1}{i+1}\right)^2 + \zeta(2) - \sum_{i=0}^{n-1}\frac{1}{(i+1)^2} + \ln^2\rho\ e_n(\alpha\rho)e^{-\alpha\rho} + 2\ln\rho\ e^{-\alpha\rho}$$
$$\times\sum_{i=0}^{n-1}\frac{e_i(\alpha\rho)}{(i+1)} + 2\ln\rho\left[-E_i(-\alpha\rho)\right]\Big\} - 2\rho^{n+1}\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+n+1)^3} \tag{3.6}$$

$$\int_1^\infty x^ne^{-\alpha x}\ln^2 x\,dx = \frac{n!}{\alpha^{n+1}}\Big\{\left[\psi(n+1) - \ln\alpha\right]^2 + \zeta(2,n+1)\Big\} - 2\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!\,(k+n+1)^3} \tag{3.7}$$

IV. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x}\ln^3 x\,dx$

A.

$$D(p,\alpha) = \int_0^\infty x^pe^{-\alpha x}\ln^3 x\,dx = \frac{\Gamma(p+1)}{\alpha^{p+1}}\Big\{\left[\psi(p+1) - \ln\alpha\right]^3 + 3\zeta(2,p+1)\left[\psi(p+1) - \ln\alpha\right] - 2\zeta(3,p+1)\Big\} \tag{1}$$

$$\alpha D(p,\alpha) = pD(p-1,\alpha) + 6C(p-1,\alpha) \tag{1.1}$$

$$\int_0^\infty e^{-\alpha x}\ln^3 x\,dx = -\frac{1}{\alpha}\left[(\gamma + \ln\alpha)^3 + 3\zeta(2)(\gamma + \ln\alpha) + 2\zeta(3)\right] \tag{1.2}$$

$$\int_0^\infty xe^{-\alpha x}\ln^3 x\,dx = -\frac{1}{\alpha^2}\left[(\gamma + \ln\alpha - 1)^3 + 3\zeta(2,2)(\gamma + \ln\alpha - 1) + 2\zeta(3,2)\right] \tag{1.3}$$

$$\int_0^\infty x^2e^{-\alpha x}\ln^3 x\,dx = -\frac{2}{\alpha^3}\left[\left(\gamma + \ln\alpha - 1 - \frac{1}{2}\right)^3 + 3\zeta(2,3)\left(\gamma + \ln\alpha - 1 - \frac{1}{2}\right) + 2\zeta(3,3)\right] \tag{1.4}$$

$$\int_0^\infty x^3 e^{-\alpha x} \ln^3 x \, dx = -\frac{6}{\alpha^4}\left[\left(\gamma + \ln\alpha - 1 - \frac{1}{2} - \frac{1}{3}\right)^3 + 3\zeta(2,4)\left(\gamma + \ln\alpha - 1 - \frac{1}{2} - \frac{1}{3}\right) + 2\zeta(3,4)\right] \tag{1.5}$$

$$\int_0^\infty x^n e^{-\alpha x} \ln^3 x \, dx = -\frac{n!}{\alpha^{n+1}}\left[\left(\gamma + \ln\alpha - \sum_{i=0}^{n-1}\frac{1}{i+1}\right)^3 + 3\zeta(2,n+1)\left(\gamma + \ln\alpha - \sum_{i=0}^{n-1}\frac{1}{i+1}\right) + 2\zeta(3,n+1)\right] \tag{1.6}$$

$$\zeta(3,n+1) = \zeta(3) - \sum_{i=0}^{n-1}\frac{1}{(i+1)^3} \tag{1.7}$$

$$\zeta(3) = \frac{\pi^3}{25.794\,36\cdots} = 1.202\,056\,9032\cdots \tag{1.8}$$

B.

$$D_1(p,\alpha,\rho) = \int_0^\rho x^p e^{-\alpha x} \ln^3 x \, dx = -6\rho^{p+1}\left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+p+1)^4} - \ln\rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+p+1)^3} + \frac{\ln^2\rho}{2}\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+p+1)^2} - \frac{\ln^3\rho}{6}\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+p+1)}\right] \tag{2}$$

$$\alpha D_1(p,\alpha,\rho) = pD_1(p-1,\alpha,\rho) + 6C_1(p-1,\alpha,\rho) - \rho^p e^{-\alpha\rho}\ln^3\rho \tag{2.1}$$

$$\int_0^\rho e^{-\alpha x}\ln^3 x\,dx = -6\rho\left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+1)^4} - \ln\rho\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+1)^3}\right] - \frac{3\ln^2\rho}{\alpha}\left\{\gamma + \ln\alpha\rho + [-E_i(-\alpha\rho)] + \frac{1}{3}\ln\rho\,(1 - e^{-\alpha\rho})\right\} \tag{2.2}$$

$$D_1(p,0,\rho) = \int_0^\rho x^p \ln^3 x\,dx = \frac{\rho^{p+1}}{(p+1)}\left[\ln^3\rho - \frac{3}{(p+1)}\ln^2\rho + \frac{6}{(p+1)^2}\ln\rho - \frac{6}{(p+1)^3}\right] \tag{2.3}$$

C.

$$D_2(p,\alpha,\rho) = \int_\rho^\infty x^p e^{-\alpha x}\ln^3 x\,dx = D(p,\alpha) - D_1(p,\alpha,\rho) \tag{3}$$

$$\alpha D_2(p,\alpha,\rho) = pD_2(p-1,\alpha,\rho) + 6C_2(p-1,\alpha,\rho) + \rho^p e^{-\alpha\rho}\ln^3\rho \tag{3.1}$$

$$\int_\rho^\infty e^{-\alpha x}\ln^3 x\,dx = 6\rho\left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+1)^4} - \ln\rho\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!\,(k+1)^3}\right] + \frac{3\ln^2\rho}{\alpha}\left\{\ln\rho + [-E_i(-\alpha\rho)] + \frac{1}{3}\ln\rho\,(1 - e^{-\alpha\rho})\right\} - \frac{1}{\alpha}\left\{(\gamma + \ln\alpha)^3 + 3(\gamma + \ln\alpha)\left[\zeta(2) - \ln^2\rho\right] + 2\zeta(3)\right\} \tag{3.2}$$

V. MISCELLANEOUS RELATED INTEGRALS

A.

$$F(n,\alpha,p) = \int_0^\infty x^p e^{-\alpha x}(\ln x)^n\,dx \tag{1}$$

$$\alpha F(n,\alpha,p) = pF(n,\alpha,p-1) + nF(n-1,\alpha,p-1) \tag{1.1}$$

$$\begin{aligned} F(n,\alpha,p) = {} & \sum_{k=0}^{n-2}\left[\sum_{l=k}^{n-2}\binom{n-1}{l}\binom{l}{k}(\ln\alpha)^{l-k}(-1)^{n-l}(n-1-l)!\,\zeta(n-l,p+1)\right]F(k,\alpha,p) \\ & + \sum_{k=0}^{n-1}\binom{n-1}{k}(\ln\alpha)^{n-k-1}\left[\psi(p+1) - \frac{n}{n-k}\ln\alpha\right]F(k,\alpha,p) \end{aligned} \tag{1.2}$$

$$\begin{aligned} F(n,\alpha,p) = {} & \frac{\Gamma(p+1)}{\alpha^{p+1}}\left[\theta^n + 2!\binom{n}{2}\left(\frac{\Phi_2}{2}\right)\theta^{n-2} - 3!\binom{n}{3}\left(\frac{\Phi_3}{3}\right)\theta^{n-3} + 4!\binom{n}{4}\left(\frac{\Phi_4}{4}+\frac{\Phi_2^2}{8}\right)\theta^{n-4}\right. \\ & - 5!\binom{n}{5}\left(\frac{\Phi_5}{5}+\frac{\Phi_3\Phi_2}{6}\right)\theta^{n-5} + 6!\binom{n}{6}\left(\frac{\Phi_6}{6}+\frac{\Phi_4\Phi_2}{8}+\frac{\Phi_3^2}{18}+\frac{\Phi_2^3}{48}\right)\theta^{n-6} - 7!\binom{n}{7} \\ & \left.\times\left(\frac{\Phi_7}{7}+\frac{\Phi_5\Phi_2}{10}+\frac{\Phi_4\Phi_3}{12}+\frac{\Phi_3\Phi_2^2}{24}\right)\theta^{n-7} + \cdots\right] \end{aligned} \tag{1.3}$$

where $\theta = \psi(p+1) - \ln\alpha$

$\Phi_r = \zeta(r,p+1)$

$$\begin{aligned} F(n,\alpha,0) = {} & \sum_{k=0}^{n-2}\left[\sum_{l=k}^{n-2}\binom{n-1}{l}\binom{l}{k}(\ln\alpha)^{l-k}(-1)^{n-l}(n-l-1)!\,\zeta(n-l)\right]F(k,\alpha,0) \\ & - \sum_{k=0}^{n-1}\binom{n-1}{k}(\ln\alpha)^{n-k-1}\left(\gamma + \frac{n}{n-k}\ln\alpha\right)F(k,\alpha,0) \end{aligned} \tag{1.4}$$

$$F(n,1,0) = (-1)^n(n-1)!\sum_{k=0}^{n-2}\frac{(-1)^k\zeta(n-k)}{k!}F(k,1,0) - \gamma F(n-1,1,0) \tag{1.5}$$

B.

$$F_1(n,\alpha,p,\rho) = \int_0^\rho x^p e^{-\alpha x}(\ln x)^n\,dx = (-1)^n n!\,\rho^{p+1}\sum_{k=0}^{n}\frac{(-\ln\rho)^k}{k!}\left[\sum_{l=0}^{\infty}\frac{(-\alpha\rho)^l}{l!\,(p+l+1)^{n-k+1}}\right] \tag{2}$$

$$\alpha F_1(n,\alpha,p,\rho) = pF_1(n,\alpha,p-1,\rho) + nF_1(n-1,\alpha,p-1,\rho) - \rho^p e^{-\alpha\rho}(\ln\rho)^n \tag{2.1}$$

$$F_1(n,0,p,\rho) = \int_0^\rho x^p(\ln x)^n\,dx = \frac{\rho^{p+1}}{(p+1)}\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}(\ln\rho)^{n-k}}{(p+1)^k} \tag{2.2}$$

C.

$$\int_\rho^\infty e^{-\alpha x}(\ln x)^n\,dx = \frac{(\ln\rho)^n e^{-\alpha\rho}}{\alpha} + \frac{n}{\alpha}\int_\rho^\infty \frac{e^{-\alpha x}(\ln x)^{n-1}}{x}\,dx \tag{3}$$

$$\int_\rho^\infty e^{-\alpha x}\ln x\,dx = \frac{\ln\rho\, e^{-\alpha\rho}}{\alpha} + \frac{1}{\alpha}\int_\rho^\infty \frac{e^{-\alpha x}}{x}\,dx = \frac{1}{\alpha}\left\{\ln\rho\, e^{-\alpha\rho} + \left[-E_i(-\alpha\rho)\right]\right\} \tag{3.1}$$

(See Sec. II, Eq. 3.2.)

$$\int_\rho^\infty e^{-\alpha x}\ln^2 x\,dx = \frac{\ln^2\rho\, e^{-\alpha\rho}}{\alpha} + \frac{2}{\alpha}\int_\rho^\infty \frac{e^{-\alpha x}\ln x}{x}\,dx \tag{3.2}$$

$$\int_\rho^\infty \frac{\ln x\, e^{-\alpha x}}{x}\,dx = \frac{1}{2}\left\{(\gamma + \ln\alpha + \ln\rho)^2 + \zeta(2) + 2\ln\rho\left[-E_i(-\alpha\rho)\right]\right\} + \sum_{k=0}^{\infty}\frac{(-\alpha\rho)^{k+1}}{k!\,(k+1)^3} \tag{3.3}$$

(See Sec. III, Eq. 3.2 and Sec. V,Eq. 3.2.)

$$\int_\rho^\infty \frac{\ln x\, e^{-\alpha x}}{x}\,dx = \left[-E_i(-\alpha\rho)\right]\ln\rho + \int_\rho^\infty \frac{\left[-E_i(-\alpha x)\right]}{x}\,dx \tag{3.4}$$

$$\int_\rho^\infty \frac{\left[-E_i(-\alpha x)\right]}{x}\,dx = \frac{1}{2}\left[(\gamma + \ln\alpha + \ln\rho)^2 + \zeta(2)\right] + \sum_{k=0}^{\infty}\frac{(-\alpha\rho)^{k+1}}{k!\,(k+1)^3} \tag{3.5}$$

(See Sec. V, Eq. 3.3 and 3.4.)

VI. INTEGRALS OF THE TYPE $\int x^p\left[-E_i(-\beta x)\right]e^{-\alpha x}\,dx$

A.

$$\begin{aligned} G(p,\alpha,\beta) = \int_0^\infty x^p e^{-\alpha x}\left[-E_i(-\beta x)\right]dx &= \frac{\Gamma(p+1)}{(p+1)(\alpha+\beta)^{p+1}}\,{}_2F_1\left(1,p+1;p+2;\frac{\alpha}{\alpha+\beta}\right) \\ &= \frac{\Gamma(p+1)}{\alpha^{p+1}}\,B_{\frac{\alpha}{\alpha+\beta}}(p+1,0) \\ &= \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}}\,\Phi\left(\frac{\alpha}{\alpha+\beta},1,p+1\right) \\ &= \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}}\sum_{k=0}^{\infty}\frac{1}{(p+k+1)}\left(\frac{\alpha}{\alpha+\beta}\right)^k \end{aligned} \tag{1}$$

$$\alpha G(p,\alpha,\beta) = pG(p-1,\alpha,\beta) - A(p-1,\alpha+\beta) \tag{1.1}$$

$$\int_0^\infty e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{1}{\alpha}\ln\left(1+\frac{\alpha}{\beta}\right) \tag{1.2}$$

$$\int_0^\infty xe^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{1}{\alpha^2}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - \frac{\alpha}{\alpha+\beta}\right] \tag{1.3}$$

$$\int_0^\infty x^2e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{2}{\alpha^3}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - 1\left(\frac{\alpha}{\alpha+\beta}\right) - \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2\right] \tag{1.4}$$

$$\int_0^\infty x^3e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{6}{\alpha^4}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - 1\left(\frac{\alpha}{\alpha+\beta}\right) - \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2 - \frac{1}{3}\left(\frac{\alpha}{\alpha+\beta}\right)^3\right] \tag{1.5}$$

$$\int_0^\infty x^ne^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{n!}{\alpha^{n+1}}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - \sum_{k=1}^{n}\frac{1}{k}\left(\frac{\alpha}{\alpha+\beta}\right)^k\right] \tag{1.6}$$

$$\int_0^\infty x^p \left[-E_i(-\beta x)\right] dx = \frac{\Gamma(p+1)}{(p+1)\beta^{p+1}} \tag{1.7}$$

$$\Phi(z,1,n+1) = -\frac{1}{z^{n+1}}\left[\ln(1-z) + \sum_{i=1}^{\infty}\frac{z^i}{i}\right] \tag{1.8}$$

B.

$$\begin{aligned} G_1(p,\alpha,\beta,\rho) = \int_0^\rho x^p e^{-\alpha x}\left[-E_i(-\beta x)\right] dx &= \frac{\left[-E_i(-\beta\rho)\right]\gamma(p+1,\alpha\rho)}{\alpha^{p+1}} + \frac{1}{\beta^{p+1}}\sum_{k=0}^{\infty}\frac{\left(-\frac{\alpha}{\beta}\right)^k\gamma(p+k+1,\beta\rho)}{k!\,(p+k+1)} \\ &= \frac{\left[-E_i(-\beta\rho)\right]\gamma(p+1,\alpha\rho)}{\alpha^{p+1}} + \rho^{p+1}\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(p+k+l+1)} \end{aligned} \tag{2}$$

$$\alpha G_1(p,\alpha,\beta,\rho) = pG_1(p-1,\alpha,\beta,\rho) - A_1(p-1,\alpha+\beta,\rho) - \rho^p e^{-\alpha\rho}\left[-E_i(-\beta\rho)\right] \tag{2.1}$$

$$\int_0^\rho e^{-\alpha x}\left[-E_i(-\beta x)\right] dx = \frac{1}{\alpha}\ln\left(1+\frac{\alpha}{\beta}\right) - \frac{1}{\alpha}\left(e^{-\alpha\rho}\left[-E_i(-\beta\rho)\right] - \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\}\right) \tag{2.2}$$

$$\begin{aligned} \int_0^\rho x e^{-\alpha x}\left[-E_i(-\beta x)\right] dx = \frac{1}{\alpha^2}\Bigg(&\ln\left(1+\frac{\alpha}{\beta}\right) - e^{-\alpha\rho}(1+\alpha\rho)\left[-E_i(-\beta\rho)\right] \\ &+ \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\} + \left(\frac{\alpha}{\alpha+\beta}\right)\left[e^{-(\alpha+\beta)\rho}-1\right]\Bigg) \end{aligned} \tag{2.3}$$

$$\begin{aligned} \int_0^\rho x^2 e^{-\alpha x}\left[-E_i(-\beta x)\right] dx = \frac{2}{\alpha^3}\Bigg(&\ln\left(1+\frac{\alpha}{\beta}\right) - e^{-\alpha\rho}e_2(\alpha\rho)\left[-E_i(-\beta\rho)\right] + \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\} \\ &+ \left(\frac{\alpha}{\alpha+\beta}\right)\left[e^{-(\alpha+\beta)\rho}-1\right] + \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2\left\{e^{-(\alpha+\beta)\rho}\left[1+(\alpha+\beta)\rho\right]-1\right\}\Bigg) \end{aligned} \tag{2.4}$$

$$\begin{aligned} \int_0^\rho x^3 e^{-\alpha x}\left[-E_i(-\beta x)\right] dx = \frac{6}{\alpha^4}\Bigg(&\ln\left(1+\frac{\alpha}{\beta}\right) - e^{-\alpha\rho}e_3(\alpha\rho)\left[-E_i(-\beta\rho)\right] + \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\} \\ &+ \left(\frac{\alpha}{\alpha+\beta}\right)\left[e^{-(\alpha+\beta)\rho}-1\right] + \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2\left\{e^{-(\alpha+\beta)\rho}e_1\left[(\alpha+\beta)\rho\right]-1\right\} \\ &+ \frac{1}{3}\left(\frac{\alpha}{\alpha+\beta}\right)^3\left\{e^{-(\alpha+\beta)\rho}e_2\left[(\alpha+\beta)\rho\right]-1\right\}\Bigg) \end{aligned} \tag{2.5}$$

$$\begin{aligned} \int_0^\rho x^n e^{-\alpha x}\left[-E_i(-\beta x)\right] dx = \frac{n!}{\alpha^{n+1}}\Bigg(&\ln\left(1+\frac{\alpha}{\beta}\right) - e^{-\alpha\rho}e_n(\alpha\rho)\left[-E_i(-\beta\rho)\right] + \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\} \\ &+ \sum_{k=1}^{n}\frac{1}{k}\left(\frac{\alpha}{\alpha+\beta}\right)^k\left\{e^{-(\alpha+\beta)\rho}e_{k-1}\left[(\alpha+\beta)\rho\right]-1\right\}\Bigg) \end{aligned} \tag{2.6}$$

$$\int_0^\rho x^p\left[-E_i(-\beta x)\right] dx = \frac{1}{(p+1)}\left\{\frac{1}{\beta^{p+1}}\gamma(p+1,\beta\rho) + \rho^{p+1}\left[-E_i(-\beta\rho)\right]\right\} \tag{2.7}$$

C.
$$G_2(p,\alpha,\beta,\rho) = \int_\rho^\infty x^p e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = G(p,\alpha,\beta) - G_1(p,\alpha,\beta,\rho) \tag{3}$$

$$\alpha G_2(p,\alpha,\beta,\rho) = pG_2(p-1,\alpha,\beta,\rho) - A_2(p-1,\alpha+\beta,\rho) + \rho^p e^{-\alpha\rho}\left[-E_i(-\beta\rho)\right] \tag{3.1}$$

$$\int_\rho^\infty e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{1}{\alpha}\left(e^{-\alpha\rho}\left[-E_i(-\beta\rho)\right] - \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\}\right) \tag{3.2}$$

$$\int_\rho^\infty xe^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{1}{\alpha^2}\left(e_1(\alpha\rho)\,e^{-\alpha\rho}\left[-E_i(-\alpha\rho)\right] - \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\}\right) - \frac{e^{-(\alpha+\beta)\rho}}{\alpha(\alpha+\beta)} \tag{3.3}$$

$$\int_\rho^\infty x^2 e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{2}{\alpha^3}\left(e_2(\alpha\rho)e^{-\alpha\rho}\left[-E_i(-\alpha\rho)\right] - \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\}\right)$$
$$- \frac{e^{-(\alpha+\beta)\rho}}{\alpha^2(\alpha+\beta)^2}\left[2(\alpha+\beta) + \alpha + \rho\alpha(\alpha+\beta)\right] \tag{3.4}$$

$$\int_\rho^\infty x^3 e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{6}{\alpha^4}\left(e_3(\alpha\rho)e^{-\alpha\rho}\left[-E_i(-\beta\rho)\right] - \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\}\right.$$
$$\left. - e^{-(\alpha+\beta)\rho}\sum_{k=1}^{3}\frac{e_{k-1}\left[(\alpha+\beta)\rho\right]}{k\left(1+\frac{\beta}{\alpha}\right)^k}\right) \tag{3.5}$$

$$\int_\rho^\infty x^n e^{-\alpha x}\left[-E_i(-\beta x)\right]dx = \frac{n!}{\alpha^{n+1}}\left(e_n(\alpha\rho)e^{-\alpha\rho}\left[-E_i(-\beta\rho)\right] - \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\}\right.$$
$$\left. - e^{-(\alpha+\beta)\rho}\sum_{k=1}^{n}\frac{e_{k-1}\left[(\alpha+\beta)\rho\right]}{k\left(1+\frac{\beta}{\alpha}\right)^k}\right) \tag{3.6}$$

$$\int_\rho^\infty x^p\left[-E_i(-\beta x)\right]dx = \frac{1}{(p+1)}\left\{\frac{1}{\beta^{p+1}}\Gamma(p+1,\beta\rho) - \rho^{p+1}\left[-E_i(-\beta\rho)\right]\right\} \tag{3.7}$$

VII. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x} \ln x\left[-E_i(-\beta x)\right]dx$

A.
$$H(p,\alpha,\beta) = \int_0^\infty x^p e^{-\alpha x}\ln x\left[-E_i(-\beta x)\right]dx = \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}}\left\{\left[\psi(p+1) - \ln(\alpha+\beta)\right]\right.$$
$$\left.\times\ \Phi\left(\frac{\alpha}{\alpha+\beta},1,p+1\right) - \Phi\left(\frac{\alpha}{\alpha+\beta},2,p+1\right)\right\} \tag{1}$$

$$\alpha H(p,\alpha,\beta) = pH(p-1,\alpha,\beta) - B(p-1,\alpha+\beta) + G(p-1,\alpha,\beta) \tag{1.1}$$

$$\int_0^\infty e^{-\alpha x}\ln x\,\left[-E_i\,(-\beta x)\right]dx = -\frac{1}{\alpha}\ln\left(1+\frac{\alpha}{\beta}\right)\left[\gamma+\ln(\alpha+\beta)\right]-\frac{1}{(\alpha+\beta)}\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,1\right) \tag{1.2}$$

$$\int_0^\infty xe^{-\alpha x}\ln x\,\left[-E_i\,(-\beta x)\right]dx = -\frac{1}{\alpha^2}\left[\ln\left(1+\frac{\alpha}{\beta}\right)-1\left(\frac{\alpha}{\alpha+\beta}\right)\right]\times\left[\gamma+\ln(\alpha+\beta)-1\right]-\frac{1}{(\alpha+\beta)^2}\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,2\right) \tag{1.3}$$

$$\int_0^\infty x^2e^{-\alpha x}\ln x\,\left[-E_i\,(-\beta x)\right]dx = -\frac{2}{\alpha^3}\left[\ln\left(1+\frac{\alpha}{\beta}\right)-1\left(\frac{\alpha}{\alpha+\beta}\right)-\frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2\right]\times\left[\gamma+\ln(\alpha+\beta)-1-\frac{1}{2}\right]-\frac{2}{(\alpha+\beta)^3}\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,3\right) \tag{1.4}$$

$$\int_0^\infty x^3e^{-\alpha x}\ln x\,\left[-E_i\,(-\beta x)\right]dx = -\frac{6}{\alpha^4}\left[\ln\left(1+\frac{\alpha}{\beta}\right)-1\left(\frac{\alpha}{\alpha+\beta}\right)-\frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2-\frac{1}{3}\left(\frac{\alpha}{\alpha+\beta}\right)^3\right]\times\left[\gamma+\ln(\alpha+\beta)-1-\frac{1}{2}-\frac{1}{3}\right]-\frac{6}{(\alpha+\beta)^4}\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,4\right) \tag{1.5}$$

$$\int_0^\infty x^ne^{-\alpha x}\ln x\,\left[-E_i\,(-\beta x)\right]dx = -\frac{n!}{\alpha^{n+1}}\left[\ln\left(1+\frac{\alpha}{\beta}\right)-\sum_{i=1}^{n}\frac{1}{i}\left(\frac{\alpha}{\alpha+\beta}\right)^i\right]\times\left[\gamma+\ln(\alpha+\beta)-\sum_{i=1}^{n}\frac{1}{i}\right]-\frac{n!}{(\alpha+\beta)^{n+1}}\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,n+1\right) \tag{1.6}$$

$$\int_0^\infty x^p\ln x\,\left[-E_i\,(-\beta x)\right]dx = \frac{\Gamma(p+1)}{(p+1)\beta^{p+1}}\left\{\left[\psi(p+1)-\ln\beta\right]-\frac{1}{p+1}\right\} \tag{1.7}$$

$$z^n\,\Phi\,(z,2,n+1) = \Phi\,(z,2,1)-\sum_{i=0}^{n-1}\frac{z^i}{(i+1)^2} \tag{1.8}$$

B.

$$H_1\,(p,\alpha,\beta,\rho) = \int_0^\rho x^pe^{-\alpha x}\ln x\,\left[-E_i\,(-\beta x)\right]dx = \frac{\ln\rho}{\alpha^{p+1}}\left[-E_i\,(-\beta\rho)\right]\gamma(p+1,\alpha\rho)-\rho^{p+1}\left[-E_i\,(-\beta\rho)\right]\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)^2}$$
$$+\frac{1}{\beta^{p+1}}\left[\ln\rho\sum_{k=0}^{\infty}\frac{\left(-\frac{\alpha}{\beta}\right)^k}{k!}\,\frac{\gamma(p+k+1,\beta\rho)}{(p+k+1)}-\sum_{k=0}^{\infty}\frac{\left(-\frac{\alpha}{\beta}\right)^k}{k!}\,\frac{\gamma(p+k+1,\beta\rho)}{(p+k+1)^2}\right]$$
$$-\rho^{p+1}\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(p+k+l+1)^2} \tag{2}$$

$$\alpha H_1\,(p,\alpha,\beta,\rho) = pH_1\,(p-1,\alpha,\beta,\rho)-B_1\,(p-1,\alpha+\beta,\rho)+G_1\,(p-1,\alpha,\beta,\rho)-\rho^pe^{-\alpha\rho}\ln\rho\,\left[-E_i\,(-\beta\rho)\right] \tag{2.1}$$

Solution (a)

$$\int_0^\rho e^{-\alpha x}\ln x\,[-E_i(-\beta x)]\,dx = -\frac{1}{\alpha}\ln\left(1+\frac{\alpha}{\beta}\right)\left(\gamma+\ln(\alpha+\beta)+\left\{-E_i[-(\alpha+\beta)\rho]\right\}\right)$$
$$-\frac{1}{\alpha}\left[\left(\frac{\alpha}{\alpha+\beta}\right)\Phi\left(\frac{\alpha}{\alpha+\beta},2,1\right)+\ln\rho\left(e^{-\alpha\rho}[-E_i(-\beta\rho)]-\{-E_i[-(\alpha+\beta)\rho]\}\right)\right]$$
$$+\frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{\alpha}{\alpha+\beta}\right)^{k+1}E_{k+2}[\rho(\alpha+\beta)]$$

Solution (b)

$$= -\frac{1}{\alpha}\left\{[\gamma+\ln(\alpha+\beta)]\ln\left(1+\frac{\alpha}{\beta}\right)+\frac{\alpha}{\alpha+\beta}\Phi\left(\frac{\alpha}{\alpha+\beta},2,1\right)\right\}-\frac{1}{\alpha}[-E_i(-\beta\rho)]$$
$$\times\left\{\ln\rho e^{-\alpha\rho}+[E_i(-\alpha\rho)]\right\}+\frac{1}{\alpha}\left(\frac{1}{2}\zeta(2)+\frac{1}{2}[\gamma+\ln(\alpha+\beta)+\ln\rho]^2\right.$$
$$\left.+\ln\rho\left\{-E_i[-(\alpha+\beta)\rho]\right\}+\sum_{k=0}^{\infty}\frac{[-(\alpha+\beta)\rho]^{k+1}}{k!\,(k+1)^3}\right)+\frac{1}{\alpha}\int_\rho^\infty\frac{e^{-\beta x}[-E_i(-\alpha x)]}{x}dx$$

Solution (c)
(good for $\alpha > \beta$)

$$= -\frac{1}{\alpha}\left\{[\gamma+\ln(\alpha+\beta)]\ln\left(1+\frac{\alpha}{\beta}\right)+\left(\frac{\alpha}{\alpha+\beta}\right)\Phi\left(\frac{\alpha}{\alpha+\beta},2,1\right)\right\}$$
$$+\frac{1}{\alpha}\left[(\gamma+\ln\beta\rho)[-E_i(-\alpha\rho)]+\frac{1}{2}\left\{(\gamma+\ln\alpha\rho)^2+[\gamma+\ln(\alpha+\beta)\rho]^2+2\zeta(2)\right\}\right.$$
$$+\sum_{k=0}^{\infty}\frac{[-\rho(\alpha+\beta)]^{k+1}}{k!\,(k+1)^3}+\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^{k+1}}{k!\,(k+1)^3}+e^{-\alpha\rho}\sum_{k=0}^{\infty}\frac{(-\beta/\alpha)^{k+1}}{(k+1)^2}e_k(\alpha\rho)$$
$$\left.-\ln\rho\left(e^{-\alpha\rho}[-E_i(-\beta\rho)]-\left\{-E_i[-(\alpha+\beta)\rho]\right\}\right)\right]$$

Solution (d)
(good for $\beta > \alpha$)

$$= -\frac{1}{\alpha}\left\{[\gamma+\ln(\alpha+\beta)]\ln\left(1+\frac{\alpha}{\beta}\right)+\left(\frac{\alpha}{\alpha+\beta}\right)\Phi\left(\frac{\alpha}{\alpha+\beta},2,1\right)\right\}$$
$$+\frac{1}{\alpha}\left[-\left\{\gamma+\ln\alpha\rho+[-E_i(-\alpha\rho)]\right\}[-E_i(-\beta\rho)]\right.$$
$$+\frac{1}{2}\left\{[\gamma+\ln\rho(\alpha+\beta)]^2-(\gamma+\ln\beta\rho)^2\right\}+\sum_{k=0}^{\infty}\frac{[-\rho(\alpha+\beta)]^{k+1}}{k!\,(k+1)^3}-\sum_{k=0}^{\infty}\frac{(-\rho\beta)^{k+1}}{k!\,(k+1)^3}-e^{-\beta\rho}$$
$$\left.\times\sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^{k+1}}{(k+1)^2}e_k(\beta\rho)-\ln\rho\left(e^{-\alpha\rho}[-E_i(-\beta\rho)]-\left\{-E_i[-(\alpha+\beta)\rho]\right\}\right)\right] \tag{2.2}$$

$$\int_0^\rho e^{-\alpha x}\ln x\left[-E_i(-\alpha x)\right]dx = -\frac{1}{\alpha}\ln\rho\left\{e^{-\alpha\rho}\left[-E_i(-\alpha\rho)\right] - \left[-E_i(-2\alpha\rho)\right]\right\}$$
$$-\frac{1}{2\alpha}\left[-E_i(-\alpha\rho)\right]^2 + \frac{1}{2\alpha}(\gamma + \ln\alpha\rho)^2 + \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{(-2\alpha\rho)^{k+1}}{k!\,(k+1)^3} \tag{2.3}$$

$$\int_0^\rho x^p\ln x\left[-E_i(-\beta x)\right]dx = \frac{1}{\beta^{p+1}(p+1)}\left(\ln\rho - \frac{1}{p+1}\right)\left\{\gamma(p+1,\beta\rho) + (\beta\rho)^{p+1}\left[-E_i(-\beta\rho)\right]\right\}$$
$$-\frac{\rho^{p+1}}{(p+1)}\sum_{k=0}^{\infty}\frac{(-\beta\rho)^k}{k!\,(p+k+1)^2} \tag{2.4}$$

$$\int_0^\rho x^n\ln x\left[-E_i(-\beta x)\right]dx = \frac{n!}{\beta^{n+1}(n+1)}\left\{\left[-E_i(-\beta\rho)\right]\left[\frac{\ln\rho\,(\beta\rho)^{n+1}}{n!} - \frac{(\beta\rho)^{n+1}}{(n+1)!} - 1\right]\right.$$
$$\left. + e^{-\beta\rho}\left[\frac{e_n(\beta\rho)}{n+1} - \ln\rho\, e_n(\beta\rho) - \sum_{k=0}^{n-1}\frac{e_k(\beta\rho)}{k+1}\right] + \left[\psi(n+1) - \frac{1}{n+1} - \ln\beta\right]\right\} \tag{2.5}$$

$$z\,\Phi(z,2,1) = L_2(z) = -\int_0^z \frac{\ln(1-u)du}{u}\ \text{(Euler's dilogarithm)} \tag{2.6}$$

$$L_2\left(\frac{1}{2}\right) = \frac{1}{2}\left[\zeta(2) - \ln^2(2)\right] \tag{2.7}$$

C.

$$H_2(p,\alpha,\beta,\rho) = \int_\rho^\infty x^p e^{-\alpha x}\ln x\left[-E_i(-\beta x)\right]dx = H(p,\alpha,\beta) - H_1(p,\alpha,\beta,\rho) \tag{3}$$

$$\alpha H_2(p,\alpha,\beta,\rho) = pH_2(p-1,\alpha,\beta,\rho) - B_2(p-1,\alpha+\beta,\rho) + G_2(p-1,\alpha,\beta,\rho) + \rho^p e^{-\alpha\rho}\ln\rho\left[-E_i(-\beta\rho)\right] \tag{3.1}$$

Solution (a)

$$\int_\rho^\infty e^{-\alpha x}\ln x\left[-E_i(-\beta x)\right]dx = \frac{1}{\alpha}\ln\rho\left(e^{-\alpha\rho}\left[-E_i(-\beta\rho)\right] - \left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\}\right)$$
$$+\frac{1}{\alpha}\ln\left(1+\frac{\alpha}{\beta}\right)\left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\} - \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left(\frac{\alpha}{\alpha+\beta}\right)^{k+1}E_{k+2}\left[\rho(\alpha+\beta)\right]$$

Solution (b)

$$= \frac{1}{\alpha}\left[-E_i(-\beta\rho)\right]\left\{\ln\rho\, e^{-\alpha\rho} + \left[-E_i(-\alpha\rho)\right]\right\} - \frac{1}{\alpha}\left(\frac{1}{2}\zeta(2) + \frac{1}{2}\left[\gamma + \ln(\alpha+\beta) + \ln\rho\right]^2\right.$$
$$\left. + \ln\rho\left\{-E_i\left[-(\alpha+\beta)\rho\right]\right\} + \sum_{k=0}^{\infty}\frac{\left[-(\alpha+\beta)\rho\right]^{k+1}}{k!\,(k+1)^3}\right) - \frac{1}{\alpha}\int_\rho^\infty\frac{e^{-\beta x}\left[-E_i(-\alpha x)\right]}{x}dx$$

Solution (c)
(good for $\alpha > \beta$)

$$= -\frac{1}{\alpha}\Bigg[(\gamma + \ln\beta\rho)\,[-E_i(-\alpha\rho)] + \frac{1}{2}\Big\{(\gamma + \ln\alpha\rho)^2 + [\gamma + \ln(\alpha + \beta)\rho]^2 + 2\,\zeta(2)\Big\}$$
$$+ \sum_{k=0}^{\infty}\frac{[-\rho(\alpha+\beta)]^{k+1}}{k!\,(k+1)^3} + \sum_{k=0}^{\infty}\frac{(-\alpha\rho)^{k+1}}{k!\,(k+1)^3} + e^{-\alpha\rho}\sum_{k=0}^{\infty}\frac{(-\beta/\alpha)^{k+1}}{(k+1)^2}\,e_k(\alpha\rho)$$
$$-\ln\rho\left(e^{-\alpha\rho}\,[-E_i(-\beta\rho)] - \Big\{-E_i\,[-(\alpha+\beta)\rho]\Big\}\right)\Bigg]$$

Solution (d)
(good for $\beta > \alpha$)

$$= -\frac{1}{\alpha}\Bigg[-\Big\{\gamma + \ln\alpha\rho + [-E_i(-\alpha\rho)]\Big\}\,[-E_i(-\beta\rho)] + \frac{1}{2}\Big\{[\gamma + \ln(\alpha+\beta) + \ln\rho]^2$$
$$-(\gamma + \ln\beta\rho)^2\Big\} + \sum_{k=0}^{\infty}\frac{[-\rho(\alpha+\beta)]^{k+1}}{k!\,(k+1)^3} - \sum_{k=0}^{\infty}\frac{(-\rho\beta)^{k+1}}{k!\,(k+1)^3} - e^{-\beta\rho}\sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^{k+1}}{(k+1)^2}\,e_k(\beta\rho)$$
$$-\ln\rho\left(e^{-\alpha\rho}\,[-E_i(-\beta\rho)] - \Big\{-E_i\,[-(\alpha+\beta)\rho]\Big\}\right)\Bigg] \tag{3.2}$$

$$\int_\rho^\infty e^{-\alpha x}\ln x\,[-E_i(-\alpha x)]\,dx = \frac{1}{\alpha}\ln\rho\Big\{e^{-\alpha\rho}\,[-E_i(-\alpha\rho)] - [-E_i(-2\alpha\rho)]\Big\} + \frac{1}{2\alpha}[-E_i(-\alpha\rho)]^2$$
$$-\frac{1}{2\alpha}\,[(\gamma + \ln 2\alpha\rho)^2 + \zeta(2)] - \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{(-2\alpha\rho)^{k+1}}{k!\,(k+1)^3} \tag{3.3}$$

$$\int_\rho^\infty x^p\ln x\,[-E_i(-\beta x)]\,dx = \frac{\Gamma(p+1)}{(p+1)\beta^{p+1}}[\psi(p+1) - \ln\beta] - \frac{\Gamma(p+1,\beta\rho)}{(p+1)^2\beta^{p+1}} - \frac{\ln\rho}{(p+1)\beta^{p+1}}\Big\{\gamma(p+1,\beta\rho)$$
$$+ (\beta\rho)^{p+1}[-E_i(-\beta\rho)]\Big\} + \frac{\rho^{p+1}}{(p+1)^2}[-E_i(-\beta\rho)] + \frac{\beta^{p+1}}{(p+1)}\sum_{k=0}^{\infty}\frac{(-\beta\rho)^k}{k!\,(p+k+1)^2} \tag{3.4}$$

VIII. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x}\ln^2 x\,[-E_i(-\beta x)]\,dx$

A.

$$I(p,\alpha,\beta) = \int_0^\infty x^p e^{-\alpha x}\ln^2 x\,[-E_i(-\beta x)]\,dx = \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}}\left(\Big\{[\psi(p+1) - \ln(\alpha+\beta)]^2 + \zeta(2,p+1)\Big\}\,\Phi\left(\frac{\alpha}{\alpha+\beta},1,p+1\right)\right.$$
$$\left.-2\,[\psi(p+1) - \ln(\alpha+\beta)]\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,p+1\right) + 2\,\Phi\left(\frac{\alpha}{\alpha+\beta},3,p+1\right)\right) \tag{1}$$

$$\alpha I(p,\alpha,\beta) = pI(p-1,\alpha,\beta) - C(p-1,\alpha+\beta) + 2H(p-1,\alpha,\beta) \tag{1.1}$$

$$\int_0^\infty e^{-\alpha x} \ln^2 x \left[-E_i(-\beta x)\right] dx = \frac{1}{\alpha}\ln\left(1+\frac{\alpha}{\beta}\right)\left\{[\gamma + \ln(\alpha+\beta)]^2 + \zeta(2)\right\}$$
$$+ \frac{2}{(\alpha+\beta)}[\gamma + \ln(\alpha+\beta)]\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,1\right) + \frac{2}{(\alpha+\beta)}\Phi\left(\frac{\alpha}{\alpha+\beta},3,1\right) \tag{1.2}$$

$$\int_0^\infty xe^{-\alpha x} \ln^2 x \left[-E_i(-\beta x)\right] dx = \frac{1}{\alpha^2}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - 1\left(\frac{\alpha}{\alpha+\beta}\right)\right]\left\{[\gamma + \ln(\alpha+\beta) - 1]^2 + \zeta(2) - 1\right\}$$
$$+ \frac{2}{(\alpha+\beta)^2}[\gamma + \ln(\alpha+\beta) - 1]\,\Phi\left(\frac{\alpha}{\alpha+\beta},2,2\right) + \frac{2}{(\alpha+\beta)^2}\Phi\left(\frac{\alpha}{\alpha+\beta},3,2\right) \tag{1.3}$$

$$\int_0^\infty x^2e^{-\alpha x} \ln^2 x \left[-E_i(-\beta x)\right] dx = \frac{2}{\alpha^3}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - 1\left(\frac{\alpha}{\alpha+\beta}\right) - \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2\right]\left\{\left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2}\right]^2\right.$$
$$\left. + \zeta(2) - 1 - \frac{1}{2^2}\right\} + \frac{4}{(\alpha+\beta)^3}\left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2}\right]\Phi\left(\frac{\alpha}{(\alpha+\beta)},2,3\right)$$
$$+ \frac{4}{(\alpha+\beta)^3}\Phi\left(\frac{\alpha}{\alpha+\beta},3,3\right) \tag{1.4}$$

$$\int_0^\infty x^3e^{-\alpha x} \ln^2 x \left[-E_i(-\beta x)\right] dx = \frac{6}{\alpha^4}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - 1\left(\frac{\alpha}{\alpha+\beta}\right) - \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2 - \frac{1}{3}\left(\frac{\alpha}{\alpha+\beta}\right)^3\right]$$
$$\times\left\{\left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2} - \frac{1}{3}\right]^2 + \zeta(2) - 1 - \frac{1}{2^2} - \frac{1}{3^2}\right\} \tag{1.5}$$
$$+ \frac{12}{(\alpha+\beta)^4}\left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2} - \frac{1}{3}\right]\Phi\left(\frac{\alpha}{\alpha+\beta},2,4\right) + \frac{12}{(\alpha+\beta)^4}\Phi\left(\frac{\alpha}{\alpha+\beta},3,4\right)$$

$$\int_0^\infty x^ne^{-\alpha x} \ln^2 x \left[-E_i(-\beta x)\right] dx = \frac{n!}{\alpha^{n+1}}\left[\ln\left(1+\frac{\alpha}{\beta}\right) - \sum_{k=0}^{n-1}\frac{1}{k+1}\left(\frac{\alpha}{\alpha+\beta}\right)^{k+1}\right]\left\{\left[\gamma + \ln(\alpha+\beta) - \sum_{k=0}^{n-1}\frac{1}{k+1}\right]^2 + \zeta(2)\right.$$
$$\left. - \sum_{k=0}^{n-1}\frac{1}{(k+1)^2}\right\} + \frac{2(n!)}{(\alpha+\beta)^{n+1}}\left[\gamma + \ln(\alpha+\beta) - \sum_{k=0}^{n-1}\frac{1}{k+1}\right]\Phi\left(\frac{\alpha}{\alpha+\beta},2,n+1\right)$$
$$+ \frac{2(n!)}{(\alpha+\beta)^{n+1}}\Phi\left(\frac{\alpha}{\alpha+\beta},3,n+1\right) \tag{1.6}$$

$$\int_0^\infty x^p \ln^2 x \left[-E_i(-\beta x)\right] dx = \frac{\Gamma(p+1)}{\beta^{p+1}(p+1)}\left(\left\{[\psi(p+1) - \ln\beta]^2 + \zeta(2,p+1)\right\} - \frac{2}{p+1}[\psi(p+1) - \ln\beta] + \frac{2}{(p+1)^2}\right) \tag{1.7}$$

$$z^n\,\Phi(z,3,n+1) = \Phi(z,3,1) - \sum_{k=0}^{n-1}\frac{z^k}{(k+1)^3} \tag{1.8}$$

B.

$$I_1(p,\alpha,\beta,\rho) = \int_0^\rho x^p e^{-\alpha x} \ln^2 x \left[-E_i(-\beta x)\right] dx = \left[-E_i(-\beta\rho)\right] \left[\frac{\gamma(p+1,\alpha\rho)\ln^2\rho}{\alpha^{p+1}} - 2\rho^{p+1}\ln\rho \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k!\,(p+k+1)^2}\right.$$

$$\left. + 2\rho^{p+1}\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)^3}\right] + \frac{1}{\beta^{p+1}}\left[\ln^2\rho \sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^k}{k!}\frac{\gamma(p+k+1,\beta\rho)}{(p+k+1)} - 2\ln\rho\sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^k}{k!}\frac{\gamma(p+k+1,\beta\rho)}{(p+k+1)^2}\right.$$

$$\left. + 2\sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^k}{k!}\frac{\gamma(p+k+1,\beta\rho)}{(p+k+1)^3}\right] + 2\rho^{p+1}\left[\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)^2}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(p+k+l+1)^2}\right.$$

$$\left. + \sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(p+k+l+1)^3} - \ln\rho\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(p+k+1)}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(p+k+l+1)^2}\right] \tag{2}$$

$$\alpha I_1(p,\alpha,\beta,\rho) = pI_1(p-1,\alpha,\beta,\rho) - C_1(p-1,\alpha+\beta,\rho) + 2H_1(p-1,\alpha,\beta,\rho) - \rho^p e^{-\alpha\rho}\ln^2\rho\left[-E_i(-\beta\rho)\right] \tag{2.1}$$

$$\int_0^\rho e^{-\alpha x}\ln^2 x\left[-E_i(-\beta x)\right]dx = \frac{1}{\alpha}\left[-E_i(-\beta\rho)\right]\left(\ln^2\rho\,(1-e^{-\alpha\rho}) - 2\ln\rho\left\{\gamma + \ln\alpha + \ln\rho + \left[-E_i(-\alpha\rho)\right]\right\}\right.$$

$$\left. - 2\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^{k+1}}{k!\,(k+1)^3}\right) + \frac{1}{\beta}\left[\ln^2\rho\sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^k}{k!\,(k+1)}\gamma(k+1,\beta\rho) - 2\ln\rho\sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^k}{k!\,(k+1)^2}\right.$$

$$\left.\times\gamma(k+1,\beta\rho) + 2\sum_{k=0}^{\infty}\frac{(-\alpha/\beta)^k}{k!\,(k+1)^3}\gamma(k+1,\beta\rho)\right] + 2\rho\left[\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+1)^2}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(k+l+1)^2}\right.$$

$$\left. + \sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+1)}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(k+l+1)^3} - \ln\rho\sum_{k=0}^{\infty}\frac{(-\alpha\rho)^k}{k!\,(k+1)}\sum_{l=0}^{\infty}\frac{(-\beta\rho)^l}{l!\,(k+l+1)^2}\right] \tag{2.2}$$

$$\int_0^\rho x^p\ln^2 x\left[-E_i(-\beta x)\right]dx = \frac{1}{\beta^{p+1}(p+1)}\left[\ln^2\rho - \frac{2\ln\rho}{(p+1)} + \frac{2}{(p+1)^2}\right]\left\{\gamma(p+1,\beta\rho) + (\beta\rho)^{p+1}\left[-E_i(-\beta\rho)\right]\right\}$$

$$- \frac{2\rho^{p+1}}{(p+1)}\left(\ln\rho - \frac{1}{p+1}\right)\sum_{k=0}^{\infty}\frac{(-\beta\rho)^k}{k!\,(p+k+1)^2} + \frac{2\rho^{p+1}}{(p+1)}\sum_{k=0}^{\infty}\frac{(-\beta\rho)^k}{k!\,(p+k+1)^3} \tag{2.3}$$

C.

$$I_2(p,\alpha,\beta,\rho) = \int_\rho^\infty x^p e^{-\alpha x}\ln^2 x\left[-E_i(-\beta x)\right]dx = I(p,\alpha,\beta) - I_1(p,\alpha,\beta,\rho) \tag{3}$$

$$\alpha I_2(p,\alpha,\beta,\rho) = pI_2(p-1,\alpha,\beta,\rho) - C_2(p-1,\alpha+\beta,\rho) + 2H_2(p-1,\alpha,\beta,\rho) + \rho^p e^{-\alpha\rho}\ln^2\rho\left[-E_i(-\beta\rho)\right] \tag{3.1}$$

$$\int_\rho^\infty e^{-\alpha x}\ln^2 x\left[-E_i(-\beta x)\right]dx = \tag{3.2}$$

(See Sec. VIII, Eq. 1.2 and 2.2.)

$$\int_\rho^\infty x^p\ln^2 x\left[-E_i(-\beta x)\right]dx = \tag{3.3}$$

(See Sec. VIII, Eq. 1.7 and 2.3.)

IX. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x} \ln^3 x \,[-E_i(-\beta x)]\, dx$

A.

$$J(p,\alpha,\beta) = \int_0^\infty x^p e^{-\alpha x} \ln^3 x \left[-E_i(-\beta x)\right] dx = \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}} \left(\left\{ \left[\psi(p+1) - \ln(\alpha+\beta)\right]^3 + 3\zeta(2,p+1)\left[\psi(p+1) - \ln(\alpha+\beta)\right] - 2\zeta(3,p+1) \right\} \Phi\left(\frac{\alpha}{\alpha+\beta},1,p+1\right) - 3\left\{\left[\psi(p+1) - \ln(\alpha+\beta)\right]^2 + \zeta(2,p+1)\right\} \times \Phi\left(\frac{\alpha}{\alpha+\beta},2,p+1\right) + 6\left[\psi(p+1) - \ln(\alpha+\beta)\right]\Phi\left(\frac{\alpha}{\alpha+\beta},3,p+1\right) - 6\,\Phi\left(\frac{\alpha}{\alpha+\beta},4,p+1\right)\right) \quad (1)$$

$$\alpha J(p,\alpha,\beta) = pJ(p-1,\alpha,\beta) - D(p-1,\alpha+\beta) + 6I(p-1,\alpha,\beta) \quad (1.1)$$

$$\int_0^\infty e^{-\alpha x} \ln^3 x \left[-E_i(-\beta x)\right] dx = -\frac{1}{\alpha}\ln\left(1+\frac{\alpha}{\beta}\right)\left\{\left[\gamma + \ln(\alpha+\beta)\right]^3 + 3\zeta(2)\left[\gamma+\ln(\alpha+\beta)\right] + 2\zeta(3)\right\} - \frac{3}{(\alpha+\beta)}\left\{\left[\gamma+\ln(\alpha+\beta)\right]^2 + \zeta(2)\right\}\Phi\left(\frac{\alpha}{\alpha+\beta},2,1\right) - \frac{6}{(\alpha+\beta)}\left[\gamma+\ln(\alpha+\beta)\right] \times \Phi\left(\frac{\alpha}{\alpha+\beta},3,1\right) - \frac{6}{(\alpha+\beta)}\Phi\left(\frac{\alpha}{\alpha+\beta},4,1\right) \quad (1.2)$$

$$\int_0^\infty x^p \ln^3 x \left[-E_i(-\beta x)\right] dx = \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}(p+1)}\left(\left\{\left[\psi(p+1) - \ln\beta\right]^3 + 3\zeta(2,p+1)\left[\psi(p+1) - \ln\beta\right] - 2\zeta(3,p+1)\right\} - \frac{3}{(p+1)}\left\{\left[\psi(p+1) - \ln\beta\right]^2 + \zeta(2,p+1)\right\} + \frac{6}{(p+1)^2}\left[\psi(p+1) - \ln\beta\right] - \frac{6}{(p+1)^3}\right) \quad (1.3)$$

B.

$$J_1(p,\alpha,\beta,\rho) = \int_0^\rho x^p e^{-\alpha x} \ln^3 x \left[-E_i(-\beta x)\right] dx \quad (2)$$

$$\alpha J_1(p,\alpha,\beta,\rho) = pJ_1(p-1,\alpha,\beta,\rho) - D_1(p-1,\alpha+\beta,\rho) + 6I_1(p-1,\alpha,\beta,\rho) - \rho^p e^{-\alpha\rho}\ln^3\rho\left[-E_i(-\beta\rho)\right] \quad (2.1)$$

C.

$$J_2(p,\alpha,\beta,\rho) = \int_\rho^\infty x^p e^{-\alpha x} \ln^3 x \left[-E_i(-\beta x)\right] dx \quad (3)$$

$$\alpha J_2(p,\alpha,\beta,\rho) = pJ_2(p-1,\alpha,\beta,\rho) - D_2(p-1,\alpha+\beta,\rho) + 6I_2(p-1,\alpha,\beta,\rho) + \rho^p e^{-\alpha\rho}\ln^3\rho\left[-E_i(-\beta\rho)\right] \quad (3.1)$$

X. MISCELLANEOUS RELATED INTEGRALS

A.

$$K(n,\alpha,\beta,p) = \int_0^\infty x^p e^{-\alpha x}(\ln x)^n \left[-E_i(-\beta x)\right] dx = \sum_{k=0}^{n} (-1)^k k! \binom{n}{k} \Phi\left(\frac{\alpha}{\alpha+\beta}, k+1, p+1\right) F(n-k, \alpha+\beta, p) \qquad (1)$$

(See Sec. V, Eq. 1.)

$$\alpha K(n,\alpha,\beta,p) = pK(n,\alpha,\beta,p-1) - F(n,\alpha+\beta,p-1) + nK(n-1,\alpha,\beta,p-1) \qquad (1.1)$$

B.

$$\alpha \int_\rho^\infty e^{-\alpha x}(\ln x)^n \left[-E_i(-\beta x)\right] dx = n \int_\rho^\infty \frac{e^{-\alpha x}(\ln x)^{n-1}\left[-E_i(-\beta x)\right] dx}{x} - \int_\rho^\infty \frac{e^{-(\alpha+\beta)x}(\ln x)^n\, dx}{x} + e^{-\alpha\rho}(\ln\rho)^n \left[-E_i(-\beta\rho)\right] \qquad (2)$$

C.

$$\int_\rho^\infty \frac{e^{-\alpha x}\left[-E_i(-\beta x)\right] dx}{x} = \frac{1}{2}\left[(\gamma + \ln\beta\rho)^2 + \zeta(2)\right] + \sum_{k=0}^{\infty} \frac{(-\beta\rho)^{k+1}}{k!(k+1)^3} + \left[-E_i(-\beta\rho)\right]\Big\{\gamma + \ln\alpha\rho + \left[-E_i(-\alpha\rho)\right]\Big\} + e^{-\beta\rho} \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^{k+1}}{(k+1)^2} e_k(\beta\rho) \qquad (3)$$

$$\int_\rho^\infty \frac{e^{-\alpha x}\left[-E_i(-\beta x)\right] dx}{x} = \left[-E_i(-\alpha\rho)\right]\left[-E_i(-\beta\rho)\right] - \int_\rho^\infty \frac{e^{-\beta x}\left[-E_i(-\alpha x)\right] dx}{x} \qquad (3.1)$$

$$\int_\rho^\infty \frac{\left[-E_i(-\alpha x)\right] dx}{x} = \int_\rho^\infty \frac{\ln x\, e^{-\alpha x}\, dx}{x} - \ln\rho\left[-E_i(-\alpha\rho)\right] \qquad (3.2)$$

(See Sec. V, Eq. 3.4.)

$$\int_\rho^\infty \frac{\left[-E_i(-\alpha x)\right] dx}{x} = \frac{1}{2}\left[(\gamma + \ln\alpha\rho)^2 + \zeta(2)\right] + \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^{k+1}}{k!(k+1)^3} \qquad (3.3)$$

(See Sec. V, Eq. 3.5.)

$$\int_\rho^\infty \frac{e^{-\alpha x}\left[-E_i(-\alpha x)\right] dx}{x} = \frac{1}{2}\left[-E_i(-\alpha\rho)\right]^2 \qquad (3.4)$$

D.

$$K_1(n,\alpha,\beta,p,\rho) = \int_0^\rho x^p e^{-\alpha x}(\ln x)^n \left[-E_i(-\beta x)\right] dx \qquad (4)$$

$$\alpha K_1(n,\alpha,\beta,p,\rho) = pK_1(n,\alpha,\beta,p-1,\rho) - F_1(n,\alpha+\beta,p-1,\rho) + nK_1(n-1,\alpha,\beta,p-1,\rho) - \rho^p e^{-\alpha\rho}(\ln\rho)^n \left[-E_i(-\beta\rho)\right] \qquad (4.1)$$

NOMENCLATURE

$\binom{a}{b}$	Binomial coefficients	$\frac{a!}{b!(a-b)!}$
$B_x(p,q)$	Incomplete Beta function	$\int_0^x t^{p-1}(1-t)^{q-1}\,dt$
$e_n(x)$	Truncated exponential	$\sum_{k=0}^{n} \frac{x^k}{k!}$
$[-E_i(-x)]$	Exponential integral	$\int_x^\infty e^{-t}t^{-1}dt$ $E_1(x)$
$E_n(x)$	Placzek function	$\int_1^\infty e^{-xt}t^{-n}dt$ $x^{n-1}\Gamma(1-n,x)$
${}_2F_1(a,b;c;z)$	Hypergeometric series	$\sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,z^n}{(c)_n\,n!}$
$L_2(z)$	Euler's dilogarithm	$z\Phi\,(z,2,1)$ $\sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^2}$
$(p)_n$		$\frac{\Gamma(p+n)}{\Gamma(p)}$
γ	Euler's constant	.577 215 6649
$\gamma(a,x)$	Incomplete Gamma function	$\int_0^x e^{-t}t^{a-1}dt$
$\Gamma(a,x)$	Incomplete Gamma function	$\int_x^\infty e^{-t}t^{a-1}dt$
$\Gamma(z)$	Gamma function	$\int_0^\infty e^{-t}t^{z-1}dt$
$\zeta(s)$	Riemann Zeta function	$\zeta(s,1)$ $\Phi\,(1,s,1)$ $\sum_{n=0}^{\infty} \frac{1}{(n+1)^s}$
$\zeta(s,v)$	Generalized Zeta function	$\sum_{n=0}^{\infty} \frac{1}{(n+v)^s}$
$\Phi(z,s,v)$		$\sum_{n=0}^{\infty} \frac{z^n}{(n+v)^s}$
$\psi(z)$	Psi function	$\frac{d\ln\Gamma(z)}{dz}$

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