Technical Report No. 32-386

Partial Mathematical Solution of the Three-Dimensional Four-Bar Linkage

R. G. Chamberlain

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R. G. Chamberlain

M. G. Comuntzis

M. G. Comuntzis, *Acting Chief*

Spacecraft Design

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ABSTRACT

This paper considers a four-bar linkage of which two adjacent hinges have skew axes and are fixed in the coordinate system. The remaining two joints are universal. Analytical relations are found between the rotations, angular velocities, angular accelerations, and torques about the fixed joints.

I. INTRODUCTION

In considering a spacecraft design having four solar panels arranged in two pairs, it was desired to investigate the possibility of utilizing a single, linear, spring actuator to unfold each pair from its launch position. Figure 1 shows a model of the spacecraft with one pair of panels folded against the spacecraft and one pair unfolded. The actuation mechanism consisted of a dual four-bar linkage, with skewed axes, attached to a single actuator. Figure 2 is a model of the linkage. The actuator is represented by the T-shaped handle on the right, and the solar panels by the section of beam and flat plate on the left. One panel has been disconnected from the linkage for clarity. A linear motion of the actuator, then, transmits, through the linkage, a torque about each hinge axis H.

Figure 3 diagrammatically depicts one-half of the linkage and defines the terms used in the analysis. Axes A and H are fixed to the spacecraft and are skewed relative to each other. Joints P and Q are universal bearings.

II. DISCUSSION OF PROBLEM

It was necessary to find the forces in the links and the torque relations between the arms. If the torques about A and H are both known, it is a simple matter to find the forces in the various links. Therefore, this paper concerns itself primarily with finding the relations between the torques about the two fixed axes. It is also intended that a dynamical analysis be facilitated, so the relations between the accelerations of the arms about the fixed axes are also determined. By-products of the investigation are the relations between the angles and rates about the axes.

Four-bar linkage solutions in the literature often consider the motion of points connected to link r. This paper does not, and the omission is part of the reason that the paper's title refers to a "partial solution" to the problem. Since the locations of P and Q are known as functions of β or δ from the present results, the loci of such points may be found without unreasonable difficulty.

III. SOLUTION

The torque relations will follow directly from the principles of Virtual Work and Conservation of Energy once the velocity relations between links c and d are known.

The velocities and accelerations may be found by successive differentiations of the angles with respect to time. Consequently, it is first necessary to geometrically relate the angles β and δ.

The location of point P can easily be found in terms of the Cartesian coordinates of Point A, the angle δ, and the length d. Similarly, the coordinates of point Q can be found from point H, the angle β, and length c. The angles β and δ may be combined in one equation by equating the distance between points P and Q to the length of link r (Eq. 1).

After this equation is solved for β (Eq. 8) or δ (Eq. 9), first and second time derivatives may be taken, giving velocity and acceleration relations. The torque calculation is inserted between the first and second differentiations, since it follows logically at that point.

If the coordinates of A and H are given in some coordinate system such as in Fig. 3, say A_x, A_y, A_z and H_x, H_y, H_z, then

$$b_x = H_x - A_x, \qquad b_y = H_y - A_y, \qquad b_z = H_z - A_z$$

The components of the location of point P, with a secondary coordinate system at A, are

$$P_x = 0, \qquad P_y = -d \cos \delta, \qquad P_z = -d \sin \delta$$

The components of the location of point Q are

$$Q_x = b_x - c \sin \alpha \cos \beta, \qquad Q_y = b_y + c \cos \alpha \cos \beta, \qquad Q_z = b_z + c \sin \beta$$

The square of the distance between P and Q is

$$(P_x - Q_x)^2 + (P_y - Q_y)^2 + (P_z - Q_z)^2 = r^2 \tag{1}$$

Expanding, Eq. (1) becomes

$$(-b_x + c \sin \alpha \cos \beta)^2 + (-d \cos \delta - b_y - c \cos \alpha \cos \beta)^2 + (-d \sin \delta - b_z - c \sin \beta)^2 = r^2 \tag{2}$$

Further expansion gives

$$\begin{aligned} & b_x^2 - 2b_x c \sin \alpha \cos \beta + c^2 \sin^2 \alpha \cos^2 \beta + d^2 \cos^2 \delta + b_y^2 \\ & + 2b_y d \cos \delta + c^2 \cos^2 \alpha \cos^2 \beta + 2\, cd \cos \alpha \cos \beta \cos \delta \\ & + 2b_y c \cos \alpha \cos \beta + d^2 \sin^2 \delta + b_z^2 + 2b_z d \sin \delta + c^2 \sin^2 \beta \\ & + 2cd \sin \beta \sin \delta + 2b_z c \sin \beta = r^2 \end{aligned} \tag{3}$$

Collecting terms in Eq. (3) gives

$$a_1 \cos \beta + a_2 \sin \beta = a_3 \tag{4}$$

or

$$b_1 \cos \delta + b_2 \sin \delta = b_3 \tag{5}$$

where

$$\begin{aligned} a_1 &= 2c\,[-b_x \sin \alpha + (b_y + d \cos \delta) \cos \alpha] \\ a_2 &= 2c\,(b_z + d \sin \delta) \\ a_3 &= r^2 - b^2 - c^2 - d^2 - 2d\,(b_z \sin \delta + b_y \cos \delta) \\ b_1 &= 2d\,(b_y + c \cos \alpha \cos \beta) \\ b_2 &= 2d\,(b_z + c \sin \beta) \\ b_3 &= r^2 - b^2 - c^2 - d^2 - 2c\,[(b_y \cos \alpha - b_x \sin \alpha) \cos \beta + b_z \sin \beta] \end{aligned}$$

Equation (4) is considered. If

$$A = \sqrt{a_1^2 + a_2^2}$$

and

$$\epsilon = \tan^{-1} \frac{a_1}{a_2}.$$

then

$$A \sin (\beta + \epsilon) = a_3 \tag{6}$$

$$\beta = \sin^{-1} \frac{a_3}{A} - \epsilon \tag{7}$$

$$\beta = \sin^{-1} \frac{a_3}{\pm \sqrt{a_1^2 + a_2^2}} - \tan^{-1} \frac{a_1}{a_2} \tag{8}$$

Equation (8) gives four, not necessarily distinct, possible solutions to Eq. (4). Physical considerations indicate that two, and only two, values will satisfy the original equation. Unless additional criteria are available by which one of the values may be eliminated, both must be carried throughout the remainder of the calculation.

Since Eq. (5) is identical in form to Eq. (4),

$$\delta = \sin^{-1} \frac{b_3}{\pm \sqrt{b_1^2 + b_2^2}} - \tan^{-1} \frac{b_1}{b_2} \tag{9}$$

Differentiation of Eqs. (8) and (9) gives the relations between the angular velocities.

$$\frac{d\beta}{dt} = \frac{1}{a_1^2 + a_2^2} \left(a_1 \frac{da_2}{dt} - a_2 \frac{da_1}{dt} \right) \pm \frac{1}{A'} \left[\frac{da_3}{dt} - \frac{a_3}{a_1^2 + a_2^2} \left(a_1 \frac{da_1}{dt} + a_2 \frac{da_2}{dt} \right) \right] \tag{10}$$

where

$$A' = \sqrt{a_1^2 + a_2^2 - a_3^2}$$

Similarly,

$$\frac{d\delta}{dt} = \frac{1}{b_1^2 + b_2^2}\left(b_1 \frac{db_2}{dt} - b_2 \frac{db_1}{dt}\right) \pm \frac{1}{B'}\left[\frac{db_3}{dt} - \frac{b_3}{b_1^2 + b_2^2}\left(b_1 \frac{db_1}{dt} + b_2 \frac{db_2}{dt}\right)\right] \tag{11}$$

where

$$B' = \sqrt{b_1^2 + b_2^2 - b_3^2}$$

Physical considerations indicate that A' and B' are real unless (1) the linkage is impossible, or (2) the oscillation range of the oscillating bar (if there is one) is exceeded.

Since β and δ are the only variable quantities in the definitions of the quantities a and b,

$$\frac{da_1}{dt} = -2cd \cos\alpha \sin\delta \frac{d\delta}{dt} \tag{12}$$

$$\frac{da_2}{dt} = 2cd \cos\delta \frac{d\delta}{dt} \tag{13}$$

$$\frac{da_3}{dt} = 2d\,(b_y \sin\delta - b_z \cos\delta) \frac{d\delta}{dt} \tag{14}$$

$$\frac{db_1}{dt} = -2cd \cos\alpha \sin\beta \frac{d\beta}{dt} \tag{15}$$

$$\frac{db_2}{dt} = 2cd \cos\beta \frac{d\beta}{dt} \tag{16}$$

$$\frac{db_3}{dt} = 2c\,[(b_y \cos\alpha - b_x \sin\alpha)\sin\beta - b_z \cos\beta] \frac{d\beta}{dt} \tag{17}$$

In order to find the torque relations at the hinges A and H, the fact that the work being put in at, say, hinge A equals the work taken out at hinge H (neglecting friction in the linkage) is used. If the torque at A is called T_A, and the torque at H called T_H, then, mathematically, this statement is

$$T_A \, d\delta = T_H \, d\beta \tag{18}$$

For generality, to avoid the problem of what happens if $d\delta = d\beta = 0$, $d\delta$ and $d\beta$ should be considered to be virtual displacements, and the works being equated are the virtual works.

By virtue of Eqs. (12) - (14), it is seen that Eq. (10) may be written

$$\frac{d\beta}{dt} = T_{A/H} \frac{d\delta}{dt} \tag{19}$$

where

$$T_{A/H} = \frac{2cd}{a_1^2 + a_2^2} (a_1 \cos \delta + a_2 \cos \alpha \sin \delta)$$
$$\pm \frac{2d}{A'} \left[b_y \sin \delta - b_z \cos \delta - \frac{a_3 c}{a_1^2 + a_2^2} (a_2 \cos \delta - a_1 \cos \alpha \sin \delta) \right] \tag{20}$$

which depends entirely upon the geometry. Similarly, Eqs. (15) - (17) used in Eq. (11) imply

$$\frac{d\delta}{dt} = T_{H/A} \frac{d\beta}{dt} \tag{21}$$

where

$$T_{H/A} = \frac{2cd}{b_1^2 + b_2^2} (b_1 \cos \beta + b_2 \cos \alpha \sin \beta),$$
$$\pm \frac{2c}{B'} \left[(b_y \cos \alpha - b_x \sin \alpha) \sin \beta - b_z \cos \beta - \frac{b_3 d}{b_1^2 + b_2^2} (b_2 \cos \beta - b_1 \cos \alpha \sin \beta) \right] \tag{22}$$

which depends entirely upon the geometry. Considering the differentials as virtual velocities, Eqs. (19) and (21) are statements that virtual power, in the absense of losses, is the same at both ends of the linkage. Then

$T_{A/H}$ = torque ratio, torque at A divided by torque at H

and

$$T_{H/A} = \text{torque ratio, torque at } H \text{ divided by torque at } A$$

A second time differentiation of the expressions for β and δ gives the relations between the angular accelerations of the arms.

$$\frac{d^2\beta}{dt^2} = \frac{2}{(a_1^2 + a_2^2)^2} \left\{ (a_2^2 - a_1^2) \frac{da_1}{dt} \frac{da_2}{dt} + a_1 a_2 \left[\left(\frac{da_1}{dt}\right)^2 - \left(\frac{da_2}{dt}\right)^2 \right] \right\}$$

$$+ \frac{1}{a_1^2 + a_2^2} \left(a_1 \frac{d^2 a_2}{dt^2} - a_2 \frac{d^2 a_1}{dt^2} \right)$$

$$\pm \frac{1}{A'} \left(\frac{1}{A'} \frac{dA'}{dt} \left[\frac{a_3}{a_1^2 + a_2^2} \left(a_1 \frac{da_1}{dt} + a_2 \frac{da_2}{dt} \right) - \frac{da_3}{dt} \right] + \frac{d^2 a_3}{dt^2} \right.$$

$$- \frac{1}{a_1^2 + a_2^2} \left\{ \frac{da_3}{dt} \left(a_1 \frac{da_1}{dt} + a_2 \frac{da_2}{dt} \right) + a_3 \left[\left(\frac{da_1}{dt}\right)^2 + \left(\frac{da_2}{dt}\right)^2 + a_1 \frac{d^2 a_1}{dt^2} + a_2 \frac{d^2 a_2}{dt^2} \right] \right.$$

$$\left. \left. - \frac{2a_3}{a_1^2 + a_2^2} \left(a_1 \frac{da_1}{dt} + a_2 \frac{da_2}{dt} \right)^2 \right\} \right) \tag{23}$$

Similarly,

$$\frac{d^2\delta}{dt^2} = \frac{2}{(b_1^2 + b_2^2)^2} \left\{ (b_2^2 - b_1^2) \frac{db_1}{dt} \frac{db_2}{dt} + b_1 b_2 \left[\left(\frac{db_1}{dt}\right)^2 - \left(\frac{db_2}{dt}\right)^2 \right] \right\}$$

$$+ \frac{1}{b_1^2 + b_2^2} \left(b_1 \frac{d^2 b_2}{dt^2} - b_2 \frac{d^2 b_1}{dt^2} \right)$$

$$\pm \frac{1}{B'} \left(\frac{1}{B'} \frac{dB'}{dt} \left[\frac{b_3}{b_1^2 + b_2^2} \left(b_1 \frac{db_1}{dt} + b_2 \frac{db_2}{dt} \right) - \frac{db_3}{dt} \right] + \frac{d^2 b_3}{dt^2} \right.$$

$$- \frac{1}{b_1^2 + b_2^2} \left\{ \frac{db_3}{dt} \left(b_1 \frac{db_1}{dt} + b_2 \frac{db_2}{dt} \right) + b_3 \left[\left(\frac{db_1}{dt} \right)^2 + \left(\frac{db_2}{dt} \right)^2 + b_1 \frac{d^2 b_1}{dt^2} + b_2 \frac{d^2 b_2}{dt^2} \right] \right.$$

$$\left. \left. - \frac{2b_3}{b_1^2 + b_2^2} \left(b_1 \frac{db_1}{dt} + b_2 \frac{db_2}{dt} \right)^2 \right\} \right) \tag{24}$$

Regarding the $\pm$ sign, the same sign is used in Eqs. (23) or (24) as was used in Eqs. (10) or (11).

Finally, the second derivatives of the a and b quantities and the derivatives of A' and B' are as follows:

$$\frac{dA'}{dt} = \frac{1}{A'} \left(a_1 \frac{da_1}{dt} + a_2 \frac{da_2}{dt} - a_3 \frac{da_3}{dt} \right) \tag{25}$$

$$\frac{d^2 a_1}{dt^2} = -2cd \cos \alpha \left[\cos \delta \left(\frac{d\delta}{dt} \right)^2 + \sin \delta \left(\frac{d^2 \delta}{dt^2} \right) \right] \tag{26}$$

$$\frac{d^2 a_2}{dt^2} = -2cd \left[\sin \delta \left(\frac{d\delta}{dt} \right)^2 - \cos \delta \left(\frac{d^2 \delta}{dt^2} \right) \right] \tag{27}$$

$$\frac{d^2 a_3}{dt^2} = 2d \left[(b_y \cos \delta + b_z \sin \delta) \left(\frac{d\delta}{dt} \right)^2 + (b_y \sin \delta - b_z \cos \delta) \left(\frac{d^2 \delta}{dt^2} \right) \right] \tag{28}$$

$$\frac{dB'}{dt} = \frac{1}{B'} \left(b_1 \frac{db_1}{dt} + b_2 \frac{db_2}{dt} - b_3 \frac{db_3}{dt} \right) \tag{29}$$

$$\frac{d^2 b_1}{dt^2} = -2cd \cos \alpha \left[\cos \beta \left(\frac{d\beta}{dt} \right)^2 + \sin \beta \left(\frac{d^2\beta}{dt^2} \right) \right] \tag{30}$$

$$\frac{d^2 b_2}{dt^2} = -2cd \left[\sin \beta \left(\frac{d\beta}{dt} \right)^2 - \cos \beta \left(\frac{d^2\beta}{dt^2} \right) \right] \tag{31}$$

$$\frac{d^2 b_3}{dt^2} = 2c \left\{ (b_y \cos \alpha - b_x \sin \alpha) \left[\cos \beta \left(\frac{d\beta}{dt} \right)^2 + \sin \beta \left(\frac{d^2\beta}{dt^2} \right) \right] \right.$$

$$\left. + b_z \left[\sin \beta \left(\frac{d\beta}{dt} \right)^2 - \cos \beta \left(\frac{d^2\beta}{dt^2} \right) \right] \right\} \tag{32}$$

NOMENCLATURE

A	"driver" axis
a_1, a_2, a_3 b_1, b_2, b_3 A, ϵ, A', B'	quantities defined for manipulative purposes
b, c, d, r	lengths of links (Fig. 3)
H	"follower" axis
P, Q	universal joints at ends of connecting rod
T_A	torque about A
T_H	torque about H
α	angle between the axes A and H
β, δ	angles locating the positions of the "follower" and the "driver" (Fig. 3)

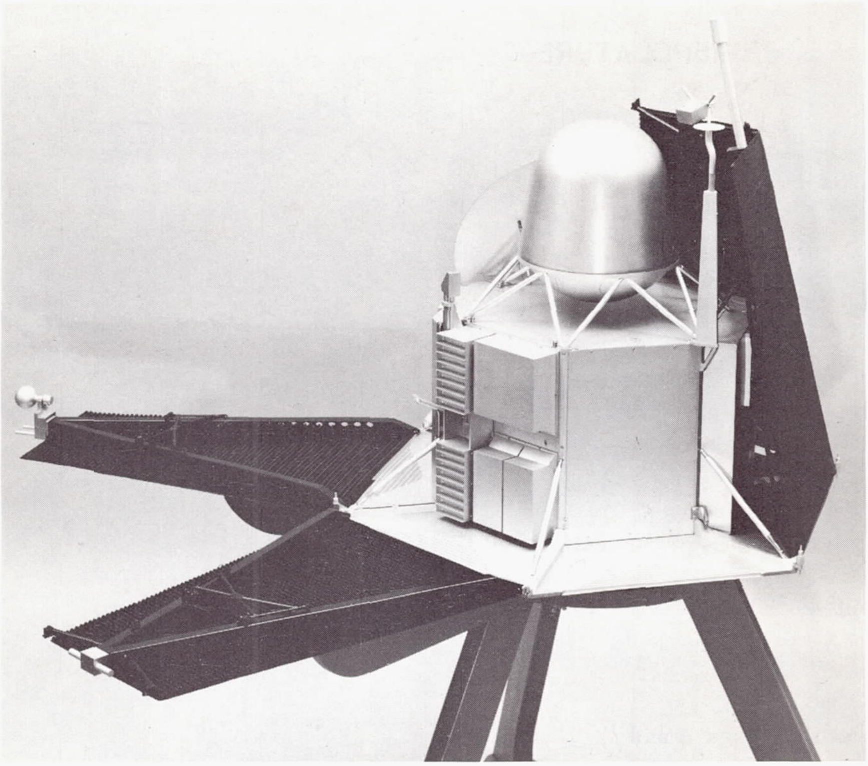


Fig. 1. Solar panel arrangement on spacecraft

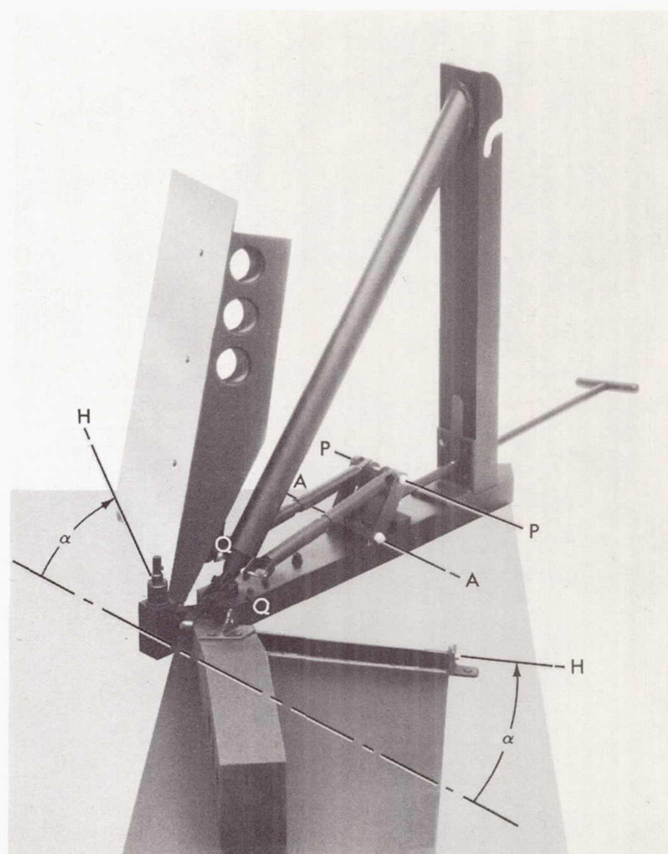
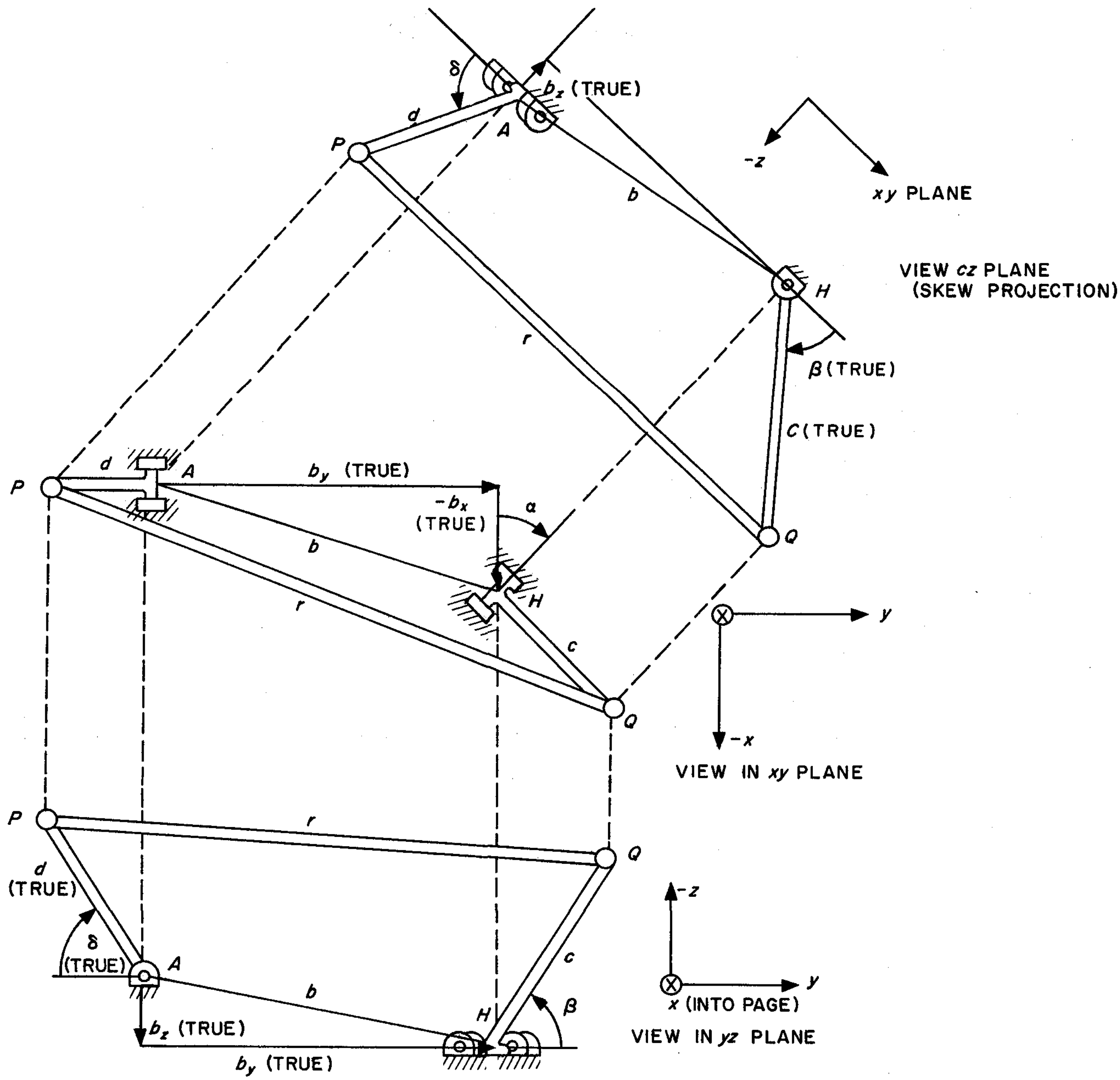


Fig. 2. Model of actuator assembly



NOTE: NO LENGTHS OR ANGLES IN THE ABOVE VIEWS ARE TRUE UNLESS THEY ARE SO INDICATED.

Fig. 3. Three-dimensional four-bar linkage

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