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a study on OPTIMAL TRAJECTORY PROGRAMMING

By

Rowland E. Burns

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A STUDY ON OPTIMAL TRAJECTORY PROGRAMMING

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ABSTRACT

The problem of determining the trajectory geometry which minimizes the propellant expenditure necessary to attain specified end conditions, for the planar ascent of a constant thrust rocket-powered vehicle *in vacuo*, is treated by the second order Euler equations of variational calculus.

A series of equations is derived which allows a rapid isolation of the initial values of certain Lagrangian multipliers (which are necessary to attain prescribed end conditions) via an approximation using a parabolic turn program.

A restricted (though *not* approximate) solution of the general result is derived which does not involve these multipliers. This solution is of a simple nature, and may find application to "on-board" adaptive guidance computers.

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Rowland E. Burns

ADVANCED FLIGHT SYSTEMS BRANCH

PROPULSION AND VEHICLE ENGINEERING DIVISION

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LIST OF SYMBOLS

C_1, C_2	Integration Constants
F	Fundamental Function Appearing in the Euler Equations
G	Universal Gravitational Constant
H	The Quantity to be Minimized (Maximized) Including all Applicable Constraints
K_1, K_2	Constants Appearing in the Parabolic Turn Program
M	Planetary Mass
m	Vehicle Mass
r	Magnitude of the Radius Vector
T	Magnitude of the Thrust Vector
t	Time
V	Magnitude of the Velocity Vector
x_i	Any of the Constrained Variables Appearing in F
y_j	The Control (State) Variable β
α	Angle of Attack; Measured from the Velocity Vector to the Thrust Vector
β	The Sum of the Angle of Attack and the Flight Path Angle; the Thrust Orientation Angle Measured from the Vertical to the Thrust Vector
ϑ	The Flight Path Angle; Measured from the Vertical to the Velocity Vector
λ_1, λ_2	Time Dependent Lagrange Multipliers
ψ	Central Angle (Longitude) Measured from Launch to the Local Radius Vector

Subscripts

f	Denotes a Final (Cutoff) Point
0	Denotes an Initial Point. Usually First Stage (Booster) Cutoff.
max	Denotes the Maximum Value of a Quantity

Time derivatives are denoted by the dot notation.

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SUMMARY

The problem of determining the trajectory geometry which minimizes the propellant expenditure necessary to attain specified end conditions, for the planar ascent of a constant thrust rocket-powered vehicle *in vacuo*, is treated by the second order Euler-Lagrange equations of variational calculus.

The normal procedure of eliminating the time dependent turn program as a function of certain Lagrangian multipliers is reversed in that the multipliers are expressed in terms of the turn program and its derivatives. This allows an isolation of the (generally unknown) initial values of the multipliers in terms of a functional approximation of the optimal turn program. The approximation presented (which is certainly neither unique or sacrosanct) assumes that the angle of attack is a parabolic function of time.

This procedure yields another valuable result. A second order differential equation governing the thrust orientation angle as a function of time which contains *no* Lagrangian multipliers is derived. This solution, a special case of the first result, contains the thrust orientation angle and its first derivative as the only variables having unknown initial values. These quantities are generally quite easily approximated by an experienced worker. Few people have an intuitive feel for the initial values of the Lagrange multipliers.

These solutions are described from the "degrees-of-freedom" viewpoint and numerical applications are indicated.

SECTION I. INTRODUCTION

Since the problem of optimal thrust programming to obtain maximum altitude for a given rocket powered vehicle was first posed by Goddard, over three hundred studies have been devoted to optimal rocket trajectories. These studies have considered optimal thrust programming and optimal trajectory geometry to obtain maximum or minimum values of some "payoff" at the end point.

This "payoff" may be almost any desired function of the end conditions such as maximum range (for ballistic missiles), maximum altitude (Goddard's problem for sounding rockets), maximum payload (for satellite carriers), maximum endurance (for rocket boosted gliders), etc. Most authors have restricted the treatment of such trajectories to two dimensions, usually commenting that an extension to three dimensions is a simple matter (this is true in theory, but less straightforward in practice). The problem of including drag effects has been treated extensively and a number of analytical drag models have been investigated. Finally, almost all writers in this field have considered simultaneous optimization of the trajectory geometry and burning program.

The situation is modified considerably if operating computational programs are considered rather than theoretical studies. These programs are usually of a preliminary design nature since the general problem has not been formulated to a degree which allows inclusion of numerous important effects. On this level, the restriction to planar flight is a reasonable assumption, especially if we consider only upper stage trajectories. Drag is usually not considered for several reasons. First of all, this would require inequality constraints on the angle of attack, dynamic pressure, angle of attack at cutoff, etc. Another restriction is that analytical drag models tend to be rather crude while table look-up is difficult to apply to variational formulations so that the actual procedure for introducing drag into the calculations is not always clear cut. Another point of departure between theory and practice is that numerical calculations seldom are involved with variable thrust along the trajectory. This specialization is due to the fact that most existing engines have a constant thrust level at the present day state-of-the-art.

Even under the above restrictions there are few calculus of variations trajectory shaping procedures actually used. This is principally because the operation of such a program requires that the initial values of certain Lagrange multipliers (which are used in the formulation of the problem) be known. The general procedure is to approximate these initial values and conduct numerical iterations until the correct initial values are obtained. This approximating technique is usually effective only if the worker is quite skilled from past experience. The length of time required to attain this ability is usually rather long.

This paper deals with a method of determining the initial values of the Lagrange multipliers without previous experience by any person familiar with a reasonable trajectory computation procedure.* Furthermore, another optimal turn program which does not involve these multipliers is derived as an important by-product of the formulation. This turn program is more general than the so-called bilinear tangent solution in that the assumption of a flat earth is not invoked in the present development.

The author wishes to express thanks to S. Peter Gary without whom this paper would never have been started, and Larry G. Singleton without whom it could never have been completed.

SECTION II. EQUATIONS OF MOTION

Let r and ψ be the polar coordinates which describe the position of the center of mass of a rocket-powered vehicle above the (spherical) earth. Let V be the magnitude of the velocity vector and ϑ the angle *from* a unit vector in the radial direction *to* a unit vector in the direction of the velocity (the so-called flight path angle). Similarly, define α (the angle of attack) as the angle *from* the velocity vector *to* the thrust vector. Denoting the magnitude of the thrust by T, the mass of the vehicle by m, and the gravitational parameter of the earth by GM, the equations of motion, for vacuum flight, are:

$$\dot{V} = \frac{T}{m} \cos \alpha - \frac{GM}{r^2} \cos \vartheta \tag{1}$$

$$\dot{\vartheta} = \frac{T}{mV} \sin \alpha + \frac{\sin \vartheta}{V} \left(\frac{GM}{r^2} - \frac{V^2}{r} \right) \tag{2}$$

$$\dot{r} = V \cos \vartheta \tag{3}$$

$$\dot{\psi} = \frac{V}{r} \sin \vartheta \tag{4}$$

* The paper is generally orientated toward maximization of payload for a satellite carrier, but has other applications as well.

Most variational formulations proceed directly from these equations; however, it will be convenient to modify this approach for our purposes.

Taking the ratio of equation (4) to equation (3) gives

$$\tan \vartheta = \frac{r\,\dot{\psi}}{\dot{r}} \tag{5}$$

Squaring equations (3) and (4), and adding the results we find

$$V = (\dot{r}^2 + r^2\,\dot{\psi}^2)^{1/2} \tag{6}$$

Now differentiating equation (3)

$$\frac{d\dot{r}}{dt} = \ddot{r} = \dot{V}\cos\vartheta - V\,\dot{\vartheta}\,\sin\vartheta \tag{7}$$

The $\dot{V}$ and $\dot{\vartheta}$ terms can now be eliminated from equation (7) via equations (1) and (2) which results in

$$\ddot{r} = \frac{T}{m}\cos(\alpha + \vartheta) - \frac{GM}{r^2} + \frac{V^2}{r}\sin^2\vartheta \tag{8}$$

As a final modification of the last equation, we now eliminate V and $\sin\vartheta$ by use of equations (5) and (6). Thus

$$\ddot{r} = \frac{T}{m}\cos(\alpha + \vartheta) - \frac{GM}{r^2} + r\,\dot{\psi}^2 \tag{9}$$

A similar differentiation of equation (4), followed by elimination of V, $\dot{V}$, ϑ, , and $\dot{\vartheta}$, gives us

$$\ddot{\psi} = \frac{T}{m\,r}\sin(\alpha + \vartheta) - \frac{2\,\dot{r}\,\dot{\psi}}{r} \tag{10}$$

It is interesting to note that the equations of motion in vehicle-fixed coordinates[equations (1) through (4)] convert to polar coordinates form [equations (9) and (10)] by simple differentiations.

In order to simplify the following algebra, we now substitute

$$\beta = \vartheta + \alpha \tag{11}$$

into equations (9) and (10) obtaining

$$\ddot{r} = \frac{T}{m}\cos\beta - \frac{GM}{r^2} + r\,\dot{\psi}^2 \tag{12}$$

$$\ddot{\psi} = \frac{T}{mr}\sin\beta - \frac{2\dot{r}\,\dot{\psi}}{r} \tag{13}$$

Our variational formulation will assume equations (12) and (13) as a starting point.

SECTION III. VARIATIONAL FORMULATION

The problem at hand is to determine β as a function of time such that the "payoff" (the quantity to be maximized) is a maximum (or, more correctly, stationary) value. The payload* will be defined to be our "payoff".

As is well known, maximizing the payload is equivalent to minimizing the propellant expenditure required to attain orbit. Denoting a final point with an f subscript, an initial point with a 0 subscript, and the expended propellant by m_p we have

$$m_p = m_0 - m_f = m_0 - (m_0 - \dot{m} t_f) = \dot{m} t_f \tag{14}$$

where t is the time and $\dot{m}$ is the (constant) mass flow rate.

Our problem is now seen to be of the Mayer type; that is, we minimize t_f subject only to the constraints of equations (12) and (13).

Defining

$$\begin{aligned} F &= \lambda_1 \left[\ddot{r} - \frac{T}{m} \cos\beta + \frac{GM}{r^2} - r\dot{\psi}^2 \right] \\ &\quad + \lambda_2 \left[\ddot{\psi} - \frac{T}{mr} \sin\beta + \frac{2\dot{r}\dot{\psi}}{r} \right] = F[r, \dot{r}, \ddot{r}, \dot{\psi}, \ddot{\psi}, \beta, t] = 0 \end{aligned} \tag{15}$$

where λ_1 and λ_2 are time-dependent Lagrange multipliers, we may now define the problem by seeking the stationary values of

$$H = \dot{m} t_f + \int_{t_0}^{t_f} F\left[r(t), \dot{r}(t), \ddot{r}(t), \dot{\psi}(t), \ddot{\psi}(t), \beta(t), t\right] dt \tag{16}$$

Application of the Mayer formulation to the above expression shows that among the necessary conditions are the following Euler equations (Ref. 1).

$$\frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{x}_i}\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}_i}\right) + \frac{\partial F}{\partial x_i} = 0 \tag{17}$$

where $x_i = (r, \psi)$ and

$$\frac{\partial F}{\partial y_j} = 0 \tag{18}$$

where y_j is a state variable, which for our case is β.

From equation (15), it may be immediately noted that ψ does not appear, and we thus obtain a first integral of the form

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{\psi}}\right) - \frac{\partial F}{\partial \dot{\psi}} + C_1 = 0 \tag{19}$$

---

* The term "payload" is used to denote gross weight in orbit. Thus our definition lumps together the structure of the last stage, residual propellant, actual payload, etc., under the general heading of payload.

where C_1 is a constant of integration.

Applying equation (18) to equation (15) with $y_j = \beta$ we find

$$\frac{T}{m}\left(\lambda_1 \sin\beta - \frac{\lambda_2}{r}\cos\beta\right) = 0 \tag{20}$$

For $\frac{T}{m} \neq 0$ and $\cos\beta \neq 0$, equation (20) now gives

$$\lambda_2 = \lambda_1 \; r \tan\beta \tag{21}$$

The variational equation for the ψ coordinate may be derived from equations (15) and (19). For the first term

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{\psi}}\right) = \dot{\lambda}_2 \tag{22}$$

and, for the second term

$$\frac{\partial F}{\partial \dot{\psi}} = -2\lambda_1 r\dot{\psi} + 2\lambda_2 \frac{\dot{r}}{r} \tag{23}$$

Inserting equations (22) and (23) into equation (19) we have

$$\dot{\lambda}_2 + 2\lambda_1 r\dot{\psi} - 2\lambda_2 \frac{\dot{r}}{r} + C_1 = 0 \tag{24}$$

Now turning to equations (15) and (17) for $x_i = r$, we find

$$\frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{r}}\right) = \ddot{\lambda}_1 \tag{25}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{r}}\right) = \frac{d}{dt}\left(\frac{2\lambda_2 \dot{\psi}}{r}\right) = 2\dot{\lambda}_2 \frac{\dot{\psi}}{r} + 2\lambda_2 \frac{\ddot{\psi}}{r} - \frac{2\lambda_2 \dot{r}\dot{\psi}}{r^2} \tag{26}$$

$$\frac{\partial F}{\partial r} = -2\lambda_1 \frac{GM}{r^3} - \lambda_1 \dot{\psi}^2 + \lambda_2 \frac{T}{mr^2}\sin\beta - 2\lambda_2 \frac{\dot{r}\dot{\psi}}{r^2} \tag{27}$$

Inserting equations (25), (26), and (27) into equation (17) gives

$$\ddot{\lambda}_1 - 2\dot{\lambda}_2 \frac{\dot{\psi}}{r^3} - 2\lambda_2 \frac{\ddot{\psi}}{r} - 2\lambda_1 \frac{GM}{r^3} - \lambda_1 \dot{\psi}^2 + \lambda_2 \frac{T}{mr^2}\sin\beta = 0 \tag{28}$$

Equation (28) may be rewritten by replacing $\dot{\lambda}_2$ by the expression obtained from equation (24), $\ddot{\psi}$ by the right-hand side of equation (13) and λ_2 by the right-hand side of equation (21). This results in

$$\ddot{\lambda}_1 + \lambda_1\left(3\dot{\psi}^2 - 2\frac{GM}{r^3}\right) - \lambda_1 \frac{T}{mr}\sin\beta\tan\beta + 2C_1 \frac{\dot{\psi}}{r} = 0 \tag{29}$$

Equations (12) and (13), the equations of motion, along with equations (24) and (29), governing the Lagrange multipliers, and equation (21), which relates the steering program to the Lagrange multipliers, are five equations in the five unknowns r, ψ, λ_1, λ_2, and β. These equations, along with the initial values r_0, $\dot{r}_0$, ψ_0, $\dot{\psi}_0$, β_0 (which are known), $(\lambda_1)_0$, $(\dot{\lambda}_1)_0$, $(\lambda_2)_0$, (which are generally unknown) and C_1 (which is constant though unknown) are sufficient for a complete solution of the problem.

SECTION IV. FURTHER DEVELOPMENT OF THE TURN PROGRAM EQUATION

We shall now develop relationships between the Lagrange multipliers and the angle β which permit an estimate of the initial values of the multipliers in terms of an approximate turn program.

Taking the time derivative of equation (21) gives

$$\dot{\lambda}_2 = \dot{\lambda}_1 \; r \tan\beta + \lambda_1 \; \dot{r} \; \tan\beta + \lambda_1 \; r \sec^2\beta \; \dot{\beta} \tag{30}$$

Eliminating $\dot{\lambda}_2$ from equation (30) via equation (24) results in

$$\dot{\lambda}_1 \, r \tan\beta - \lambda_1 \; \dot{r} \tan\beta + \lambda_1 \; r \, \dot{\beta} \sec^2\beta + 2 \, \lambda_1 \, r \; \dot{\psi} + C_1 = 0 \tag{31}$$

Differentiating equation (31) now produces

$$\begin{aligned} &\ddot{\lambda}_1 \, r \tan\beta + 2 \, \dot{\lambda}_1 \, r \, (\dot{\psi} + \dot{\beta} \; \sec^2\beta) \\ &+ \lambda_1 [- \ddot{r} \tan\beta + 2 \dot{r} \, \dot{\psi} + 2 \, r \, \ddot{\psi} + r \, (\ddot{\beta} \sec^2\beta + 2 \, \dot{\beta}^2 \sec^2\beta \; \tan\beta)] = 0 \end{aligned} \tag{32}$$

$\ddot{\lambda}_1$, $\dot{\lambda}_1$, $\ddot{r}$, and $\ddot{\psi}$ may now be eliminated from equation (32) through the use of equations (29), (31), (12), and (13), respectively. Substitution from these equations and algebraic manipulation of the resulting form give

$$\begin{aligned} \cos^2\beta \Bigg\{ \lambda_1 \Bigg[3 \; \frac{GM}{r^2} \; \sin^2\beta - 4 \, r \, \dot{\psi}^2 + \frac{T}{m} \; \sin\beta \tan\beta \\ + r \, (\ddot{\beta} \; \tan\beta - 2 \, \dot{\beta}^2 - 6 \, \dot{\beta} \; \dot{\psi}) + 2 \, \dot{r} \, \dot{\beta} \tan\beta \Bigg] \\ - 2 \, C_1 \, (\dot{\psi} + \dot{\beta}) \Bigg\} = 0 \end{aligned} \tag{33}$$

The possible solutions and implications of equation (33) will now be discussed.

SECTION V. TURN PROGRAMS

Equations (12), (13), (21), (24), and (29) correspond to the results obtained when variational methods are applied to the study of rocket-powered vehicle trajectories. Equation (33) is a converse solution, insofar as the multipliers λ_1 and C_1 (or at least their ratio) are determined, if the turn program is known as a function of time. As will now be shown, a combination of the two points of view yields more information than either approach produces separately.

Several possible solutions to equation (33) immediately present themselves. The first of these, found by setting $\cos\beta = 0$, was formally excluded by the derivation of equation (21). We proceed directly to the following possibility:

$$\begin{aligned} \lambda_1 \Bigg[3 \; \frac{GM}{r^2} \sin^2\beta - 4 \, r \, \dot{\psi}^2 + \frac{T}{m} \sin\beta \tan\beta + r \, (\ddot{\beta} \; \tan\beta - 2 \, \dot{\beta}^2 - 6 \, \dot{\beta} \, \dot{\psi}) \\ + 2 \, \dot{r} \, \dot{\beta} \tan\beta \Bigg] - 2 \, C_1 \; (\dot{\psi} + \dot{\beta}) = 0 \end{aligned} \tag{34}$$

OPTION ONE

The most general case for a solution of this equation assumes that

$$\lambda_1 \neq 0 \tag{35}$$

and

$$C_1 \neq 0 \tag{36}$$

If equations (35) and (36) are fulfilled, we now have a method of determining all initial values of the Lagrange multipliers.

To illustrate, the pertinent equations of Sections III and IVwill be gathered together. These are:

$$\ddot{r} = \frac{T}{m} \cos \beta - \frac{GM}{r^2} + r \dot{\psi}^2 \tag{12}$$

$$\ddot{\psi} = \frac{T}{mr} \sin \beta - \frac{2 \dot{r} \dot{\psi}}{r} \tag{13}$$

$$\ddot{\lambda}_1 + \lambda_1 \left(3 \dot{\psi}^2 - 2 \frac{GM}{r^3}\right) - \lambda_1 \frac{T}{mr} \sin \beta \tan \beta + 2 C_1 \frac{\dot{\psi}}{r} = 0 \tag{29}$$

$$\dot{\lambda}_2 + 2 \lambda_1 r \dot{\psi} - 2 \lambda_2 \frac{\dot{r}}{r} + C_1 = 0 \tag{24}$$

$$\lambda_2 = \lambda_1 r \tan \beta \tag{21}$$

$$\dot{\lambda}_1 = \lambda_1 \left[\frac{\dot{r}}{r} - \dot{\beta} \sec \beta \csc \beta - 2 \dot{\psi} \cot \beta\right] - \frac{C_1}{r} \cot \beta \tag{37}$$

$$m = m_0 - \dot{m} t \tag{38}$$

where equation (37) was derived from equation (31) and equation (38) gives an explicit relationship between the vehicle mass and time.*

Once the initial values of the physical variables r, $\dot{r}$, ψ, $\dot{\psi}$, T and m (or t) are specified, the only available choice of initial conditions to obtain desired end conditions are $(\lambda_1)_0$, $(\dot{\lambda}_1)_0$, $(\lambda_2)_0$, C_1, and the burning time $t_f - t_0$. The choice of these parameters then allows a numerical integration of equations (12), (13), (21), (24), (29), and (38) to be carried out. One of these values is arbitrary, however, as can be seen by noticing that equations (21), (24), and (29) are homogeneous in the multipliers (including C_1). Thus, we are left with one arbitrary choice, three unknown choices, and the burning time (which may conveniently be used to specify energy or velocity). The choice of which initial value is arbitrary is actually more important than what value is chosen. Numerical work performed with this set of equations indicates that $(\lambda_1)_0$ is a good choice since the end conditions are rather insensitive to variations in this parameter (with another initial value fixed).**

---

* It is unnecessary to include equation (38) as a constraint in the formulation of equation (15) since we assume an explicit time dependence in the F function.

** A good choice for upper stage trajectories of large modern vehicles is $(\lambda_1)_0 = 50.0$.

Proceeding on the assumption that λ_1 is arbitrarily fixed, we now further assume that the initial values of β_0, $\dot{\beta}_0$, and $\ddot{\beta}_0$ are known at least approximately. From equation (34) we may now determine the value of C_1. Once C_1 has been found, equation (37) can be employed to find the initial value of $(\dot{\lambda}_1)_0$. The remaining unknown initial condition, $(\lambda_2)_0$, follows immediately from equation (21) whether or not the previous two initial conditions have been found.

The problem of determining the optimal trajectory has now been reduced to the isolation of the initial values of β, $\dot{\beta}$, and $\ddot{\beta}$, and some considerations will now be directed toward their relationships to the Lagrange multipliers.

It should be kept in mind that in this option, the use of the thrust orientation angle (β) and its first two derivatives is something of an artifice, and is used only to derive values of the multipliers at the initial point. Indeed, once these values have been found it is no longer necessary to obtain values of β or its derivatives unless we desire to do so for physical considerations.*

To cement ideas, let us consider a concrete example wherein a circular orbit of prespecified altitude is desired. This may be assured if circular energy, zero radial velocity, and altitude are all matched at the end point. Thus, we have only three end conditions that must be met, while at the initial point we have four degrees of freedom — C_1, $(\dot{\lambda}_1)_0$, $(\lambda_2)_0$, and burning time. These first three correspond to initial choices of β, $\dot{\beta}$, and $\ddot{\beta}$. Thus, there is one degree of freedom available, and we might fix this by any of several possibilities. We may for instance, require that β be continuous across staging. We could also determine the initial value of β such that the angle of attack at cutoff is zero or some other prespecified value. Finally, we could determine the initial value of β such that our payoff function (payload, for instance) is a maximum with respect to this value, etc. In this case the necessary end point constraints associated with the Mayer problem are applicable to the determination of β at the initial point, or we could apply a numerical isolation procedure.

The situation is slightly more complicated when we consider the isolation of C_1 and $(\dot{\lambda}_1)_0$ by approximating techniques with equations (34) and (37), respectively. Once β_0 and $(\lambda_1)_0$ have been chosen, all terms of equation (34) are known (from booster cutoff conditions) except $\dot{\beta}_0$, $\ddot{\beta}_0$, and C_1. $\dot{\beta}_0$ and $\ddot{\beta}_0$ are inserted from the approximate turn program and the value of C_1 determined. It should be noted, however, that this equation is a quadratic in $\dot{\beta}_0$ and we can thus have *two* values of $\dot{\beta}_0$ that yield the same C_1. Furthermore, the value of $\ddot{\beta}_0$ could, at this point, be arbitrarily set and a proper choice of C_1 still found by variation of $\dot{\beta}_0$. On the other hand, $(\dot{\lambda}_1)_0$ is a linear function of $\dot{\beta}$ and C_1, but is independent of $\ddot{\beta}$. It would seem that we are not immediately guaranteed a unique pair $[\dot{\beta}_0, \ddot{\beta}_0]$ for a given pair $[(\dot{\lambda}_1)_0, C_1]$.

It is interesting to note that if $(\dot{\lambda}_1)_0$ is arbitrary (i.e., only two end conditions are of interest), then there are two values of $\dot{\beta}$ which give equivalent end results once the variational trajectory is complete. These two values may not both be physically realistic. For example, several cases were found where the value of $\dot{\beta}_0$ was negative, presumably indicating that a retrograde orbit** was ahead. Nevertheless, normal orbits were attained just as if $\dot{\beta}$ had been initially positive.

A few further comments are in order. The first of these is that if a multi-stage vehicle is

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* Once the variational turn program has been initiated, equations (12), (13), (21), (24), (29), and (38) are integrated numerically. Equation (21) may be used to eliminate β from equations (12), (13), and (29). From this time point, we may then use equation (21) to determine β, equa- (37) to determine $\dot{\beta}$, and equation (34) to find $\ddot{\beta}$ at any point along the trajectory. The angle of attack, α, may be found by use of equations (5) and (11) if it is of interest.

** Assuming that the vehicle was initially fired with the planetary rotation.

considered we have violated the initial assumptions used in the formulation of the Mayer problem since there now exist discontinuities in the mass and thrust of the vehicle which introduce discontinuities into the Lagrange multipliers. The interested reader is referred to Reference 4.

Secondly, insofar as magnitudes of the multipliers are concerned, a few general statements can be made. The signs of $(\dot{\lambda}_1)_0$ and C_1 are negative for orbits that are with the Earth's rotation while $(\lambda_2)_0$ is positive. The magnitude of $(\dot{\lambda}_1)_0$ is usually between -0.1 and -0.5, but the actual value is critical. The magnitude of C_1 is generally bounded by $-10^5 < C_1 < -10^6$.* The values of β_0, $\dot{\beta}_0$, and $\ddot{\beta}_0$ are far more easily available from an intuitive viewpoint. For example, it is evident that a rate of 5°/sec (or even 1°/sec) is extremely high, whereas a value of 50° is not unreasonable for β at booster cutoff.

It may be seen that most realistic end conditions can be met without the assumption of coasting between stages. This is often not the case with other turn programs. The gravity turn, for instance, is shown to require such coast periods in Reference 3. It is also of interest to note that it is shown in this source that better performance results when no coast periods occur.

Finally, it is usually possible to optimize the cutoff conditions of a non-optimal booster program with respect to this turn program. Such a procedure has been found to yield significant performance increases. Such an optimization is also possible in the alternative option treated below.

OPTION TWO

Another possible solution to equation (34) (which is actually a special solution of Option One), is found by setting $C_1 = 0$ and assuming that $\lambda_1(t) \neq 0$ for all values of t. For this case we have

$$3 \frac{GM}{r^2} \sin^2 \beta - 4 r \dot{\psi}^2 + \frac{T}{m} \sin \beta \tan \beta + r (\ddot{\beta} \tan \beta - 2 \dot{\beta}^2 - 6 \dot{\beta} \dot{\psi})$$

$$+ 2 \dot{r} \dot{\beta} \tan \beta = 0 \qquad (39)$$

Equations (12), (13), and (39) are sufficient for the complete solution of the ascent problem. We may solve equation (39) for β and simultaneously integrate these three equations which *do not involve any Lagrange multipliers.*

It should be remembered that equation (39) is a **special** *solution of the optimal turn program, but it is not an* **approximate** *solution except insofar as the basic assumptions limit all solutions.*

The problem of discontinuities in the multipliers across staging does not arise in this option.

These advantages are partially offset by the loss of one degree of freedom. We now have only β_0, $\dot{\beta}_0$, and the burning time at our disposal since $\ddot{\beta}_0$ is specified by equation (39). These three values are sufficient to attain a circular orbit of specified altitude, but a free choice β_0 may not always be realistic. In this case we can perform certain other tricks to obtain more than two end conditions. We could shape the booster trajectory such that no discontinuity need occur in the thrust orientation angle, or use coast periods of unspecified lengths between one or more stages to accomplish the desired ends.

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* C_1 may range down to zero (see Option Two), but the probability of obtaining values between -10^5 and 0 is small.

Equation (39) may have a closed form solution, but none has been discovered thus far. The solution

$$\beta = C_2 - \psi , \tag{40}$$

where C_2 is a constant of integration, is irritatingly close to a general solution of this equation. It is a solution only for the limiting case when the mass of the primary body vanishes ($M = 0$); in this limit, however, the physical meaning of the solution becomes questionable.

A recent derivation of Leitmann (Ref. 5) obtained a solution to a quite different problem which is similar in form to equation (40). This solution, based originally on Reference 2, was first obtained by Lurie (Ref. 6). It is interesting to note that Leitmann's solution uses a technique similar to that employed in the derivation of equation (34).*

As a final comment, one often finds variational treatments of optimal trajectories which are formulated using only equations (1), (2), and (3). That is, the "range" constraint is deliberately excluded. While the Lagrange multipliers of this report do *not* correspond (directly) with those used in such formulations, Option Two is equally as general and gives a much more useful resulting form.

OTHER OPTIONS

Equation (34) may have other possible solutions. The first such possibility that would seem to be of interest is obtained by setting $\lambda_1 (t) \equiv 0$, and requiring that $C_1 \neq 0$. Equation (34) may then be satisfied if equation (40) is fulfilled. Further probing into the basic set of equations of motion and the necessary Euler conditions indicates that further assumptions are necessary. The result of these further assumptions is that only unthrusted radial decent can satisfy all restrictions.

If $\lambda_1 (t) \equiv 0$ and $C_1 = 0$ we lose all information about the steering program.

No other solutions to equation (34) have been found at the present time.

SECTION VI. AN APPROXIMATE TURN PROGRAM

The usefulness of Option One is a function of how accurately the optimal turn program can be approximated. This problem is, by its very nature, rather iterative and the following material is simply a fair first guess.

We shall assume that some trajectory analysis has given an approximation of the time to orbit and the maximum angle of attack that was necessary to obtain orbit. Using these data we now "curve fit" a symmetric parabola to the data as follows:

Assume that $\alpha (t)$ is given by

$$\alpha = K_1 \left(t - \frac{t_0 + t_f}{2} \right)^2 + K_2 \tag{41}$$

where K_1 and K_2 are constants, t_0 and t_f are the initial and final times over which the turn program is applied, and t is the time along the trajectory. We now proceed to determine K_1 and K_2.

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* The author was unaware of this reference during the development of the present theory.

Assuming that $\alpha = \alpha_{max}$ at $t = \frac{t_0 + t_f}{2}$, we find

$$K_2 = \alpha_{max} \tag{42}$$

and, now,

$$\alpha = \alpha_{max} + K_1 \left(t - \frac{t_0 + t_f}{2}\right)^2 \tag{43}$$

If the program is initiated with $\alpha = 0$ at $t = t_0$,* as is often the case, we have

$$\alpha_{max} + K_1 \left(\frac{t_f - t_0}{2}\right)^2 = 0 \tag{44}$$

Thus

$$K_1 = -\alpha_{max} \left(\frac{2}{t_f - t_0}\right)^2 \tag{45}$$

Finally

$$\alpha = \alpha_{max} \left[1 - \left(\frac{2}{t_f - t_0}\right)^2 \left(t - \frac{t_0 + t_f}{2}\right)^2\right] \tag{46}$$

This approximate turn immediately requires that $\beta_0 = \vartheta_0$, since α_0 is zero, and

$$\beta_0 = \alpha_0 + \vartheta_0 \tag{47}$$

The first two derivatives of equation (46), at the initial point, are

$$\dot{\alpha} \Big|_{t = t_0} = 4 \frac{\alpha_{max}}{t_f - t_0} \tag{48}$$

$$\ddot{\alpha} \Big|_{t = t_0} = -8 \frac{\alpha_{max}}{(t_f - t_0)} \tag{49}$$

The initial values of $\dot{\beta}_0$ and $\ddot{\beta}_0$ may now be evaluated from

$$\dot{\beta}_0 = \dot{\alpha}_0 + \dot{\vartheta}_0 \tag{50}$$

and

$$\ddot{\beta}_0 = \ddot{\alpha}_0 + \ddot{\vartheta}_0 \tag{51}$$

The value of $\dot{\vartheta}_0$, from equations (2), (5), and (6), recalling that $\alpha_0 = 0$, is

$$\dot{\vartheta}_0 = \frac{\dot{\psi}_0}{r_0} \left[\frac{GM - r_0(\dot{r}_0^2 + r_0^2 \dot{\psi}_0^2)}{(\dot{r}_0^2 + r_0^2 \dot{\psi}_0^2)}\right] \tag{52}$$

---

* This also forces α to be zero at $t = t_f$ by symmetry.

By differentiation of equation (2) and similar substitutions, we find

$$\ddot{\vartheta}_0 = \frac{T}{m_0 \sqrt{\dot{r}_0^2 + r_0^2 \dot{\psi}_0^2}} \left\{ 4 \frac{\alpha_{\max}}{t_f - t_0} - \dot{\psi}_0 \left[1 + \frac{GM}{r_0 (\dot{r}_0^2 + r_0^2 \dot{\psi}_0^2)} \right] \right\}$$

$$+ \frac{2 \dot{r}_0 \dot{\psi}_0}{r_0} \left\{ \left[\frac{GM}{r_0 (\dot{r}_0^2 + r_0^2 \dot{\psi}_0^2)} \right]^2 - \left[\frac{GM}{r_0 (\dot{r}_0^2 + r_0^2 \dot{\psi}_0^2)} \right] + 1 \right\} \qquad (53)$$

This parabolic approximation could be supplanted by first employing Option Two and obtaining a solution for Option One, with the results thus obtained. In this case, C_1 would be changed to a non-zero value with a simultaneous modification of α_0 or coast time toward their desired values.

A gravity turn approximation (Ref. 3) may also be substituted for the parabolic function given above.

SECTION VII. CONCLUSIONS

The foregoing development has led to some rather interesting conclusions about optimal planar trajectory programming of a constant thrust vehicle *in vacuo*. As pointed out in the introduction, these assumptions are usually employed in numerical studies of optimal upper stage trajectories.

Option One is the most generally applicable case and the approximate turn program of Section VI allows a rapid isolation of the initial values of the Lagrange multipliers that usually bottleneck optimal turn program studies.

Option Two is of interest from at least two points of view. First of all, it *is* an optimal turn program that involves *no* Lagrange multipliers; to the best of the author's knowledge it has not been demonstrated previously. Secondly, it is probably simple enough to find application to adaptive guidance schemes for hardware vehicles.

Both of the above solutions were investigated by numerical integration techniques. In all cases, they were found to agree with standard variational procedures.

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APPROVAL

A STUDY ON OPTIMAL TRAJECTORY PROGRAMMING

By

Rowland E. Burns

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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