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THEORETICAL STUDY OF AUTOMATIC FLIGHT

CONTROL OF AIRCRAFT

By F. Haus Aug. 1956 360 p refs

Translation ~~of~~ from "Étude Théorique du Pilotage Automatique des Avions." Memoires de la Societé Royale Belge des Ingenieurs et des Industriels, Série B, No. 1, 1950. no. 1

2 ca Research Center, Langley Station

~~LANGLEY AERONAUTICAL LABORATORY~~

August 1956

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

THEORETICAL STUDY OF AUTOMATIC FLIGHT CONTROL OF AIRCRAFT*

By F. Haus

VOLUME I

INTRODUCTION

The purpose of the present report is the theoretical study of automatic flight control of aircraft.

The report starts with a presentation of definitions, conventions, and generalities; it is important to have them determined before broaching the subject. This presentation comprises the first four chapters.

Chapter V is devoted to a summary investigation of the effect of the controls on the conditions of equilibrium of an airplane for steady state.

Following, the present report takes up the classical theory of the motion of airplanes which is based on the linearized equations. In the summary of that theory which is contained in chapters VI to XI we attempted to show to what high degree employment of dimensionless coefficients systematizes the problem, and we have stressed the physical aspect of the phenomena which in the greater part of previous reports has been made only insufficiently clear.

We believe, besides, that the exposition we give here of the linearized theory, which is today classical, contains a certain number of new original points, notably the explanation of maneuvers of an actual pilot by means of the Duhamel integral, and the ensuing graphical construction (chapter XII).

Once the essential points of the theory of motion of airplanes are established, it becomes possible to go into the study of automatic pilots. The properties of airplanes provided with ideal equipment acting without inertia and following simple laws, are easily obtained by a simple generalization of the linear theory.

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*"Étude Théorique du Pilotage Automatique des Avions." Memoires de la Société Royale Belge des Ingénieurs et des Industriels, Série B, No. 1, 1950.

This investigation, presented in chapters XV and XVI, leads to valuable indications. It was made during the occupation, and the conclusions reached were not published. After the liberation we learned to our pleasure that similar work, resulting in analogous conclusions, had been carried out during the war in Great Britain and in the United States.

One of our general conclusions was that the properties of an automatic pilot for use with a prescribed airplane must depend on the characteristics of the motion which the airplane performs with controls fixed.

This idea now seems to be universally admitted; consequently, an importance it did not have formerly is attributed to the study of the relative motions of the airplane.

However, the "idealized" theory of automatic flight control constitutes only a first approximation and the indications it furnishes require extension.

Taking into account the inertia and frictions, one may set up the equations of motion of the automatic-flight-control instrument, and combine them with the equations of motion of the airplane.

However, setting-up the equations according to the conventional methods of mechanics leads to insurmountable complications; to continue the investigation, it is necessary to resort to methods analogous to those used in the field of electricity. The most efficient approach seems to be the study of the response of the system when it is subjected to harmonic or sinusoidal excitations.

The last chapters of this report serve to indicate the possibilities offered by this method of investigation, that is, the study of the frequency response.

The present investigation is, above all, of theoretical character. As regards the practical verifications, we are forced to refer to tests which have been carried out abroad and made the object of publications there.

An attempt made before the war (with highly valuable collaborations) to establish experimental methods permitting a kinematic analysis of the trajectories was stopped by the events and could not be taken up again. This attempt has shown that the means necessary for experimentation in flight considerably exceed everything which we should have been able to set up.

Information we acquired regarding the magnitude of the means used in research centers abroad entirely confirms this point of view.

Our contribution to the solution of the problem of automatic flight control is presently limited to theoretical work. Nevertheless we believe that the publication of the present investigation might attract the attention of organizations who have at their disposal the means for investigation in flight, taking advantage of theoretical studies related to the dynamics of flight.

In the course of discussions we held with French engineers on the problems which form the object of our study we have found that certain airplane designers shared our opinion regarding the usefulness of a simultaneous investigation of the airplane and of the apparatus for automatic flight control.

We wish to express here our gratitude to the officials of the "Research Center for the Mechanics of Flight" for regularly keeping us up-to-date on their work. We also thank the National Society of Aeronautic Constructions of the South-East and the Airplane Society Bréguet for the information their technicians were authorized to communicate to us.

TABLE OF THE PRINCIPAL NOTATIONS USED

A,B,C	without subscript: moments of inertia of an airplane
A	with subscript: coefficients of the characteristic nondimensional equation
C	with subscript: aerodynamic coefficient
D	diameter of the propeller
D	duration for decrease to half-value
G	weight of airplane
I	moment of inertia of a propeller
J	total acceleration
K	with subscript: coefficients of the characteristic equation in dimensional form
L,M,N	components of the aerodynamic moment
Q	engine torque
S	lifting surface

T	propeller thrust
T	period of an oscillation
V	velocity
W	power
X,Y,Z	components of an aerodynamic force
$\underline{L},\underline{M},\underline{N}$	control-surface hinge moments
a,b,c,d,e,h,k	with subscripts: aerodynamic coefficients
b	without subscript: wing span
c	without subscript: wing chord
c	with subscript: chord of a movable surface (control)
g	acceleration of gravity
h	distance
i	$\sqrt{-1}$
$\underline{i}$	intensity of a current
k	real part of the root x
l	distance from the tail unit to the center of gravity
ln	Napierian log
m	mass of airplane
n	velocity of engine rotation expressed in number of revolutions per unit time
p	symbolic variable
p,q,r	components of angular velocity
r	radius of gyration
s	semispan of wing
s	imaginary part of the root x

t	time
u,v,w	velocity components
x	root of characteristic equation, in dimensional form
x	general expression for an input signal
z	general expression for an output signal
α	angle of incidence
β	angle of sideslip
γ	index of propeller operation
δ	general expression for an increment
ϵ	angle of deflection of current
ϵ	angle of inclination between one direction with respect to another
ϵ	error or difference
φ,θ,ψ	angles determining the inclination of the airplane
λ	root of the characteristic equation in nondimensional form
κ	real part of the root λ
μ	density of airplane
$\tilde{\omega},\chi\ \rho$	reduced expressions of angular velocities
ρ	specific mass of air
σ	imaginary part of the root λ
σ	symbolic expression for the degree of throttling
τ	angle of inclination of the flight path
τ	aerodynamic time
ξ,η,ζ	deflection of the control surfaces

ω	angular velocity of engine
ω	frequency of an excitation
Ω	angular velocity of airplane

CHAPTER I

KINEMATICS OF THE AIRPLANE

1. Fixed Axes Referred to the Ground

Let a system of axes TX_0, TY_0, TZ_0 be prescribed, originating from a point T fixed to the ground.

The axis TZ_0 will be oriented vertically, positive upward.

The axes TX_0 and TY_0 may be chosen arbitrarily, with the one condition that the orthogonal system of axes should be right-hand rotational.

The orthogonal system of axes TX_0, Y_0, Z_0 constitutes a system of fixed axes, called in what follows, geodetic orthogonal system of axes.

2. Axes with Origin at the Center of Gravity of the Airplane

We shall utilize two systems of axes fixed to the airplane issuing from an origin O which coincides with the center of gravity of the airplane.

The first system is a system of axes OX_0, OY_0, OZ_0, restricted to remaining parallel to the axes of the geodetic orthogonal system of axes. This system constitutes the geoparallel orthogonal system of axes. Its origin is the only one which is involved in the translational motion of the airplane.

The second system is a system of axes OX, OY, OZ fixed to the airplane and involved at the same time in the motions of translation and of rotation of the machine.

This system constitutes the dynamic orthogonal system of axes.

Every airplane possesses a plane of symmetry. We shall agree to place the axes OX and OZ in the plane of symmetry, attaching the axis OX to a significant direction of the airplane, defining its logitudinal axis. By convention, we shall direct OX forward.

The orthogonal system of axes will be right-hand rotational, and the positive directions of the axes will be as follows:

Forward for the axis OX

Upward for the axis OZ

Toward the left for the axis OY.

It remains to select the significant direction along which we shall place the axis OX.

We may choose one of the following directions:

Direction of the propeller axis

Direction of the geometric chord of the wing (in its plane of symmetry)

Direction of the chord corresponding to zero lift

Direction of the central axis of inertia nearest to the directions designated above.

In fact, these directions form between them, only angles of a few degrees.

If we choose the direction of the chords, we facilitate the expression of aerodynamic forces along the axes, but we complicate the equations of motion.

If we choose the direction of the axis of inertia, we simplify the equations of motion but we impose upon ourselves the transformations necessary for referring the expressions of the aerodynamic forces to axes not fixed to the external forms of the airplane but to the distribution of masses in its interior.

Since our purpose is to study the equations of motion, we shall choose the second method.

The dynamic axes then coincide with axes of inertia.

The principal moments of inertia will be A, B, C.

All three products of inertia are zero:

D and F due to the existence of a plane of symmetry

E as a consequence of our choosing the direction OX.

Remark: The positive directions of the orthogonal system of axes are the ones generally used in investigations published in French language. Each of these axes is opposite to that of the orthogonal system of axes employed in English-speaking countries.

3. Position of the Airplane in Space

The position of an airplane in space will be defined:

(1) By the three coordinates of its center of gravity O, referred to the geodetic orthogonal system of axes TX_0, Y_0, Z_0

(2) By the orientation of the dynamic orthogonal system of axes O, X, Y, Z, referred to the geoparallel orthogonal system of axes OX_0, Y_0, Z_0.

This orientation will be defined by means of three rotations ψ, θ, φ to which one must subject the geoparallel orthogonal system of axes in order to transform it into the dynamic orthogonal system of axes.

The rotations we use are not those utilized by Euler. We shall actually accomplish:

(1) An amplitude rotation ψ about OZ_0, changing the axes OX_0 and OY_0 into OX' and OY' whereas the auxiliary axis OY' is, according to definition, the intersection of the plane X_0, OY_0 with the plane Z, O, Y

(2) An amplitude rotation θ about the axis OY', changing OZ_0 into OZ', and OX' into OZ

(3) An amplitude rotation φ around the axis OX, changing OY' into OY, and OZ' into OZ.

The positive sense of these rotations is:

$\psi > 0$ makes the airplane turn to the left

$\theta > 0$ causes nose down

$\varphi > 0$ causes inclination to the right.

The direction cosines of the three axes OX, OY, OZ with respect to the orthogonal system of axes OX_0, Y_0, Z_0 are given by the table.

	OX	OY	OZ
OX_0	$\cos\theta \cos\psi$	$-\cos\varphi \sin\psi + \sin\varphi \sin\theta \cos\psi$	$\sin\varphi \sin\psi + \cos\varphi \sin\theta \cos\psi$
OY_0	$\cos\theta \sin\psi$	$\cos\varphi \cos\psi + \sin\varphi \sin\theta \sin\psi$	$-\sin\varphi \cos\psi + \cos\varphi \sin\theta \sin\psi$
OZ_0	$-\sin\theta$	$\sin\varphi \cos\theta$	$\cos\varphi \cos\theta$

Knowledge of the direction cosines permits an easy setting up of the transformation formulas that might be required.

Let us note that the three angles ψ, θ, φ are fixed to the parameters customarily used for characterizing the position of an airplane.

If $\theta = \varphi = 0$ ψ defines the azimuth.

If $\varphi = 0$ θ defines the trim.

If $\theta = 0$ φ defines the lateral inclination.

4. Motion of the Airplane

The motion of the airplane is, at every instant, determined:

(1) By the velocity V of its center of gravity

(2) By the angular velocity Ω about an axis of rotation going through its center of gravity.

If the atmosphere in which the airplane flies is stationary, that is, is not in motion due to air currents, V expresses at the same time the velocity with respect to the ground and the velocity with respect to the surrounding medium.

A motionless observer, that is, fixed to the ground, will define the motion by the projections V and Ω on the system of fixed axes attached to the ground.

An observer placed in the airplane will define the motion by the projections V and Ω on the dynamic axes.

Assume:

u, v, w to be the projections of V on the dynamic axes.

p, q, r to be the projections of Ω on the dynamic axes.

One has necessarily:

$$V^2 = u^2 + v^2 + w^2$$

$$\Omega^2 = p^2 + q^2 + r^2$$

These six projections define actually the motion of the dynamic orthogonal system of axes referred to itself, that is, with respect to the instantaneous position it occupies. Since it is being continually displaced, the motion of the airplane in space is not always described in a convenient manner by these six projections.

However, the problem we are dealing with is, in fact, the study of the motion as perceived by the pilot, not that of the motion as seen by an observer on the ground.

Thus we shall always refer to the motion of the dynamic orthogonal system of axes, and shall continuously make use of the projections u, v, w and p, q, r.

5. Relations Between the Angular Velocities and the Attitude Angles

The three components p, q, r of the angular velocity correspond to motions determined as follows:

The component p, about the axis OX, constitutes the motion of rolling.

The component q, about the axis OY, constitutes the motion of pitching.

The component r, about the axis OZ, constitutes the motion of yawing.

The resultant angular velocity may be defined either by its components p, q, r about the axes OX, OY, and OZ, or by the three components:

$$\frac{d\varphi}{dt} \text{ around the axis } OX$$

$$\frac{d\theta}{dt} \text{ around the axis } OY'$$

$$\frac{d\psi}{dt} \text{ around the axis } OZ_0$$

Projecting on the axes OX, OY, OZ a vector the components of which along the axes OX, OY' and OZ_0 are $\frac{d\varphi}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, one obtains:

$$p = \frac{d\varphi}{dt} - \frac{d\psi}{dt} \sin \theta$$

$$q = \frac{d\theta}{dt} \cos \varphi + \frac{d\psi}{dt} \cos \theta \sin \varphi$$

$$r = \frac{d\psi}{dt} \cos \theta \cos \varphi - \frac{d\theta}{dt} \sin \varphi$$

These purely geometrical relations occur in the equations defining the motion.

Written so as to express explicitly $\frac{d\varphi}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, they become:

$$\frac{d\varphi}{dt} = p + \frac{\sin \theta}{\cos \theta}(q \sin \varphi + r \cos \varphi)$$

$$\frac{d\theta}{dt} = q \cos \varphi - r \sin \varphi$$

$$\frac{d\psi}{dt} = \frac{1}{\cos \theta}(q \sin \varphi + r \cos \varphi)$$

Remark: If $\varphi = 0$ (wings horizontal), one obtains:

$$q = \frac{d\theta}{dt} \qquad r = \frac{d\psi}{dt} \cos \theta$$

Pitching modifies the longitudinal inclination of the airplane, the motion of yawing displaces the airplane in azimuth.

The relation between the motions of the airplane relative to the air, q and r, and its displacement with respect to the ground is orthogonal.

When $\varphi = \pi/2$, that is, when the wings are vertical:

$$q = \frac{d\psi}{dt} \cos \theta \qquad r = - \frac{d\theta}{dt}$$

The usual relation between the motions of the airplane relative to the air and its displacement with respect to the horizon is reversed; the pitching modifies the azimuth of the airplane whereas the yawing motion modifies the longitudinal inclination.

6. Equivalent Representation

Utilization of the six components

$$u,v,w \ \left(\text{dimensions} \quad LT^{-1}\right)$$
$$p,q,r \ \left(\text{dimensions} \quad T^{-1}\right)$$

presents difficulties sometimes. It can be useful to employ only one single dimensional characteristic, the velocity, and to define the five other elements of the motion by dimensionless parameters.

For this purpose, one may define the motion with respect to the dynamic orthogonal system of axes as follows:

(1) Instead of using the three projections of the vector V, one characterizes the translational motion by the numerical value of the resultant V, and the orientation of this vector with respect to the dynamic orthogonal system of axes. This orientation will be characterized by two angles; the angles of attack and of sideslip the exact definition of which is given in the following section.

(2) Instead of using the components p, q, r of the angular velocity, it is of advantage to use the three dimensionless quantities called "rotational velocity ratios."[1]

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[1]ρ is used also to designate the specific mass of the air. Nevertheless, we think it possible to use ρ to denote two essentially different quantities since no confusion whatsoever could arise.

$$\tilde{\omega} = \frac{ps}{V} \qquad \chi = \frac{ql}{V} \qquad \rho = \frac{rs}{V}$$

In these expressions s represents a transverse dimension, l a longitudinal dimension of the airplane.

The rotational velocity ratios then express the relationship that exists between the linear velocities of a point at a prescribed distance s or l from the center of gravity due to the rotation considered, and the velocity V of the center of gravity.

As a matter of fact, we will take:

s = semispan of the airplane.

l = lever arm of the horizontal tail surfaces, that is, the distance separating the centroid of the tail plane from the center of gravity of the airplane.

7. Axes Dependent on the Flight Path and on the Airplane

Assume an axis Ox coinciding with the velocity, an axis Oz, defined by the intersection of the plane originating from O and perpendicular to Ox with the plane of symmetry ZOX - this axis Oz will be directed upward - and an axis Oy, perpendicular to the preceding ones and directed toward the left.

These axes define an orthogonal system of axes, called aerodynamic orthogonal system. Let V_s be the projection of the velocity on the plane of symmetry.

The angle measured in the plane of symmetry and comprised between the directions V_s and Ox is called the angle of attack α.

The angle measured in the plane V_sOV and comprised between the directions V_s and V is the angle of sideslip β.

According to definition:

The angle of attack α will be positive if V_s is directed below OX.

The angle of sideslip β is positive if V is directed at the left of OY.

One has, under these conditions:

$$u = V \cos \alpha \cos \beta$$

$$v = V \sin \beta$$

$$w = -V \sin \alpha \cos \beta$$

and knowledge of the velocity V and the two angles of α and β is quite equivalent to knowledge of the three projections.

Two angular quantities, the angles α and β are sufficient for a complete determination of the respective position of the aerodynamic and dynamic orthogonal systems of axes since, according to the convention for determining Oz, the two orthogonal systems are not completely independent of each other.

Let us note that $-\frac{w}{u} = \tan \alpha$ and $\frac{v}{u} = \tan \beta \cos \alpha$

The aerodynamic orthogonal system is frequently used. In wind-tunnel tests, the aerodynamic reactions are always determined by means of balances which measure the components in directions invariably fixed to the airstream.

These directions are usually:

The direction $-Ox$; that is, that of the airstream

The direction Oz.

The direction Oy.

It is frequently necessary to refer to the dynamic axes forces which are defined by their components along the aerodynamic axes.

The transformation formulas can be immediately set up as soon as the direction cosines of one system of axes with respect to the other one are known.

These direction cosines of the three axes Ox, Oy, Oz with respect to the dynamic orthogonal system of axes $OXYZ$ are given by the table.

	Ox	Oy	Oz
OX	$\cos \alpha \cos \beta$	$-\cos \alpha \sin \beta$	$\sin \alpha$
OY	$\sin \beta$	$\cos \beta$	0
OZ	$-\sin \alpha \cos \beta$	$-\sin \alpha \sin \beta$	$\cos \alpha$

CHAPTER II

DYNAMICS OF THE AIRPLANE

1. Forces and Moments

The external actions affecting the airplane are:

(a) The force of gravity applied at the center of gravity

(b) The aerodynamic forces and moments exerted on the airplane

(c) The forces exerted by the propeller and transmitted to the airplane by the motor mounts

These actions admit the following components, along or around the dynamic axes.

Weight.- The projections of the weight G along the three dynamic axes are the projections of a continually vertical force. One has therefore:

$$G_x = G \sin \theta$$

$$G_y = -G \sin \varphi \cos \theta$$

$$G_z = -G \cos \theta \cos \varphi$$

The projections of the force of gravity depend therefore on the angles θ and φ.

Aerodynamic actions exerted on the airframe.- The aerodynamic actions exerted by the airframe consist in a resultant F, applied at the center of gravity, and a moment C.

We shall call X, Y, Z the components of F along the three dynamic axes, and L, M, N the components of C around the three dynamic axes.

The positive directions are necessarily those of the forces acting along the positive direction of the axes, and those of the moments tending to produce positive rotations p, q, r.

Effects exerted by the propeller.- The forces transmitted to the airplane by the motor mounts comprise:

(a) The aerodynamic reactions exerted by the surrounding medium on the propeller

(b) The internal forces of the power plant, such as gyroscopic moments. These last may be considered as external actions as far as their effect on the motion of the airplane is concerned.

The reactions comprised in (a) are the thrust T of the propeller and a torque Q acting around that axis.

The thrust acts precisely along the propeller axis only when the velocity V coincides with the propeller axis. It may show components perpendicular to the axis if the forward speed forms an appreciable angle with the axis.

On the other hand, the propeller axis is not necessarily parallel to one of the axes of the airplane. Under these conditions, the thrust possesses, generally, three components T_x, T_y, T_z along the axes fixed to the airplane.

If the straight line along which the thrust is acting does not pass through the center of gravity, the thrust exerts a moment the components of which around the three axes will be called L_h, M_h, N_h.

The moment Q is equal to the engine torque.[2] It depends therefore on the throttle setting selected by the pilot. It possesses, as a rule, three components: Q_x, Q_y, Q_z.

The propeller exerts a gyrostatic moment which, for certain maneuvers, is not negligible.

Let I be the moment of inertia of the gyrostat (propeller), ω its angular velocity.

If the gyrostat is driven by a forced rotation Ω, it exerts a moment of reaction $I\omega\Omega$ the components of which are

$$L_g = I(\omega_y r - \omega_z q)$$

$$M_g = I(\omega_z p - \omega_x r)$$

$$N_g = I(\omega_x q - \omega_y p)$$

[2]When the engine is not geared down. If there is a reduction gear, the gear ratio must be taken into account.

calling ω_x, ω_y, ω_z the projections of the vector ω on the dynamic axes. This general expression may be simplified, however.

Sum of the external actions.- Let us call ΣX, ΣY, ΣZ the sums of the projections of the various external forces along the three axes, and ΣL, ΣM, ΣN the sums of the projections of the external moments acting around these axes.

One then has, as a rule:

$$\Sigma X = X + T_x + G \sin \theta$$

$$\Sigma Y = Y + T_y - G \cos \theta \sin \varphi$$

$$\Sigma Z = Z + T_z - G \cos \theta \cos \varphi$$

$$\Sigma L = L + L_h + Q_x + L_g$$

$$\Sigma M = M + M_h + Q_y + M_g$$

$$\Sigma N = N + N_h + Q_z + N_g$$

Certain terms may be neglected, however. The axis of the propellers is, in fact, parallel to the plane of symmetry so that one may put:

$$T_y = 0 \qquad Q_y = 0$$

$$L_h = 0 \qquad \omega_y = 0$$

When the airplane is symmetrical, one has $N_h = 0$; however, there exists one important case: that of a multiengined airplane flying with one outboard engine stopped where one has

$$N_h \neq 0$$

One can approximate ω_x with ω and neglect ω_y and ω_z. Hence the gyroscopic moment possesses only two components:

$$M_g = -I\omega r$$

$$N_g = I\omega q$$

2. Factors at the Disposal of the Pilot

The motion of an airplane may be studied in the following cases:

(a) Nonpiloted aircraft, flying with controls fixed

(b) Nonpiloted aircraft, flying with free controls (if the latter are reversible)

(c) Piloted aircraft, with the controls being manipulated according to a certain law, either by the pilot or by a mechanical device called automatic pilot.

This will lead us to an analysis of the means the pilot has at his disposal for influencing the motion of the airplane.

We should like to remark right now that the second case is a particular case of the third: the displacement of the controls then is the one which occurs in the course of the maneuvers of the airplane if the force applied to them is zero.

3. Equation of Motion of an Airplane Flying with Controls Fixed

The motion of an airplane is determined as a function of the external actions, by the six fundamental equations of dynamics.

Assume m to be the mass of the airplane; A, B, C its principal moments of inertia.

By virtue of the selection of axes, the product of inertia: $E = 0$. Due to the symmetry: $D = F = 0$.

Referring the motion to the axes fixed to the airplane, the equations of motion are written:

$$m\left(\frac{du}{dt} + qw - rv\right) = \Sigma X$$

$$m\left(\frac{dv}{dt} + ru - pw\right) = \Sigma Y$$

$$m\left(\frac{dw}{dt} + pv - qu\right) = \Sigma Z$$

$$A \frac{dp}{dt} + qr(C - B) = \Sigma L$$

$$B \frac{dq}{dt} + rp(A - C) = \Sigma M$$

$$C \frac{dr}{dt} + pq(B - A) = \Sigma N$$

where m represents the mass of the airplane; A, B, C its three moments of inertia.

Since the projections of weight are functions of φ and θ, the preceding six equations constitute a system connecting the linear velocities u, v, w, the angular velocities p, q, r, and the angles of orientation φ and θ with the independent variable t.

We must therefore complete the system by means of equations which connect the angles with speeds of rotation. We have at our disposal the relations:

$$\frac{d\varphi}{dt} = p + \frac{\sin \theta}{\cos \theta}(p \sin \varphi + r \cos \varphi)$$

$$\frac{d\theta}{dt} = q \cos \varphi - r \sin \varphi$$

$$\frac{d\psi}{dt} = \frac{1}{\cos \theta}(q \sin \varphi + r \cos \varphi)$$

The two first ones are sufficient for completing the system. If one wants, moreover, the azimuth, one must utilize the third equation which then introduces the variable ψ.

The motion is thus characterized by nine equations with nine dependent variables.

The instantaneous values of the external actions must be introduced in these equations, and we shall have to investigate to what extent these actions are known as functions of:

(a) The instantaneous values of the variables u, v, w, p, q, r (or V, α, β, $\tilde{\omega}$, χ, ρ) which define the motion of the airplane

(b) Their derivatives

(c) In certain cases, of the previous history of the motion of the airplane.

One difficulty arises in connection with the power plant: With A the constant throttle setting, we may assume the engine torque Q_m to be constant, but the resistant propeller torque depends on its speed of rotation ω and on the forward speed of the machine:

$$Q = f(\omega,V)$$

The speed of rotation of the power plant is therefore connected with the translational velocity of the airplane by a relation:

$$I \frac{d\omega}{dt} = Q_m - f(\omega,V)$$

where I designates the moment of inertia of the propeller.

If one wants to proceed rigorously, one has to add to the system a new variable, the rotational speed of the propeller, and also a new equation, and must then combine the study of the motion of the airplane in space with the study of the rotational motion of the power plant.

This mode of procedure would increase the complexity of the system still more. We point it out only for the record. The system of the nine equations must, on the contrary, be simplified in order to lead to practical conclusions.

We shall make use of artifices which permit avoiding the introduction of variations from the power plant regime into the equations of motion of the airplane.

4. Separation of the Equations

The investigation of the motion of airplanes will be facilitated, in numerous cases, by the possibility of splitting up the system of equations into two systems independent of one another which define the longitudinal and the transverse motion, respectively.

The longitudinal motion contains the displacements along the axes OX and OZ, and the rotations around the axis OY; it corresponds to the equations:

$$m\left(\frac{du}{dt} + pw - rv\right) = X + T_x + P \sin\theta$$

$$m\left(\frac{dw}{dt} + pv - qu\right) = Z + T_z - P \cos\theta \cos\varphi$$

$$B\frac{dq}{dt} = M + M_h + M_g$$

$$\frac{d\theta}{dt} = q$$

This motion is considered to take place with constant values of v, p, r.

The external effects are functions of the above constant values and of the magnitude of the variable quantities u, w, q, θ (equivalent, as we have seen, to V, α, X, and θ).

The transverse motion comprises the displacements along the axis OY and the rotations around the axes OX and OZ. It corresponds to the equations:

$$m\left(\frac{dv}{dt} + ru - pw\right) = Y - P \cos\theta \sin\varphi$$

$$A\frac{dp}{dt} + qr(C - B) = L + Q_x$$

$$C\frac{dr}{dt} + pq(B - A) = N + N_h + N_g + Q_z$$

$$\frac{d\varphi}{dt} = p + (r \cos\varphi + q \sin\varphi)\frac{\sin\theta}{\cos\theta}$$

$$\frac{d\psi}{dt} = r\frac{\cos\varphi}{\cos\theta} + q\frac{\sin\varphi}{\cos\theta}$$

It is considered to take place with constant values of u, w, q, θ.

The external effects are functions of the above constant values and of the magnitude of the variable quantities v, p, r, φ, and ψ (equivalent to β, $\tilde{\omega}$, ρ, φ, and ψ).

5. The Accelerations

Let us note that the three equations of translation may be written:

$$m\left[\frac{du}{dt} + (qw - g \sin \theta)\right] = T_x + X$$

$$m\left[\frac{dv}{dt} + (ru - pw) + g \sin \varphi \cos \theta\right] = Y$$

$$m\left[\frac{dw}{dt} + (pv - qu) + g \cos \theta \cos \varphi\right] = T_z + Z$$

If we represent the quantities between brackets by J_x, J_y, J_z, we obtain:

$$mJ_x = T_x + X$$

$$mJ_y = Y$$

$$mJ_z = T_z + Z$$

The quantities J_x, J_y, J_z are the sum:

Of the linear accelerations

Of the centripetal accelerations

Of the projection of the acceleration of the force of gravity.

They may be considered as constituting the components of the total acceleration and are hence in close relation with the sensations experienced by the pilot and the passengers.

The quantities J_x, J_y, J_z may receive another physical interpretation. Every one among them is equal to the effect which the apparent gravity exerts along the negative direction of the corresponding axis.

6. Equations of Motion of an Airplane When the Controls Are Manipulated

In these cases the preceding systems of equations must be completed by the relations giving the deflection of the control surfaces as a function of time; the external actions must be expressed as functions of the deflections of the control surfaces.

These questions will be developed further on.

CHAPTER III

THE EXTERNAL ACTIONS

1. Utilization of Dimensionless Factors

We take up again the enumeration we made in the first section of the preceding chapter regarding the various external actions.

In the course of this chapter we shall describe the means utilized for investigating, defining and predicting the magnitude of the external actions.

The necessity of making comparisons between airplanes showing considerable differences in weight, dimensions, and speed, has led to defining all external effects acting upon an airplane by dimensionless factors.

We shall set up here the indispensable definitions.

2. Density of the Airplane

In the investigation of the motion the mass of the airplane will be replaced to advantage by the dimensionless relation:

$$\mu = \frac{2m}{\rho Sc}$$

called the density of the airplane.

In this expression:

ρ denotes the specific mass of the surrounding air

S the lifting-surface area

c a characteristic dimension, the wing chord

The density of the airplane is the ratio between the mass of the airplane and the mass of an air volume equal to half the product of its lifting-surface area and the length of the chord.

It results from this definition that the density of the airplane varies with the altitude.

3. Projection of the Aerodynamic Actions on the Axes Fixed to the Airplane

The components X, Y, Z, L, M, N will be expressed as functions of:

The lifting-surface area S

The mean chord c of that surface

The span b

The dynamic pressure $\rho V^2/2$

Six dimensionless factors.

We shall put:

$$X = C_X S \frac{\rho V^2}{2} \qquad L = C_L Sb \frac{\rho V^2}{2}$$

$$Y = C_Y S \frac{\rho V^2}{2} \qquad M = C_M Sc \frac{\rho V^2}{2}$$

$$Z = C_Z S \frac{\rho V^2}{2} \qquad N = C_N Sb \frac{\rho V^2}{2}$$

The factors C_X, C_Y, C_Z, C_L, C_M, C_N are ordinarily called coefficients. The nomenclature "coefficient" seems to indicate that these factors are constant. This is not true, however; these factors are essentially variable and their value depends on a great number of variables.

4. Projections of the External Actions on the Axes Referred to the Relative Wind

It is clear that one can refer the external actions also to the axes Ox, Oy, Oz constituting the aerodynamic orthogonal system of axes.

We shall call:

Resistance Fx or drag, the projection on the direction of the relative wind, that is, along the negative direction of Ox.

Transverse force Fy, the projection on the axis Oy.

Lift Fz, the projection on the axis Oz.

These components may be defined by factors analogous to the preceding ones, generally denoted in the technical literature in French language by C_x, C_y, C_z, with the subscripts written as lower-case letters.

The components of the moment around the three axes Ox, Oy, Oz are defined by corresponding factors which one may write C_l, C_m, C_n.

5. Actions Exerted by the Propeller

Any propeller is defined by its exterior form. Among the parameters on which this form depends, there is the angle at which the blades are set, which determines the pitch.

With the exterior form (and consequently the pitch) fixed, it will be possible to determine uniquely the thrust T and the moment Q as a function of the peripheral velocity $\omega D/2$ of the propeller and of the velocity V of the airplane, or as a function of one of these quantities and of their ratio.

We shall put:

$$\gamma = \frac{V}{nD}$$

where n denotes the rotational speed, expressed in revolutions per second, and D the diameter of the propeller.

The ratio of the velocity of the airplane and the peripheral velocity of the propeller is indicated, except for the factor $1/\pi$, by this characteristic γ:

$$\frac{2V}{\omega D} = - \frac{2V}{2\pi nD} = \frac{1}{\pi} \gamma$$

In practice, the thrust T and the moment Q are expressed, as functions of one of the two velocities, by dimensionless factors the numerical values of which are functions of the advance ratio γ.

One may utilize one or the other of the two series of factors K or C.

If one expresses T and Q as functions of the rotational speed, one has:

$$T = K_T \rho n^2 D^4$$

$$Q = K_Q \rho n^2 D^5$$

If one expresses these quantities as functions of the velocity of the airplane, it will be of advantage to use as ratios the dynamic pressure and the surface of the disk swept by the propeller. One then obtains:

$$T = C_T \frac{\rho V^2}{2} \frac{\pi D^2}{4}$$

$$Q = C_Q \frac{\rho V^2}{2} \frac{\pi D^2}{4} D$$

For a propeller of prescribed form the factors $K_T K_Q$ and $C_T C_Q$ are uniquely functions of γ.

The factors C_T and C_Q express the action of the propeller in a manner analogous to the one utilized for denoting the force exerted on the airframe; they show the inconvenience of being represented as functions of γ by curves which possess, for small values of γ, an infinite branch.

The following relation exists between the factors C and the factors K:

$$C = K \frac{8}{\pi \gamma^2}$$

The moments M_h and N_h exerted by the thrust of the propeller if its axis does not pass through the center of gravity will be expressed as functions of T and of the corresponding linear dimension. One has, for instance:

$$M_h = Th = C_T h S_H \rho V^2/2$$

with S_H denoting the surface of the disk swept by the propeller.

Remarks: (1) It may be useful to complete the notations, showing, how one can express the power absorbed by the propeller.

One has necessarily:

$$W = Q\omega$$

Hence:

$$W = K_W \rho n^3 D^5$$

or

$$W = C_W S_H \rho V^3/2$$

with, according to definition:

$$K_W = 2\pi K_Q$$

$$C_W = K_W \frac{8}{\pi\gamma^3}$$

(2) The effective power is equal to the absorbed power multiplied by the efficiency η of the propeller. Thus it will be possible to define the effective power by the factors C_{wu} or K_{wu}, uniquely functions of γ and analogous to the previous ones.

(3) The factors C_T, C_Q, C_W or K_T, K_Q, K_W characterize a given propeller and are functions of the advance ratio γ.

If the angle at which the blades are set is variable, there exist as many relations (or curves) K_T, K_Q, K_W or C_T, C_Q, C_W as functions of γ as there are different blade-angle settings.

The described propellers of variable pitch are therefore characterized by a family of these curves.

6. Variables Defining the Steady-State External Forces

The forces exerted by the surrounding medium on an airplane in motion with controls fixed are functions of a large number of variables.

Having agreed to represent the forces by dimensionless factors, one must now necessarily determine on what variables these factors depend, and how they depend on them.

The conventional hypothesis of similitude consists in admitting that the factors C_X . . . C_N are independent:

(a) Of the physical characteristics of the atmosphere

(b) Of the aerodynamic velocity

(c) Of the absolute dimensions of the aircraft.

The external forces are then determined for steady-state conditions, by factors which are uniquely functions:

Of the angles of attack and of sideslip α and β

Of the rotation ratio $\tilde{\omega}$, χ, ρ, and of the ratio γ which determines the effect of the slipstream of the propeller on the aircraft.

Actual experiences show that the aerodynamic forces depend in addition on a certain number of other variables.

Certain characteristics of the surrounding medium - viscosity, turbulence - exert a considerable effect on the aerodynamic forces in steady state.

In setting up the reactions an element dependent on the airplane, the roughness of the surface in contact with the atmosphere, also plays a role which must not be neglected.

In the present report we shall make the assumption that it is always possible to determine the aerodynamic reactions in steady state either by means of laboratory tests or by means of theoretical or empirical calculations. This implies that when our knowledge is based on laboratory tests, carried out on scaled-down models under conditions of viscosity, turbulence, and roughness different from those existing for full-scale models, the development of the factors C_X . . . C_N, as functions of these characteristics, is supposed to be known.

The determination of the aerodynamic forces in steady state is completely outside of the scope of the present report, and we shall discuss the test methods only when this will be useful in making the mechanical significance of one or the other characteristic understood.

Besides, the experimental possibilities of investigation are not the same if the state of motion consists of a pure translation or of a translation accompanied by a rotation.

7. Tests in Pure Translation

The tests corresponding to motions comprising only translation can be carried out in wind tunnels. We shall here not enlarge upon test technique.

The number of wind tunnels throughout the world is such that for any new aircraft project, one can arrive at an experimental determination of the factors $C_X \ldots C_N$ for the principal steady states of translation, and for different configurations of the airplane.[3]

Wind tunnels are generally equipped for measurement of the forces along the trihedron Oxyz (taking into account the remark made before regarding the positive sense of the drags).

If one wants to introduce experimental results into the foregoing equations of motion, it is necessary to make use of the transformation formulas which permit passing from one system to another.

We have given before the table of the direction cosines.

It is well to remark also that the direction used most which serves as reference for the definition of the angles of attack is not the same in the wind tunnel and for the actual aircraft. Since the angle of attack in the wind tunnel is referred to a chord fixed to the profile, one has:

$$\alpha = \alpha_s + \epsilon$$

when the axis of inertia is raised with respect to the said chord.

When Cx and Cz are defined as functions of this angle of attack in the wind tunnel, the usual case, one obtains for example for zero sideslip:

$$C_X = -Cx \cos(\alpha_s + \epsilon) + Cz \sin(\alpha_s + \epsilon)$$

$$C_Z = -Cx \sin(\alpha_s + \epsilon) + Cz \sin(\alpha_s + \epsilon)$$

The tests yield serviceable results only when the models are provided with electric motors driving the propellers at speeds determined for each test by the conditions of similitude (equality of the values of γ).

This leads us to say a few words regarding the moments M and $M_h = T\,h$.

If M is the longitudinal moment of the aerodynamic forces exerted on the airframe:

$M_h = T\,h$ is the moment exerted about the center of gravity by the thrust of the propeller, and constitutes the direct effect of the propeller on the longitudinal moment.

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[3]One calls configuration of the airplane the external form corresponding to a prescribed position of the movable elements which will be discussed in the following chapter.

In fact, the propeller modifies by its slipstream the velocity and the direction of the airstream striking the horizontal tail surfaces and certain parts of the wing. It acts upon the aerodynamic reactions and exerts an indirect effect on the moment M.

It is almost impossible to isolate the direct effect from the indirect effect. The latter is frequently of opposite sense, and one is inevitably led to visualize a total moment:

$$M_t = M + Th$$

defined by a total-moment coefficient:

$$C_{M_t} = C_M + C_T \frac{S_H h}{Sc}$$

It is easy to determine in the wind tunnel the C_{M_t} for all combinations of angle of attack α and advance ratio γ (for a given blade-angle setting) so that the C_{M_t} can be characterized by an experimental diagram the shape of which is represented in figure 12 or in figure 13.

The following considerations permit an interpretation of these diagrams.

Any change in the magnitude of γ modifies the thrust exerted and alters the slipstream.

At small values of γ, the propeller operates in the neighborhood of static thrust. The thrust developed is large, and the ratio of the slipstream velocity of the propeller to the aerodynamic velocity is maximum.

Both the thrust exerted and the relative magnitude of the slipstream decrease when γ increases.

For a certain value of γ the propeller does not exert any thrust, and the propeller slipstream does not exert any influence on this moment. The figures are plotted under the hypothesis that the thrust becomes zero for $\gamma = 1.7$.

If $\frac{dC_{M_t}}{d\gamma} > 0$ (fig. 12), everything takes place as if the preponderant effect were the direct effect exerted by a propeller the axis of which passes below the center of gravity.

If $\frac{dC_{M_t}}{d\gamma} < 0$ (fig. 13), the sense of the phenomenon is that exerted by the direct effect of a propeller the thrust axis of which passes above the center of gravity.

In view of the prescribed fact that C_{M_t} is a function of α and γ, and that in the course of maneuver which changes the angle of attack the velocity of the airplane generally varies, one has necessarily:

$$\frac{dC_{M_t}}{d\alpha} = \frac{\partial C_{M_t}}{\partial \alpha} + \frac{\partial C_{M_t}}{\partial \gamma} \frac{d\gamma}{d\alpha}$$

where the derivative $d\gamma/d\alpha$ has to take into account the manner in which the advance ratio of the propeller varies in the course of the maneuver considered.

8. Tests in Translation Accompanied by Rotation

The tests reproducing steady-state conditions which comprise at the same time a translation and a rotation can be performed in the wind tunnel if the radius of rotation is small (spin), as whirling-arm tests if the radius is large.

Actually, only the second case is of interest to us.

The aerodynamic whirling-arm test is a means of investigation utilized at the beginning of aviation which had, however, practically disappeared toward 1925-1930. It has been taken up again these last years, and a modern whirling-arm apparatus has been constructed at the N.P.L. at Teddington.

We shall describe a possible experiment which shows the effect of a continuous and constant rotation χ upon the factor C_M.

Assume a model, the moment coefficient C_M of which has been measured, for given angle of attack α and control-surface deflection η, in the course of a tunnel test where the relative motion contains only translation.

The same model is placed at the extreme end of a whirling arm of the length R, with the model axis OY being parallel to the axis of the whirling arm.

The rotational speed of the latter is Ω. Hence, the velocity V of the model is ΩR, its angular velocity $q = \Omega$, and the ratio:

$$\chi = \frac{ql}{V} = \frac{l}{R}$$

The aircraft is placed on the arm, in such a manner that it presents itself at the angle of attack α. Dynamometers which permit measurement of the moment C_M are placed on the arm.

This experiment leads to the statement that the moment C_M, realized on the whirling arm with a rotation ratio χ is different from the one found in simple translation, for the same angle of attack α and the same control-surface deflection η.

The cause of this difference is easily found. Even though the angle of attack of the wings is the same in the two tests, this does not hold true for the angle of attack of the tail surfaces.

Let α be the angle of attack of the tail surfaces during the translation test.

In the whirling-arm test, this angle is altered by the effect of the angular velocity and becomes $\alpha' - \chi$.

The rotation $q = \Omega$ actually subjects all points of the airplane to complementary velocities:

$$\Delta u = zq \qquad \Delta w = -xq$$

The distance x of the tail surfaces is negative and equal to $a - l$; the incremental velocity is equal to ql; it gives rise, by combination with the translation V, to an incremental angle of attack:

$$-\frac{\Delta w}{V} = -\frac{ql}{V} = -\chi$$

There results an incremental reaction on the tail surfaces:

$$\Delta Z' = -S'\rho V^2/2 \, \frac{\partial C'_z}{\partial \alpha'} \, \chi$$

which in turn produces an incremental moment:

$$\Delta M = l\Delta Z'$$

One obtains finally:

$$\frac{dC_M}{d\chi} = -\frac{S'l}{Sc}\,\frac{dC'_z}{d\alpha'}$$

which determines the order of magnitude of the effect of the rotation on the C_M.

For instance, for $S' = \frac{1}{6} S$

$$l = 3c$$

since the gradient of the lift coefficient of the tail surfaces with reference to the actual angle of attack of the tail surfaces is

$$\frac{dC'_z}{d\alpha'} = 3$$

one obtains

$$\frac{dC_M}{dX} = -\frac{9}{6} = -1.5$$

The minus sign indicates that one deals here with a damping effect: a moment acting in a sense opposite to that of the rotation.

The other parts of the airplane - fuselage, wings - exert, for their part, an effect which contributes to increase this damping. Although it is not quite as easy to roughly evaluate this effect, one may say that, with a normal machine, the dC_M/dX due to these elements is of the order of -0.20 to -0.15.

One must add this effect to that of the tail surfaces.

In steady state, the development of the C_N could be determined as a function of the ratio ρ, with the aid of a whirling-arm test where the axis OZ of the model is placed parallel to the axis of rotation of the whirling arm. This simple remark is sufficient for explaining the significance of the derivative $\partial C_N/\partial\rho$ which defines the damping of the motions of yaw.

9. Aerodynamic Derivatives

Since the forces and moments realized in steady state are continuous functions of the variables u, v, w, p, q, r, they possess derivatives. We shall have to use the latter constantly in calculations later on and it will be convenient to discuss them right away.

In the investigation of logitudinal motion, we shall encounter derivatives such as $X'_u \ldots M'_q$ which we shall define by dimensionless factors $a'_1 \ldots c_3$ in accordance with the table:

$$X'_u = -a'_1 SV \frac{\rho}{2} \quad \text{whence} \quad a'_1 = -\frac{2X'_u}{\rho SV}$$

$$X'_w = -b_1 SV \frac{\rho}{2} \qquad b_1 = -\frac{2X'_w}{\rho SV}$$

$$X'_q = -c_1 S l V \frac{\rho}{2} \qquad c_1 = -\frac{2X'_q}{\rho S l V}$$

$$Z'_u = -a_2 SV \frac{\rho}{2} \qquad a_2 = -\frac{2Z'_u}{\rho SV}$$

$$Z'_w = -b_2 SV \frac{\rho}{2} \qquad b_2 = -\frac{2Z'_w}{\rho SV}$$

$$Z'_q = -c_2 S l V \frac{\rho}{2} \qquad c_2 = -\frac{2Z'_q}{\rho S l V}$$

$$M'_u = -a_3 S c V \frac{\rho}{2} \qquad a_3 = -\frac{2M'_u}{\rho S c V}$$

$$M'_w = -b_3 S c V \frac{\rho}{2} \qquad b_3 = -\frac{2M'_w}{\rho S c V}$$

$$M'_q = -c_3 S c l V \frac{\rho}{2} \qquad c_3 = -\frac{2M'_q}{\rho S c l V}$$

These factors are determined as functions of the coefficients C_X, C_Z, C_M and their derivatives with respect to the variables α and χ.

For the derivatives with respect to the linear velocities, one obtains:

$$X'_u = \left(\frac{\partial C_X}{\partial u} V^2 + \frac{\partial V^2}{\partial u} C_X\right) S \frac{\rho}{2}$$

$$= \left(\frac{\partial C_X}{\partial u} V + 2C_X\right) SV \frac{\rho}{2}$$

$$= -a'_1 SV \frac{\rho}{2}$$

now:

$$\frac{\partial C_X}{\partial u} = \frac{\partial C_X}{\partial \alpha}\frac{d\alpha}{du} = \frac{w}{u^2}\frac{\partial C_X}{\partial \alpha} = -\frac{\alpha}{V}\frac{\partial C_X}{\partial \alpha}$$

whence

$$a'_1 = \alpha\frac{\partial C_X}{\partial \alpha} - 2C_X$$

$$X'_w = \left(\frac{\partial C_X}{\partial w} V\right) SV\rho/2 = -b_1 SV\frac{\rho}{2}$$

now:

$$\frac{\partial C_X}{\partial w} = \frac{\partial C_X}{\partial \alpha}\frac{d\alpha}{dw} = -\frac{1}{u}\frac{\partial C_X}{\partial \alpha}$$

taking into account that $u = V$, one obtains:

$$b_1 = \frac{\partial C_X}{\partial \alpha}$$

Likewise, the determination of Z'_u and Z'_w leads to:

$$a_2 = \alpha\frac{\partial C_Z}{\partial \alpha} - 2C_Z$$

$$b_2 = \frac{\partial C_Z}{\partial \alpha}$$

Finally:

$$M'_{t,u} = \left(\frac{\partial C_{M_t}}{\partial u} V^2 + \frac{\partial V^2}{\partial u} C_{M_t}\right) Sc\frac{\rho}{2}$$

$$= \left(\frac{\partial C_{M_t}}{\partial u} V + 2C_{M_t}\right) ScV\frac{\rho}{2}$$

$$= -a_3 ScV\rho/2$$

When the propeller is of fixed pitch, C_{M_t} depends on two variables α and γ. Both are functions of the velocity of translation V. One obtains therefore:

$$\frac{\partial C_{M_t}}{\partial u} = \frac{\partial C_{M_t}}{\partial \alpha}\frac{d\alpha}{du} + \frac{\partial C_{M_t}}{\partial \gamma}\frac{d\gamma}{du}$$

$$= \frac{\partial C_{M_t}}{\partial \alpha}\frac{w}{u^2} + \frac{\partial C_{M_t}}{\partial \gamma}\frac{d\gamma}{du}$$

however, in order to determine $d\gamma/du$, one must know, in addition, the variations of the velocity of rotation of the engine.

If this velocity of rotation were constant, one would have:

$$\frac{d\gamma}{du} = \frac{1}{nD} = \frac{\gamma}{V}$$

An expedient, studied in appendix I, permits taking this variation in velocity into account by introduction of a dimensionless factor n', so that one obtains finally:

$$a_3 = \alpha\frac{\partial C_{M_t}}{\partial \alpha} - \gamma n'\frac{\partial C_{M_t}}{\partial \gamma} - 2C_{M_t}$$

When the propeller is of variable pitch, the variation in velocity u modifies the propeller blade-angle setting, but the number of revolutions is constant. The a_3 can still be determined but knowledge of a certain number of characteristics is necessary.

The determination of $M'_{t,w}$ is easy and leads immediately to:

$$b_3 = \frac{\partial C_{M_t}}{\partial \alpha}$$

As to the derivatives with respect to the angular velocity q, one assumes generally that:

$$X'_q = 0$$

$$Z'_q = 0$$

Thus c_3 is the only coefficient to be determined. One obtains

$$M'_q = \left(\frac{\partial C_M}{\partial q} V\right) ScV \frac{\rho}{2} = -c_3 Sc l V \frac{\rho}{2}$$

now:

$$\chi = ql/V \quad \text{whence} \quad d\chi/dq = lV$$

$$\frac{\partial C_M}{\partial q} V = \frac{\partial C_M}{\partial \chi} \frac{d\chi}{dq} V = l \frac{\partial C_M}{\partial \chi}$$

and

$$c_3 = -\frac{\partial C_M}{\partial \chi}$$

In the investigation of the transverse motion, we shall encounter derivatives such as $Y'_v \ldots N'_r$. Since the two investigations are made separately, there is no inconvenience in utilizing the letters $a_1 \ldots c_3$ for defining them, in accordance with the table:

$$Y'_v = -a_1 SV \frac{\rho}{2} \qquad a_1 = -\frac{2Y'_v}{\rho SV}$$

$$Y'_p = -b_1 SsV \frac{\rho}{2} \qquad b_1 = -\frac{2Y'_p}{\rho SsV}$$

$$Y'_r = -c_1 SsV \frac{\rho}{2} \qquad c_1 = -\frac{2Y'_r}{\rho SsV}$$

$$L'_v = a_2 SbV \frac{\rho}{2} \qquad a_2 = -\frac{2L'_v}{\rho SbV}$$

$$L'_p = b_2 SbsV \frac{\rho}{2} \qquad b_2 = -\frac{2L'_p}{\rho SbsV}$$

$$L'_r = c_2 SbsV \frac{\rho}{2} \qquad c_2 = -\frac{2L'_r}{\rho SbsV}$$

$$N'_v = a_3 SbV \frac{\rho}{2} \qquad a_3 = -\frac{2N'_v}{\rho SbV}$$

$$N'_p = b_3 SbsV \frac{\rho}{2} \qquad b_3 = -\frac{2N'_p}{\rho SbsV}$$

$$N'_r = c_3 SbsV \frac{\rho}{2} \qquad c_3 = -\frac{2N'_r}{\rho SbsV}$$

The factors $a_1 \ldots c_3$ corresponding to the transverse motion are determined as functions of the coefficients C_Y, C_L, C_N and of their derivatives with respect to the variables β, $\tilde{\omega}$, ρ. They are calculated in the same manner, but since $\beta = v/u$ instead of $\alpha = -w/u$, the quantities a_1, a_2, and a_3 are preceded by the minus sign.

One assumes likewise that the derivatives of the force Y with respect to the angular velocities p and r are zero.

Calculation of the others leads to:

$$a_1 = -\frac{\partial C_Y}{\partial \beta}$$

$$a_2 = -\frac{\partial C_L}{\partial \beta} \qquad b_2 = -\frac{\partial C_L}{\partial \tilde{\omega}} \qquad c_2 = -\frac{\partial C_N}{\partial \tilde{\omega}}$$

$$a_3 = -\frac{\partial C_N}{\partial \beta} \qquad b_3 = -\frac{\partial C_L}{\partial \rho} \qquad c_3 = -\frac{\partial C_N}{\partial \rho}$$

Let us recall here a conventional result.

An elementary calculation shows that the role of the wings in the terms b_2, c_2, b_3, and c_3 of the transverse motion is of the order of magnitude:

$$\frac{\partial C_L}{\partial \tilde{\omega}} = -b_2 = -\frac{1}{8}\frac{\partial C_z}{\partial \alpha}$$

$$\frac{\partial C_N}{\partial \tilde{\omega}} = -b_3 = \frac{1}{8}\frac{\partial C_x}{\partial \alpha}$$

$$\frac{\partial C_L}{\partial \rho} = -c_2 = -\frac{1}{4} C_z$$

$$\frac{\partial C_N}{\partial \rho} = -c_3 = -\frac{1}{4} C_x$$

Added to this effect, of course, is that of the other surfaces of the aircraft.

Remark: The factors a_1, b_1, etc., have the same significance as the factors x_u, x_w or y_p etc., defined in "Nomenclature for Stability Coefficients" R. & M. 1801, but they differ from them sometimes by a constant factor 2 and by the sign. This last difference results from the use of a dynamic trihedron the axes of which are differently oriented.

We did not want to use here the English notations in order to avoid the confusion which would result from these differences.

10. Influence of the Attitude Parameters on the Aerodynamic Effects

The aerodynamic forces and reactions are independent of the angles φ, θ, ψ which define the orientation of the airplane in space.

On an airplane flying with controls fixed, the derivatives M'_θ, L'_φ, and L'_ψ, N'_φ, and N'_ψ are necessarily zero. However, we shall see that they can cease to be zero if the airplane is provided with an automatic pilot so that it will be useful for symmetry of the calculations to define the following notations immediately.

Longitudinal Motion

$$M'_\theta = -d_3ScV^2\frac{\rho}{2} \qquad d_3 = -\frac{\partial C_M}{\partial \theta}$$

Transverse Motion

$$L'_\varphi = -d_2SbV^2\frac{\rho}{2} \qquad d_2 = -\frac{\partial C_L}{d\varphi}$$

$$L'_\psi = -d_3SbV^2\frac{\rho}{2} \qquad d_3 = -\frac{\partial C_L}{\partial \psi}$$

$$N'_\varphi = -e_2SbV^2\frac{\rho}{2} \qquad e_2 = -\frac{\partial C_N}{d\varphi}$$

$$N'_\psi = -e_3SbV^2\frac{\rho}{2} \qquad e_3 = -\frac{\partial C_N}{\partial \psi}$$

11. Derivatives of the Propeller Thrust

In the course of the calculations we shall have to use the derivatives of the thrust T with respect to the variables u, v, w, p, q, r.

In a first treatment of the problem, only T'_u is assumed not to be zero.

Putting:

$$T'_u = -a''_1 SV \frac{\rho}{2} \qquad a''_1 = -\frac{2T'_u}{\rho SV}$$

one must calculate a''_1.

In the case of the constant-pitch propeller one has:

$$T'_u = S_h \frac{\rho V^2}{2} \frac{dC_T}{du} + 2C_T S_h \frac{\rho}{2} V$$

It is shown in the appendix I that one may write:

$$V \frac{dC_T}{du} = V \frac{dC_T}{d\gamma} \frac{d\gamma}{du} = \frac{dC_T}{\partial\gamma} n'\gamma$$

whence

$$a''_1 = \frac{S_h}{S}\left(\frac{\partial C_T}{\partial\gamma} n'\gamma - 2C_T\right)$$

The calculations may be carried further, and a''_1 may be expressed, if desired, as a function of K_T instead of C_T.

Remark: In the calculations, the quantities a'_1 and a''_1 are always added. We shall put therefore:

$$a_1 = a'_1 + a''_1$$

12. External Forces in Unsteady-State Motion

Investigation of the various states of motion requires knowledge of the aerodynamic forces for unsteady-state conditions.

Theoretical and experimental research has been carried out with a view to determination of the transitory phenomena resulting from changes:

(a) Of the aerodynamic velocity

(b) Of the angle of attack α

(c) Of the angular velocities p, q, r

The situation is as follows:

(a) Accelerations dV/dt.- It has been possible to determine theoretically the effect of the accelerations for bodies presenting only drag as well as for lifting elements.

(b) Changes of angle of attack.- The transitory phenomena which accompany a change in angle of attack have been studied theoretically. (Theories of Küssner, of Kármán and Sears, etc.)

In the domain where flow theory applies, the increase in lift corresponding to an abrupt increase in angle of attack $\Delta\alpha$ is not instantaneous.

The circulation, and hence, the lift corresponding to the new angle of attack α is established gradually.

On the other hand, the experiments have proved an important fact relating to the flows at angles of attack in the neighborhood of maximum lift. In the case of a rapid increase in angle of attack, the theoretical state of flow is established in accordance with the theory, but the flow separates after having been established, if the final angle of attack is near that of maximum lift or exceeds it. As a result, the lift, under these conditions, is apt to attain transitorily a value exceeding the one it has at the same angle of attack in steady state.

(c) Effect of variable angular velocities p, q, r.- It has been possible to establish in the tunnel data regarding the effect of a variable angular velocity by making the models oscillate.

The motion of a model with the moment of inertia I, oscillating freely about its transverse axis, satisfies the equation:

$$I \frac{d^2\theta}{dt^2} + J \frac{d\theta}{dt} + K\theta = 0$$

where I is the moment of inertia of the model, K a coefficient of the restoring moment, and J a coefficient proportional to the damping moment.

The damping moment $J \frac{d\theta}{dt}$ is due to the effect of the rotation q. The experimental determination of the oscillation characteristics permits, when I and K are known, to determine J, that is, a factor proportional to $dC_M\, d\chi$, defining the effect of the angular velocity on the pitching moment.

More complex oscillation methods, using especially forced oscillations, may be contrived with a view to establishing the effect of any one of the angular velocities p, q, r on any one of the moments L, M, N.

The methods utilizing forced oscillations yield much more accurate results than those utilizing free oscillations; they are the only ones actually in use.

The derivatives such as $dC_M/d\chi$, $dC_N/d\rho$, obtained for unsteady-state conditions by the method of oscillations, differ considerably from those obtained under steady-state conditions by means of whirling-arm tests. They correspond, in fact, to a physically different phenomenon.[4]

13. Introduction of Our Knowledge of Unsteady-State Phenomena into the Calculations

We think that a step forward would be made by introducing into the calculations our knowledge of unsteady-state phenomena, if we could add to the expression of the aerodynamic forces as a function of the instantaneous values of the variables u, v, w, p, q, r realized in steady state, factors expressing the influence of the derivative of each of these variables.

We show in chapter II what are the extensions to be made to the methods of calculation using the flight conditions.

The ideal procedure would be to introduce into the calculations the expressions of the external forces which take into account the entire previous history of the motion. This result has not yet been attained at the present time.

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[4] It is impossible for us to treat this question here in detail. Its investigation could give occasion to a complete report, independent of the one given here.

14. Introduction of the Mach Number into the Theory

One can study the behavior of aircraft controlled by automatic pilots, flying at speeds of 300 to 500 km per hour without taking into consideration the changes in the aerodynamic forces which are produced when the critical value of the Mach number is approached.

At the values indicated above, the abrupt change of the aerodynamic forces is not yet present.

However, there is no getting away from the fact that the flight investigations of aircraft using automatic pilots are frequently intended to predict the behavior of guided missiles, travelling at speeds reaching sonic velocity and even surpassing it.

However, establishment of a complete theory of automatic flight in largely subsonic regions seems to us a preliminary condition, realization of which is necessary before it can become possible to undertake in a useful manner investigations of flight in the transonic and supersonic regions.

In the present report, we limit ourselves to the flight in the subsonic region and we do not attempt to introduce into the calculations the influence of the changes in the external forces due to the variations of the Mach number regarding which our knowledge is still rudimentary.

The study of automatic flight in completely subsonic regions is in itself of sufficient interest, owing to the development taken by this type of flight, to justify the present report; besides, we reserve the right to supplement it later on by introduction of the effects due to the compressibility of the air.

CHAPTER IV

THE EFFECTS OF THE PILOT

1. Intervention of the Pilot

The motion of an airplane in space is generally not that of an indeformable solid body free in space. The airplane is guided by a pilot whose intervention makes itself felt in flight by various actions. The principal ones, involved in the handling of the aircraft, are:

The maneuver of power-setting for the engine

The deformations voluntarily imposed on the airplane.

The changes of throttle setting produce variations of the engine torque. They must be balanced by a corresponding variation of the torque-opposing moment.

If the propeller has constant pitch, an increase in engine torque can be balanced by an increase in the torque-opposing moment only when the rotational speed increases.

If the propeller is adjustable which imposes a reasonably constant speed of rotation, an increase in engine torque will be balanced by an increase in pitch.

In both cases, a change of throttle setting which increases the engine torque produces an increase in thrust force.

The modification of the conditions for operation of the propeller (modification of the parameter γ or of the pitch) exerts an influence on the slipstream of the propeller, and the reactions X, Z, M exerted on the airframe may be altered by this fact.

The modification of these reactions constitutes a secondary effect.

The changes in the external configurations of a glider modify either one of the aerodynamic moments L, M, or N, or one of the components of the reaction X, Y, Z, or several among them.

The pilot possesses means of action upon the external forces and moments applied to the airplane, which means, he is able to affect the flight path.

2. Principal Controls

The surfaces, displacement of which produces a modification of moment, are the controls.

In an aircraft of conventional form the pilot can apply moments about each of the three axes; he has for this purpose the following three controls at his disposal:

(a) The ailerons

(b) The rudder

(c) The elevator

which constitute the main controls.

(a) The ailerons are intended to produce moments about the longitudinal axis. They are situated on both sides of the wing and their motion is generally linked together. Their deflection, defined by the angle ξ, will be considered positive when the left aileron is lowered while the right aileron is raised by an equivalent amount.

(b) The elevator produces a moment about the lateral axis. The deflection, represented by η, is considered positive if it is made downward, for an elevator situated at the rear (usual case).

(c) The rudder is intended to produce moments about the yaw axis. The deflection ζ will be positive if the rudder is deflected to the left.

The movable surfaces of the controls are rather small in proportion to the wing surface and it is generally assumed that their displacement exerts only an insignificant effect on the forces.

It would be desirable that a maneuver performed with the purpose of exerting a moment about one of the axes should not have any effect about the two other axes. This is not always the case. Maneuvering of the ailerons exerts a secondary effect about the yaw axis which, generally, cannot be suppressed.

The control mechanism of the principal controls is reversible on small and medium airplanes.

3. Magnitude of the Control Forces

When a control surface is displaced, it modifies one or the other forces or moments exerted on the airplane. The derivative of the force or moment exerted with respect to the displacement or deflection, characterizes the effectiveness of the control surface.

A. Effect of throttle setting.- The throttle setting, represented symbolically by the variable σ, exerts on the forces applied to the airplane effects represented by the factors s_1, s_2, and s_3.

The change in drag opposes the change in thrust and its effect may be incorporated in s_1

$$T'_\sigma = s_1 SV^2 \frac{\rho}{2}$$

The modifications in the forces Z and M are given by:

$$Z'_\sigma = s_2 SV^2 \frac{\rho}{2}$$

$$M'_\sigma = ScV^2 \frac{\rho}{2}$$

so that:

$$s_1 = \frac{dC_T}{d\sigma}$$

$$s_2 = \frac{dC_Z}{d\sigma}$$

$$s_3 = \frac{dC_M}{d\sigma}$$

For the performance of numerical calculations, σ must be given concrete significance. A useful definition of σ may be given by the manifold pressure.

B. Effect of the elevator.- The deflection η of the elevator exerts on the total moment M_t an effect represented by h_3:

$$M'_\eta = h_3 ScV^2 \frac{\rho}{2}$$

so that

$$h_3 = \frac{\partial C_M}{\partial \eta}$$

C. Effect of the lateral controls.- We characterize them by:

$$L'_\xi = h_2 bSV^2 \frac{\rho}{2} \quad \text{or} \quad h_2 = \frac{\partial C_L}{\partial \xi}$$

$$L'_\zeta = k_2 bSV^2 \frac{\rho}{2} \qquad k_2 = \frac{\partial C_L}{\partial \zeta}$$

$$N'_\xi = h_3 bSV^2 \frac{\rho}{2} \qquad h_3 = \frac{\partial C_N}{\partial \xi}$$

$$N'_\zeta = k_3 bSV^2 \frac{\rho}{2} \qquad k_3 = \frac{\partial C_N}{\partial \zeta}$$

4. Control-Hinge Moments

Knowledge of the control-hinge moments is essential in any study of handling qualities.

We shall denote by $\underline{L}$, $\underline{M}$, $\underline{N}$ the moment exerted by the aerodynamic reactions about the hinge of every one of the three control surfaces.

These hinge moments will be defined by the coefficients C_{cl}, C_{cm}, C_{cn}, by means of the following relations:

For the ailerons:

$$\underline{L} = C_{cl} S_m c_m \frac{\rho V^2}{2}$$

For the elevator:

$$\underline{M} = C_{cm} S'_m c'_m \frac{\rho V^2}{2}$$

For the rudder:

$$\underline{N} = C_{cn} S''_m c''_m \frac{\rho V^2}{2}$$

where S_m, S'_m, S''_m represent the areas of the movable elements, c_m, c'_m, c''_m the chord of these movable elements.

Knowledge of the hinge moments permits finding the corresponding force to be exerted by the pilot, taking into account the mechanical advantage of control linkage and in some cases the weight of the movable surfaces if their center of gravity is not situated on the axis.

For the ailerons, the moment $\underline{L}$ is positive when it tends to raise the left aileron, that is, when the control stick tends to be displaced toward the left.

For the elevator, $\underline{M}$ is positive when it tends to raise the movable surface, that is, when the control stick tends to push forward in the hand of the pilot.

For the rudder, $\underline{N}$ is positive when it tends to oppose a deflection to the left, that is, when the rudder bar pushes against the left foot of the pilot.

The coefficients C_{cl}, C_{cm}, C_{cn} are functions not only of the aileron deflection angles ξ, η, ζ, but also of the angles of attack and of sideslip α and β, and even of the rotation ratios $\tilde{\omega}$, χ, ρ.

They can be measured in the wind tunnel for all the cases corresponding to steady states of translation.

While the control surface is in the process of being deflected, the coefficients are functions of the rate of change of deflection, that is, of $d\xi/dt$, $d\eta/dt$, $d\zeta/dt$.

Even though we can take this fact into account in setting up the equations, it is unfortunately difficult to fix the numerical values for this effect.

5. Compensating Devices

When the dimensions of the aircraft lead, under certain conditions, to excessive values of the moments $\underline{L}$, $\underline{M}$, $\underline{N}$, it is necessary to use compensating devices the purpose of which is a reduction of the coefficients C_{cl}, C_{cm}, C_{cn}.

These compensating devices can be utilized if one requires achievement of a hinge moment zero, for a given condition. Most frequently they are made by providing at the trailing edge of the movable control surface a supplementary degree of deformation, controlled by the pilot.

The portion of the horizontal tail surface, considered as fixed, may in general be adjustable, with its setting controlled by a worm gear. It constitutes in this case an auxiliary control of the pitching moments.

Likewise, the action of the rudder can be modified or reinforced by the trimming of the fin which is sometimes adjustable in flight.

Finally, there exist aerodynamic surfaces the displacement of which by the pilot has the purpose of modifying the aerodynamic forces. Properly speaking, these parts of the airplane are not control surfaces:

(1) Certain airplanes are equipped with aerodynamic brakes, permitting modification of the component X.

(2) Of greater importance are the lifting devices permitting, at equal velocity, increasing of the component Z.

These devices generally consist of flaps extending over considerable portions of the wing span. The majority of lifting flaps affect equally the component X and modify the drag; the latter does, generally, not constitute an inconvenience.

It is always desirable that the maneuvering of the surfaces intended to modify the forces should have as small an effect as possible on the moments.

In our investigation of aircraft motion, we shall content ourselves with studying the motions resulting from the displacement of the reversible controls which we have called the principal controls.

We shall assume that the deflection of the irreversible controls modifies the airplane once and for all and defines in some way another airplane which could be investigated, if necessary, by the same means as the first one.

8. Specialization of the Controls

We divided the investigation of motion into two distinct problems: longitudinal motion and lateral motion.

On the other hand, we retain four controls as fundamental controls.

The longitudinal motion will be studied as a function of the displacement of two controls:

The deflection η of the elevator

The power setting of the engine which we represent symbolically by σ.

The lateral motion will be determined by the movement of the two other control surfaces:

The deflection ξ of the ailerons

The deflection ζ of the elevator

9. Mechanical Deformations

In our investigation, we take into account only the voluntarily produced deflections ξ, η, ζ and we assume the rigidity of the airplane to be sufficient to enable us to neglect any other structural deformation.

It is important to draw attention to this hypothesis because elastic deformations of the airframe always take place under the action of external forces.

Every time the forces to which an aircraft is subjected show a variation, its frame undergoes deformations. The amplitude of the elastic deformations depends on the rigidity of the construction.

One may reduce the amplitude of the elastic deformations, but one cannot suppress them completely. A perfectly rigid aircraft does not exist.

The elastic deformations play a large role in the vibrations of airplane frames.

Between the vibrations of two components of an airplane, for instance, between wing and aileron, there may exist an aerodynamic coupling which can increase considerably the amplitude of certain deformations and which leads no longer to simple oscillations, but to real flapping, corresponding to the phenomenon generally called "flutter".

The elastic deformations of the frame, whether they are oscillatory or not, may play a role in the determination of the trajectory of the airplane.

The hypothesis we made consists in assuming this effect to be sufficiently weak to be neglected.

CHAPTER V

EFFECT OF THE CONTROLS ON THE CONDITIONS OF EQUILIBRIUM

1. Action of the Controls

An investigation of the motion of airplanes which forms the object of the present report, assumes several facts resulting from the conditions of equilibrium to be firmly established.

We shall recall below the essential principles defining the effect of the controls in the course of flights in steady state.

2. Condition of Longitudinal Equilibrium for a Rectilinear Flight Path

Using for this particular problem axes fixed to the flight path, one may write the equations of equilibrium:

$$T \cos \alpha_t - C_x S \frac{\rho}{2} V^2 - G \sin \tau = 0$$

$$T \sin \alpha_t - C_z S \frac{\rho}{2} V^2 - G \cos \tau = 0$$

$$M_t \; . \; . \; . \; . \; . \; . \; . \; . = 0$$

where α_t is the angle between the thrust axis and the trajectory, and τ the slope of the trajectory, assumed as positive when the airplane climbs.

Writing

$$\cos \alpha_t = 1$$

$$\cos \tau = 1$$

neglecting $T \sin \alpha_t$ in the presence of G, one obtains, multiplying the first equation by V:

$$TV - C_x S \frac{\rho}{2} V^3 - GV \sin \tau = 0$$

$$C_z S \frac{\rho}{2} V^2 = G$$

$$C_{M_t} Sc \frac{\rho}{2} V^2 = 0$$

The first condition expresses the equilibrium of power. The second condition expresses the equilibrium of lift. The third condition expresses the equilibrium of moment.

The moment coefficient C_{M_t} is a known function of the angle of attack α, of the deflection η of the rudder, and of the advance ratio of the propeller $\gamma = V/nD$.

The lift coefficient is uniquely a function of α.

Eliminating α between the second and third equations, one obtains a relation:

$$V = F(\eta)$$

which may be written also:

$$\alpha = f(\eta)$$

The control surface called the elevator determines the flight velocity or the angle of attack at which a state of equilibrium is possible. It constitutes the control for the flight speed.

Let us now examine the first and second equations.

We put $TV = W_u$ useful power

$V \sin \tau = V_v$ rate of climb, positive when the airplane goes up.

We eliminate V between these equations and obtain:

$$W_u = GV_v + G \frac{C_x}{C_z^{3/2}} G \sqrt{G/S} \sqrt{2/\rho}$$

The ratio $\dfrac{C_x}{C_z^{3/2}}$ is a known function of the angle of attack and, indirectly, of the velocity.

The effective power W_u depends on the power setting σ, on the flight velocity, and on the density of the air.

One may plot the curves of effective power as functions of the velocity, for different values of σ.

Let us plot as a function of the velocity, for prescribed weight and altitude, the curve of:

$$\frac{C_x}{{C_z}^{3/2}} G \sqrt{G/S} \sqrt{2/\rho}$$

which determines the power required for horizontal flight.

These curves permit determination of the excess power ΔW or GV_v.

Hence, they determine the rate of climb.

It is found that, at constant velocity, the only means of influencing the rate of climb, that is, the slope of the trajectory, consists in increasing the engine power.

It is the engine power which makes the airplane climb, and the throttle setting is the control which directly affects this power.

However, maneuvering of the elevator exerts an indirect effect.

Let us suppose that an aircraft flies horizontally, at the speed V_A, with the power setting σ_2.

If the pilot places the airplane in equilibrium at a speed $V_B < V_A$, by means of a deflection $-\eta$, he frees a certain excess of power which permits the airplane to climb.

This secondary effect of the elevator justifies the name given to this control.

When $\dfrac{\partial C_M}{\partial \eta} > 0$ (normal effect of the elevator)

$\dfrac{\partial C_M}{\partial \alpha} > 0$ (aircraft having static stability)

and when one examines operating points at a speed higher than the minimum necessary for lift (so-called high-speed flight), all effects are in accord.

A negative displacement $\Delta\eta$ exerts a tail-down moment and determines a position of equilibrium at a larger angle of attack.

At this new position of equilibrium, the power required for horizontal flight is smaller, and a power margin, "unfrozen" as it were, allows the aircraft to maintain an equilibrium of power on an ascending trajectory.

If one of the preceding conditions is not satisfied, either because the airplane is statically unstable $\left(\frac{\partial C_M}{\partial \alpha} < 0\right)$ or because the intended point of operation lies at a velocity smaller than that for maximum power (slow regime), at least one of the effects is reversed.

3. Turning Flight

Let us assume an airplane describing a curve with angular velocity about a vertical axis.

The vector Ω, directed upward, defines according to our conventions a curve to the left.

This vector is projected on the following dynamic axes:

$$p = -\Omega \sin \theta$$

$$q = \Omega \sin \varphi \cos \theta$$

$$r = \Omega \cos \varphi \cos \theta$$

We shall suppose that the airplane axis OX is sufficiently close to the horizontal to permit neglecting $\Omega \sin \theta$.

One sees that an airplane in a regular turn is subject to permanent angular velocities q and r.

q comes into play in the conditions of longitudinal equilibrium and acts upon M.

r comes into play in the conditions of lateral equilibrium.

Let us briefly investigate the latter.

4. Conditions of Lateral Equilibrium in Turns

We shall assume:

(a) That the angle of attack and the velocity, known and determined by the conditions of longitudinal equilibrium, constitute the given factors of the problem

(b) That the lift coefficient realized at this angle of attack is independent of the sideslip β

(c) That the effect of the propeller slipstream and that of the engine torque have been compensated

(d) That the thrust axis lies in the plane of symmetry.

The pilot may influence the selection of the position of equilibrium by means of two controls, that of the ailerons and that of the rudder. He modifies thus the factors C_L and C_N which define the aerodynamic forces acting upon the airplane.

$\frac{\partial C_L}{\partial \xi}$ characterizes the principal effect of the ailerons

$\frac{\partial C_N}{\partial \xi}$ characterizes the secondary effect of the ailerons

$\frac{\partial C_N}{\partial \zeta}$ characterizes the principal effect of the elevator

$\frac{\partial C_L}{\partial \zeta}$ characterizes the secondary effect of the elevator

We write the secondary effect of each control as a fraction x or z of the principal effect:

$$\frac{\partial C_N}{\partial \xi} = x \frac{\partial C_L}{\partial \xi}$$

$$\frac{\partial C_L}{\partial \zeta} = z \frac{\partial C_N}{\partial \zeta}$$

The conditons of lateral equilibrium number three. For steady-state conditions they become, if one makes $p = 0$:

$$\Sigma Y - mg \sin \varphi = mrV$$

$$\Sigma L = qr(C - B)$$

$$\Sigma N = 0$$

or

$$\Sigma Y - mg \sin \varphi = mV\Omega \cos \varphi$$

$$\Sigma L = \Omega^2 \sin \varphi \cos \varphi (C - B)$$

$$\Sigma N = 0$$

In fact, the numerical value of Ω is small, and the second equation may in an approximate study be written:

$$\Sigma L = 0$$

In the third equation, we shall neglect the gyrostatic moment N_g which enters in ΣN.

Replacing $\Omega \cos \varphi = r$ by $\rho \frac{V}{s}$ and writing the equations in non-dimensional form, we obtain:

$$C_Y = \frac{\mu c g}{V^2}\left(\frac{V^2 s}{g} \rho \cos \varphi + \sin \varphi\right)$$

$$C_L = 0$$

$$C_N = 0$$

When the aerodynamic coefficients vary linearly with β, ξ, ζ, and ρ, one has:

$$C_Y = \frac{\partial C_Y}{\partial \beta} \beta + \frac{\partial C}{\partial \zeta} \zeta$$

because of the lateral component on the rudder, produced by the deflection, and

$$C_L = \frac{\partial C_L}{\partial \beta} \beta + \frac{\partial C_L}{\partial \xi} \xi + \frac{\partial C_L}{\partial \zeta} \zeta + \frac{\partial C_L}{\partial \rho} \rho$$

$$C_N = \frac{\partial C_N}{\partial \beta} \beta + \frac{\partial C_N}{\partial \xi} \xi + \frac{\partial C_N}{\partial \zeta} \zeta + \frac{\partial C_N}{\partial \rho} \rho$$

The conditions of equilibrium are finally written:

$$\frac{\partial C_Y}{\partial \beta} \beta + \frac{\partial C_Y}{\partial \zeta} \zeta = \frac{\mu c g}{V^2}\left(\frac{V^2 s}{g} \rho \cos \varphi + \sin \varphi\right)$$

$$\frac{\partial C_L}{\partial \beta} \beta + \frac{\partial C_L}{\partial \xi} \xi + \frac{\partial C_L}{\partial \zeta} \zeta + \frac{\partial C_L}{\partial \rho} = 0$$

$$\frac{\partial C_N}{\partial \beta} \beta + \frac{\partial C_N}{\partial \xi} \xi + \frac{\partial C_N}{\partial \zeta} \zeta + \frac{\partial C_N}{\partial \rho} = 0$$

This system depends on five variables:

The angle of sideslip β

The lateral inclination φ

The angular velocity of rotation replaced by the ratio of roll ρ

The deflection of the ailerons ξ

The deflection of the elevator ζ.

There exists therefore an infinite square number of solutions. But two of the variables, namely the deflections, are actually degrees of freedom at the disposal of the pilot.

For any arbitrarily selected value of the deflections ξ and ζ, the variables β, φ, and Ω are determined.

Remark: This conclusion is valid only as long as $p = 0$, that is, only for turns about a vertical axis in the course of which the axis OX remains horizontal.

5. Discussion of the Equations

Taking the previous restriction into account, we shall investigate the conditions of equilibrium for three different types of turns, characterized respectively by:

$\beta = 0$: perfect turn, sideslip zero

$\zeta = 0$: turn with action of the ailerons only

$\xi = 0$: turn with action of the elevator only

<u>A. Perfect turn</u>.- The two equations of moment give:

$$\xi \frac{\partial C_L}{\partial \xi} = -\rho\left(\frac{\partial C_L}{\partial \rho} - z\frac{\partial C_L}{\partial \rho}\right)\frac{1}{1 - xz}$$

$$\zeta \frac{\partial C_N}{\partial \zeta} = -\rho\left(\frac{\partial C_N}{\partial \rho} - x\frac{\partial C_N}{\partial \rho}\right)\frac{1}{1 - xz}$$

On the usual aircraft,

The $\frac{\partial C_L}{\partial \beta}$ and $\frac{\partial C_N}{\partial \beta}$ are positive

The $\frac{\partial C_L}{\partial \rho}$ and $\frac{\partial C_N}{\partial \rho}$ are negative

For $\rho > 0$, when x and z are small, deflections ξ and ζ are positive.

In order to maintain the airplane in a perfect turn, one must apply continually a deflection of the rudder in the direction of the turn, and one must hold up the inside wing.

These facts are easily understood. The $\frac{\partial C_N}{\partial \rho} \rho$ is a resistance to the turn. In order to overcome it and to maintain the turn, one must apply a continuous moment C_N which is done chiefly by the deflection of the rudder.

The $\frac{\partial C_L}{\partial \rho}$ is a secondary or disturbance moment which results from the decrease of lift on the inside wing, and tends to depress it.

In order to overcome this disturbance moment, one must hold up the inside wing which is done chiefly by the deflection of the ailerons.

The exact inclination φ will be calculated by means of the translational equation of equilibrium according to OY, possibly taking into account the lateral reaction developed by the deflection of the control surface.

B. Turn effected under action of the ailerons only.- A steady-state turn can be maintained by means of deflecting one of the two lateral controls while maintaining the other in neutral position.

The turn ceases to be perfect, a sideslip being necessary in order to maintain it.

We make $\zeta = 0$ in the preceding equations and eliminate β between the two equations of equilibrium of moment.

We obtain:

$$\xi \frac{\partial C_L}{\partial \xi} = -\rho \frac{D}{\frac{\partial C_N}{\partial \beta} - x \frac{\partial C_L}{\partial \beta}}$$

with

$$D = \frac{\partial C_N}{\partial \beta} \frac{\partial C_L}{\partial \rho} - \frac{\partial C_L}{\partial \beta} \frac{\partial C_N}{\partial \rho}$$

On an airplane possessing static stability, the factors $\frac{\partial C_L}{\partial \beta}$ and $\frac{\partial C_N}{\partial \beta}$ are both positive, the factors $\frac{\partial C_L}{\partial \rho}$ and $\frac{\partial C_N}{\partial \rho}$ are both negative.

The quantity D resulting from the differences of the two products may be positive, zero, or negative according to the respective values of the preceding factors. It constitutes an important characteristic.

Since the secondary effect is small, $\frac{\partial C_N}{\partial \beta} \text{-x} \frac{\partial C_L}{\partial \beta} > 0$, and the deflection ξ which ensures maintenance of a correct turn will be

Negative when $D > 0$

Zero when $D = 0$

Positive when $D < 0$.

Let us assume a turn for which $\rho > 0$, that is, a turn to the left.

On an airplane of the first type, one must lower the left wing, that is, lower the wing on the inside of the turn.

On an airplane of the second type, one need not do anything: a turn once started maintains itself even if the two lateral controls do not undergo any deflection

On an airplane of the third type, one must hold up the inside wing.

The system of equations permits also the determination of β. The calculation of β shows that the turns effected under action of the ailerons alone can constitute a state of equilibrium only if they are accompanied by a continual sideslip β toward the inside.

This sideslip develops the moments L_β and N_β which replace the moments L_ξ and N_ζ produced in the preceding case by the deflection of the two controls. However, in the preceding case, the two deflections were independent and could be adjusted in such a manner as to produce separately moments which equilibrate exactly the opposing moment N_ρ and the disturbance moment L_ρ.

Here, the sideslip β can only accidentally produce moments L_β and N_β exactly equal at the same time to L and to N. This occurs when $D = 0$.

This is only rarely the case. In general, $D \neq 0$. When the N_β due to the sideslip opposes and compensates the resistance to the turn N_ρ, the produced L_β does not have the exact value required for the rolling equilibrium: one must adjust the rolling moment by a deflection ξ in one or the other direction.

C. Turn effected under action of the rudder alone.- Let us make $\xi = 0$ in the equations and calculate ζ. We obtain:

$$\zeta \frac{\partial C_N}{\partial \zeta} = \rho \frac{D}{\frac{\partial C_L}{\partial \beta} - z \frac{\partial C_N}{\partial \beta}}$$

Since z is small, the denominator of the second number will be positive, and the deflection ζ necessary for the turn to the left will be

Positive when $D > 0$

Zero when $D = 0$

Negative when $D < 0$.

On an airplane of the first type, one must deflect the rudder toward the side of the turn which has to be maintained.

On an airplane of the second type, the turn maintains itself without any deflection of the lateral controls.

On an airplane of the third type, one must deflect the rudder in the direction opposed to the turn which has to be maintained.

Here also steady-state conditions are not possible unless the airplane shows a continual sideslip toward the inside. This sideslip furnishes the largest portion of the necessary yawing and rolling moments. The rudder deflection is applied only to adjust the moments in such a manner that the airplane may simultaneously satisfy both conditions of equilibrium of moment.

6. Numerical Application

It will be useful to illustrate the previous conclusions by a numerical example.

We assume an aircraft flying at 50 m/second and describing a turn of 450 m radius.

Rolling moment due to the turn $\frac{\partial C_L}{\partial \beta}$

Inherent stability in roll $\frac{\partial C_N}{\partial \beta}$

Inherent stability in yaw $\frac{\partial C_L}{\partial \rho}$

The execution of a turn takes:

$$\frac{2\pi \times 450}{50} = 56.6 \text{ seconds}$$

and

$$\Omega = \frac{2\pi}{56.5} = 0.111$$

The lateral inclination in a perfect turn is 29.30° whence $\cos \varphi = 0.87$

$$r = 0.111 \times 0.87 = 0.0965$$

For an airplane of a span of 20 m:

$$\rho = \frac{r \times 10}{V} = 0.0193$$

This ratio would be $\rho = 0.02$ for a span of 20.35 m.

$\rho = 0.02$ is the value for which we shall perform the calculation.

We shall suppose that:

$$\frac{\mu c g}{V^2} = 0.60$$

and shall consider three airplanes characterized by certain common values and certain differing values.

(a) Elements common to the three airplanes:

Lateral force due to the sideslip: $\dfrac{\partial C_Y}{\partial \beta} = -0.008$

Lateral force due to the deflection: $\dfrac{\partial C_Y}{\partial \zeta} = -0.0024$

Principal effect of the ailerons: $\dfrac{\partial C_L}{\partial \xi} = +0.003$

Principal effect of the rudder: $\dfrac{\partial C_N}{\partial \zeta} = +0.0012$

Secondary effects of the controls: $x = z = 0$.

(b) Elements varying between the different airplanes:

	Airplane No. 1	Airplane No. 2	Airplane No. 3
	0.0010	0.0012	0.0014
	0.0008	0.00075	0.0007
	-0.18	-0.16	-0.14
Resistance to the turn $\dfrac{\partial C_N}{\partial \rho}$	-0.08	-0.10	-0.12
Characteristic D	-6.4×10^{-5}	0	$+7 \times 10^{-5}$

The numerical values of the derivatives with respect to the above angles are expressed by taking the degree as unit.

They must be multiplied by 57.3 in order to obtain the derivatives of the forces and of the moments if the angles are expressed in radians.

The problem amounts to investigating which are the values of the deflections and of the sideslip which maintain a turn at $\rho = 0.02$.

The result of the calculation is as follows, with all angles expressed in degrees:

	Airplane No. 1	Airplane No. 2	Airplane No. 3
Perfect turn			
ξ =	$+1^{\circ}20$	$+1^{\circ}06$	$+0^{\circ}93$
ζ =	$+1^{\circ}32$	$+1^{\circ}67$	$+2^{\circ}00$

	Airplane No. 1	Airplane No. 2	Airplane No. 3
Turn maintained by ailerons alone			
	$\beta = +2^{\circ}00$	$+2^{\circ}66$	$+3^{\circ}46$
	$\xi = +1^{\circ}07$	0°	$-0^{\circ}66$
Turn maintained by rudder alone			
	$\beta = +3^{\circ}60$	$+2^{\circ}66$	$+2^{\circ}00$
	$\zeta = -1^{\circ}07$	0°	$+0^{\circ}835$

The inclination φ which ensures the translational equilibrium can be calculated. In the turns due to the action of one control, this inclination exceeds the one necessary for a perfect turn by 2 degrees to 3 degrees.

We state two important facts:

1. The deflections required for maintaining a continuous turn are very small whatever the type of turn which is adopted.

2. Besides the perfect turn, without sideslip, there exist turns somewhat more inclined than the perfect turn, and accompanied by a slight sideslip toward the inside.

The effect of this deflection is to force the airplane into the turn, and to decrease the deflections to be applied by the pilot.

These turns may be maintained by the operation of one single control. From the viewpoint of piloting, these last turns may be considered correct, but the airplanes require continual deflections the direction of which varies according to the sign of the characteristic D.

If one is content with qualitatively observing the position of the control surfaces in flight, it is impossible to establish the distinction between the different types of turns described above. It is not without reason that instructors tell their pupils: "Once the turn is started, put the controls back in neutral position . . ."

Nevertheless, a distinction between these types of turns, in flight, may be made by means of appropriate measuring instruments, and the analysis of the conditions of turns is of considerable importance in the investigation of automatic flight control.

The turn with insufficient inclination and with sideslip toward the outside is, on the contrary, entirely faulty.

We return to the numerical example and calculate the deflections and the sideslip, prescribing a lateral inclination of 24° and of 18° (with the inclination of a correct turn being $29^\circ30$); we find in fact the following results:

Deflection ξ of the ailerons

Inclination φ	1st airplane	2nd airplane	3rd airplane
$29^\circ30$	$1^\circ2$	$1^\circ06$	$0^\circ93$
24°	4°	$4^\circ4$	$4^\circ8$
18°	$7^\circ1$	$8^\circ1$	9°

Deflection ζ of the rudder

$29^\circ30$	$1^\circ32$	$1^\circ67$	2°
24°	$6^\circ9$	$6^\circ9$	$6^\circ9$
18°	13°	$12^\circ50$	$12^\circ01$

Corresponding sideslip β

$29^\circ30$	0°	0°	0°
24°	$-8^\circ40$	$-8^\circ40$	$-8^\circ40$
18°	$-17^\circ65$	$-17^\circ50$	$-17^\circ35$

It is clear that a turn with insufficient inclination is dangerous because of the large increase in drag due to the sideslip.

7. Initiation of the Turn

For the start of a turn, it is necessary to:

(1) Incline the airplane toward the center of the turn by the angle φ

(2) Impart to the airplane the angular velocity r about the axis OZ.

One may attain this result by several methods:

1. Acting simultaneously upon the ailerons and upon the rudder, that is, acting simultaneously upon control stick and rudder pedal

2. Using first the control stick and then the rudder pedal, that is, first inclining the airplane and only afterwards beginning to make it turn

3. Using first the rudder pedal and only afterwards the control stick, that is, beginning to make the airplane turn before inclining it.

Theoretically, the first maneuver is the best. By skillfully combining the movements of control stick and rudder pedal it is possible to effect the entire initiation of the turn without a sideslip of the airplane.

The lateral inclination will be obtained by pushing the control stick toward the inside. On the other hand, we shall see that, once the state of rotation is attained, the pilot must hold up the lower wing. An inversion of the lateral movements of the control stick is therefore inevitable if one wants to produce a perfect turn by the conventional maneuver of the two lateral controls.

The angular velocity r will be produced by the deflection of the rudder.

The pilot must overcome the inertia of the airplane. If he wants to obtain the motion of a turn rather quickly, he will accelerate the initiation of rotation of the airplane by giving transitorily to the rudder a larger deflection than is needed to maintain the turn once it has been started. The pilot will also perform a reverse motion of the rudder pedal.

The second method of inducing the turn is frequently used. It consists in starting the turn by a sideslip (that is a skid toward the interior) the effect of which on the airplane contributes to putting it into the turn.

The amplitude of the maneuver to be carried out with the rudder pedal is thereby decreased.

Carrying this method of piloting to the limit, that is, accentuating the maneuvering of the control stick with a view to reducing that of the rudder pedal, one would arrive at putting the airplane into the turn by means of the ailerons alone. This method of action is conceivable in view of the fact that a continual turn can be maintained by a maneuvering of the ailerons.

The third maneuver induces a skid of the airplane toward the outside. Owing to the static stability of the airplane, this skid produces a rolling moment which tends to incline the airplane toward the inside.

The amplitude of the maneuver to be applied to the control stick is then reduced.

This method of piloting is, theoretically, very bad. Skidding toward the outside constitutes fundamentally a serious fault in piloting to be avoided under any circumstances.

CHAPTER VI

THE LONGITUDINAL MOTION

1. The Equations of the Longitudinal Motion

Let (1) be the system of equations of the varied motion:

$$\left.\begin{aligned} m\left(\frac{du}{dt} + qw - rv\right) &= X + T_x + G \sin\theta \\ m\left(\frac{dw}{dt} + pv - qu\right) &= Z + T_z - G\cos\theta\cos\varphi \\ B\,\frac{dq}{dt} &= M_t + I\omega r \\ \frac{d\theta}{dt} &= +q \end{aligned}\right\} \qquad (1)$$

The external forces and moments T_x, T_z, X, Z, and M_t are supposed to be known as functions of:

(1st) The characteristics of the motion:

u, w, q, variables

v, p, r, supposed constants

(2nd) The parameters:

η deflection

σ power setting dependent on action of the pilot

The general problem consists in calculating the motion as a function of time, that is, in determining

$$\left.\begin{aligned} u &= F_1(t) \\ w &= F_2(t) \\ q &= F_3(t) \\ \theta &= F_4(t) \end{aligned}\right\} \qquad (2)$$

if one knows the initial conditions of the motion and the actions carried out by the pilot, defined as functions of time by the functions:

$$\eta = \varphi_1(t)$$

$$\sigma = \varphi_2(t)$$

In the development that we make here, we shall assume the surrounding medium to be excited by invariable displacement motions.

The theory may be generalized and extended to include the case of variable displacement motions.

The equations of the motion may be written in the form:

$$\left.\begin{aligned} \frac{du}{dt} &= f_1(u,w,q,\theta,\eta,\sigma) \\ \frac{dw}{dt} &= f_2(u,w,q,\theta,\eta,\sigma) \\ \frac{dq}{dt} &= f_3(u,w,q,\theta,\eta,\sigma) \\ \frac{d\theta}{dt} &= f_4(u,w,q,\theta,\eta,\sigma) \end{aligned}\right\} \qquad (3)$$

The two principal cases to be studied are:

A. Aircraft flying with controls fixed ($\eta = C^t$, $\sigma = C^t$) the motion of which has, however, undergone an initial disturbance, defined by the value of the variables u, w, q, θ, at the instant $t = 0$ at which the disturbance is assumed to have occurred.

The motion then is a return motion toward the initial state, and the problem is that of the stability of a motion.

B. Aircraft subjected to actions of the pilot.

The most elementary action is the following: a deflection passing abruptly, at the time t_0, from

$$\eta \text{ to } \eta + \Delta\eta$$

and the power setting of the engine changed from

$$\sigma \text{ to } \sigma + \Delta\sigma$$

The desired motion is the response of the airplane, that is, the motions the airplane carries out in order to attain the state determined by the new values of deflections. The problem is that of the maneuverability of the airplane.

The solution of the two problems is facilitated by the process of linearization of the equations.

2. Linearization of the Equations

We assume as a hypothesis that there is an equilibrium condition possible. This will be, in general, the condition which exists before the time t_0.

Assume $\bar{u}$, $\bar{w}$, $\bar{q}$, $\bar{\theta}$ to be the values of the variables corresponding to this condition and $\bar{\eta}$, $\bar{\sigma}$ to be the positions of the controls.

One has necessarily:

$$\left.\begin{aligned} f_1(\bar{u},\bar{w},\bar{q},\bar{\theta},\bar{\eta},\bar{\sigma}) &= 0 \\ f_2(\bar{u},\bar{w},\bar{q},\bar{\theta},\bar{\eta},\bar{\sigma}) &= 0 \\ f_3(\bar{u},\bar{w},\bar{q},\bar{\theta},\bar{\eta},\bar{\sigma}) &= 0 \\ f_4(\bar{u},\bar{w},\bar{q},\bar{\theta},\bar{\eta},\bar{\sigma}) &= 0 \end{aligned}\right\} \tag{4}$$

<u>A. Let us examine the case of motions with controls fixed</u>.- In the course of such a maneuver, the variables take on the values:

$$\left.\begin{aligned} u &= \bar{u} + \delta u \\ w &= \bar{w} + \delta w \\ q &= \bar{p} + \delta q \\ \theta &= \bar{\theta} + \delta\theta \end{aligned}\right\} \tag{5}$$

$\bar{u}$, $\bar{w}$, $\bar{q}$, $\bar{\theta}$ are the values corresponding to a steady state. δu, δw δq, $\delta\theta$ represent the difference between the instantaneous value and the value corresponding to this steady state.

The system (3) may be written:

$$
\left.
\begin{aligned}
\frac{du}{dt} &= \frac{d\delta u}{dt} = f_1(\bar{u} + \delta u,\ \bar{w} + \delta w,\ \bar{q} + \delta q,\ \bar{\theta} + \delta\theta) \\
\frac{dw}{dt} &= \frac{d\delta w}{dt} = f_2(\bar{u} + \delta u,\ \bar{w} + \delta w,\ \bar{q} + \delta q,\ \bar{\theta} + \delta\theta) \\
\frac{dq}{dt} &= \frac{d\delta q}{dt} = f_3(\bar{u} + \delta u,\ \bar{w} + \delta w,\ \bar{q} + \delta q,\ \bar{\theta} + \delta\theta) \\
\frac{d\theta}{dt} &= \frac{d\delta\theta}{dt} = f_4(\bar{u} + \delta u,\ \bar{w} + \delta w,\ \bar{q} + \delta q,\ \bar{\theta} + \delta\theta)
\end{aligned}
\right\} \qquad (6)
$$

The differences δu, δw, δq, $\delta\theta$ become the variables of the system.

We shall proceed by means of Tayler's formula, stopping at the first derivatives.

With (4) taken into account, the derivation is reduced to:

$$
\left.
\begin{aligned}
\frac{d\delta u}{dt} - \frac{\partial f_1}{\partial u}\delta u - \frac{\partial f_1}{\partial w}\delta w - \frac{\partial f_1}{\partial q}\delta q - \frac{\partial f_1}{\partial \theta}\delta\theta &= 0 \\
\frac{d\delta w}{dt} - \frac{\partial f_2}{\partial u}\delta u - \frac{\partial f_2}{\partial w}\delta w - \frac{\partial f_2}{\partial q}\delta q - \frac{\partial f_2}{\partial \theta}\delta\theta &= 0 \\
\frac{d\delta q}{dt} - \frac{\partial f_3}{\partial u}\delta u - \frac{\partial f_3}{\partial w}\delta w - \frac{\partial f_3}{\partial q}\delta \imath - \frac{\partial f_3}{\partial \theta}\delta\theta &= 0 \\
\frac{d\delta\theta}{dt} - \frac{\partial f_4}{\partial u}\delta u - \frac{\partial f_4}{\partial w}\delta w - \frac{\partial f_4}{\partial q}\delta q - \frac{\partial f_4}{\partial \theta}\delta\theta &= 0
\end{aligned}
\right\} \qquad (7)
$$

The partial derivatives are those corresponding to the values $\bar{u}$, $\bar{w}$, $\bar{q}$, $\bar{\theta}$ of u, w, q, θ, (they have constant values and the system has become a system of linear equations with constant coefficients,) in which the increments δu, δw, δq, $\delta\theta$ about the equilibrium condition are the variables.

<u>B. Action of the pilot</u>.- In the course of a flight in equilibrium, the elevator and the throttle undergo displacements $\Delta\eta$ and $\Delta\sigma$ at the instant t_0; from this instant onward at which the displacements are applied, the δu, δw, δq, $\delta\theta$ originate.

$\theta + \Delta\theta$, $\sigma + \Delta\sigma$ must be introduced into the system (6). Under these conditions, the system (7) is written:

$$\left.\begin{aligned}
&\frac{d\delta u}{dt} - \frac{\partial f_1}{\partial u}\delta u - \frac{\partial f_1}{\partial w}\delta w - \frac{\partial f_1}{\partial q}\delta q - \frac{\partial f_1}{\partial \theta}\delta\theta = \frac{\partial f_1}{\partial \eta}\Delta\eta + \frac{\partial f_1}{\partial \sigma}\Delta\sigma \\
&\frac{d\delta w}{dt} - \frac{\partial f_2}{\partial u}\delta u - \frac{\partial f_2}{\partial w}\delta w - \frac{\partial f_2}{\partial q}\delta q - \frac{\partial f_2}{\partial \theta}\delta\theta = \frac{\partial f_2}{\partial \eta}\Delta\eta + \frac{\partial f_2}{\partial \sigma}\Delta\sigma \\
&\frac{d\delta q}{dt} - \frac{\partial f_3}{\partial u}\delta u - \frac{\partial f_3}{\partial w}\delta w - \frac{\partial f_3}{\partial q}\delta q - \frac{\partial f_3}{\partial \theta}\delta\theta = \frac{\partial f_3}{\partial \eta}\Delta\eta + \frac{\partial f_3}{\partial \sigma}\Delta\sigma \\
&\frac{d\delta \theta}{dt} - \frac{\partial f_4}{\partial u}\delta u - \frac{\partial f_4}{\partial w}\delta w - \frac{\partial f_4}{\partial q}\delta q - \frac{\partial f_4}{\partial \theta} = 0
\end{aligned}\right\} \quad (8)$$

We can immediately write 0 in the fourth equation because of the particular form of f_4 which gives necessarily:

$$\frac{\partial f_4}{\partial \eta} = 0 \qquad \frac{\partial f_4}{\partial \sigma} = 0$$

We have here a system of linear equations with second term.

If the $\Delta\eta$ and $\Delta\sigma$ are constant (independent of time), the integration of the system presents hardly any difference from that of the preceding system.

If the $\Delta\eta$ and $\Delta\sigma$ are arbitrary functions of time, the problem may be solved analytically, without insurmountable difficulties, for certain particular forms (sinusoidal or exponential) of the functions.

If the latter are of any other form, graphical methods or methods of iteration still permit arriving at the solution.

The device which permits a replacement of the variables u, w, q, θ by their increments δu, δw, δq, $\delta\theta$ about a position of equilibrium $\bar{u}$, $\bar{w}$, $\bar{q}$, $\bar{\theta}$, is called linearization.

Under the assumption that in linearizing one writes the aerodynamic actions as functions of the instantaneous values of the variables by means of a term proportional to the increment, the linearization assumes either that the second derivatives of the forces with respect to the variables are zero, or that the increments are sufficiently small to make their effect negligible in the terms of higher order where they appear as the square, cube, etc.

The derivation shows explicitly that the method can be used only under the supposition that the aerodynamic actions are constantly determined by the instantaneous value of the variables of the problem.

Nevertheless, we shall show in chapter XI that it is also possible to apply the method if these actions are at the same time functions of the said variables and of their first derivatives.

3. Integration of the Linear Equations

A. Equations without second term.- The conventional theory states that the general solution of a system of linear equations has the following form[5]:

$$\left.\begin{aligned}
\delta u &= C_1 e^{x1t} + C_2 e^{x2t} + C_3 e^{x3t} + C_4 e^{x4t} \\
\delta w &= l_1 C_1 e^{x1t} + l_2 C_2 e^{x2t} + l_3 C_3 e^{x3t} + l_4 C_4 e^{x4t} \\
\delta q &= m_1 C_1 e^{x1t} + m_2 C_2 e^{x2t} + m_3 C_3 e^{x3t} + m_4 C_4 e^{x4t} \\
\delta \theta &= n_1 C_1 e^{x1t} + n_2 C_2 e^{x2t} + n_3 C_3 e^{x3t} + n_4 C_4 e^{x4t}
\end{aligned}\right\} \qquad (9)$$

Lagrange's method permits determination of the x and of the factors l, m, n.

The four values of x are the roots of:

$$\begin{vmatrix}
\frac{\partial f_1}{\partial u} - x & \frac{\partial f_1}{\partial w} & \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial \theta} \\
\frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial w} - x & \frac{\partial f_2}{\partial q} & \frac{\partial f_2}{\partial \theta} \\
\frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial w} & \frac{\partial f_3}{\partial q} - x & \frac{\partial f_3}{\partial \theta} \\
\frac{\partial f_4}{\partial u} & \frac{\partial f_4}{\partial w} & \frac{\partial f_4}{\partial q} & \frac{\partial f_4}{\partial \theta} - x
\end{vmatrix} = 0 \qquad (10)$$

---

[5]The exponents of e must read x_1t, x_2t, x_3t, x_4t. Physical difficulties have prevented the numbers appearing as subscripts.

The same remark applies to similar expressions occurring later on in the text.

This expression is an equation of the fourth degree in x.

$$x^4 + K_1x^3 + K_2x^2 + K_3x^1 + K_4 = 0 \tag{11}$$

and constitutes the characteristic equation of the system.

The four groups of factors l_1, m_1, n_1, l_4, m_4, n_4 will be obtained by means of any three of the four equations of the system (12) where one introduces successively the four roots x_1, x_2, x_3, x_4 of the characteristic.

$$\left.\begin{aligned}
\left(\frac{\partial f_1}{\partial u} - x\right) + l\,\frac{\partial f_1}{\partial w} + m\,\frac{\partial f_1}{\partial q} + n\,\frac{\partial f_1}{\partial \theta} &= 0\\
\frac{\partial f_2}{\partial u} + l\left(\frac{\partial f_2}{\partial w} - x\right) + m\,\frac{\partial f_2}{\partial q} + n\,\frac{\partial f_2}{\partial \theta} &= 0\\
\frac{\partial f_3}{\partial u} + l\,\frac{\partial f_3}{\partial w} + m\left(\frac{\partial f_3}{\partial q} - x\right) + n\,\frac{\partial f_3}{\partial \theta} &= 0\\
\frac{\partial f_4}{\partial u} + l\,\frac{\partial f_4}{\partial w} + m\,\frac{\partial f_4}{\partial q} + n\left(\frac{\partial f_4}{\partial \theta} - x\right) &= 0
\end{aligned}\right\} \tag{12}$$

The four factors C_1, C_2, C_3, C_4 are integration constants which one determines by introducing the initial conditions of the movement considered into the general solution. It is possible to calculate them as functions of the values $(\delta u)_0$, $(\delta w)_0$, $(\delta q)_0$, $(\delta\theta)_0$, of the initial disturbance at the time $t = 0$ when the factors $l_1 \ldots n_4$ have been preliminarily determined.

The roots of the characteristic equation may be real or complex quantities.

Each pair of complex roots defines an oscillatory motion.

In the case of the longitudinal motion, the four roots are generally complex. When such is the case, the total motion results from the superposition of two oscillatory motions.

B. Equations with constant second term.- We shall visualize only the case of abrupt deflection of the elevator $\Delta\eta$. The effect of a change in the power setting would be established by an analogous argument.

It is known that the general integral of a linear system with second term is equal to the general integral of the same system without second term plus a particular solution of the equations with second term.

In the visualized case, the second term of each of the equations is a constant.

$$\frac{\partial f_1}{\partial \eta} \Delta\eta = \frac{1}{m} \frac{\partial C_X}{\partial \eta} S\rho \frac{V^2}{2} \Delta\eta = \frac{\Delta X_0}{m}$$

$$\frac{\partial f_2}{\partial \eta} \Delta\eta = \frac{1}{m} \frac{\partial C_Z}{\partial \eta} S\rho \frac{V^2}{2} \Delta\eta = \frac{\Delta Z_0}{m}$$

$$\frac{\partial f_3}{\partial \eta} \Delta\eta = \frac{1}{B} \frac{\partial C_M}{\partial \eta} Sc\rho \frac{V^2}{2} \Delta\eta = \frac{\Delta M_0}{B}$$

Since the principal effect of the deflection is to produce a modification of the moment, one may take:

$$\frac{\partial C_X}{\partial \eta} = 0 \qquad \frac{\partial C_Z}{\partial \eta} = 0$$

Only the third equation possesses a constant term at the time of the displacement of the control surface.

In order to determine the particular solution, one notes that there exists necessarily a system of constant values Δu, Δw, Δq, $\Delta\theta$ which satisfies:

$$\frac{\partial f_1}{\partial u} \Delta u + \frac{\partial f_1}{\partial w} \Delta w + \frac{\partial f_1}{\partial q} \Delta q + \frac{\partial f_1}{\partial \theta} \Delta\theta = 0$$

$$\frac{\partial f_2}{\partial u} \Delta u + \frac{\partial f_2}{\partial w} \Delta w + \frac{\partial f_2}{\partial q} \Delta q + \frac{\partial f_2}{\partial \theta} \Delta\theta = 0$$

$$\frac{\partial f_3}{\partial u} \Delta u + \frac{\partial f_3}{\partial w} \Delta w + \frac{\partial f_3}{\partial q} \Delta q + \frac{\partial f_3}{\partial \theta} \Delta\theta = \frac{\Delta M_0}{B}$$

$$\frac{\partial f_4}{\partial u} \Delta u + \frac{\partial f_4}{\partial w} \Delta w + \frac{\partial f_4}{\partial q} \Delta q + \frac{\partial f_4}{\partial \theta} \Delta\theta = 0$$

Application of these Δu, Δw, Δq, $\Delta\theta$ to the aircraft would have the effect of placing in equilibrium the aerodynamic forces and moments produced by the deflection. These quantities constitute the desired particular solution, and the general solution of the equations may be written:

$$\delta u = \Delta u + C_1 e^{x_1 t} + C_2 e^{x_2 t} + C_3 e^{x_3 t} + C_4 e^{x_4 t}$$

$$\delta w = \Delta w + l_1 C_1 e^{x_1 t} + l_2 C_2 e^{x_2 t} + l_3 C_3 e^{x_3 t} + l_4 C_4 e^{x_4 t}$$

$$\delta q = \Delta q + m_1 C_1 e^{x_1 t} + m_2 C_2 e^{x_2 t} + m_3 C_3 e^{x_3 t} + m_4 C_4 e^{x_4 t}$$

$$\delta \theta = \Delta \theta + n_1 C_1 e^{x_1 t} + n_2 C_2 e^{x_2 t} + n_3 C_3 e^{x_3 t} + n_4 C_4 e^{x_4 t}$$

The terms Δu, Δw, Δq, $\Delta \theta$ represent the difference between the final state and the initial state.

The terms in e^{xt} represent the transient part of the response.

The integration constants will be determined by writing that at the time $t = 0$ the $(\delta u)_0$, $(\delta w)_0$, $(\delta q)_0$, $(\delta \theta)_0$ defined by the general solution are zero, that is, by calculating the C_1, C_2, C_3, C_4 corresponding to:

$$-\Delta u = C_1 e^{x_1 t} + C_2 e^{x_2 t} + C_3 e^{x_3 t} + C_4 e^{x_4 t}$$

$$-\Delta w = l_1 C_1 e^{x_1 t} + l_2 C_2 e^{x_2 t} + l_3 C_3 e^{x_3 t} + l_4 C_4 e^{x_4 t}$$

$$-\Delta q = m_1 C_1 e^{x_1 t} + m_2 C_2 e^{x_2 t} + m_3 C_3 e^{x_3 t} + m_4 C_4 e^{x_4 t}$$

$$-\Delta \theta = n_1 C_1 e^{x_1 t} + n_2 C_2 e^{x_2 t} + n_3 C_3 e^{x_3 t} + n_4 C_4 e^{x_4 t}$$

It amounts, in fact, to considering the final state as steady state and to writing that at the initial instant, after application of the deflection $\Delta\eta$, when the variables still have the values characterizing the former state, everything happens as if the airplane would deviate from its new state of equilibrium by an initial perturbation equal to $-\Delta u$, $-\Delta w$, $-\Delta \theta$.

Remarks: 1. It is clear that the result obtained is not due to the fact that we have taken $\frac{\partial f_1}{\partial \eta} = \frac{\partial f_2}{\partial \eta} = 0$. We should have arrived at the same conclusion if we had kept these derivatives $\neq 0$.

2. The case of a modification of the power setting could be investigated in the same way, but here the principal effect would be a modification of thrust, and $\frac{\partial f_1}{\partial \sigma}$ would become the important term.

C. Equations the second term of which is a function of t.- The general solution is also formed from the general solution of the equations without second term plus a particular solution of the system with a second term which is a function of t.

Although the integration is possible in certain particular cases, we can avoid this investigation, since a general formula, Duhamel's integral which we study in chapter XII, gives us the possibility of calculating the response to any maneuver of the pilot, whatever the law of deflection may be, as soon as we know the response for a constant deflection of unity.

4. Types of Motion

The motions determined by e^{xt} are aperiodic when the roots x are real; they are oscillatory when the roots are complex.

In the equations of the longitudinal motion, the four roots are, in general, complex.

$$x_{1,2} = k_{1,2} \pm s_{1,2}i$$

$$x_{3,4} = k_{3,4} \pm s_{3,4}i$$

The transient part of the solution is formed from the superposition of two oscillatory motions.

The investigation of the stability of the motion of the airplane appears as follows:

1. If one attempts to determine uniquely whether a motion is dynamically stable, that is, whether the airplane tends toward its state of equilibrium, it suffices to make sure that all the e^{xt} decrease when the time increases, whatever the factors $l_1 \ldots n_4$ and the integration constants may be.

It is not necessary to solve, for this purpose, the equation of the fourth degree. One must make sure (and this is sufficient) that the roots are negative when they are real, or that their real part is negative when they are imaginary.

Routh has shown that if an equation of the fourth degree is written in the form:

$$x^4 + K_1x^3 + K_2x^2 + K_3x + K_4 = 0$$

the roots will have a negative real part, or will be entirely negative, if the following conditions are satisfied:

$$K_1 > 0$$

$$K_2 > 0$$

$$K_3 > 0$$

$$K_4 > 0$$

$$R = K_2 - \frac{K_3}{K_1} - \frac{K_1K_4}{K_3} > 0$$

These conditions constitute a criterion of dynamic stability.

2. If one desires to know the characteristic period and damping of the motion which results after a perturbation without determining the amplitudes, it is necessary (and sufficient) to solve the characteristic equation.

The values k and s determine the periods and the damping.

For any oscillatory motion, the period T is given by:

$$T = \frac{2\pi}{s}$$

The duration D required for the amplitudes to decrease to one half (or to double) is:

$$D = \frac{ln0.5}{k} = \frac{0.692}{k}$$

The logarithmic decrement δ depends on s and k. In fact:

$$\delta = ln\frac{u_2}{u_1}$$

or

$$\frac{u_2}{u_1} = e^{kT} = e^{2\pi k/s}$$

$$\delta = 2\pi \frac{k}{s}$$

From the practical view point, one uses sometimes, instead of the decrement, the ratio:

$$R = T/D$$

This ratio is equal to:

$$\frac{2\pi}{l n 0.5} \frac{k}{s} = \frac{6.2832}{0.692} \frac{k}{s}$$

It is connected with the logarithmic decrement by the relation:

$$R = \frac{\delta}{l n 0.5} = 1.4\delta$$

In all cases, the period and the damping are independent of the performed maneuver and of the initial conditions.

3. If one desires to know the amplitude and the phase displacement of the various motions which follow a prescribed initial perturbation or a unit maneuver of the controls, one must determine the factors l, m, n and the integration constants C_1, C_2, C_3, C_4.

The factors l, m, n are independent of the considered initial perturbation, they depend on the aerodynamic characteristics of the airplane.

In contrast, the integration constants $C_1 \ldots C_4$ depend in every case on the initial perturbation visualized.

If the four roots are complex, the transient part of the solution may be written[6]:

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[6]In order to simplify the notation, everything connected with the pair of complex roots 1, 2 is represented without subscript, and everything connected with the pair 3, 4, is provided with the sign '.

$$\delta u = e^{kt}A_u \sin(st + \varphi_u) + e^{k't}A'_u \sin(s't + \varphi'_u)$$

$$\delta w = e^{kt}A_w \sin(st + \varphi_w) + e^{k't}A'_w \sin(s't + \varphi'_w)$$

$$\delta q = e^{kt}A_q \sin(st + \varphi_q) + e^{k't}A'_q \sin(s't + \varphi'_q)$$

$$\delta\theta = e^{kt}A_\theta \sin(st + \varphi_\theta) + e^{k't}A_\theta \sin(s't + \varphi'_\theta)$$

The factors A represent the largest possible amplitude.

The terms φ represent the phase displacements.

The calculations connecting the sixteen factors A and φ with the sixteen factors C, l, m, n have been placed in appendix II so as not to encumber the derivation.

Important remarks: 1. Since the equations are linear, the amplitude of all motions is proportional to the causes which produce them (initial perturbations or deflections).

2. For the same reason, the motion produced by several simultaneous causes is equal to the sum of the motions which would be produced by each of these causes acting separately.

3. The method of integration is simple in theory but leads to very long numerical calculations.

It is useful to replace it, in practice, by a method derived from operational calculus.

These procedures are investigated in chapter XVIII.

4. The preceding problem is frequently treated by taking as variables $\delta u/V$ and $\delta w/V$ instead of δu and δw.

The variable $\delta w/V$ is practically equal to $-\delta\alpha$.

This does not introduce any change in the characteristic determinant.

5. The problem may also be treated by writing the equations of equilibrium of the forces along the flight path and normal to the flight path.

The variables then are:

The velocity V

The slope of the flight path τ (generally supposed to be positive when the airplane climbs)

The angle of attack α

The angular velocity $q = -\frac{d(\alpha + \tau)}{dt}$ if one maintains the definition of the positive sense of the pitching rotations.

The angle of trim $\theta = -(\alpha + \tau)$ is no longer one of the fundamental variables.

This manner of notation permits introduction of the derivatives of the lift and of the drag, and does not require the transformation of these forces into components along the axes fixed to the airplane.

It is, however, less suitable to the goal we have set ourselves, the investigation of automatic flight control, since the reference employed there most frequently is precisely the angle of trim θ, and not the inclination of the flight path τ which occurs only in devices intended to produce an entirely automatic landing, not yet in general use.

CHAPTER VII

THE LONGITUDINAL MOTION

1. Transformation of the General Equations

We take up again the general equations:

$$\frac{d\delta u}{dt} - \frac{\partial f_1}{\partial u}\delta u - \frac{\partial f_1}{\partial w}\delta w - \frac{\partial f_1}{\partial q}\delta q - \frac{\partial f_1}{\partial \theta}\delta\theta = \frac{\partial f_1}{\partial \eta}\Delta\eta + \frac{\partial f_1}{\partial \sigma}\Delta\sigma$$

$$\frac{d\delta w}{dt} - \frac{\partial f_2}{\partial u}\delta u - \frac{\partial f_2}{\partial w}\delta w - \frac{\partial f_2}{\partial q}\delta q - \frac{\partial f_2}{\partial \theta}\delta\theta = \frac{\partial f_2}{\partial \eta}\Delta\eta + \frac{\partial f_2}{\partial \sigma}\Delta\sigma$$

$$\frac{d\delta q}{dt} - \frac{\partial f_3}{\partial u}\delta u - \frac{\partial f_3}{\partial w}\delta w - \frac{\partial f_3}{\partial q}\delta q - \frac{\partial f_3}{\partial \theta}\delta\theta = \frac{\partial f_3}{\partial \eta}\Delta\eta + \frac{\partial f_3}{\partial \sigma}\Delta\sigma$$

$$\frac{d\delta \theta}{dt} - \frac{\partial f_4}{\partial u}\delta u - \frac{\partial f_4}{\partial w}\delta w - \frac{\partial f_4}{\partial q}\delta q - \frac{\partial f_4}{\partial \theta}\delta\theta = 0$$

It is clear that:

$$\frac{\partial f_1}{\partial u} = \frac{1}{m}(T'_u + X'_u) \qquad \frac{\partial f_2}{\partial u} = -\bar{q} + \frac{1}{m}Z'_u$$

$$\frac{\partial f_1}{\partial w} = -\bar{q} + \frac{1}{m}(X'_w + T'_w) \qquad \frac{\partial f_2}{\partial w} = \frac{1}{m}Z'_w$$

$$\frac{\partial f_1}{\partial q} = -\bar{w} + \frac{1}{m}(X'_q + T'_q) \qquad \frac{\partial f_2}{\partial q} = \bar{u} + \frac{1}{m}Z'_q$$

$$\frac{\partial f_1}{\partial \theta} = g\cos\theta \qquad \frac{\partial f_2}{\partial \theta} = g\sin\theta$$

$$\frac{\partial f_1}{\partial \eta} = \frac{1}{m}X'_\eta \qquad \frac{\partial f_2}{\partial \eta} = \frac{1}{m}Z'_\eta$$

$$\frac{\partial f_1}{\partial \sigma} = \frac{1}{m}T'_\sigma \qquad \frac{\partial f_2}{\partial \sigma} = \frac{1}{m}Z'_\sigma$$

$$\frac{\partial f_3}{\partial u} = \frac{1}{B} M'_u \qquad \frac{\partial f_4}{\partial u} = 0$$

$$\frac{\partial f_3}{\partial w} = \frac{1}{B} M'_w \qquad \frac{\partial f_4}{\partial w} = 0$$

$$\frac{\partial f_3}{\partial q} = \frac{1}{B} M'_q \qquad \frac{\partial f_4}{\partial q} = + 1$$

$$\frac{\partial f_3}{\partial \theta} = \frac{1}{B} M'_\theta \qquad \frac{\partial f_4}{\partial \theta} = 0$$

$$\frac{\partial f_3}{\partial \eta} = \frac{1}{B} M'_\eta \qquad \frac{\partial f_4}{\partial \eta} = 0$$

$$\frac{\partial f_3}{\partial \sigma} = \frac{1}{B} M'_\sigma \qquad \frac{\partial f_4}{\partial \sigma} = 0$$

As we visualize the stability for rectilinear motion,

$$\bar{q} = 0$$

(It would not be the same if we wanted to investigate the longitudinal stability during turns.)

The derivatives X'_q, T'_q, Z'_q are small and we shall assume them to be zero.

It will be the same with regard to T'_w.

If one wanted, nevertheless, to take these derivatives into account, it would suffice to add the corresponding terms to the set of equations.

As to the terms of the second member which define the action of the controls, we shall suppose that the elevator acts exclusively upon the moment, whence:

$$X'_\eta = Z'_\eta = 0$$

The derivatives of the aerodynamic actions $X'_u \ldots M'_\eta$ have been defined in chapters III and IV, by dimensionless factors $a'_1 \ldots h_3$.

Let us write the mass and the moment of inertia as functions of the density of the airplane as a nondimensional quantity:

$$m = \mu Sc \frac{\rho}{2}$$

$$B = \mu r^2 Sc \frac{\rho}{2}$$

Let us put, moreover:

$$c_1 = -\frac{\partial f_1}{\partial q} \times \frac{\mu c}{V} = w \frac{\mu c}{V}$$

$$d_1 = -\frac{\partial f_1}{\partial \theta} \times \frac{\mu c}{V} = -g \cos \theta \frac{\mu c}{V}$$

$$c_2 = -\frac{\partial f_2}{\partial q} \times \frac{\mu c}{V} = -u \frac{\mu c}{V}$$

$$d_2 = -\frac{\partial f_2}{\partial \theta} \times \frac{\mu c}{V} = -g \sin \theta \frac{\mu c}{V}$$

$$c_4 = -\frac{\partial f_4}{\partial q} \times \frac{\mu c}{V} = -\frac{\mu c}{V}$$

We have furthermore: $a = a'_1 + a''_1$.

The equations of motion become, when these substitutions have been made:

$$\frac{\mu c}{V}\frac{d\delta u}{dt} + a_1 \delta u + b_1 \delta w + c_1 \delta q + d_1 \delta \theta = s_1 V \Delta \sigma$$

$$\frac{\mu c}{V}\frac{d\delta w}{dt} + a_2 \delta u + b_2 \delta w + c_2 \delta q + d_2 \delta \theta = s_2 V \Delta \sigma$$

$$\frac{\mu c}{V}\frac{d\delta q}{dt} + \frac{c}{r^2} a_3 \delta u + \frac{c}{r^2} b_3 \delta w + \frac{cl}{r^2} c_3 \delta q + \frac{c}{r^2} V d_3 \delta \theta = \frac{c}{r^2}\left(h_3 V \Delta \eta + s_3 V \Delta \sigma\right)$$

$$\frac{\mu c}{V}\frac{d\delta \theta}{dt} + 0 + 0 + c_4 \delta q + 0 = 0$$

In these equations all terms appearing in the first and the second equation have the dimensions LT^{-1}, in the third: the dimensions T^{-1}, in the fourth: the dimensions 0.

Let us note, furthermore, that:

c_1 and c_2 have the dimensions L

d_1 and d_2 have the dimensions LT^{-1}

c_4 has the dimension T

2. Factors Depending on the Aerodynamic Characteristics of the Airplane

We have seen that it was possible to calculate the factors $a_1 \ldots c_3$ starting from the aerodynamic coefficients found in wind tunnel tests.

Let us recall the expressions:

$$a_1 = a'_1 + a''_1$$

$$a'_1 = \alpha \frac{\partial C_x}{\partial \alpha} - 2C_x$$

$$a''_1 = \alpha \frac{S_h}{S}\left(\frac{\partial C_T}{\partial \gamma} n'\gamma - 2C_T\right)$$

$$b_1 = \frac{\partial C_x}{\partial \alpha}$$

$$a_2 = \alpha \frac{\partial C_z}{\partial \alpha} - 2C_z$$

$$b_2 = \frac{\partial C_z}{\partial \alpha}$$

$$a_3 = \alpha \frac{\partial C_{M_t}}{\partial \alpha} - \frac{\partial C_{M_t}}{\partial \gamma} n'\gamma - 2C_{M_t}$$

$$b_3 = \frac{\partial C_{M_t}}{\partial \alpha}$$

$$c_3 = - \frac{\partial C_{M_t}}{\partial \chi}$$

One would also have:

$$d_3 = - \frac{\partial C_M}{\partial \theta}$$

if the angle of trim would exert a direct effect on the moment M.

This effect is zero because the aerodynamic reactions depend only on the linear and angular velocities. Nevertheless we maintain the term in d_3 in the equations in order to give them the generality necessary for the investigation of automatic flight control.

Finally, the factors characterizing the effect of the controls are:

$$s_1 = \frac{\partial C_Z}{\partial \sigma}$$

$$s_2 = \frac{\partial C_Z}{\partial \sigma}$$

$$s_3 = \frac{\partial C_M}{\partial \gamma} \frac{\partial \gamma}{\partial \sigma}$$

$$h_3 = \frac{\partial C_M}{\partial \eta}$$

3. The Aerodynamic Time

An examination of the equations shows that a new simplification may be used.

So far, the unit of time U_t was the second.

Let us express the time by means of a new unit U_τ related to the second by

$$U_\tau = \frac{\mu c}{V} U_t$$

An interval of time equal to t seconds will be expressed in the new system by a number

$$\tau = t \frac{V}{\mu c}$$

The quantity τ is dimensionless. It constitutes the aerodynamic time.

Replacing in the first term of each of the first members of the equations

$$\frac{\mu c}{V} \frac{d}{dt} \quad \text{by} \quad \frac{d}{d\tau}$$

we obtain:

$$\frac{d\delta u}{d\tau} + a_1\delta u + b_1\delta w + c_1\delta q + d_1\delta\theta = s_1 V\Delta\sigma$$

$$\frac{d\delta w}{d\tau} + a_2\delta u + b_2\delta w + c_2\delta q + d_2\delta\theta = s_2 V\Delta\sigma$$

$$\frac{d\delta q}{d\tau} + \frac{c}{r^2}\, a_3\delta u + \frac{c}{r^2}\, b_3\delta w + \frac{cl}{r^2}\, c_3\delta q + \frac{c}{r^2}\, Vd_3\delta\theta = \frac{c}{r^2}\, h_3 V\Delta\eta + \frac{c}{r^2}\, s_3 V\Delta\sigma$$

$$\frac{d\delta\theta}{d\tau} + c_4\delta q = 0$$

The characteristic determinant of the first members of the system becomes:

$$\begin{vmatrix} a_1 + \lambda & b_1 & c_1 & d_1 \\ a_2 & b_2 + \lambda & c_2 & d_2 \\ \frac{c}{r^2}\, a_3 & \frac{c}{r^2}\, b_3 & \frac{cl}{r^2}\, c_3 + \lambda & \frac{c}{r^2}\, Vd_3 \\ 0 & 0 & c_4 & + \lambda \end{vmatrix} = 0$$

The roots of the equation

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_u = 0$$

permit writing the solution of the differential system without second member, by means of four expressions of the form:

$$\delta u = C_1 e^{\lambda_1\tau} + C_2 e^{\lambda_2\tau} + C_3 e^{\lambda_3\tau} + C_4 e^{\lambda_4\tau}$$

with, (in the general case):

$$\lambda_{1,2} = \kappa_{1,2} \pm \sigma_{1,2}i$$

$$\lambda_{3,4} = \kappa_{3,4} \pm \sigma_{3,4}i$$

Since this quantity λ is dimensionless, the period T and the duration D required for reduction to one half of the amplitudes given by:

$$T = \frac{2\pi}{\sigma} \qquad D = \frac{\ln 0.5}{\kappa}$$

are expressed in the unit of aerodynamic time.

They must be multiplied by $\mu c/V$ to find T and D expressed in seconds.

It is important to remark that the change from t to τ made at the end of the calculation does not affect the unit of time by means of which the velocities u, w, q are measured. One continues measuring these velocities utilizing the second as unit of time.

4. Development of the Determinant

Developing the determinant as a function of the last line, we obtain:

$$\begin{vmatrix} a_1 + \lambda & b_1 & c_1 \\ a_2 & b_2 + \lambda & c_2 \\ \frac{c}{r^2} a_3 & \frac{c}{r^2} b_3 & \frac{cl}{r^2} c_3 + \lambda \end{vmatrix} - c_4 \begin{vmatrix} a_1 + \lambda & b_1 & d_1 \\ a_2 & b_2 + \lambda & d_2 \\ \frac{c}{r^2} a_3 & \frac{c}{r^2} b_3 & \frac{c}{r^2} Vd_3 \end{vmatrix} = 0$$

Let, arranging with respect to terms of subscript 3:

$$A_1 = \left(a_1 + b_2\right) + \frac{cl}{r^2} c_3$$

$$A_2 = \left(a_1 b_2 - a_2 d_1\right) + a_3 \frac{c}{r^2}\left(-c_1\right) + b_3 \frac{c}{r^2}\left(-c_2\right) + c_3 \frac{cl}{r^2}\left(a_1 + b_2\right) + d_3 \frac{c}{r^2}\left(-c_4 V\right)$$

$$A_3 = a_3 \frac{c}{r^2}\left(b_1 c_2 - b_2 c_1 + d_1 c_4\right) + b_3 \frac{c}{r^2}\left(a_2 c_1 - a_1 c_2 + d_2 c_4\right) + c_3 \frac{cl}{r^2}\left(a_1 b_2 - a_2 b_1\right) + d_3 \frac{c}{r^2}\left(-c_4 V\right)\left(a_1 + b_2\right)$$

$$A_4 = a_3 \frac{c}{r^2}\left(b_2 c_4 d_1 - b_1 d_2 c_4\right) + b_3 \frac{c}{r^2}\left(a_1 c_4 d_2 - a_2 d_1 c_4\right) + d_3 \frac{c}{r^2}\left(-c_4 V\right)\left(a_1 b_2 - a_2 b_1\right)$$

We replace the quantities c_1, d_1, c_2, d_2, c_4 by their values, and obtain:

$$A_1 = \left(a_1 + b_2\right) + \frac{cl}{r^2} e_3$$

$$A_2 = \left(a_1 b_2 - a_2 b_1\right) + a_3 \frac{c^2}{r^2} \mu \left(- \frac{\overline{w}}{V}\right) + b_3 \frac{c^2}{r^2} \mu \left(\frac{\overline{u}}{V}\right) + c_3 \frac{cl}{r^2}\left(a_1 + b_2\right) + d_3 \frac{c^2}{r^2} \mu$$

$$A_3 = + a_3 \frac{c^2}{r^2} \mu \left(-b_1 \frac{\overline{u}}{V} - b_2 \frac{\overline{w}}{V} + \cos \theta \frac{g\mu c}{V^2}\right) + b_3 \frac{c^2}{r^2} \mu \left(a_2 \frac{\overline{w}}{V} + a_1 \frac{\overline{u}}{V} + \right.$$

$$\left. \sin \theta \frac{g\mu c}{V^2}\right) + c_3 \frac{cl}{r^2}\left(a_1 b_2 - a_2 b_1\right) + d_3 \frac{c^2}{r^2} \mu \left(a_1 + b_2\right)$$

$$A_4 = + a_3 \frac{c^2}{r^2} \mu \frac{g\mu c}{V^2}\left(b_2 \cos \theta - b_1 \sin \theta\right) +$$

$$b_3 \frac{c^2}{r^2} \mu \frac{g\mu c}{V^2}\left(a_1 \sin \theta - a_2 \cos \theta\right) + b_3 \frac{c^2}{r^2} \mu \left(a_1 b_2 - b_2 a_1\right)$$

It can be immediately verified that:

$$\frac{g\mu c}{V^2} = Cz = CZ - \frac{w}{V} = \alpha + \frac{\overline{u}}{V} = 1$$

The transformations may be carried still further; however, the preceding expressions suffice for finding the essential facts.

The coefficients A_1, A_2, A_3, A_4 depend:

(1) On seven purely aerodynamic parameters, functions of the angle of attack, namely: four parameters dependent on the derivatives of the forces a_1, b_1, a_2, b_2; three parameters dependent on the derivatives of the moments a_3, b_3, c_3

The parameter d_3 is zero because the aerodynamic actions are independent of the orientation of the airplane in space, but it is provisionally maintained in the equations with a view to a subsequent generalization of the theory:

(2) On the density of μ of the airplane

(3) On the distribution of masses in the airplane, characterized by c/r.

(4) On a purely geometric ratio l/c which is introduced only because we used in the definition of the pitching coefficient χ a unit of length l other than c

(5) Considered flight regime, characterized by:

$w/V = -\alpha$ (α = angle of attack)

The angle θ

The lift coefficient $Cz = \frac{g\mu c}{V^2}$

5. Discussion of the Characteristic

It is difficult to establish, by discussion of the characteristic equation, how the solutions λ will vary when each of the parameters mentioned above varies, and it is completely impossible to determine how the terms which define the amplitudes of the motions will vary in the general solution of the differential system.

We collide here with the practical inconvenience of a complicated expression. The result depends on particular numerical values.

Consequently, we obtain from this calculation only information regarding the periods and the damping of the motion for the particular values assigned to the characteristic equations.

If we find that the system is unstable or possesses undesirable characteristics, the calculation does not provide us with any immediate indication of the manner for making it stable or satisfactory.

Actually, these inconveniences are not too serious. The number of characteristics which the designer can influence is very limited.

Certain elements such as the $b_2 = dC_z/d\alpha$, are fixed by the general properties of the flows, and only the discussion of the density μ and of the ratio c/r as functions of the three derivatives of the moment (that is, a_3, b_3, c_3) gives results of direct interest for the designer.

Numerous reports, consisting of calculations of numerical examples where one of the elements varies systematically, have given very definite indications on the direction of the development of the phenomena.

There exists already a considerable documentation regarding this subject.

The derivatives

$$\frac{\partial C_M}{\partial \alpha} = b_3 \quad \frac{\partial C_M}{\partial \chi} = c_3$$

are the most important ones.

$b_3 > 0$ expresses that the airplane, which is supposed to oscillate around an axis fixed in space, has the tendency to maintain its equilibrium angle of attack. A positive sign defines, in fact, a diving moment for an increase in angle of attack.

A similar tendency is presented by an airplane possessing the characteristic called static stability, that is to say, weather-vane stability.

The procedures are perfectly well known which permit making an airplane statically stable and even giving it a predetermined degree of static stability. The displacement of the center of gravity from the rear toward the front is the factor having the greatest effect.

$c_3 > 0$ indicates that a positive speed of rotation gives rise to a negative moment, proportional to that speed. The term c_3 defines the damping in pitch.

One may carry the investigation up to the three stages described in section 4 of the preceding chapter.

(a) Verification of the criterion of stability.- One fixes one or several values of a_3. For each of them, the A_1, A_2, . . . are linear functions of b_3 and c_3.

Directing the b_3 and c_3 along the axes, one plots the straight lines:

$$A_1 = 0$$

$$A_2 = 0$$

$$A_3 = 0$$

$$A_4 = 0$$

and the curve

$$R = A_2 - \frac{A_3}{A_1} - \frac{A_1 A_4}{A_3} = 0$$

which separates the regions of stability from those of instability.

It is easy to recognize which sides of the lines correspond to stability and to instability: one determines which is the boundary common to all regions of stability.

(b) Investigation of periods and damping.- The solution of the equations is necessary. There exists, however, an approximation method which, though not giving the exact roots, furnishes approximate roots knowledge of which is perfectly sufficient to define the nature of the phenomenon.

The equation

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$$

can, in fact, be put in the form

$$\left(\lambda^2 + A_1\lambda + A_2\right)\left[\lambda^2 + \left(\frac{A_3}{A_2} - \frac{A_1 A_4}{A_2^2}\right)\lambda + \frac{A_4}{A_2}\right] = 0$$

Each of the equations of the second degree defines an oscillatory motion: one is a motion of short period, rapidly damped; the other is a motion of long period, slightly damped.

(c) Calculation of the amplitudes.- If one carries the calculation as far as determination of the amplitudes, one sees that the short-period motion consists primarily of an oscillation about the center of gravity whereas the long-period oscillation corresponds to a succession of rises and falls in the trajectory of the center of gravity.

These motions are accompanied by considerable variations in the velocity V or u; the airplane accelerates in descent, and vice versa.

This completely differentiates the slow oscillation from the rapid oscillation which takes place without appreciable modification of the speed.

An analysis of the motions is easily made by examining the solutions of the system of equations, put in the form of curves.

As an example, we give such diagrams for an airplane the characteristics of which would be:

$$a_1 = +0.125 \qquad b_1 = +0.345 \qquad c_3 = +1.37$$

$$a_2 = -0.80 \qquad b_2 = +3.75 \qquad \mu = 28.8$$

$$a_3 = 0 \qquad b_3 = 0.344 \qquad c/r = 1.53$$

$$l/c = 2.6$$

for a flight regime corresponding to:

$$\alpha = 0 \qquad \theta = 0 \qquad C_z = 0.40$$

The calculated motions are those that follow:

(a) An initial perturbation corresponding to the effect of a horizontal gust coming from in front

$$(\delta \mathbf{u})_0 = 0.25V$$

(b) An initial perturbation:

$$(\delta w)_0 = -0.20V$$

which corresponds to an ascending gust producing at the initial instant an increment in angle of attack:

$$(\delta\alpha)_0 = +0.2 \text{ rad}$$

(c) An initial perturbation formed by the superposition of:

$$(\delta\alpha)_0 = +0.2 \text{ rad}$$

$$(\delta\theta)_0 = -0.2 \text{ rad}$$

that is, an angular displacement in space of the aircraft.

(d) The application of an abrupt deflection producing

$$\Delta C_M = -0.015$$

which is intended to establish the airplane on a flight path ascending more steeply, travelled at larger angle of attack and at lower speed.

(e) The suppression of the thrust due to stopping of the engine - extreme case of a variation in throttle setting.

One can find the following facts in the diagrams:

(a) Effect of $(\delta u)_0 > 0$.- The first consequence of the perturbation is the appearance of an excess of lift. Under its effect the trajectory becomes ascending.

The inclination of the trajectory has the effect of diminishing the angle of attack.

The aircraft which due to its static stability tends to maintain a constant angle of incidence will perform a nose-up motion depending on the rapid oscillation.

The airplane which has very nearly found again its incidence of equilibrium condition now follows an ascending trajectory where the conditions of equilibrium of power are not satisfied. It will find its power equilibrium again through the effect of the slow oscillation.

(b) Initial perturbation $(\delta w)_0$.- The perturbation $(\delta w)_0$, of negative value, becomes manifest by an increase in angle of attack, at the instant $t = 0$; this increment in angle of attack tends to decrease through the diving motion due to the static stability, and through the undulation of the trajectory of the airplane resulting from the excess of lift.

The diving motion is the more energetic the greater the static stability. It depends on the rapid oscillation.

When the equilibrium of moment has been reestablished and that oscillation has ended, the airplane follows, however, a descending trajectory. The conditions of power equilibrium are not satisfied, and the airplane accelerates. An oscillation of long period originates; its amplitude is the larger the greater the inclination of the airplane has been in the course of these phenomena, that is, the greater the static stability.

(c) Abrupt pull-up motion.- An initial pull-up motion modifies at the same time $\delta\alpha$ and $\delta\theta$ as long as the trajectory itself has not been modified.

The diagram shows that the initial perturbation is opposed partly by the diving motion (which will increase with the static stability and which depends on the rapid oscillation), partly by a modification in the inclination of the trajectory.

This inclination creates a lack of equilibrium of power which in turn causes the beginning of the slow oscillation.

(d) Effect of the deflection of the elevator.- The initial effect is a motion which tends to make the airplane nose up. However, this motion which depends on the rapid oscillation is visible only at the beginning of the diagram of the angle of attack.

Since the airplane is required to settle itself on a trajectory which greatly differs from the initial trajectory as to the speed and the trim of the airplane, motions which depend on the slow oscillation are produced.

Inspite of a considerable degree of static stability, the slow oscillation does not arise strictly at constant angle of attack.

At a constant deflection η, one has $dC_M = 0$, but

$$dC_M = \frac{\partial C_M}{\partial \alpha} d\alpha + \frac{\partial C_M}{\partial \chi} d\chi + \frac{\partial C_M}{\partial \gamma} \frac{d\gamma}{dV} dV$$

If $\partial C_M / \partial \gamma = 0$, the amplitudes of $d\alpha$ and $d\chi$ will be in the ratio:

$$\frac{d\alpha}{d\chi} = - \frac{\partial C_M / \partial \chi}{\partial C_M / \partial \alpha}$$

The oscillation in α, opposed to that of q, that is, at the derivative of θ, will lag behind that of θ by $\pi/2$.

(e) Suppression of the thrust corresponding to the stopping of the engine.- Since the calculations have been carried out with $\partial C_M / \partial \gamma = 0$, the suppression of the thrust does not produce any direct effect on the equilibrium of moments. The airplane will slow up and the motions which will be produced will all depend on the slow oscillation because they result from the airplane's pursuit of power-equilibrium conditions.

The complete stoppage of the engine corresponds to an initial perturbation:

$$\delta\theta = -\text{arc tg}\,\frac{C_X}{C_Z}$$

It is well to remark that, in the case b, the phenomenon is greatly schematized. The diagram corresponds to the roughest calculation one can possibly make; it supposes, in fact:

That the gust arises abruptly

That the airplane is, from the first instant onward, in its entirety subjected to this gust.

6. The Total Damping

The motion of the airplane results from the superposition of two motions:

The rapid oscillation is strongly damped; the duration D of decrease to half-amplitude is a fraction of a second.

The slow oscillation is slightly damped; D is of the order of 30 seconds.

The damping of each oscillation is proportional to the real part (necessarily negative) of the corresponding root.

Now the sum of the roots = $-A_1$.

The coefficient A_1, with changed sign, may therefore be regarded as the total of available damping.

One can influence the total damping only by means of the parameters a_1, b_2, and c_3.

The other parameters do not affect the total of available damping. They can influence only the distribution of the damping between the two components of the motion.

If one succeeds, by the effect of a_3, b_3, or d_3, in increasing the damping of the slow oscillation, one diminishes that of the rapid oscillation.

Let us recall

$$\lambda = \kappa \pm \sigma i$$

$$\lambda' = \kappa' \pm \sigma' i$$

where the pair of roots 3, 4 relative to the slow oscillation, takes on the sign prime.

Since $\kappa > \kappa'$ when we make a small quantity n pass from κ to κ', D' which has become $\frac{ln 0.5}{\kappa - n}$ is little modified, whereas D' which has become $\frac{ln 0.5}{\kappa' + n}$ is strongly reduced.

Any alteration in the airplane which increases κ' at the expense of κ is favorable, since the damping of the rapid oscillation can generally be somewhat diminished without disadvantage.

Remark: It is clear that the method of approximate solution pointed out above does not give exact results since it leads to attributing to the rapid oscillation a damping equal to the total available damping.

7. Remarks on the Expressions A_1, A_2, A_3, A_4

A. The factor μ always multiplies the terms in a_3 and b_3, but does not multiply c_3:

b_3 is a restoring moment

c_3 is a damping moment.

The increase of the airplane density μ will have an unfavorable effect; an airplane of high density (that is, with a high wing loading) will necessitate a larger damping coefficient $\partial C_M/\partial \chi$ (in absolute value) than a machine with small loading.

B. At normal angles of incidence, the factors a_1, b_1, a_2, b_2 have such values as to make the term in $a_1b_2 - a_2b_1$ appearing in A_2 positive.

On the other hand, the factors which multiply a_3, b_3, c_3 in A_2, A_3, or A_4 are generally positive.

The only factor which can become negative is the one which multiplies b_3 in the expression A_3.

Ascending trajectories may lead to values such that $\sin\theta$ gives its sign (negative) not only to the term in b_3 but also to the entire expression A_3.

This fact explains that the stability of ascending flight paths is always more precarious than that of horizontal or descending flight paths.

Pursuit planes endowed with normal characteristics become frequently unstable in the case of steeply ascending trajectories at full engine speed because the term in b_3 which appears in A_3 has become negative and gives its sign to the entire A_3.

One finds therefore that the increase in static stability of the air frame can only increase the instability of the motion.

The instability considered above always affects the slow oscillation.

C. The sense in which T, D, T' and D' develop if a_3, b_3 or c_3 are altered can be established in a general manner only by treating numerical examples and the conclusions are, on principle, of value only in the particular case considered.

Anyway, we shall indicate the conclusions at which we have arrived (airplane with the same characteristics as the one to which refer the curves of figure 24).

Effect of a_3.

Let us recall that a_3 can express:

$$a_3 = \frac{\partial C_{M_t}}{\partial \alpha}\alpha - \frac{\partial C_{M_t}}{\partial \gamma} n'\gamma - 2C_{M_t}$$

The parameter a_3 exerts only an insignificant effect on the rapid oscillation, but a considerable one on the slow oscillation.

At small angles of attack (α in the neighborhood of zero) its variations stem exclusively from the effect

$$-\frac{\partial C_{M_t}}{\partial \gamma} n'\gamma$$

An increase in the velocity of the aircraft diminishes the thrust of the propeller.

When the propeller axis passes above the center of gravity, the thrust reduction produces a nose-up moment. An increase of V produces a negative ΔC_M; the airplane is characterized by:

$$\frac{\partial C_{M_t}}{\partial \gamma} < 0$$

Such a characteristic is favorable for the stability. As a result of the form of A_4 a machine which is statically unstable for slow conditions may be made dynamically stable by:

$$\frac{\partial C_{M_t}}{\partial \gamma} < 0$$

It must, however, be noted that the stability produced by this means does not correspond to very desirable flight-path characteristics. The period of the slow oscillation decreases whereas the duration D increases: oscillations of this type may become inconvenient.

Inversely, an airplane in which the thrust axis passes below the center of gravity is generally characterized by:

$$\frac{\partial C_{M_t}}{\partial \gamma} > 0$$

Such a characteristic tends to make the slow oscillation unstable. If the flight path of an aircraft has become, for any reason whatsoever, a descending one, the plane will necessarily accelerate. The moment M becomes positive, that is, nose-down and tends to oppose the levelling-out of the flight path and to maintain or accentuate the diving condition.

A statically stable airplane may become dynamically unstable if:

$$\frac{\partial C_{M_t}}{\partial \gamma} > 0$$

The calculations show from what value of a_3 onward this instability may manifest itself, taking into account the value of the other parameters.

It is interesting to note that a machine characterized by $\partial C_{M_t}/\partial\gamma < 0$ has necessarily the tendency to nose-up when the engine is stopped. A machine characterized by $\partial C_{M_t}/\partial\gamma > 0$ tends to become nose-down. This last reaction is favorable for safety.

The airplanes fall into different classes regarding the effect of the $\partial C_{M_t}/\partial\gamma$ according to whether one considers the phenomena occurring when the motor turns normally or those which accompany an abrupt stopping.

Effect of $b_3 = \frac{\partial C_{M_t}}{\partial\alpha}$.

The period of the rapid oscillation (motion of rotation about the center of gravity) is necessarily linked to the magnitude of the static stability, or the restoring moment.

This period decreases when b_3 increases.

It increases when b_3 decreases, and, for a low degree of static stability, the rapid motion ceases to be oscillatory and becomes the sum of two aperiodic motions.

Nevertheless the roots λ_1 and λ_2 remain negative and the corresponding aperiodic motions remain stable, even for a negative static stability.

The mechanical cause of this phenomenon is easily found; it is due to the undulations of the flight path produced by the increase of lift $(b_2 > 0)$ which accompanies any increase in angle of attack.

The damping of the rapid motion as long as the latter maintains its oscillatory character is independent of b_3.

The period of the slow oscillation decreases also when b_3 increases. Its damping also decreases.

When b_3 decreases, the period increases; then the motion ceases to be oscillatory.

When $b_3 = 0$, the instability limit $A_4 = 0$ is easily surmounted.

This instability goes back to one of the components of the slow oscillation and the duration of amplification of the perturbations is long.

The flight of an airplane affected by a slight instability of the slow motion is no longer possible with controls fixed but remains possible when the pilot makes the necessary corrections.

Effect of $c_3 = \dfrac{\partial C_M}{\partial \chi}$.

The factor c_3 uniquely exercises an effect on the damping. It increases the total damping, but the increase relates almost entirely to the root having to do with the rapid motion.

8. The Accelerations

One of the quantities to which the occupants of an airplane are the most sensitive is the normal component J_z of the total acceleration, opposed to the apparent gravity.

In the course of a varied motion

$$J_z = \overline{J}_z + \delta J_z$$

where $\overline{J}_z = g \cos \theta \cos \varphi$ represents the steady-state component and δJ_z the increment.

The hypotheses adopted permit us to write:

$$\begin{aligned} \delta J_z &= \left[\frac{d\delta w}{dt} - \left(q\delta u + u\delta q\right) - g \sin \theta \ \delta\theta\right] \\ &= \frac{1}{m} Z'_u\delta u + \frac{1}{m} Z'_w\delta w + \frac{1}{m} Z'_q\delta q \end{aligned}$$

Since q and Z'_q are supposed to be zero, we can adopt either one of the expressions:

$$\delta J_z = \frac{d\delta w}{dt} - u\delta q - g \sin \theta \ \delta\theta$$

or

$$\delta J_z = \frac{1}{m}\left(Z'_u\delta u + Z'_w\delta w\right)$$

As soon as we know the development of the δu, δw, δq, $\delta\theta$ in the course of unsteady motion, we can determine at every instant the increment of acceleration δJ_z.

CHAPTER VIII

LINEARIZATION OF THE EQUATIONS OF THE LATERAL MOTION

1. What Lateral Stability Consists of

The lateral motion is determined by:

The equilibrium of the forces following the transverse axis OY

The equilibrium of moments along each of the axes OX and OZ.

Just as in the case of longitudinal stability, the destruction of one of the states of equilibrium gives rise to forces and moments which in turn act upon the other equilibria.

The sequence of phenomena which take place may in certain cases be established by simple reasoning.

Let us assume an airplane to which one has imparted the two static stabilities of roll and yaw, satisfying the two conditions:

$$\frac{\partial C_L}{\partial \beta} > 0 \qquad \frac{\partial C_N}{\partial \beta} > 0$$

We suppose that the machine is inclined toward the left $\delta\varphi < 0$. The machine will deviate in this direction, under the effect of gravity. and a perturbation $\delta\beta > 0$ will originate.

This perturbation constitutes a lateral translation which will have two effects:

(1st) The airplane will have a tendency to level out under the action of $\partial C_L/\partial\beta$; the skidding to the left will tend to incline the machine to the right.

(2nd) The airplane will have a tendency to veer to the left since it behaves like a wind vane, with $\partial C_N/\partial\beta$ being positive.

Thus a new perturbation arises, a turn $\delta r > 0$. In this motion the right wing will be at the outside of the turn and will be displaced more rapidly than the left wing.

The lift will be stronger and will tend to increase still more, thus to emphasize the lateral inclination. The moment L is, in fact, a function of r.

The derivative $\partial C_L/\partial r$ is negative and the turn δr has a tendency of straightening out the outer wing, in the case considered the right wing.

Two opposite effects are produced and, according to the proportions of the machine, one or the other predominates.

If the effect of the static stability of roll prevails, it will be possible that after a turn the airplane will resume its initial state; nevertheless it will fly in another direction than before the initial perturbation.

If the effect of the angular velocity r on the rolling moment L is larger than the effect of the skidding β, the second effect prevails. The inclination of the machine increases, the airplane starts on a more and more inclined turn and describes a spiral trajectory.

The machine is then dynamically unstable; the instability affecting it is called spiral instability.

The dynamic study shows that a machine which satisfies separately each of the two conditions of static stability (of flight path and of yaw) may be unstable if the first stability is too highly developed in proportion to the second.

Likewise the motion of an airplane may present unfavorable characteristics if the rolling stability is too high in proportion to the flight-path stability.

Let us imagine that the machine skids to the left, with the axis OX of the plane oriented to the right with respect to the flight path.

If $\partial C_L/\partial\beta$ is high, the machine will be forcibly inclined to the right, and the rolling will be positive. The tendency toward a leftward turn will on the contrary be slight since $\partial C_N/\partial\beta$ is, by hypothesis, supposed to be small. The secondary rolling moment which might develop due to this turn will tend to incline the machine to the left, but it will remain weak since the turn is little pronounced.

The motion of the positive rolling, to the right, will predominate. Since nothing opposes its action, the airplane will lean to the right. The resultant of forces along the transverse axis will at this moment make the airplane skid to the right, and the same phenomena, in the inverse sense, will occur.

One can see how there arises the possibility of a yawing motion on which a continuous balancing is superposed. This motion becomes unstable and the amplitudes will increase for too small values of $\partial C_N/\partial\beta$.

2. Setting-Up of Equations by the Method of Linearization of Equations

The equations of the variable motion are:

$$m\left(\frac{dv}{dt} + ru - pw\right) = Y - G \sin \varphi$$

$$C \frac{dp}{dt} + pq(B - A) = \Sigma N$$

$$A \frac{dr}{dt} + qr(C - B) = \Sigma L$$

One must add two geometric relations connecting the angular velocities p and r with the derivatives $d\varphi/dt$ and $d\psi/dt$ and resulting from the definition of the rotations:

$$p = \frac{d\varphi}{dt} - \frac{d\psi}{dt} \sin \theta$$

$$r = \frac{d\psi}{dt} \cos \theta \cos \varphi - \frac{d\theta}{dt} \sin \varphi$$

which may be written:

$$\frac{d\varphi}{dt} = p + \frac{\sin \theta}{\cos \theta}(q \sin \varphi + r \cos \varphi)$$

$$\frac{d\psi}{dt} = \frac{1}{\cos \theta}(q \sin \varphi + r \cos \varphi)$$

We shall be able to assume $q = 0$ when we investigate the lateral stability of a rectilinear motion.

The system of the five equations has the form:

$$\frac{dv}{dt} = f_1(v,p,r,\varphi,\psi)$$

$$\frac{dp}{dt} = f_2(v,p,r,\varphi,\psi)$$

$$\frac{dr}{dt} = f_3(v,p,r,\varphi,\psi)$$

$$\frac{d\varphi}{dt} = f_4(v,p,r,\varphi,\psi)$$

$$\frac{d\psi}{dt} = f_5(v,p,r,\varphi,\psi)$$

It can be linearized as in the study of longitudinal stability, with the perturbations δv, δp, δr, $\delta\varphi$, $\delta\psi$ now becoming the variables.

The integral system depends after linearization on an algebraic equation of the fifth degree in x, instead of an equation of the fourth degree.

This equation will be:

$$\begin{vmatrix}
\frac{\partial f_1}{\partial v} - x & \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial \psi} \\
\frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial p} - x & \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial \psi} \\
\frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial p} & \frac{\partial f_3}{\partial r} - x & \frac{\partial f_3}{\partial \varphi} & \frac{\partial f_3}{\partial \psi} \\
\frac{\partial f_4}{\partial v} & \frac{\partial f_4}{\partial p} & \frac{\partial f_4}{\partial r} & \frac{\partial f_4}{\partial \varphi} - x & \frac{\partial f_4}{\partial \psi} \\
\frac{\partial f_5}{\partial v} & \frac{\partial f_5}{\partial p} & \frac{\partial f_5}{\partial r} & \frac{\partial f_5}{\partial \varphi} & \frac{\partial f_5}{\partial \psi} - x
\end{vmatrix} = 0$$

One notices immediately that the derivatives of the five functions with respect to the variable ψ are zero.

The equation of the fifth degree admits a zero root:

$$x_5 = 0$$

This circumstance facilitates the analytical investigation since the characteristic equation becomes one of the fourth degree if one eliminates this particular solution, and the mathematical investigation will be carried out by methods similar to those used for the study of the longitudinal motion.

The existence of this particular root corresponds to a well-determined mechanical fact.

In the study of the longitudinal motion one states that certain projections of the external forces depend on the angle θ.

If θ is not zero, the axis of the airplane is inclined upward or downward, with respect to the horizon, and the gravity exerts along the axis OX a component which is to be subtracted from or added to the propeller thrust.

When an aircraft is dynamically stable, it reverts, after a series of oscillations, to its initial state. The forces acting upon it must reassume their initial value. This result can be obtained only if the airplane recovers its original trim.

In the study of the lateral motion we shall encounter two angular quantities φ and ψ.

The component of the forces along the axis OY depends on the angle φ since the gravity exerts a lateral component when the airplane is inclined. If the airplane is dynamically stable, it reverts after any perturbation whatsoever to its initial state and must therefore assume again its initial inclination.

This does not apply to the angle ψ. Whatever the final position of the airplane may be, the projections of the weight on the axes are independent of the azimuth ψ; no force and no moment exists which would be a function of ψ.

If, consequently, a dynamically stable aircraft returns after a perturbation to its initial state, it has to reassume a motion characterized by the same velocities and the same angles θ and φ as the initial motion, but not necessarily by the same angle ψ.

After a perturbation, a machine does not possess any stability of heading. The existence of a solution $x_5 = 0$ is only the mathematical consequence of this fact.

Momentarily setting aside this solution, we write the characteristic equation as for the longitudinal stability:

$$x^4 + A_1x^3 + A_2x^2 + A_3x + A_4 = 0$$

The integral system will be written:

$$\delta v = C_1e^{x1t} + C_2e^{x2t} + C_3e^{x3t} + C_4e^{x4t}$$

$$\delta p = l_1C_1e^{x1t} + l_2C_2e^{x2t} + l_3C_3e^{x3t} + l_4C_4e^{x4t}$$

$$\delta r = m_1C_1e^{x1t} + m_2C_2e^{x2t} + m_3C_3e^{x3t} + m_4C_4e^{x4t}$$

$$\delta\varphi = n_1C_1e^{x1t} + n_2C_2e^{x2t} + n_3C_3e^{x3t} + n_4C_4e^{x4t}$$

To obtain the perturbation $\delta\psi$, will be possible only by the integration:

$$\delta\psi = \int \frac{d\delta\psi}{dt}\,dt = \int \frac{1}{\cos\theta}(\delta q \sin\varphi + \delta r \cos\varphi)dt$$

The characteristic equation in x always admits in the study of the lateral motion one pair of imaginary roots $x_{1,2}$ and two real roots x_3 and x_4.

The motion will therefore result from the superposition of an oscillation and of two aperiodic motions.

3. Motion Effected by the Action of the Lateral Controls

The moments produced by the lateral controls will be introduced in the second term of the equations, and the solution of the system with a second term permits determination of the effect of the ailerons or of the rudder on the lateral motion.

The principle of the method is the same as for the longitudinal motion, but it is well to point out immediately that the integration can be accomplished much more easily by the operational method described in chapter XVIII than by the classical integration procedure.

CHAPTER IX

THE LATERAL MOTION

1. Transformation of the General Equations

Let us write the general equations in the linearized form:

$$\frac{d\delta w}{dt} - \frac{\partial f_1}{\partial v}\delta v - \frac{\partial f_1}{\partial p}\delta p - \frac{\partial f_1}{\partial r}\delta r - \frac{\partial f_1}{\partial \varphi}\delta\varphi - \frac{\partial f_1}{\partial \psi}\delta\psi = \frac{\partial f_1}{\partial \xi}\Delta\xi + \frac{\partial f_1}{\partial \zeta}\Delta\zeta$$

$$\frac{d\delta p}{dt} - \frac{\partial f_2}{\partial v}\delta v - \frac{\partial f_2}{\partial p}\delta p - \frac{\partial f_2}{\partial r}\delta r - \frac{\partial f_2}{\partial \varphi}\delta\varphi - \frac{\partial f_2}{\partial \psi}\delta\psi = \frac{\partial f_2}{\partial \xi}\Delta\xi + \frac{\partial f_2}{\partial \zeta}\Delta\zeta$$

$$\frac{d\delta r}{dt} - \frac{\partial f_3}{\partial v}\delta v - \frac{\partial f_3}{\partial p}\delta p - \frac{\partial f_3}{\partial r}\delta r - \frac{\partial f_3}{\partial \varphi}\delta\varphi - \frac{\partial f_3}{\partial \psi}\delta\psi = \frac{\partial f_3}{\partial \xi}\Delta\xi + \frac{\partial f_3}{\partial \zeta}\Delta\zeta$$

$$\frac{d\delta \varphi}{dt} - \frac{\partial f_4}{\partial v}\delta v - \frac{\partial f_4}{\partial p}\delta p - \frac{\partial f_4}{\partial r}\delta r - \frac{\partial f_4}{\partial \varphi}\delta\varphi - \frac{\partial f_4}{\partial \psi}\delta\psi = 0$$

$$\frac{d\delta \psi}{dt} - \frac{\partial f_5}{\partial v}\delta v - \frac{\partial f_5}{\partial p}\delta p - \frac{\partial f_5}{\partial r}\delta r - \frac{\partial f_5}{\partial \varphi}\delta\varphi - \frac{\partial f_5}{\partial \psi}\delta\psi = 0$$

The derivatives of the functions f have the form:

$$\frac{\partial f_1}{\partial v} = \frac{1}{m} Y'_v \qquad \frac{\partial f_1}{\partial \varphi} = -g \cos \varphi$$

$$\frac{\partial f_1}{\partial p} = \bar{w} + \frac{1}{m} T'_p \qquad \frac{\partial f_1}{\partial \psi} = 0$$

$$\frac{\partial f_1}{\partial r} = -\bar{u} + \frac{1}{m} Y'_r \qquad \frac{\partial f_1}{\partial \xi} = \frac{1}{m} Y'_\xi$$

$$\frac{\partial f_1}{\partial \zeta} = \frac{1}{m} Y'_\zeta$$

$$\frac{\partial f_2}{\partial v} = \frac{1}{A} L'_v \qquad \frac{\partial f_3}{\partial v} = \frac{1}{C} N'_v$$

$$\frac{\partial f_2}{\partial p} = \frac{1}{A} L'_p \qquad \frac{\partial f_3}{\partial p} = \frac{1}{C} N'_p - q(B - A)$$

$$\frac{\partial f_2}{\partial r} = \frac{1}{A} L'_r - \overline{q}(C - B) \qquad \frac{\partial f_3}{\partial r} = \frac{1}{C} N'_r$$

$$\frac{\partial f_2}{\partial \varphi} = \frac{1}{A} L'_\varphi \qquad \frac{\partial f_3}{\partial \varphi} = \frac{1}{C} N'_\varphi$$

$$\frac{\partial f_2}{\partial \psi} = \frac{1}{A} L'_\psi \qquad \frac{\partial f_3}{\partial \psi} = \frac{1}{C} N'_\psi$$

$$\frac{\partial f_2}{\partial \xi} = \frac{1}{A} L'_\xi \qquad \frac{\partial f_3}{\partial \xi} = \frac{1}{C} N'_\xi$$

$$\frac{\partial f_2}{\partial \zeta} = \frac{1}{A} L'_\zeta \qquad \frac{\partial f_3}{\partial \zeta} = \frac{1}{C} N'_\zeta$$

$$\frac{\partial f_4}{\partial v} = 0 \qquad \frac{\partial f_5}{\partial v} = 0$$

$$\frac{\partial f_4}{\partial p} = 1 \qquad \frac{\partial f_5}{\partial p} = 0$$

$$\frac{\partial f_4}{\partial r} = \frac{\sin\theta}{\cos\theta}\cos\varphi \qquad \frac{\partial f_5}{\partial r} = \frac{\cos\varphi}{\cos\theta}$$

$$\frac{\partial f_4}{\partial \varphi} = -\overline{r}\,\frac{\sin\theta}{\cos\theta}\sin\varphi \qquad \frac{\partial f_5}{\partial \varphi} = -\overline{r}\,\frac{\sin\varphi}{\cos\theta}$$

$$\frac{\partial f_4}{\partial \psi} = 0 \qquad \frac{\partial f_5}{\partial \psi} = 0$$

We have moreover:

$$\frac{\partial f_4}{\partial \xi} = \frac{\partial f_4}{\partial \zeta} = \frac{\partial f_5}{\partial \xi} = \frac{\partial f_5}{\partial \zeta} = 0$$

which have not even been written in the preceding equations.

The following calculations are devoted to the study of the rectilinear system for which $q = 0$.

If one wanted to investigate the dynamic stability of a curvilinear trajectory, one would have to maintain this term.

The derivatives of the aerodynamic actions $Y'_p \ldots N'_r$ have been defined in chapters III and IV by factors $a_1 \ldots k_3$.

We shall assume that the derivatives Y'_p and Y'_r are zero, likewise Y'_ξ and Y'_ζ.

The moments L and N are independent of the attitude of the airplane in space. The factors d_2, d_3, e_2, e_3 are normally zero. However, one can, by means of instruments for automatic flight control, make the moments functions of the angles, and we will temporarily keep the terms $d_2 \ldots e_3$ in the equations.

Let us again replace:

$$m \text{ by } \mu S c \frac{\rho}{2}$$

$$A \text{ by } m r^2_a = \mu S c \frac{\rho}{2} r^2_a$$

$$C \text{ by } m r^2_c = \mu S c \frac{\rho}{2} r^2_c$$

When all substitutions have been carried out, the equations of the motion, written with a second term, become:

$$\frac{\mu c}{V}\frac{d\delta v}{dt} + a_1\delta v + b_1\delta p + c_1\delta r + d_1\delta\varphi + e_1\delta\psi = 0$$

$$\frac{\mu c}{V}\frac{d\delta p}{dt} + \frac{b}{r^2_a} a_2\delta v + \frac{bs}{r^2_a} b_2\delta p + \frac{bs}{r^2_a} c_2\delta r + \frac{bV}{r^2_a} d_2\delta\varphi + \frac{bV}{r^2_a} e_2\delta\psi = \frac{bV}{r^2_a} h_2\delta\xi + \frac{bV}{r^2_a} k_2\delta\zeta$$

$$\frac{\mu c}{V}\frac{d\delta r}{dt} + \frac{b}{r^2_c} a_3\delta v + \frac{bs}{r^2_c} b_3\delta p + \frac{bs}{r^2_c} c_3\delta r + \frac{bV}{r^2_c} d_3\delta\varphi + \frac{bV}{r^2_c} e_3\delta\psi = \frac{bV}{r^2_c} h_3\delta\xi + \frac{bV}{r^2_c} k_3\delta\zeta$$

$$\frac{\mu c}{V}\frac{d\delta\varphi}{dt} + a_4\delta v + b_4\delta p + c_4\delta r + d_4\delta\varphi + e_4\delta\psi = 0$$

$$\frac{\mu c}{V}\frac{d\delta\psi}{dt} + a_5\delta v + b_5\delta p + c_5\delta r + d_5\delta\varphi + e_5\delta\psi = 0$$

in which:

$$a_4 = 0 \quad \text{because} \quad \frac{\partial f_4}{\partial v} = 0$$

$$a_5 = 0 \qquad\qquad \frac{\partial f_5}{\partial v} = 0$$

$$b_1 = -\bar{w}\,\frac{\mu c}{V}$$

$$c_1 = +\bar{u}\,\frac{\mu c}{V}$$

$$b_4 = -\frac{\mu c}{V}\frac{\partial f_4}{\partial p} = -\frac{\mu c}{V}$$

$$b_5 = 0 \quad \text{because} \quad \frac{\partial f_5}{\partial p} = 0$$

$$d_1 = +g \cos \frac{\mu c}{V}$$

$$d_4 = +\frac{\mu c}{V}\,\bar{r}\,\frac{\sin\theta}{\cos\theta}\sin\varphi$$

$$d_5 = +\frac{\mu c}{V}\,\bar{r}\,\frac{\sin\theta}{\cos\theta}$$

$$c_4 = -\frac{\mu c}{V}\frac{\sin\theta}{\cos\theta}\cos\varphi$$

$$c_5 = -\frac{\mu c}{V}\frac{\sin\theta}{\cos\theta}$$

If the aerodynamic characteristics are not modified by a mechanical device:

$$d_2 = 0 \qquad e_2 = 0$$

$$d_3 = 0 \qquad e_3 = 0$$

Finally, $e_1 = e_4 = e_5 = 0$, since the derivatives of the corresponding functions with respect to ψ are zero.

d_4 and c_4 are zero for a horizontal trim since $\sin\theta = 0$.

Practically there remain only seven factors connected with the aerodynamic characteristics of the airplane and four factors characterizing the action of the controls, namely:

$$a_1 = -\frac{\partial C_Y}{\partial \beta}$$

$$a_2 = -\frac{\partial C_L}{\partial \beta} \qquad a_3 = -\frac{\partial C_N}{\partial \beta}$$

$$b_2 = -\frac{\partial C_L}{\partial \tilde{\omega}} \qquad b_3 = -\frac{\partial C_N}{\partial \tilde{\omega}}$$

$$c_2 = -\frac{\partial C_L}{\partial \rho} \qquad c_3 = -\frac{\partial C_N}{\partial \rho}$$

Finally:

$$h_2 = \frac{\partial C_L}{\partial \xi} \qquad h_3 = \frac{\partial C_N}{\partial \xi}$$

$$k_2 = \frac{\partial C_L}{\partial \zeta} \qquad k_3 = \frac{\partial C_N}{\partial \zeta}$$

2. Characteristic Determinant

We shall make the same transformation of the unit of time as in the study of the longitudinal motion and search for the solution in τ.

Let us write the characteristic determinant of the equation system without second term. We obtain:

$$\begin{vmatrix} a_1 + \lambda & b_1 & c_1 & d_1 & 0 \\ \frac{b}{r^2_a} a_2 & \frac{bs}{r^2_a} b_2 + \lambda & \frac{bs}{r^2_a} c_2 & \frac{bV}{r^2_a} d_2 & \frac{bV}{r^2_a} e_2 \\ \frac{b}{r^2_c} a_3 & \frac{bs}{r^2_c} b_3 & \frac{bs}{r^2_c} c_3 + \lambda & \frac{bV}{r^2_c} d_3 & \frac{bV}{r^2_c} e_3 \\ 0 & b_4 & c_4 & d_4 + \lambda & 0 \\ 0 & 0 & c_5 & d_5 & \lambda \end{vmatrix} = 0$$

Let us assume an equation of the fifth degree in λ:

$$\lambda^5 + B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5 = 0$$

However, if one takes into consideration only the aerodynamic characteristics of the airframe, without equipment for automatic flight control:

$$d_2 = d_3 = e_2 = e_3 = 0$$

and the equation is reduced to:

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$$

It is no longer possible to write the A_1, A_2, A_3, A_4 in the form of linear functions of the six factors a_2, b_2, c_2 and a_3, b_3, c_3 which define the derivatives of the moments of roll and yaw.

In certain terms of the development the derivatives of one moment are multiplied by those of the other.

For $c_4 = d_4 = 0$ (horizontal trim),

$$d_2 = d_3 = e_2 = e_3 = 0$$

one may write the development (replacing b by $2s$, in order to avoid the coexistence of the letter b, span, and of the b's with subscript which designate the derivatives of C_L):

$$A_1 = a_1 + 2\frac{s^2}{r^2_a} b_2 + 2\frac{s^2}{r^2_c} c_3$$

$$A_2 = a_2\, 2\frac{sc}{r^2_a}\frac{w}{V}\mu - a_3\, 2\frac{sc}{r^2_c}\frac{u}{V}\mu + b_2\, 2\frac{s^2}{r^2_a} a_1 +$$

$$c_3\, 2\frac{s^2}{r^2_c} a_1 + \left(b_2c_3 - b_3c_2\right)\frac{4s^4}{r^2_a r^2_c}$$

$$A_3 = -a_2\frac{2sc}{r^2_a}\cos\varphi\,\frac{g\mu c}{V^2}\mu + a_2\frac{4s^3c}{r^2_a r^2_c}\left(\frac{u}{V}b_3 + \frac{w}{V}c_3\right)\mu -$$

$$a_3\frac{4s^3c}{r^2_a r^2_c}\left(\frac{w}{V}c_2 + \frac{u}{V}b_2\right)\mu + \left(b_2c_3 - b_3c_2\right)\frac{4s^4}{r^2_a r^2_c} a_1$$

$$A_4 = \left(a_3c_2 - a_2c_3\right)\frac{4s^3c}{r^2_a r^2_c}\mu^2\frac{gc}{V^2}$$

Let us note that all factors c (chord) appearing in the expressions are multiplied by μ. They stem from substitutions of μ for m.

One could write:

$$c\mu = s\nu$$

which would eliminate the factor c and would replace it everywhere by s, under the condition that the density μ be replaced by another density equal to:

$$\nu = \mu\frac{c}{s} = \frac{2m}{\rho Ss}$$

In this case every airplane would be characterized by two densities:

The one, μ, utilized in the study of the longitudinal motion

The other, ν, utilized in the study of the lateral motion.

We have preferred using only one single expression for the density.

We state that the A_1, A_2, A_3, A_4 are functions:

(1) Of dimensionless factors $a_1 \ldots c_3$ dependent on the aerodynamic characteristics of the airplane

(2) Of geometrical characteristics such as s/r_a or s/r_c dependent on the moments of inertia

(3) Of the angle of attack $-w/V$ and of the lift coefficient:

$$Cz = \frac{\mu gc}{V^2}$$

(4) Of the density μ of the airplane.

Let us remark that the aerodynamic factors $a_1 \ldots c_3$ vary, for a given airplane, with the angle of attack.

3. Characteristics of the Motion

The discussion ought to be carried out as a function of six quantities. One can proceed with it only by treating series of numerical examples. On the other hand, this investigation should be made for different angles of attack.

Finally, it would be useful to investigate the effects of variations of the density μ and of modification in the mass distribution - which determine modifications of s/r_a or s/r_c.

We shall be content with recalling the essential facts which have become classical.

The solution of the characteristic equation contains always a pair of complex roots, determining an oscillatory motion, and two real roots, determining two aperiodic motions.

Following, one of those two motions will be called spiral motion, the other strongly damped motion.

We shall write the subscript 3 for the root λ corresponding to the spiral motion, and the subscript 4 for the one corresponding to the motion called "strongly damped."

These roots are easily distinguished. The root determining the strongly damped motion is of large absolute value and always negative. The root determining the spiral motion is much smaller in absolute value, and may be positive or negative.

Because of the large number of variables we shall investigate first the influence of the derivatives a_2 and a_3 which constitute the static-stability coefficients.

We shall examine successively:

(a) The criterion of stability

(b) The period and the damping of the motions

(c) The amplitudes.

These three problems correspond to the stages we pointed out in the study of the longitudinal motion.

(A) Criterion of stability.- We shall write the coefficients A as functions of a_2 and a_3 for normal and constant values of the other parameters, and trace, taking a_2 and a_3 as axes, the lines:

$$A_1 = 0 \quad A_2 = 0 \quad A_3 = 0 \quad A_4 = 0$$

$$R = A_2 - \frac{A_3}{A_1} - \frac{A_1 A_4}{A_3} = 0$$

separating regions of the diagram where each of these expressions is $>$ or < 0.

In fact, the curves $A_1 = 0$ and $A_2 = 0$ pass outside of the trimming limits, the useful part of the diagram is always on the stable side, and only the lines $A_3 = 0$, $A_4 = 0$, $R = 0$ have to be considered.

There is one region where these three expressions are all simultaneously > 0.

The airplanes, the static-stability coefficients of which fall into this region, are all dynamically stable.

The condition $A_4 > 0$ is nothing else but:

$$a_3 c_2 - a_2 c_3 > 0$$

or

$$\frac{\partial C_N}{\partial \beta}\frac{\partial C_L}{\partial \rho} - \frac{\partial C_L}{\partial \beta}\frac{\partial C_N}{\partial \rho} > 0$$

which is precisely the expression $D > 0$ of chapter V.

If $D = 0$, an airplane which is making a turn when the two lateral controls are neutral ($\xi = 0$; $\zeta = 0$), continues with this turn indefinitely.

If $D < 0$, a similar turn will have the tendency of becoming more pronounced because the inner wing must be supported in order to maintain the lateral inclination and to prevent it from increasing.

The machine presents spiral instability. Any flight path travelled with $\xi = 0$ and $\zeta = 0$ is finally transformed into a spiral.

If $D > 0$, a turn without control deflection has a tendency to stop because the airplane must be maintained in the turn by an appropriate deflection of one of the controls. The rectilinear flight path constitutes the stable trajectory.

Since the $\partial C_N/\partial \beta$ and $\partial C_L/\partial \beta$ are habitually positive, and $\partial C_N/\partial \rho$ and $\partial C_L/\partial \rho$ are habitually negative, the stability condition may be written:

$$\frac{\partial C_N}{\partial \beta} < \frac{\dfrac{\partial C_L}{\partial \beta}\dfrac{\partial C_N}{\partial \rho}}{\dfrac{\partial C_L}{\partial \rho}}$$

The static stability about the axis OZ must be inferior at a certain limit which is a function of the static stability about the axis OX.

We recall that the principal factor producing the first is the magnitude of the vertical tail surfaces and that the one producing the second is the dihedral of the wing.

These two characteristics are connected to one another.

However, the diagram shows us certain unexpected facts. The region of stability extends sometimes below the axis of the abscissas.

A directionally unstable airplane may be dynamically stable if the instability remains slight. Finally, the area of stability may present a pointed region (of very much reduced area) corresponding at the same time to a slight static directional instability and to a slight static rolling instability.

In the figure the line A_4 is a straight line because we plotted it under the assumption that the derivatives $\partial C_L/\partial \rho$ and $\partial C_N/\partial \rho$ are constant. In reality, it would be difficult to vary the static directional stability in significant proportions without exerting an influence on the area or the lever arm of the vertical tail surfaces.

Any increase of $\partial C_N/\partial\beta$ then entails an increase, in absolute value, of $\partial C_N/\partial\rho$. In order to keep to effectively realizable cases, one would have to take into account the modifications of $\partial C_N/\partial\rho$ accompanying those of $\partial C_N/\partial\beta$.

The line $A_4 = 0$ will cease to be a straight line and will become a curve.

NUMERICAL VALUES USED FOR THE EXAMPLES CONCERNING THE LATERAL MOTION

	Chapter V			Chapter IX, Sections 3 and 4			Chapter IX, Section 5		
Airplane	1	2	3	A	B	C	A'	B'	C'
$a_1 = -\dfrac{\partial C_Y}{\partial\beta}$	+0.45	+0.45	+0.45	+0.60	+0.60	+0.60	+0.40	+0.40	+0.40
$a_2 = -\dfrac{\partial C_L}{\partial\beta}$	-0.0573	-0.069	-0.080	-0.080	-0.040	-0.020	-0.120	-0.060	-0.120
$b_2 = -\dfrac{\partial C_L}{\partial\tilde{\omega}}$	-------	------	------	+0.24	+0.24	+0.24	+0.42	+0.42	+0.42
$c_2 = -\dfrac{\partial C_L}{\partial\rho}$	+0.18	+0.16	+0.14	+0.056	+0.056	+0.056	+0.06	+0.06	+0.06
$a_3 = -\dfrac{\partial C_N}{\partial\beta}$	-0.046	-0.043	-0.040	-0.040	+0.010	-0.040	-0.048	-0.024	-0.040
$b_3 = -\dfrac{\partial C_N}{\partial\tilde{\omega}}$	-------	------	------	-0.017	-0.017	-0.017	-0.03	-0.03	-0.03
$c_3 = -\dfrac{\partial C_N}{\partial\rho}$	+0.08	+0.10	+0.12	+0.045	+0.045	+0.045	+0.072	+0.048	+0.072
C_p		0.62			0.40			0.20	
μ		------			28.2			10b/c	
$r^2{}_a/s^2$		------			0.10			0.06	
$r^2{}_c/s^2$		------			0.24			0.12	

$\dfrac{\partial C_L}{\partial\xi}$ in degrees: 0.003 $\qquad$ $\dfrac{\partial C_N}{\partial\zeta}$ in degrees: 0.012

(B) Period and damping of the motions.- Solution of the characteristic is necessary. Performing the calculations for three airplanes called A, B, and C in the table on the preceeding page, we state:

(1) That the root λ_4 maintains a practically constant value

(2) That the root λ_3 is positive on one side of the condition $A_4 = 0$ and negative on the other. It determines satisfactorily the spiral stability

(3) That the complex root determines a sufficiently damped oscillatory motion when the static directional stability is normal, but that the period increases and the damping diminishes when the directional stability decreases.

Exceeding the limit $R = 0$ makes the instability of the oscillatory motion manifest.

(C) Amplitudes.- One can obtain a conception of the amplitudes by examining the return motion of the airplane toward its initial state after an initial perturbation of each of the four variables δv, δp, δr, and $\delta\varphi$.

In making these calculations, one finds the following facts:

(1) A perturbation of the lateral inclination fosters especially the spiral motion. It excites the oscillation only slightly and exerts practically zero effect on the damped motion.

(2) An initial perturbation of skidding excites the oscillation in a high degree, the spiral motion in a low degree. The amplitude of the damped motion is negligible.

(3) An initial perturbation of the angular velocity p goes back almost entirely to the motion called "strongly damped." The perturbation δp stops quickly but leaves the airplane with a lateral inclination $\delta\varphi$ which, in turn, may be considered as the initial perturbation which excites the return motion.

(4) An initial perturbation of the angular velocity δr produces a skidding in the same manner as a perturbation δp creates the lateral inclination.

It does not exert any effect on the damped motion and is afterwards reabsorbed like the motion of skidding it had created.

Actually only the initial perturbations of skidding and of lateral inclination are typical.

The motions to which they give rise are indicated as functions of time in figure 26 for the three airplanes considered before.

Knowing the amplitudes one can state that the oscillatory motion constitutes a real balancing, formed by the superposition of a rolling and of a yawing motion; it is called "Dutch roll" by Anglo-Saxon engineers.

4. Effect of the Other Parameters

Numerous points ought to be investigated.

We shall point out some well-established important facts:

(a) For a given airplane the aerodynamic characteristics vary when the sustained angle of attack increases, but the variation takes place in a sense always unfavorable to stability.

(b) When the density of the airplane increases as do the radii of gyration (relation such that s/r_a becomes smaller) one finds that the region of stability shrinks considerably because the curve $R = 0$ rises.

On modern aircraft it becomes more and more difficult to realize degrees of static stability (and to endow the airplane with them) which ensure stable trajectories for controls fixed.

Even though for several years now there has been a tendency to consider spiral instability as a rather serious defect, at present one is forced to admit it if there is no way of avoiding it.

In case of a slight spiral instability, the speed of increase in initial perturbations is tolerable and leaves the pilot sufficient time for intervening.

(c) Nevertheless, it is useful for the continuation of this study to investigate the effect of modifications of the parameters b_2, c_2, b_3, and c_3, that is, of the derivatives of the moments in proportion to the angular velocities.

Increment in b_2:

$$b_2 = - \frac{\partial C_L}{\partial \tilde{\omega}}$$

The parameter is a damping factor. Any increment in b_2 augments the coefficient A_1, that is, the total of available damping.

The greatest part of this increment goes to the root λ_4; the two other roots receive only a very small part of it.

Increment in b_3:

$$b_3 = -\frac{\partial C_N}{\partial \tilde{\omega}}$$

The factor b_3 does not affect A_1. It can produce only an exchange between the different roots. This effect is absolutely insignificant.

Increment in c_2:

$$c_2 = -\frac{\partial C_L}{\partial \rho}$$

The factor c_2 does not affect A_1, but it produces an exchange between the damping of the oscillatory motion and that of the spiral motion.

It does not exert any influence on λ_4.

The exchange results from the form of A_4:

$$A_4 = a_3 c_2 - a_2 c_3$$

where the values of a_3 and a_2 are frequently of the same order of magnitude.

Since the stability condition is written (with consideration of the signs):

$$c_2 < c_3 \frac{a_2}{a_3}$$

or

$$\frac{\partial C_L}{\partial \rho} > \frac{\partial C_N}{\partial \rho} \frac{\partial C_L}{\partial \beta} \frac{\partial C_N}{\partial \beta}$$

one sees that a reduction of c_2 $\left(\text{increase of } \partial C_L/\partial \rho\right.$ in absolute value$\left.\right)$ is favorable to the stability of the spiral motion. This improvement will be obtained by borrowing from the real part of 1.2, that is to say, by borrowing from the damping of the oscillation.

Increment in c_3:

$$c_3 = - \frac{\partial C_N}{\partial \rho}$$

The coefficient A_1 depends on c_3. Any increment of c_3 augments the total damping available.

This increment never applies to λ_4 but is distributed between the oscillatory and the spiral motions.

5. Motion of the Airplane Under the Effect of a Control

If we place in the second term the moment produced by the deflection of the ailerons or of the elevator, it is possible to determine the motion caused by this deflection.

We give on the diagrams which follow the result of the calculation for the three airplanes A', B', and C' the characteristics of which are indicated in the table.

Each of the controls is supposed not to exert any secondary effect.

The moment applied by the ailerons tends to lower the left wing; it is taken equal to:

$$C_L = -0.006$$

which corresponds to a deflection of -2 degrees when the efficiency of the control $\partial C_L/\partial \xi$ is equal to:

0.172 (angles expressed in radians)

or

0.003 (angles expressed in degrees)

The moment applied by the rudder tends to cause rotation to the left; it is taken equal to:

$$C_N = +0.0024$$

which corresponds to a deflection of +2 degrees when the efficiency of the control $\partial C_N/\partial\zeta$ is equal to:

$$0.069 \text{ (angles in radians)}$$

or

$$0.0012 \text{ (angles in degrees)}$$

Under the action of the rolling moment developed by the ailerons, the airplane immediately is inclined to the left.

However, it starts out by turning to the right, due to the effect of N'_p, but this motion changes rapidly its sense and the airplane then turns to the left.

The rotation to the left is due to the skidding toward the left which originates as a consequence of the lateral inclination. This skidding produces a yawing moment due to the directional static stability with which the considered airplanes are supposed to be endowed.

Under the action of the yawing moment toward the left, the airplane starts a rotation r to that side; but since the trajectory is not immediately modified, the airplane skids toward the right, that is, toward the outside of the turn.

Under the effect of this skidding the airplanes which possess lateral static stability are inclined toward the inside of the turn. An airplane which is characterized by $\partial C_L/\partial\beta = 0$ is not subjected to this rolling moment.

The diagrams show that, if one wants to start a turn by means of aileron action alone, the airplane with spiral instability is the one with the most rapid changes of course $\delta\psi$.

If one wants to induce a turn uniquely by maneuvering of the rudder, the airplane with spiral instability is the one which turns most unsatisfactorily because it is not inclined toward the inside under the action of the skid toward the outside.

One can understand immediately that the simple maneuvers consisting of an invariable deflection of one or the other of the two lateral controls do not lead to placing the airplane into a regular turn.

We studied in chapter V the equilibrium-type conditions of turns. If the flight path corresponds to $\theta = 0$, one has necessarily $p = 0$ once the steady state has been established.

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CHAPTER X

EXPERIMENTAL RESULTS

1. Development of the Preceding Theory

The linearized theory of the motion of an airplane is due to the research of G. H. Bryan whose report: "Stability in Aviation" constitutes a major work.

Published in 1911, at the era of the "Paris-Madrid," "Paris-Rome," etc., races the theory went completely unnoticed and did not exert any influence on the development of aircraft. This development occurred in a semiempirical manner, up to the time in 1916-1918 when Bairslow in Great Britain and Hunsacker in the United States took up again Bryan's theory and introduced into it the numerical values of the derivatives which the progress of aerodynamic knowledge permitted one to evaluate.

One became then aware that the linearized theory permits an explanation, (along general lines), of the particularities presented by the motion of airplanes.

From the period 1920-1923 onward, experimenters were busy recording the trajectories of aircraft flying with controls fixed, that is, behaving in accordance with the fundamental hypothesis of the preceding theory.

The experiments made in the United States by Norton, Warner, and Allen, in the course of the years following the first World War, gave a qualitative and partial confirmation of the theory.

For a long time, however, it was not possible to observe the rapid oscillation depending on the longitudinal motion. This is not at all surprising. It is sufficient to visualize the aspect an oscillation assumes when its damping is such that the duration D for decrease to half-value becomes 1/20 of the period, in order to perceive that such an oscillation must lose its customary appearance (fig. 28).

A quantitative study of a particular type of aircraft, the Bristol Fighter, was begun in Great Britain, toward 1920. Laboratories attempted measurement of the aerodynamic derivatives by means of a series of models: several airplanes of this type were subjected to experimental verifications. These studies lasted until 1926; they enhanced the repute of the theory but were not reviewed. No other type of aircraft was ever more the object of a complete investigation with the aim of a determination of the derivatives and verification of the theory.

The laboratory tests, carried out for the first time on a Bristol Fighter, gave a result which, since then, always has been reconfirmed.

In the neighborhood of maximum lift, the derivative $\partial C_M/\partial \chi$ decreases considerably and becomes zero (sometimes even changes its sign). Under these conditions the rapid oscillation must become perceptible. And that is what actually happens.

Recordings (R. & M., Nr. 1367) of the flight path of the Bristol Fighter, flying at approximately maximum lift, permitted recordings of the rapid oscillation (period of 3" on this type of airplane).

Toward 1930, the NACA was able to announce that the theory of longitudinal motion had formed the object of quantitative verifications in level flight (with the T' and D' of the slow oscillation corresponding to those deduced from the theory); however, the correspondence of results was not attained for flight with engine in operation.

The reason for this lay probably in insufficient knowledge of the effect of the propeller on the aerodynamic actions undergone by the airplane. These effects are still only incompletely known, and an important part of the experimental work of the American wind tunnels during these last years aimed precisely at an investigation of this action.

2. Usefulness of the Theory

For many years the methods of calculation which permit forseeing the motion of an airplane did not play a part in airplane-design studies. One did not attempt to numerically predict the characteristics D and T, and one even did not always make measurements of those existing once the airplane had been built.

For the designer, a theory can be of use only when it permits fixing in advance the characteristics to be obtained.

The American NACA and the technical authorities of several countries posed themselves the question: "What are the dynamic characteristics which must be realized?"

In order to solve this problem as far as the longitudinal motion is concerned, the NACA proceeded by statistical means. It had a series of pilots test airplanes with known characteristics T' and D' of the slow oscillation, and asked their, necessarily qualitative, opinion regarding the flight properties.

The answers did not permit establishing any correlation between the valuations of the pilots and the characteristics T' and D', with controls fixed.

Other factors determined the judgment of the pilots; these factors referred, above all, to the magnitude of the reactions of the controls, to the flight properties with free controls and to the response of the airplane to the action of the pilot in the neighborhood of maximum lift.

The determination of the desirable characteristics regarding the lateral motion has likewise given rise to divergent opinions.

At one time one tried to avoid at all costs spiral instability even though the instability, or even the insufficient damping, of the swinging motion constitutes a much more serious inconvenience.

Nevertheless, the theory has been valuable explaining the motions and making the character of the phenomena comprehensible which otherwise would have remained mysterious.

3. Study of Flight With Free Controls

The theory of flight with free controls may be established by investigating the flight of an airplane subjected to a particular law of deflection, namely that which ensures constantly a zero hinge moment.

This problem does not constitute the object of this report; nevertheless we shall show how its study could be undertaken, and we shall find that it constitutes a particular development of the preceding theory.

In 1936 the development of our knowledge had progressed so far that, for the first time, the specification of the conditions of stability and maneuverability to be realized in a new airplane contained requirements relative to the dynamic characteristics.

Millikan's publication of the technical specifications which the Douglas DC 4 had to satisfy (the studies of which began at that time) constituted, from this point of view, an important event.

Aside from stipulations relative to the static conditions (for instance, deflection of the controls and their reactions in flight) the following requirements were to be found:

(a) The airplane will be placed in a dive until its speed has increased by 40 km per hour, and at that moment the control will be released.

The oscillations which will then be produced will have a period of at least 35 seconds and their damping will have to be such that the amplitude will be reduced to 20 percent of the initial value in four cycles.

(b) With the course (azimuth) of the airplane modified by 10° by a deflection of the rudder, the control will be released and the airplane will stabilize itself according to an azimuth varying by not more than 5° from the one at which the airplane was at the moment where the control was released.

The amplitude of the first oscillation in azimuth must not exceed 12°, and after three cycles this amplitude will be reduced to a maximum of 3°. The period must be at least 20 seconds; the ailerons are used during the maneuver for constantly maintaining a horizontal position of the transverse axis.

(c) The airplane will be inclined laterally 15° by means of the ailerons, and these will then be released.

The airplane must recover, and its inclination must be less than 2° after 15 seconds. The rudder is maneuvered so as to cancel the skidding.

4. Airplane with Simplified Flight Control

An American engineer, Mr. Weick, has devoted himself to the realization of a simplified flight control, using only one organ of lateral control instead of two. After years of studies and tests, he succeeded in realizing an airplane where the aerodynamic derivatives are proportioned in such a manner that the pilot can make the airplane perform correct maneuvers, using only one single organ of lateral control - a linkage the displacement of which determines at the same time the motion of the ailerons and that of the rudder; the respective deflections

$$\xi = f_1(x) \qquad \zeta = f_2(x)$$

are two different functions f_1 and f_2 of the motion realized by the pilot but determined once and for all. The airplane in question is the Ercoupe, well known in Belgium.

It is impossible for the pilot of the Ercoupe to hold the plane level in the turn, that is, to hold up the inside wing. The connection existing between the two controls prevents the pilot from making a flat turn.

The turns are probably carried out with a very slight sideslip (a few degrees) toward the inside which may be considered as practically correct.

The realization of an airplane carrying out such maneuvers with a single control organ for the two lateral controls should, in our opinion, be credited to the theory of the lateral motion.

An examination of the successive achievements of Mr. Weick shows clearly that they were not simply a matter of chance but the result of patient work.

Like several others, Mr. Weick had at first tried to completely do away with one of the lateral controls (the rudder). However, he has obtained results only by maintaining the latter but doing away with the independent control organ.

5. Extension of the Method of Linearization

A. Action of the derivatives.- The four preceding chapters have been devoted to the study of the motion with controls fixed under the hypothesis that the aerodynamic actions are dependent only on the fundamental variables

$$u,\ w,\ q$$

$$v,\ p,\ r$$

The range of the linearization method can be extended. The introduction of supplementary terms permits taking into account what occurs when the external actions depend on the derivatives of the fundamental variables.

In the following chapter the argument will be presented for the longitudinal motion but it could be integrally reproduced in the case of lateral motion.

This extension of the method is of interest only when the manner in which the external actions depend on the said derivatives is known. In fact, this dependence is little known, and only the effect of lag in attaining the deflection is easily evaluated.

Most of the recent presentations of the investigation of the longitudinal motion incorporate this effect in the equations set up initially.

However, we have preferred to carry out first a study of the motion in the simplest case and to show then, in a general manner, how the effect of all the derivatives u', w', q' on the external actions may be introduced into the calculations.

An elementary presentation is sufficient to give an understanding of the general behavior of the airplane in flight.

The introduction of the effect of the derivatives of the fundamental variables on the external actions into the calculation is, on the contrary, indispensable if one wants to carry out a complete comparison between the conclusions of the theory and the numerical characteristics noted in flight.

We should like to remark, however, that such an investigation can be undertaken only by organizations which have elaborate test facilities at their disposal.

The study of the effect of the derivatives of the fundamental variables on the aerodynamic reactions is, at any rate, interesting with a view to automatic flight control under the assumption that it is easy to imagine devices sensitive to the derivatives of the fundamental variables, and for application of known aerodynamic moments under the action of these derivatives.

B. Inertia of the engine.- We have indicated, ever since chapter II, that we assume that the engine instantaneously reached its steady speed.

This hypothesis permitted us to write T as a function of V.

It is evident that a more detailed analysis of the phenomena should lead us to take into account also the transient phenomena due to the inertia of the propeller, since the speed of revolution of the engine is connected with the velocity of translation V, but does not, in the case of variation, take on immediately the magnitude it possesses in steady state.

At the present time, the absence of exact data concerning the effect of the propeller on the external aerodynamic actions makes such an improvement in the calculations rather impractical.

If we possessed the necessary data, it would be possible to introduce them in the form of the effect of dV/dt, and it seems that it would be possible to avoid the introduction of a supplementary equation of a moment about the propeller axis into the system of equations.

CHAPTER XI

MOTIONS WHEN THE AERODYNAMIC ACTIONS DEPEND ON THE DERIVATIVES

1. General Theory

We shall now abandon the hypothesis we have made so far: that the forces are completely determined by the instantaneous values of the variables u, w, q, θ. We shall now investigate what becomes of the motion when the aerodynamic reactions depend on the derivatives du/dt, dw/dt, dq/dt.

The reactions are assumed to be expressed as functions of the velocities u, w, q; but to the functions used previously, supplementary terms have to be added which express the effect of the derivatives on the aerodynamic actions.

We shall simplify the notation by representing the derivatives by u', w', q'.

The system of equations of motion has the form

$$u' = g_1(\delta u, \delta w, \delta q, \delta\theta, u', w', q')$$

$$w' = g_2(\delta u, \delta w, \delta q, \delta\theta, u', w', q')$$

$$q' = g_3(\delta u, \delta w, \delta q, \delta\theta, u', w', q')$$

$$\theta' = \delta q$$

After linearization, and taking into consideration that the derivatives of the functions g are equal to those of the functions f defined before:

$$\frac{\partial g_1}{\partial u} = \frac{\partial f_1}{\partial u} \quad . \ . \ . \ . \ . \quad \frac{\partial g_4}{\partial \theta} = \frac{\partial f_4}{\partial \theta}$$

one obtains:

$$u' = \frac{\partial f_1}{\partial u}\delta u + \frac{\partial f_1}{\partial w}\delta w + \frac{\partial f_1}{\partial q}\delta q + \frac{\partial f_1}{\partial \theta}\delta\theta + \frac{\partial g_1}{\partial u'}u' + \frac{\partial g_1}{\partial w'}w' + \frac{\partial g_1}{\partial q'}q'$$

$$w' = \frac{\partial f_2}{\partial u}\delta u + \frac{\partial f_2}{\partial w}\delta w + \frac{\partial f_2}{\partial q}\delta q + \frac{\partial f_2}{\partial \theta}\delta\theta + \frac{\partial g_2}{\partial u'}u' + \frac{\partial g_2}{\partial w'}w' + \frac{\partial g_2}{\partial q'}q'$$

$$q' = \frac{\partial f_3}{\partial u}\delta u + \frac{\partial f_3}{\partial w}\delta w + \frac{\partial f_3}{\partial q}\delta q + \frac{\partial f_3}{\partial \theta}\delta\theta + \frac{\partial g_3}{\partial u'}u' + \frac{\partial g_3}{\partial w'}w' + \frac{\partial g_3}{\partial q'}q'$$

$$\theta' = \delta q$$

We write:

$$\frac{\partial f_1}{\partial u}\delta u + \frac{\partial f_1}{\partial w}\delta w + \frac{\partial f_1}{\partial q}\delta q + \frac{\partial f_1}{\partial \theta}\delta\theta = F_1(\delta u,\delta w,\delta q,\delta\theta)$$

$$\frac{\partial f_2}{\partial u}\delta u + \frac{\partial f_2}{\partial w}\delta w + \frac{\partial f_2}{\partial q}\delta q + \frac{\partial f_2}{\partial \theta}\delta\theta = F_2(\delta u,\delta w,\delta q,\delta\theta)$$

$$\frac{\partial f_3}{\partial u}\delta u + \frac{\partial f_3}{\partial w}\delta w + \frac{\partial f_3}{\partial q}\delta q + \frac{\partial f_3}{\partial \theta}\delta\theta = F_3(\delta u,\delta w,\delta q,\delta\theta)$$

and the system becomes:

$$u' = F_1(\delta u,\delta w,\delta q,\delta\theta) + \frac{\partial g_1}{\partial u'}u' + \frac{\partial g_1}{\partial w'}w' + \frac{\partial g_1}{\partial q'}q'$$

$$w' = F_2(\delta u,\delta w,\delta q,\delta\theta) + \frac{\partial g_2}{\partial u'}u' + \frac{\partial g_2}{\partial w'}w' + \frac{\partial g_2}{\partial q'}q'$$

$$q' = F_3(\delta u,\delta w,\delta q,\delta\theta) + \frac{\partial g_3}{\partial u'}u' + \frac{\partial g_3}{\partial w'}w' + \frac{\partial g_3}{\partial q'}q'$$

Solving with respect to u', w', q', θ', one obtains:

$$u' = \frac{d\delta u}{dt} = h_1(\delta u,\delta w,\delta q,\delta\theta)$$

$$w' = \frac{d\delta w}{dt} = h_2(\delta u,\delta w,\delta q,\delta\theta)$$

$$q' = \frac{d\delta q}{dt} = h_3(\delta u,\delta w,\delta q,\delta\theta)$$

$$\theta' = \frac{d\delta\theta}{dt} = h_4(\delta u,\delta w,\delta q,\delta\theta)$$

The functions h_1, h_2, h_3, h_4 are linear functions and of the first degree in δu, δw, δq, $\delta\theta$. The system may therefore be integrated by the same methods as those applied before.

2. Solution when the Aerodynamic Actions Depend on the Velocity of Variation of Incidence

We shall treat the case where the effects X, Z, M depend, aside from the usual variables, on the velocity of variation of incidence $d\alpha/dt$, that is, on the derivative $dw/dt = w'$.

An argument analogous to the one that follows here could be established if the aerodynamic forces depended on u'.

In the case considered, the system is reduced to

$$u' = F_1(\delta u,\delta w,\delta q,\delta\theta) + \frac{\partial g_1}{\partial w'} w'$$

$$w' = F_2(\delta u,\delta w,\delta q,\delta\theta) + \frac{\partial g_2}{\partial w'} w'$$

$$q' = F_3(\delta u,\delta w,\delta q,\delta\theta) + \frac{\partial g_3}{\partial w'} w'$$

with

$$\frac{\partial g_1}{\partial w'} = \frac{1}{m} X'_{w'} \quad \text{and} \quad X'_{w'} = -SV\frac{\rho}{2} z_1$$

$$\frac{\partial g_2}{\partial w'} = \frac{1}{m} Z'_{w'} \qquad Z'_{w'} = -SV\frac{\rho}{2} z_2$$

$$\frac{\partial g_3}{\partial w'} = \frac{1}{B} M'_{w'} \qquad M'_{w'} = -\frac{Sc}{r^2} V\frac{\rho}{2} z_3$$

whence:

$$\frac{\partial g_1}{\partial w'} = \frac{-V}{\mu c} z_1$$

$$\frac{\partial g_2}{\partial w'} = \frac{-V}{\mu c} z_2$$

$$\frac{\partial g_3}{\partial w'} = \frac{-c}{r^2}\frac{V}{\mu c} z_3$$

These expressions will be useful only in so far as we can express the effect of the accelerations by means of the factors z_1, z_2, z_3.

We obtain after solution for u', w', q', and θ'

$$u' = F_1 + F_2 \frac{-z_1}{(\mu c/V) + z_2}$$

$$w' = F_2 + F_2 \frac{-z_2}{(\mu c/V) + z_2}$$

$$q' = F_3 + F_2 \frac{-(c/r^2)z_3}{(\mu c/V) + z_2}$$

Putting

$$b_x = \frac{-z_1}{(\mu c/V) + z_2}$$

$$b_z = \frac{-z_2}{(\mu c/V) + z_2}$$

$$b_m = \frac{z_3}{(\mu c/V) + z_2}$$

we shall write

$$u' = F_1 + b_x F_2$$

$$w' = F_2 + b_z F_2$$

$$q' = F_3 + b_m \frac{c}{r^2} F_2$$

Thus the system to be integrated is written when there is no second term (case of an initial perturbation)

$$\frac{d\delta u}{dt} - \left(\frac{\partial f_1}{\partial u} + b_x \frac{\partial f_2}{\partial u}\right) \delta u - \left(\frac{\partial f_1}{\partial w} + b_x \frac{\partial f_2}{\partial w}\right) \delta w -$$

$$\left(\frac{\partial f_1}{\partial q} + b_x \frac{\partial f_2}{\partial q}\right) \delta q - \left(\frac{\partial f_1}{\partial \theta} + b_x \frac{\partial f_2}{\partial \theta}\right) \delta\theta = 0$$

$$\frac{d\delta w}{dt} - \left(\frac{\partial f_2}{\partial u} + b_z \frac{\partial f_2}{\partial u}\right) \delta u - \left(\frac{\partial f_2}{\partial w} + b_z \frac{\partial f_2}{\partial w}\right) \delta w -$$

$$\left(\frac{\partial f_2}{\partial q} + b_z \frac{\partial f_2}{\partial q}\right) \delta q - \left(\frac{\partial f_2}{\partial \theta} + b_z \frac{\partial f_2}{\partial \theta}\right) \delta\theta = 0$$

$$\frac{d\delta q}{dt} - \left(\frac{\partial f_3}{\partial u} + b_m \frac{c}{r^2} \frac{\partial f_2}{\partial u}\right) \delta u - \left(\frac{\partial f_3}{\partial w} + b_m \frac{c}{r^2} \frac{\partial f_2}{\partial w}\right) \delta w -$$

$$\left(\frac{\partial f_3}{\partial q} + b_m \frac{c}{r^2} \frac{\partial f_2}{\partial q}\right) \delta q - \left(\frac{\partial f_3}{\partial \theta} + b_m \frac{c}{r^2} \frac{\partial f_2}{\partial \theta}\right) \delta\theta = 0$$

$$\frac{d\delta\theta}{dt} - \delta q = 0$$

The characteristic determinant for the solution in terms of τ is written, with all terms multiplied by $\mu c/V$

$$\begin{vmatrix} \left(a_1 + b_x a_2\right) + \lambda & \left(b_1 + b_x b_2\right) & \left(c_1 + b_x c_2\right) & \left(d_1 + b_x d_2\right) \\ \left(a_2 + b_z a_2\right) & \left(b_2 + b_z b_2\right) + \lambda & \left(c_2 + b_z c_2\right) & \left(d_2 + b_z d_2\right) \\ \frac{c}{r^2}\left(a_3 + b_m a_2\right) & \frac{c}{r^2}\left(b_3 + b_m b_3\right) & \frac{c}{r^2}\left(l c_3 + b_m c_3\right) + \lambda & \frac{c}{r^2}\left(d_3 V + b_m d_2\right) \\ 0 & 0 & c_4 & + \lambda \end{vmatrix} = 0$$

This determinant may be developed without difficulty and leads to an equation of the fourth degree in λ.

It is necessary to replace b_x, b_z, b_m by expressions which take into account the effect of the aerodynamic phenomenon on the motion which we are investigating.

The modifications of w which take place at constant V correspond to modifications of the incidence α.

Since

$$\alpha = - \frac{w}{V}$$

$$\frac{d\alpha}{dt} = - \frac{dw}{dt}\frac{1}{V}$$

$$w' = - V\alpha'$$

one states that

$$z_1 = + \frac{\partial C_x}{\partial \alpha'}$$

$$z_2 = + \frac{\partial C_z}{\partial \alpha'}$$

$$z_3 = + \frac{\partial C_z}{\partial \alpha'}$$

which permits determining the b_x, b_z, b_m.

The z_1, z_2, z_3 have the dimensions T just like $\mu c/V$.

3. Effects of the Velocity of Increase in Incidence on the Aerodynamic Forces

The following effects exist:

(a) Lag in the establishment of the lift and of the deflection.

When the incidence undergoes an increase, the corresponding circulation is not immediately established, and the lift develops only with a certain lag.

In a phenomenon where the incidence increases progressively, the instantaneous lift will be smaller than the one corresponding to the same instantaneous incidence, realized in steady state.

These phenomena have been studied theoretically by Küssner, Kuethe, Sears, etc.

The presence of a systematic lag in the establishment of the lift permits predicting that in the course of a phenomenon where the incidence varies progressively, the instantaneous lift will be a function not only of the instantaneous incidence but also of its speed of increase.

The C_M to which an airplane is subjected, the incidence of which varies, will be influenced by the phenomenon described above, therefore:

(1) By the lag in the establishment of the lift of the wing

(2) By the lag in the establishment of the lift of the tail

(3) By a lag in the deflection which necessarily accompanies the lag in the establishment of the lift

(b) Lag in the break down of the flow.

In the neighborhood of maximum lift there appears, aside from the lag in the establishment of the theoretical circulation, a lag in the break down of the flow.

A rapid increase of incidence may transitorily carry the lift up to a value exceeding the maximum value that it has in steady state.

However, the flow is unstable and deteriorates rapidly, causing a reduction of lift.

Taking as a function of the incidence the instantaneous lift realized in the neighborhood of and beyond the critical incidence, at different speeds of variation of incidence, the NACA has obtained a series of different curves.

The phenomenon investigated here is different from the previous one. It occurs only at large incidences whereas the preceding one may take place at all incidences.

A positive speed of increase of incidence leads to a higher transitory lift than the steady-state lift at the same incidence, whereas it produces a transitory lift lower than the steady-state lift as in the case of the previous phenomenon.

The lag in the break down of the flow is a true hysteresis phenomenon.

If one obtains a polar starting from an incidence larger than that at maximum lift and descending, one obtains frequently in the neighborhood of the maximum lift a polar different from the one obtained in ascending.

(c) Lag in the onset of downwash at the tail.

The lag in the onset of downwash at the tail constitutes a phenomenon the mechanism of which is clear and indisputable. It is easy to find for it the physical cause and the analytical expression.

4. Effect of the Lag in the Establishment of Lift

Let us suppose that an increase in incidence has occurred corresponding under static conditions to an increase of C_z equal to one.

The diagram of Sears indicates how the lift increment varies effectively as a function of time - this time being estimated by means of a unit of time s equal to the time taken by the airplane for traversing half the chord.

$$\text{unit} \quad s = \frac{2V}{c} \text{ seconds}$$

Between the measurements of the same time interval there exists the relation

$$\Delta s = \frac{c}{2V} \Delta t$$

$$\Delta t = \frac{c}{2V} \Delta s$$

By means of the graphical construction of Carson (defined in the following chapter) we can investigate the development of the lift as a function of the time s when we suppose that the increment $\Delta\alpha$ has been realized progressively, with the duration of the increment being, respectively, equal to

$$\Delta s = 10 \quad \text{or} \quad \Delta t = 10\ c/2V \text{ seconds}$$
$$\Delta s = 20 \quad \text{or} \quad \Delta t = 20\ c/2V \text{ seconds}$$
$$\Delta s = 30 \quad \text{or} \quad \Delta t = 30\ c/2V \text{ seconds}$$

We find that during the period of increase in incidence the C_z has a smaller value than it would have if the lift would correspond to the instantaneous incidence. The order of magnitude of the phenomenon is taken into account by replacement of the instantaneous curve C_z by a straight line.

Thus one finds a constant difference of

$$\Delta C_z = 0.11 \quad \text{for} \quad \Delta t = 15 \ c/V \ \text{seconds}$$

$$\Delta C_z = 0.165 \ \text{for} \quad \Delta t = 10 \ c/V \ \text{seconds}$$

$$\Delta C_z = 0.33 \quad \text{for} \quad \Delta t = 5 \ c/V \ \text{seconds}$$

The correction of lift ΔC_z to be applied to the C_z corresponding to the instantaneous incidence, in order to have the C_z real, is negative and its absolute value is inversely proportional to the duration which is necessary to bring the wing to the final incidence. It is therefore proportional to the speed of increase in incidence $d\alpha/dt$

$$\Delta C_z = -K \frac{d\alpha}{dt}$$

When the incidence corresponding to $C_z = 1$ is attained during a time $\Delta t = 15 \ c/V$ seconds, the speed of variation in incidence is

$$\frac{d\alpha}{dt} = \frac{1}{\partial C_z/\partial\alpha} \times \frac{1}{15 \ c/V}$$

the difference in lift is $\Delta C_z = -0.11$.

This permits fixing the value of K

$$K = 1.65 \frac{\partial C_z}{\partial\alpha} \frac{c}{V}$$

$\partial C_z/\partial\alpha$ is of the order of 4.

For $C = 3$ meters and $V = 100$ m/sec, one would have

$$K = +0.20$$

whence

$$\partial C_z/\partial\alpha' = -0.20$$

The previously given relations between the lift and the force Z show that one may write almost without error

$$C_z = C_Z$$

whence

$$z_2 = \frac{\partial C_z}{\partial \alpha'} = \frac{\partial C_Z}{\partial \alpha'}$$

which permits calculation of the factors b_x, b_z, b_m.

Remark: The preceding argument constitutes an attempt to introduce into the frame work of the linear equations a typical value for the lag, in the establishment of the lift.

The factor b_z will be >1 when $\mu c/V$ is larger than $-z_2$ which is the general case.

The result obtained is therefore rather paradoxical; the term $b_2 = \partial C_Z/\partial\alpha$ of the determinant is multiplied by a factor > 1 although it would rather seem that the lag in the establishment of the lift should be expressed by a reduction of b_2.

The calculations have been checked several times in order to make sure whether this result was not due to an error in sign, but the sign found has always been confirmed.

$$\text{Since} \quad C_m = C_{mo} + 0.25C_z$$

because of the wing and independently of any action of the tail, one should admit at the same time

$$z_3 = \frac{\partial C_m}{\partial \alpha'} = -0.05$$

5. Effect of the Lag in the Break Down of the Flow

This phenomenon occurs in the neighborhood of maximum lift.

The method of linearization of the equations is basically no longer applicable in this case, in view of the fact that the second derivatives may no longer be neglected.

The investigation of flight in the neighborhood of maximum lift must be excluded from the present report.

There is therefore no reason for visualizing here the consequences of the lag in the appearance of the separation.

6. Effect of the Lag in the Arrival of the Downwash at the Tail

One can easily find the physical cause and the analytical expression of this phenomenon.

Let us try to express it by a simple argument. In the static investigation of the moment C_M, one takes for the angle of attack of the tail α_e:[7]

$$\alpha_e = \alpha + \delta - \epsilon$$

where α is the incidence of the wing

ϵ the downwash due to the wing

δ the final decalage fixed by the design

This relation serves as the basis for the calculation of the C_M in steady state, but it ceases to be exact when the airplane undergoes a pitching motion in the course of which the angle of attack varies.

At an instant t the stream lines which strike the tail have actually not been deflected by the angle ϵ_t realized at that moment but by the angle ϵ_t' which existed at a previous instant t'.

$$t' = t - l/V$$

where l/V is the time for the flow to traverse the distance which separates the wing from the tail.

The real angle of attack of the tail at the time t is therefore

$$\alpha\epsilon_t = (\alpha - \epsilon + \delta)_t + \epsilon_t - \epsilon_t'$$

---

[7]Since $\alpha' = \partial\alpha/dt$ is to designate the derivative of the incidence of the wing, we shall use in what follows:
α_e for the real angle of attack of the tail
C_{ze} for the lift of the tail

that is, it is equal to the angle of attack given by the static relation, plus a correction $\epsilon_t - \epsilon_t'$.

Now

$$\epsilon_t - \epsilon_t' = \frac{d\epsilon}{dt}(t - t') = \frac{d\epsilon}{dt}\frac{l}{V}$$

or else

$$\epsilon_t - \epsilon_t' = \frac{d\epsilon}{d\alpha}\frac{d\alpha}{dt}\frac{l}{V}$$

Introducing this correction of the angle of attack into the static expression of the moment, we see that there exists actually a supplementary moment, proportional to the derivative $\partial\alpha/dt$ and equal to

$$C_M = \frac{S'l}{Sc}\frac{\partial C_{z,e}}{\partial\alpha_e}\frac{l}{V}\frac{d\epsilon}{d\alpha}\alpha'$$

Hence

$$z_3 = \frac{\partial C_M}{\partial\alpha'} = \frac{S'l}{Sc}\frac{\partial C_{z,e}}{\partial\alpha_e}\frac{l}{V}\frac{d\epsilon}{d\alpha}$$

and one finds thus that the lag in the arrival of the downwash at the tail has the effect of bringing about a $\partial C_M/d\alpha'$ which can be easily calculated.

The phenomenon considered here does not produce any effect on the C_x and C_z. If one limits oneself to the effect of the lag in the arrival of the downwash, one has therefore

$$z_1 = 0 \qquad z_2 = 0$$

It can be easily verified that the sign of the expression $\partial C_M/\partial\alpha'$ is correct.

When α increases, the calculation carried out with the static value ϵ_t leads to a too large downwash, that is, to an angle of attack α_e which is too small. The correction must augment α_e, that is, the normal reaction on the tail surfaces and the diving moment.

The correction must be positive.

We introduce into the system the values of z_1, z_2, and z_3.

We obtain

$$b_x = 0$$
$$b_z = 0$$
$$b_m = -\frac{V}{\mu c}\frac{\partial C_M}{\partial \alpha'}$$

In the characteristic determinant which gives the solution in τ only the third line is modified.

The term $\frac{c}{r^2} a_3$ becomes $\frac{c}{r^2}\left(a_3 + b_m a_2\right)$

$$\frac{c}{r^2} b_3 \qquad \frac{c}{r^2}\left(b_3 + b_m b_2\right)$$

$$\frac{cl}{r^2} c_3 \qquad \frac{cl}{r^2}\left(c_3 + b_m c_2/l\right)$$

$$\frac{c}{r^2} d_3 V \qquad \frac{c}{r^2}\left(d_3 V + b_m d_2\right)$$

with $d_3 = 0$.

What happened was:

(1) The terms a_3, b_3, and c_3 underwent appreciable modifications

(2) The fourth term of the third line ceased to be always zero

Let us examine first the quantity which has been just added to c_3.

We have

$$c_3 = -\frac{\partial C_M}{\partial \chi} \qquad c_2 = -V\frac{\mu c}{V} \qquad \text{since} \quad u = V$$

Hence

$$c_3 + \frac{b_m c_2}{l} = -\frac{\partial C_M}{\partial \chi} + \frac{S' l}{Sc}\frac{\partial C_{z,e}}{\partial \alpha_e}\frac{d\epsilon}{d\alpha}$$

For steady conditions, an important part of the $\partial C_M/\partial \chi$ is produced by the lift on the tail; one had

$$- \frac{\partial C_M}{\partial \chi} = + \frac{S'l}{Sc} \frac{\partial C_{z,e}}{\partial \alpha_e}$$

One finds that everything takes place as if the part due to the horizontal tail in the damping of the pitching were multiplied by $1 + (\partial\epsilon/d\alpha)$.

Since $\partial\epsilon/d\alpha$ is of the order of magnitude of 0.5 and, since on the other hand, the effect of the tail is predominant in the damping of the pitching, the correction which is to be introduced into the calculations if one wants to take the effect investigated into account, is important.

The correction terms to be added to a_3 and b_3 are less important but they are, nevertheless, not negligible.

Actually, what happens is

$$\begin{array}{lll} a_3 & \text{became} & a_3 + b_m a_2 \\ b_3 & & b_3 + b_m b_2 \end{array}$$

It is easy to evaluate the order of magnitude of certain factors. If one takes

$$l/c = 3 \qquad S'/S = 1/5$$

one obtains

$$b_m = -1/\mu \times 3 \times 1/5 \times 3 \times 3 \times 1/2 = -2.7/\mu$$

On the other hand, $a_2 = \alpha(\partial C_z/\partial\alpha) - 2C_z$ is of the order of magnitude of -1.

$b_2 = \partial C_z/\partial\alpha$ is of the order of magnitude of +4.

The factor μ, on the other hand, is always represented by a high number, of the order of 100.

Hence, everything happens as though:

a_3 was increased by a quantity of the order of a hundredth

b_3 was reduced by a quantity which could reach several hundreths

Since the term a_3 is itself very small, and the term $b_3 = \partial C_M/\partial\alpha$ is normally of the order of a tenth, it can be necessary to take the corrections indicated into account.

Finally, there appears in the fourth column a term $c/r^2(d_3V + b_m d_2)$ but, $d_2 = -g \sin\theta \times \mu c/V$ so that, if the trim of the airplane is horizontal, this term remains zero.

7. Solution of the System when the Moments Depend on the Linear Acceleration dV/dt or on the Angular Acceleration

It is possible to repeat the calculations of the sections 2 and 4 with the assumption that the aerodynamic actions depend on the derivative u' or q'.

We shall be satisfied to indicate the formulas pertaining to the effect of u'.

We assume three coefficients which characterize the effect of the derivative

$$x_1 = -\frac{\partial C_x}{\partial u'} V$$

$$x_2 = -\frac{\partial C_z}{\partial u'} V$$

$$x_3 = -\frac{\partial C_M}{\partial u'} V$$

These coefficients lead to the factors

$$a_x = \frac{-x_1}{(\mu c/V) + x_1}$$

$$a_z = \frac{-x_2}{(\mu c/V) + x_1}$$

$$a_m = \frac{-x_3}{(\mu c/V) + x_1}$$

which are introduced into the characteristic determinant in the same manner as the b_x, b_z, b_m.

If only the moment depends on u'

$$a_x = 0 \qquad a_z = 0 \qquad a_m = -(V/\mu c)x_3$$

and only the third line of the determinant is modified.

It is written

$$\frac{c}{r^2}\left(a_3 + a_m a_1\right) \qquad \frac{c}{r^2}\left(b_3 + a_m b_1\right)$$

$$\frac{cl}{r^2} c_3 + \frac{c}{r^2} a_m c_1 + \lambda \qquad \frac{c}{r^2}\left(d_3 V + a_m d_1\right)$$

We shall see later on that the fourth term may exert a notable effect in the case where d_1 is different from zero.

The formulas concerning the effect of q' would be established in the same manner.

8. Lateral Motion

It is obvious that analogous calculations could be carried out in the course of calculation of the lateral motion.

However, these calculations would be useless (except in the case of automatic stabilizers) in view of the fact that nothing is known about the aerodynamic effects exerted.

CHAPTER XII

EFFECT OF THE CONTINUOUS DISPLACEMENT OF A CONTROL

1. Continuous Displacements

In the preceding chapters we calculated the motion which follows upon the abrupt displacement of a control.

The general expressions set up before determine the motion of the airplane when a control is being deflected by $\Delta\eta$ (longitudinal motion), by $\Delta\xi$, or by $\Delta\zeta$ (lateral motion), and maintained in its new position.

In theory, the problem is solved in the same manner for a change $\Delta\sigma$ of the power setting.

In fact, the pilot acts by continuous and progressive displacement of $d\xi$, $d\eta$, and $d\zeta$ as a function of t or τ, and it is necessary to determine the resulting motions of the airplane.

2. Duhamel's Integral

The response of the airplane to a variable movement of a control may be determined by Duhamel's integral.

For an abrupt deflection equal to unity applied at the time $t = 0$, the general solution of the system is

$$\delta u = \Delta u + \Sigma C e^{xt} = F_1(t)$$

$$\delta w = \Delta w + \Sigma / C e^{xt} = F_2(t)$$

$$\delta q = \Delta q + \Sigma m C e^{xt} = F_3(t)$$

$$\delta \theta = \Delta \theta + \Sigma n C e^{xt} = F_4(t)$$

the C, x, l, m, and n are successively provided with the subscripts 1 to 4.

If the deflection η_0 applied at the origin of the time scale is different from unity, we shall have

$$\delta u = \eta_0 F_1(t)$$

and so forth.

At the instant t_a, for example, we shall have

$$\delta u_a = \eta_0 F_1(t_a)$$

We shall write only the expression of one single variable which will permit us to eliminate the subscript 1 from F_1.

Let us assume an instant t_b subsequent to t_a. If the deflection has not changed, we shall have

$$\begin{aligned}\delta u_b &= \eta_0 F(t_b)\\ &= \eta_0 \Delta u + \eta_0 \Sigma C e^{xt_b}\end{aligned}$$

If, in addition, the deflection changes at the time t_a and assumes at that moment the value $\eta_0 + d\eta$, we obtain

$$\delta u_b = \eta_0 \Delta u + \eta_0 \Sigma C e^{xt_b} + d\eta\ \Delta u + d\eta \Sigma C e^{x(t_b - t_a)}$$

or else

$$\delta u_b = \eta_0 F(t_b) + d\eta\ F\left(t_b - t_a\right)$$

If the deflection, instead of undergoing one single increase $d\eta$ at the instant t_a, is modified regularly from the instant t_a to the instant t_b, according to a law $\eta = f(t)$, an increment $d\eta$ will be added at every one of the points between t_a and t_b, and one will have

$$\delta u_b = \eta_0 F(t_b) + \int_{t_a}^{t_b} \frac{d\eta}{dt_a} F\left(t_b - t_a\right) dt_a$$

in this expression t_a is the variable of integration.

On the other hand, if the deflection is varied starting from the origin, the lower limit of integration $t_a = 0$.

We can finally omit the subscript a and we obtain, representing the deflection by $f(t)$

$$\delta u_b = f(0) F(t_b) + \int_0^{t_b} f'(t) F\left(t_b - t\right) dt$$

This formula constitutes Duhamel's integral. It permits calculation of the response of the airplane to a deflection developing according to an arbitrary law $f(t)$, if the response of the airplane $F(t)$ to unit deflection is known.

Duhamel's integral determines the value of the variables, such as δu, at the arbitrary instant t_b by an integration carried out between the limits 0 and t_b.

Remarks: 1. The same argument permits determination of the effect of external perturbations, such as gusts arising no longer in an abrupt manner, but progressively.

2. The argument is independent of the selection of the unit of time. We made it assuming the solution of the differential equations written in the system x and t. Obviously it is also applicable to the solution containing λ and τ.

3. Other Expressions of Duhamel's Integral

The application of the formula

$$\int u \, dv = uv - \int v \, du$$

shows that the two expressions

$$\delta u_b = f(0)F(t_b) + \int_0^{t_b} f'(t)F(t_b - t)dt$$

and

$$\delta u_b = f(t_b)F(t_0) + \int_0^{t_b} f(t)F'(t_b - t)dt$$

are equivalent.

One may furthermore write the integral in the forms

$$\delta u_b = f(0)F(t_b) + \int_0^{t_b} f'(t_b - t)F(t)dt$$

and

$$\delta u_b = f(t_b)F(0) + \int_0^{t_b} f(t_b - t)F'(t)dt$$

4. Graphical Construction

A simple graphical construction permits us to find the value of

$$\int_0^{t_b} f'(t)F(t_b - t)dt$$

Let us plot, by means of a first system of axes, the curve (1), defining F as a function of t.

With a second system of axes, placed with respect to the first as indicated in the drawing, let us plot the curve (2) which defines $f(t)$ as a function of t.

Let us now choose the value t_b for which we calculate δu_b. Let us register this time t_b on the time scale of the two diagrams.

Let us assume, for instance, $t_b = 10$ seconds. Let us then divide the interval contained between 0 and t_b into n equal parts (10 on the drawing considered).

Let us plot an auxiliary curve (3) which connects the values of $F(t_b - t)$ with the values of $f'(t)$ realized at the same instant.

This curve (3) is plotted on a system of axes where the values $F(t_b - t)$ represent the abscissas and the $f(t)$ the ordinates.

It is constructed by associating:

the point (n) of the curve (1) with the point (0) of the curve (2);

the point (n - 1) of the curve (1) with the point (1) of the curve (2), and so forth.

It is clear that under these conditions:

$$\int_0^{t_b} F(t_b - t)\frac{df}{dt}\, dt$$

is represented by the area under the auxiliary curve.

One must therefore carry out a quadrature in order to find the δu at the time t_b.

Since one attempts to find the curve δu as a function of time, it will be necessary to repeat the construction for a sufficient number of points.

Finally, one generally wants to know δw and $\delta\theta$ as well; the construction will have to be applied to these variables.

The application of the graphical method takes a relatively long time even though artifices are rapidly discovered (employment of movable tracings, for instance) which permit systematizing and greatly accelerating the constructions.

5. Actual Piloting

The theoretical study permits an explanation of certain peculiarities of actual piloting.

A. Piloting with respect to elevation.- The pilot desires to pass from the rectilinear flight condition R_1 (V_1, α_1, and θ_1 are determined) to the rectilinear flight condition R_2 (V_2, α_2, and θ_2 are determined) by maneuvering of the elevator alone, without modification of the throttle setting.

The characteristics of the airplane are such that to a given displacement

$$\eta_2 - \eta_1 = \Delta\eta$$

there correspond

$$V_2 - V_1 = -2/10\ V_1$$

$$\alpha_2 - \alpha_1 = +3^\circ$$

$$\theta_2 - \theta_1 = -5^\circ$$

If this deflection $\Delta\eta$ is applied abruptly, the δV, $\delta\alpha$, and $\delta\theta$ vary as functions of time according to the expressions given previously which are produced by the curves given in the plate 24 of chapter VII.

A skillful pilot will try to rapidly attain the flight condition R_2 and then to stabilize the airplane in the corresponding position, avoiding the long-period oscillations.

He will not apply the deflection $\Delta\eta$ abruptly but will utilize another law of deflection. He will start from η_1 at $t = 0$, and will necessarily terminate at η_2, but within a determined time and after having followed an indetermined path.

The graphical method permits us a trial-and-error investigation of the laws of displacement which stabilize more or less rapidly the variables δV, $\delta\alpha$, or $\delta\theta$ at their final value.

The piloting will be precise when the pilot, in an intentional change of flight condition, rapidly attains the final value of the variables δV, $\delta\alpha$, and $\delta\theta$, without oscillation.

An examination of the diagram shows us that the motions δV, $\delta\alpha$, and $\delta\theta$, produced by an abrupt deflection, are out of phase during their transitory period. The δV and $\delta\theta$ are obviously squared.

Thus it can be predicted that it will be impossible to suppress, by a continuous variation of the deflection, simultaneously both deviations δV and $\delta\theta$. Different laws of deflection will be applicable according to the variable the pilot wants to establish first at its final value.

The inclination θ is the variable the modifications of which the pilot can most easily appreciate in visual piloting.

Conducting step by step the search of the η which brings $\delta\theta$ rapidly to approximately its final value, we find that the law of deflection represented by (1) imposes on $\delta\theta$ a variation represented by (2).

The variables δV and $\delta\alpha$ then undergo an imposed variation, corresponding to the curves 3 and 4. They tend to reach much less rapidly their final value, but the oscillatory character of the behavior corresponding to an abrupt deflection has disappeared.

One could, again by trial and error, find laws other than (1), to improve the shape of the curve (2).

The various possible laws satisfy the following characteristics: The deflection to be applied at the start of the motion is greater than the one corresponding to the final state (this is done with a view to accelerate the starting of the rotation of the airplane); then, after a very brief time interval, the control must be deflected in the opposite sense in order to avoid overshooting of the position of equilibrium.

Then the control should be brought to the final position of equilibrium by a slow progressive displacement.

The pilot must therefore carry out a double motion in order to avoid overshooting the position of equilibrium.

B. Starting a turn.- Let us take up again the considerations of chapter V sections 5 and 6 on the conditions of equilibrium in turns.

Let us investigate, for the airplanes examined in chapter IX section 5, the conditions of equilibrium during a turn to the left, effected with a lateral inclination φ of the order of 39°.

We shall assume that the semispan of the airplane has a length of 10 m, that the airplane flies at a speed of 80 m/second, and that it traverses a circle of 800 m radius.

We then have

$$\Omega = 0.1 \qquad r = \Omega \cos \varphi = 0.08 \qquad \rho = 0.01$$

The conditions of equilibrium show us that the theoretical turn corresponds, on the three airplanes considered, to

$$\begin{aligned} \xi &= +0.2^{\circ} & \beta &= 0 \\ \zeta &= +0.5^{\circ} & \varphi &= -38^{\circ}40 \\ & & &= -0.67 \text{ rad} \end{aligned}$$

The response in β, $\tilde{\omega}$, φ, ρ, and ψ, under the effect of an aileron deflection -2° and of a rudder deflection $+2^{\circ}$ is given for the three airplanes in plate 27.

The maneuver to be carried out, if one wants to realize in practice the theoretical turn, consists in manipulating the two controls according to the laws

$$\begin{aligned} \xi &= f_1(t) \quad \text{or} \quad f_1(\tau) \\ \zeta &= f_2(t) \qquad\quad\; f_2(\tau) \end{aligned}$$

such that after the shortest possible time the motion corresponds to the constant values.

$$\beta = 0 \qquad \tilde{\omega} = 0 \qquad \varphi = -38^{\circ}40 = -0.67 \text{ rad} \qquad \rho = +0.01$$

for the final values of deflection

$$\xi = +0.5^\circ \qquad \zeta = +0.5^\circ$$

The calculation of this maneuver may be made by trial and error by means of graphical construction.

In practice the pilot performs only maneuvers which lead approximately to the desired result.

The conditions of equilibrium under the action of one single control have been given in order to show that they are the attainable-accuracy limits.

Figure 33 indicates the result of the graphical construction carried out for the airplane B' under the assumption of a priori fixed laws of deflection as functions of time.

The motion of the airplane has not been stabilized at the desired values, but it is not far remote from them.

By trial and error, one would arrive at finding laws of deflection which would lead more exactly to the desired turn.

The figure shows that the motion of the rudder is a little too important with respect to that of the ailerons, for it produced temporarily a sideslip to the right, toward the outside.[8]

We have indicated previously that the conditions of equilibrium in the course of a turn, realized with a steady deflection of a single control, would differ only slightly from those corresponding to a theoretically correct turn.

For the airplane B', the conditions of equilibrium during a turn, at an angular velocity $\rho = 0.01$, under the effect of one control, are

Aileron deflection	-0.07	0
Rudder deflection	0	+0.4
Sideslip	+0.95	+0.3
Inclination	-39.50	-39.0

These results show that the conditions under which a turn can take place are not invariably fixed but that there exists, on the contrary, a certain domain of variables and of possible maneuvers.

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[8]<u>Important note</u>.- In the figure 33, the positive sideslip β is shown directed upward. A positive β corresponds to a sideslip to the left, not to the right, as erroneously indicated in the cliché.

CHAPTER XIII

THE AUTOMATIC PILOT

1. Statement of the Problem

The pilots who, after laborious and prolonged tests, first achieved instrument flying, without external visibility, did perhaps not suspect that they demonstrated the possibility of automatic flight.

Yet, giving this matter a very little thought is sufficient to realize that, if the readings of one or several instruments, combined according to a law established once for all, uniquely determine the control deflections capable of producing under all circumstances correct flying, there is only one more step to designing a machine "ad hoc."

This step has been undertaken, and numerous devices for automatic flight control have appeared.

For reasons which will become clear in section 3, these apparatus are sometimes called "automatic stabilizers."

2. Essential Elements of the Apparatus for Automatic Flight Control

Any device of automatic flight control comprizes one or several detectors of perturbation, actuating a control through the intermediary of a servomotor.

At the beginning, certain apparatus have been constructed with detectors sufficiently powerful to actuate the corresponding control, without servomotor. This is the case of the Constantin wind vane and the Éteve anemometer wind vane.

However, this solution has been completely abandoned, and the servomotor has become an indispensable organ.

This servomotor can be:

A compressed-air device

A hydraulic device

An electrical device.

In most cases the signal given by the detector is not even sufficiently powerful to directly control the servomotor; it must be amplified.

The flight parameters which one can use as references are:

1. The variables which define the kinematic flight elements, that is, the fundamental variables:

Velocity V

Angle of attack α

Trim θ

Angular velocity of pitch q

for the longitudinal motion,

Sideslip β

Lateral inclination φ

Course or azimuth ψ

Angular velocity of roll p

Angular velocity of yaw r

for the lateral motion.

2. The derivatives of these variables.

3. Measurable characteristics which are direct functions of these variables, such as the components of the apparent weight.

4. Elements dependent on the position of the airplane, as functions of terrestrial reference points.

One can see immediately that the possibility of resorting to several flight variables, of utilizing servomotors of different types, of adding amplifiers if necessary, will give rise to various types of apparatus for automatic flight control.

If we want to avoid an excessive extension of the present report, we cannot study all suggested or tested combinations or even describe the apparatus which have been actually put to use.

Let us be content to point out among the actually utilized apparatus:

In the United States: Sperry, Bendix, and Honeywell devices

In Great Britain: Smith

In France: Alkan

In Germany: Siemens, Patin, and Askama

3. Program

The automatic pilot is required to perform a certain number of tasks, concerning

A. The improvement of the flight paths

B. The passing from one flight condition to another

C. Action in case of engine failure

D. Flight following a beam.

A. Improvement of the flight paths.- When an airplane has been made the object of careful aerodynamic study, the designer is generally able to make it dynamically stable. This airplane then presents flight characteristics which may be considered normal but which exhibit nevertheless certain unavoidable defects. These defects are:

(a) Failure to hold the course

(b) Insufficient damping of the longitudinal long-period oscillation

(c) Decrease in the damping of the short-period longitudinal oscillation in flight at very reduced speed

(d) In certain special cases, for instance tailless airplanes, insufficient damping of this oscillation in normal flight conditions

(e) Major vertical accelerations in flight in bad weather

(f) Insufficient maintenance of the three parameters θ, φ, ψ which must keep up rigorously constant values in military airplanes which constitute a gunnery platform

One can require of an instrument for automatic flight control to improve these characteristics; this consists, in some measure, in modifying the "natural" reactions of the airplane.

If the aerodynamic study has left more defects than the unavoidable ones in existence, one may extend the task of the automatic pilot and assign to it the masking of these abnormal characteristics. However, this seems to us admissible only if an improvement in the "natural" flight path were attained at the cost of a considerable reduction of performance.

Besides, we should like to note that in our opinion there is no hope of reducing the accelerations due to vertical gusts by employment of instruments acting upon the controls. Only more deep-seated changes brought to bear upon the structure of the airplane can lead to an improvement in this respect.

B. Passing from one flight condition to another.- The automatic flight-control devices should execute the changes in flight condition which the pilot prescribes in manipulating the buttons.

The changes in flight condition consist in:

Making the airplane climb

Making it descend

Making it turn.

Manipulation of the push buttons consists, in fact, in changing the adjustment of the instrument: by shifting the zero point.

(a) Longitudinal motion.- Assume an airplane provided with an automatic stabilizer dependent on the angle of trim, ensuring for instance the relationship:

θ	η
0°	0°
-2°	$+1^\circ$
-4°	$+2^\circ$
-6°	$+3^\circ$

We have necessarily

$$dC_M = \frac{\partial C_M}{\partial \eta} d\eta + \frac{\partial C_M}{\partial \alpha} d\alpha$$

Let us suppose

$$\frac{\partial C_M}{\partial \eta} = 0.018$$

$$\frac{\partial C_M}{\partial \alpha} = 0.006$$

At the deflection $\eta = 0$, the airplane flies horizontally ($\tau = 0$) at the angle of attack $\alpha = 0$, and because of the trim condition $\theta = 0$.

The pilot desires, without changing the throttle setting, to establish an ascending trajectory characterized by

$$\Delta\tau = 2^{\circ} \qquad \Delta\alpha = 3^{\circ} \qquad \Delta\theta = -5^{\circ}$$

values which satisfy the condition of power equilibrium.

Realization of the equilibrium of moment demands, in the final state, a deflection $\Delta\eta = -1^{\circ}$.

One must therefore modify the adjustment of the automatic pilot in such a manner as to realize the relation

θ	η
-5°	-1°

This can be done by shifting of the zero point, and entails then:

0°	-3.5°
-2°	-2.5°
-4°	-1.5°
-6°	-0.5°

In the preceding chapter, we determined an example for a deflection law, leading to a progressive realization of the desired flight condition.

One sees that abrupt passing from one adjustment to another would entail an excessive deflection at the initial instant.

The automatic pilot should therefore be conceived in such a manner that the modification of adjustment would be carried out progressively and would lead, if possible, to laws of deflection reminiscent of the one which the application of Duhamel's integral had led us to consider as favorable. One can see immediately that one may come close to the desired design by adding a component $\eta = k\,q$ which will oppose the previous one, that is, by making the automatic pilot sensitive to the angular pitching velocity.

(b) Lateral motion.- The amount of adjustment of the automatic stabilizer required for putting an airplane into a turn, will likewise be determined by the equilibrium conditions in steady turn and by the necessity of proceeding progressively.

Let us assume an airplane provided with an automatic stabilizer, dependent on the parameters φ and ψ, regulated in such a manner as to produce, in the course of a rectilinear flight, the deflections

$$\Delta\xi = -0.5\Delta\varphi \qquad \Delta\zeta = -0.5\Delta\psi$$

The rectilinear flight constitutes the initial state.

The circular flight, at constant angular velocity, constitutes the final state.

It is clear that throughout the entire duration of the flight corresponding to the final state:

(a) The connection between the angle ψ and the deflection ζ must be interrupted.

(b) The relationship between the angle φ and the deflection ξ must be completely changed since the airplane may settle itself, in banking, in a condition where φ is of the order of magnitude of 30 to 40° (for the examples calculated above) whereas the steady-state deflections are insignificant.

The aerodynamic characteristics of the airplane permit determination of the deflections necessary for the execution of the turn and, consequently, choice of the setting imposed on the instrument.

For this, we may act with a certain latitude because there exists a whole series of possible turns, of the same radius, corresponding to inclinations which differ relatively little from one another, and to deflections which are still small, owing to the tolerance which permits the consideration that a slight sideslip toward the inside does not prevent the turn from being correct.

Whatever the solution adopted, the unbalance of the aileron control will be reasonably equal to the inclination of the desired turn. Abrupt application of such an unbalance would lead, starting from the moment when the airplane is made to turn, to excessive deflections which would no longer have anything in common with the motions which Carson's construction permits us to consider as normal.

The designer of instruments for automatic flight control should in some way add a special device intended to introduce a sufficiently progressive action in the modification of the adjustment. The result to which this device should lead has been determined at the end of the preceding chapter by graphical constructions.

The modern automatic pilots utilize as flight reference conditions not only the angles φ and ψ but also the angular velocities of roll p and the angular velocities of yaw r.

The introduction of the components

$$\xi = -kp \qquad \zeta = -kr$$

acts on the deflections during the period of starting a turn and tends to approximate the actual law of deflection to the desired law.

(c) Actions in case of engine failure.- Formerly, at the time when the single-engined airplanes were those in use most widely, it was demanded that, in case of engine failure, the stabilizer should rapidly put the airplane into a descending attitude.

This requirement led to recommending the employment of stabilizers sensitive to the velocity V, or possessing at least a component sensitive to the difference in velocity. It delayed the employment of stabilizers sensitive to θ which do not satisfy it.

At the present time, in multiengined aircraft, the failure of one engine does no longer impose the immediate execution of a maneuver involving the longitudinal motion, but instead that of a maneuver involving the lateral motion. In fact, one must oppose immediately the moment of yaw which accompanies the stopping of an outboard engine.

The control of the rudder seems effective against a perturbation $\Delta\psi$, from this point of view.

(d) Flight following a beam.- The development of automatic flight led to the requirement that the piloting equipment should make the airplanes follow flight paths materialized in space by electromagnetic fields produced by means of radio beams or radio beacons placed on the ground.

This objective poses new problems which will be examined specifically in chapter XVII.

4. Points to be Studied

We propose to study more particularly the improvement in maintaining the flight path.

This problem comprizes:

A. The study of the instruments of detection

B. The calculation of the motion of the telecontrol, transforming the indication received into a deflection

C. Investigation of the reaction of the airplane under the effect of the deflection

The point A will be summarily examined in the following section.

The preceding chapters have furnished all the elements for a solution of the problem C.

It remains therefore to be examined to what an extent it is necessary to set up, and to know the theory of, the control mechanisms.

5. The Detectors

The study we are making here postulates the existence of instruments which permit detecting any difference between the actual values and the mean values one attempts to realize, by means of measuring the instantaneous values of the variables of reference.

These differences must be transmitted to the utilizing apparatus, without decrease in precision of the instruments of measurement by the necessity of conveying a certain energy to the control apparatus.

The operation of the measuring instruments calls for the following comments:

(A) Velocity.- The apparatus which are sensitive to the velocity furnish a measure proportional to the density of the air and to the square of the velocity. They may set an appreciable energy in action.

(B) Angle of attack and sideslip.- The position of the airplane with respect to its flight path is determined by the angles of attack and of sideslip. Both may be measured:

(1) By wind vanes

(2) By the pressure difference along a spindle-shaped body

1. A wind vane articulated around an axis has a tendency of establishing itself in a position which is constant with respect to the direction of the airstream lines.

It may make the direction of these airstream lines with respect to the airplane perceptible if the friction about the axis is sufficiently small. Since this is an apparatus where in equilibrium position the moment is zero, the indications of the wind vane are independent of the velocity.

The wind vane requires employment of a relay removing from the detector as little energy as possible: it has the main disadvantage of detecting the direction of the airstream lines at the location where it is placed; this direction may differ from the flow direction, at infinity (opposed to the relative speed of the airplane).

2. Since the distribution of the pressures along a spindle-shaped body is a known function

Of the velocity

Of the density of the air

Of the angle formed by the axis of the spindle-shaped body and the wind direction,

one may measure the angle of attack or of sideslip by comparison of the measurements carried out at conveniently located points.

This method gave rise to the creation of visual indicators; their use has not become general, however.

(C) Orientation of the airplane in space.- The three parameters to be measured are the angles θ, φ, and ψ.

Trim θ: The conventional means for determination of the angle of trim consists in using a cardan-mounted gyroscope which thus constitutes a gyrostat.

The axis of this gyrostat may be placed along the axes OZ or OX. In both cases it will permit detection of a change in trim θ corresponding to one rotation about the axis OY.

The arrangement employed most frequently is that of the gyrostat with vertical axis.

Whatever the chosen solution may be, the gyrostat moves slowly relative to the airplane even when no change at all occurs in the trimming angle of the airplane.

(a) If the instrument remains in a given place, the axis has a tendency to describe, within 24 hours, the cone corresponding to the motion of the fixed star which is to be found in its extension.

(b) The terrestrial coordinates of a fixed star, at a given point, are functions of the time but they vary also if one changes position on the surface of the globe.

If the airplane which carries the gyrostat goes from one point to another, there occurs, as a result, in addition to the apparent displacement defined above another one which is due to the variation of the terrestrial coordinates of the star considered.

(c) In addition to these apparent motions of the axis which result in fact from the displacement of the case of the apparatus, there occur the actual displacements of the axis produced by precission, that is, by the disturbing moments to which the gyrostat is actually subject because of the imperfections in its construction.

It results from these phenomena that one must always control the direction of a free gyroscope and bring it back to the selected position.

If the direction to be maintained is the vertical, one has an important reference point at disposal: the gravity. The problem to be solved is: bringing the axis of the gyrostat back into the direction of the latter if it has deviated from it.

The difficulty stems from the impossibility of determining, in a mobile device, the direction of the true gravity: all apparatus sensitive to gravity indicate the apparent gravity.

One must therefore make use of a correction, applying in a continuous manner and with a very small power, and one must count on it that its action, determined by the mean direction of the apparent gravity, will produce after a certain time the same result as if this correction had depended, at every instant, on the true gravity.

Every gyrostat must be brought back to its position by such a mechanism, applying a slight moment in the direction determined by the laws of the gyroscopic effect.

We shall here not describe the devices used for this purpose; we indicate only that they are indispensable.

Lateral inclination φ: A cardan-mounted gyrostat the axis of which lies in the direction along the axes OZ or OY is capable of detecting a change in the lateral inclination φ.

The gyrostat with vertical axis is the one used most frequently: the same apparatus which then permits measurement of the angles θ and φ.

Every gyroscope stabilized according to the integral of the differences with respect to the apparent gravity presents an important property:

If there exists a difference of constant direction between the apparent and the true gravity during a sufficiently long time (for instance in the course of a series of turns in a given direction), the axis of the gyrostat wanders in the direction of this apparent gravity, and the instrument becomes deviated.

Azimuth ψ: A gyrostat with horizontal axis, placed in motion in any direction whatsoever, will continue to indicate this direction with respect to the device on which it is mounted, but this property is subject to the previous restrictions, and the axis must be stabilized with respect to a physical reference which can be nothing else but the horizontal component of the terrestrial magnetic field.

Since the functioning of compasses (magnetic) is subject to many irregularities, one can stabilize the gyrostat only with respect to the mean indication of a compass, taken over a sufficiently long time interval.

This problem has obtained numerous practical solutions. The most recent ones make use of an electrical compass called "fluxgate" which we cannot describe here and which seems to eliminate part of the irregularities of the magnetic compasses.

Let us indicate another important point.

The cardan-type gyrostat, being a zero apparatus, furnishes an indication independent of its rotational velocity. However, one cannot oppose its displacement by an opposing moment without falsifying its indications.

Employment of a cardan-type gyroscope necessitates, therefore, employment of a relay absorbing infinitesimal power. Great ingenuity has been used for accomplishing this.

(D) Angular velocities.- The angular velocities p, q, r are measured with the aid of gyrometers.

The gyrometer is an apparatus based on the properties of a gyroscope subjected to a forced precission.

The rotor is no longer cardan-mounted but undergoes the rotation to be detected: it measures this rotation by the magnitude of the opposing moment it provides.

Since the sensitivity of the instrument is proportional to the angular velocity of the gyrostats, this velocity must be controlled.

Remark: Though one needs to know for the realization of an automatic pilot at the same time the angular deviation about an axis and the corresponding angular velocity, it is not necessary to use both instruments described above: gyrostat and gyrometer. One alone is sufficient under the condition that, if the gyrostat is employed, a differentiator must be added; if the gyrometer is used, an integrator.

(E) Function of the preceding variables.- The preceding variables are the intrinsic variables of the motion of the airplane. They are independent of terrestrial references and of entrained motions of the surrounding medium.

Certain of their functions are easy to measure, and their employment should be considered from the start in setting up a program for automatic flight control. These functions are the three components of the apparent gravity which are indicated by pendulums or accelerometers.

(F) Radioelectric reference coordinates.- By means of electromagnetic fields, it becomes possible to produce in the atmosphere reference lines fixed with respect to points on the earth, not situated at infinity.

By means of special receivers, the airplane can evaluate its distance from the reference lines mentioned.

Let us prescribe for the airplane a rectilinear flight path coinciding with an electromagnetic reference line.

Assume z to be its vertical distance, y its horizontal distance, with respect to this reference line.

Let us suppose, in order to simplify the treatment, that the reference line is horizontal and directed along the origin of the azimuths.

One will have necessarily

$$dz = V \sin \tau \, dt$$

$$dy = V \sin(\psi + \beta)dt$$

with

$$\tau = -(\theta + \alpha)$$

One has therefore

$$z = \int -V \sin(\theta + \alpha)dt$$

$$y = \int V \sin(\psi + \beta)dt$$

and if we deflect the controls as a function of the indications z or y, we effect, in fact, a piloting as a function of the integral of the differences of the elementary variables θ, α or ψ and β.

Piloting as a function of the integral of the differences leads to special properties, the study of which forms the object of chapter XVII.

6. The Mechanical Automatic Servocontrol

The purpose of automatic servocontrol is to produce a deflection of a control which will be a predetermined function of the indication of the deviation detector, or of a combination of the indications of several detectors.

Let us visualize this problem from a general point of view.

We shall call "input signal" an angular quantity x, varying as a function of time; "output signal" an angular position z of a secondary axis, which we call the controlled axis; "automatic servocontrol" the mechanism which imposes on the secondary axis displacements such that z is a definite function of x.

In fact, the chosen function will be simple and it will often be required that the output signal follow the input signal as exactly as possible, except for a factor of proportionality.

In order to produce the displacement of the controlled axis, it may become necessary to overcome opposing moments proportional to z, inertia moments, and friction moments.

The automatic control system will therefore utilize work furnished by a local-energy source, the servomotor.

To arrive at this result, the servomotor actuating the controlled axis can be controlled as a function of this deviation.

$$\epsilon = x - z$$

One frequently improves the functioning of the system by adding to a control proportional to the deviation ϵ, an action proportional

to the derivative of this deviation $\frac{d\epsilon}{dt} = \frac{d(x - z)}{dt}$

to the derivative of the response $-dz/dt$ or

to the integral $\int \epsilon \, dt$ of the deviation.

The automatic servocontrol may, on the other hand, be conceived in several different ways:

(a) One may visualize the employment of a servomotor turned on and developing its maximum moment (or its full power) as soon as the deviation ϵ, its derivative or its integral attains a sufficient value for actuating a relay and controlling the servomotor.

As an extreme, (and this arrangement is incorporated in the Honeywell automatic pilot) one may imagine a servomotor rotating constantly and a relay actuating a clutch in one direction or the other.

(b) One may visualize a servomotor the moment (or the power) of which would be a continuous function of the deviation, its derivative, or its integral.

This continuous function may be a proportionality. A servocontrol of this type will be linear.

Between the on and off automatic servocontrol and the linear automatic servocontrol, one may imagine intermediate cases where the moment follows the deviation ϵ but varies by steps.

7. Classical Theories of the Linear Automatic Servocontrol

The theory of automatic servocontrol is easily set up if one can assume that the engine torque applied to the controlled axis is strictly proportional to the deviation, to the derivatives, or to the integral of the deviation.

Let:

J be the moment of inertia of the controlled axis

f a friction coefficient

k a factor defining the resisting moment of the controlled axis, proportional to the displacement.

The motion of the proportional automatic servocontrol is determined by

$$J \frac{d^2z}{dt^2} + f \frac{dz}{dt} + kz = K(x - z)$$

The calculation of the motion toward the position of equilibrium corresponding to a constant increment Δx, applied abruptly, is a classical problem.

It is well to remark that, starting from an equilibrium position for which $x = 0$ and $z = 0$, it is impossible - when k is not equal to zero - to make z remain equal to x when x varies. The necessity of developing a steady moment kz in all equilibrium positions other than $x = 0$ and $z = 0$ makes the presence of a steady deviation $\epsilon = x - z$ unavoidable.

The automatic servocontrol, proportional to the deviation, permits z to follow x only if the resisting moment is always zero.

If one adds to the moment proportional to the deviation a moment varying with the derivative of the deviation

$$K' \frac{d(x - z)}{dt}$$

or with the derivative of the position of the controlled axis alone

$$-K' \frac{dz}{dt}$$

the oscillations of the system are reduced.

The addition of a moment proportional to the integral of the deviation

$$K' \int \epsilon dt$$

permits the realization of equilibrium positions characterized by $z = x$ even when a resisting moment, proportional to the displacement, opposes the motion of the controlled axis.

The value of these calculations depends on the degree of accuracy of the hypotheses on which they are based, and it is therefore necessary to see how certain controls are actually realized.

8. The Sperry A3 Automatic Servocontrol

Let us examine the Sperry A3 automatic pilot. This instrument attempts to produce a control-surface deflection proportional to the inclination of the airplane. It comprises a gyrostat or rotor, cardan-mounted, placed in an enclosure (case), kept at very low pressure by a vacuum pump.

The rotor is shaped in such a manner as to form blades, and its rotation is obtained by directing toward these blades the jets of air which enter the case.

Outside the rotor but inside the case there is a casing E perforated by several apertures and capable of undergoing angular displacements under the action of the "follow-up" cables.

Two of the openings made in the casing are pressure inlets and terminate on two sides of a membrane.

The rotor - through the agency of a stop-valve system connected with it, and represented in the diagram by the blocks B - closes more or less the pressure-inlet openings in the casing. When the rotor occupies its mean position, it closes these openings equally, but it closes them differentially, when displaced with respect to the casing one way or the other from its mean position. Hence, any displacement of the gyrostat produces a pressure difference on the two faces of the membrane and a deformation of it.

The membrane, in being deformed, displaces, by means of a rod, the piston valve of a distributor which controls the entrance of the oil under pressure into the control servomotor. When the membrane occupies the mean position, the piston valve cuts off any communication between the oil pump and the servomotor; however, when this piston valve leaves this position, in one direction or the other, it connects one side of the piston of the servomotor with the oil pumps, and this piston is displaced as long as the connection is not cut.

In the position shown in the figure, the airplane has a nose-up position, a pressure difference has been produced on the membrane, and the distributor has led the oil pressure onto the front (left) face of the servomotor.

The piston will displace itself in the desired direction; however, its motion will have to be stopped when the control surface will have displaced itself by an angle corresponding to the relative displacement of the gyrostat. A connection between the casing E and the control surface permits this result to be attained.

The motion of the control will displace the casing in such a manner to bring the uncovered aperture back into contact with the shut-off device. Therefore the control surface, by its displacement, reacts on the sensing mechanism in a manner tending to establish equal pressure on the two forces of the membrane, and to stop the motion. This "follow-up" mechanism permits, in short, impressing on the control surface a displacement proportional to that of the gyroscope.

Any displacement of the casing produced by a control independent of the follow-up mechanism (not shown in the figure), constitutes a change in adjustment which modifies the law linking, as an end result, the position of the control surface with that of the gyro, with respect to an axis fixed to the airplane.

Such a control at the disposition of the pilot allows him to modify the flight path of the airplane through the agency of the automatic pilot.

The Sperry A_3 automatic pilot utilizes two gyrostats for operating the three controls by means of three pneumatic relays and three servomotors.

The first gyrostat, with vertical axis, detects the deviations of trim θ and of lateral inclination φ; the second gyrostat detects the deviations of azimuth ψ.

This apparatus is used very much and it functions regularly. It requires, of course, filters and accessories not represented on the diagram of its operating principles.

Moreover, it must be noted that the practical construction of the casing and the shut-off device is slightly different from the fundamental description given above.

It suffices to examine the operation of this apparatus for understanding that it does not yield a linear automatic servocontrol.

The input signal x is here the angle of trim.

The output signal z is the deflection β which reproduces itself in the angular motion of the casing B. The distribution of the pressures on the membrane does depend on the deviation $\theta - \beta$, but the description shows that the pressure finally exerted on the piston of the servomotor will by no means be proportional to $\theta - \beta$.

If one neglects the load losses which vary according to the degree of opening of the apertures of the distributor and according to the flow of oil, the pressure acting on the piston of the servomotor will be the pressure of the oil pump, and one can see that the automatic servocontrol will act in a manner which is much more nearly "on or off" than according to the linear law.

9. Linear Automatic Servocontrol

We shall now describe a classical mechanism having linear characteristics as long as a certain limit of moment corresponding to the saturation of the magnetic cores has not been attained.

A control crank (1), in being displaced by x, makes a contact (2) slide on a potentiometer powered by an independent source.

The driven axis (4) is actuated by a motor M_1. Any rotation of this axis produces, by means of a return device, the displacement of another contact (3) along the same potentiometer.

The connections are such that when the response (3) of the axis (4) occurs in the direction corresponding to the requirement x, the contact (3) tends toward (2).

This return permits proportioning of the displacement z in accordance with x.

Let us designate equally by:

x the position of the slider 2

z the position of the slider 3

The electromotive force $K(x - z)$ which one finds between 2 and 3 is used with a view to actuating the motor M_1; but, since it is insufficient, it must be amplified. For this purpose it makes a current i_0 pass into the excitation circuit of a dynamo D driven at constant speed by a completely independent motor M_0.

Under the action of the electromotive force produced in this dynamo, there originates a current i_1. This current is sent into the motor M_1.

Since the intensity i_1 is considerably higher than i_0, the apparatus constitutes a power relay.

One can assume that the motor torque developed by M_1 is proportional to i_1.

One sees immediately that, if the motor M_1 is to overcome a resisting moment proportional to z, it is not possible to realize the equality between z and x since, in order to maintain a deflection z, one must apply a motor torque equal to the resisting moment. One must, therefore, maintain an intensity i_1 different from zero which implies that the values x and z cannot coincide.

The Sperry A_{12} automatic pilot utilizes an electric control, applying the principle described above; however, it is complemented by various organs, one of which is a control for compensation of the hinge moment. The input signal x governs simultaneously the motor M_1 actuating the control, and a motor M_2 of small power actuating - somewhat more slowly - the compensator for the control-surface hinge moment; this compensator is displaced in such direction that, for the desired deflection z, the hinge moment becomes again zero.

CHAPTER XIV

THE AIRPLANE AND THE AUTOMATIC SERVOCONTROL

1. Equations of the Linear Automatic Servocontrol

The description just given shows us that the input signal x and output signal z are connected by equations more complicated than those we have given in section 7 of the preceding chapter.

We shall establish here a more exact theory of the electrical servocontrol.

The excitation circuit of the dynamo offers the impedance

$$R + L \frac{d}{dt}$$

It is subjected to an electromotive force $K(x - z)$, and the current i_0 is determined by

$$K(x - z) = Ri_0 + L \frac{di_0}{dt}$$

The circuit which produces the excitation of the dynamo and the motor M_1 are subjected to an electromotive force Ci_0 and to an electromotive force $A(dz/dt)$; A is a coefficient dependent on the windings, and dz/dt is the angular velocity measured in units of the displacement of the contact 3.

If R_1 is the resistance of the circuit, L_1 its inductance, one has as equilibrium condition of the electromotive forces

$$Ci_0 = R_1 i_1 + L_1 \frac{di_1}{dt} + A \frac{dz}{dt}$$

Finally, with M_1 furnishing a motor torque Bi_1 proportional to i_1 (motor with independent excitation), one has as the equilibrium condition for the moments about the controlled axis:

$$Bi_1 = J \frac{d^2z}{dt^2} + f \frac{dz}{dt} + kz$$

with

J moment of inertia of the axis 4

f friction coefficient

k factor defining the resisting moment assumed to be proportional to the displacement

We shall write $dz/dt = z'$.

We find that the output signal z is determined, as a function of the input signal x, by four linear equations

$$L \frac{di_0}{dt} + Ri_0 + Kz = Kx$$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + Az' - Ci_0 = 0$$

$$J \frac{dz'}{dt} + fz' + kz - Bi_1 = 0$$

$$\frac{dz}{dt} - z' = 0$$

We find here again a system of equations analogous to those which determine the longitudinal motion of an airplane; here the variables are i_0, i_1, z', and z.

When no control order is given, $x = 0$ and the system will be in stable equilibrium if the characteristic determinant satisfies Routh's criterion.

If x undergoes, at the time $t = 0$, an abrupt increment Δx, the motion of the axis $z = f(t)$ can be determined - when the position of equilibrium is known - by the method used to calculate the motion of the airplane under the effect of an abrupt displacement of a control surface.

2. Development of the Equations of the Linear System

Let us consider again the system

$$\frac{di_0}{dt} = - \frac{R}{L} i_0 + \frac{K}{L} (x - z)$$

$$\frac{di_1}{dt} = + \frac{C}{L_1} i_0 - \frac{R_1}{L_1} i_1 - \frac{A}{L_1} z'$$

$$\frac{dz'}{dt} = \frac{B}{J} i_1 - \frac{f}{J} z' - \frac{k}{J} z$$

$$\frac{dt}{dz} = z'$$

which we shall write

$$\frac{di_0}{dt} + a_1 i_0 + b_1 i_1 + c_1 z' + d_1 z = d_1 x$$

$$\frac{di_1}{dt} + a_2 i_0 + b_2 i_1 + c_2 z' + d_2 z = 0$$

$$\frac{dz'}{dt} + a_3 i_0 + b_3 i_1 + c_3 z' + d_3 z = 0$$

$$\frac{dz}{dt} + a_4 i_0 + b_4 i_1 + c_4 z' + d_4 z = 0$$

One has therefore

$$\begin{array}{llll}
a_1 = \frac{R}{L} & b_1 = 0 & c_1 = 0 & d_1 = \frac{K}{L} \\
a_2 = - \frac{C}{L_1} & b_2 = \frac{R_1}{L_1} & c_2 = \frac{A}{L_1} & d_2 = 0 \\
a_3 = 0 & b_3 = - \frac{B}{J} & c_3 = \frac{F}{J} & d_3 = \frac{k}{J} \\
a_4 = 0 & b_4 = 0 & c_4 = -1 & d_4 = 0
\end{array}$$

The system possesses a characteristic equation

$$\begin{vmatrix} a_1 + \lambda & 0 & 0 & d_1 \\ a_2 & b_2 + \lambda & c_2 & 0 \\ 0 & b_3 & c_3 + \lambda & d_3 \\ 0 & 0 & c_4 & \lambda \end{vmatrix} = 0$$

The development in the form of

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$$

gives

$$A_1 = a_1 + b_2 + c_3$$

$$A_2 = a_1b_2 + a_1c_3 + b_2c_3 - c_4d_3 - b_3c_2$$

$$A_3 = a_1b_2c_3 - a_1c_4d_3 - b_2c_4d_3 - a_1b_3c_2$$

$$A_4 = -a_2b_3c_4d_1 - a_1b_2c_4d_3$$

where we have $c_4 = -1$.

Routh's stability conditions permit us to verify whether or not a given system characterized by particular values of each one of the 11 characteristics L, R, L_1, R_1, A, B, C, J, K, k, and f, is stable, but they do not lend themselves to a general discussion.

Let us remark, however, that the condition

$$R = A_2 - \frac{A_3}{A_1} - \frac{A_1A_4}{A_3} > 0$$

leads immediately to an important conclusion. Assuming that it is satisfied for the four primary conditions $A_1 > 0 \ldots A_4 > 0$, the sensitivity K cannot exceed a certain value.

In fact, only A_4 contains d_1 or K/L. When K increases A_4 must increase and the fifth condition of stability will cease to be satisfied.

There will always arrive a moment where the servocontrol described above will be unstable if one increases its sensitivity by an increase in the coefficient K (for instance, increasing the voltage of the battery).

Just as the longitudinal motion of the airplane results from the superposition of the irregularities of the flight path (switch back) with the oscillations of the airplane about its center of gravity, the motion of an automatic control system consists, in the most general case, of the superposition of the oscillatory motions of the amplifying system and of those of the so-called mechanical control system, resulting from the equilibrium of the moments about the controlled axis.

In general, the oscillations of the amplifying system have a shorter period than those of the controlled axis,and, although the two oscillations necessarily interact, one on the other, it is possible to find in the solutions the influence of each of the component motions.

3. Control of the Applied Moment by the Angular Velocity

In the simplified study made in section 6 of the preceding chapter, we pointed out that the characteristics of the effective motion of the controlled axis improve if the moment applied to the axis is a conveniently selected function of the angular velocity z' of displacement of this axis.

This result is easily found also in a study taking into account the existence of the amplifying stage.

Let us briefly investigate what happens when the voltage applied to the first circuit becomes $K(x - z) - K'z'$ instead of being equal to $K(x - z)$.

Since the application of a command $\Delta x > 0$ tends to produce a $\Delta z > 0$, there appears also a $z_0 > 0$ as soon as the response has started.

The complementary term reduces, at this instant, the applied voltage which helps prevent the new position of equilibrium from being exceeded.

In the general theory to write

$$K(x - z) - K'z'$$

in the second term of the first equation amounts to taking $c_1 = +K'/L$ instead of $c_1 = 0$

A simple examination of the determinant shows that only a single new term has been introduced, namely the term

$$+a_2 b_3 c_1 \lambda$$

which modifies the value of the single coefficient A_3.

Since $a_2 b_3 c_1$ is > 0, the coefficient A_3 is increased and the danger of instability, from too great a value of $A_1 A_4 / A_3$ due to an excessive sensitivity, is reduced.

In spite of the danger of having R become negative through an increase of A_3/A_1, making the moment a function of the displacement velocity of the controlled axis is favorable.

4. System of the Third Degree

The problem of servo control has been expounded by several authors; they took the characteristics of the amplification device into consideration but assumed $L_1 = 0$ in the circuit of the servocontrol motor.

The characteristic equation is then of the third degree.

One has in this case

$$L \frac{di_0}{dt} + Ri_0 = +K(x - z)$$

$$Az' + R_1 i_1 = Ci_0$$

$$J \frac{dz'}{dt} + fz' + Kz = Bi_1$$

$$\frac{dz}{dt} = z'$$

The second equation gives $i_1 = 1/R_1\left(Ci_0 - Az'\right)$ which we introduce into the third equation so that we obtain

$$\frac{di_0}{dt} = \frac{-R}{L} i_0 - \frac{K}{L}(x - z)$$

$$\frac{dz'}{dt} = + \frac{BC}{JR_1} i_0 - \left(\frac{f}{J} + \frac{AB}{R_1 J}\right) z' - \frac{k}{J} z$$

$$\frac{dz}{dt} = -c_4 z'$$

We can now write the characteristic equation. Maintaining the same notations as before, we obtain

$$\frac{B}{R_1} = - \frac{a_2}{b_2}$$

$$\frac{A}{R_1} = \frac{c_2}{b_2}$$

and the determinant is written

$$\begin{vmatrix} a_1 + \lambda & 0 & d_1 \\ b_3 \dfrac{a_2}{b_2} & \left(c_3 - \dfrac{c_2}{b_2} b_3\right) + \lambda & d_3 \\ 0 & c_4 & +\lambda \end{vmatrix} = 0$$

where

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$$

with

$$A_1 = a_1 + \left(c_3 - \frac{c_2}{b_2} b_3\right)$$

$$A_2 = a_1 c_3 - a_1 \frac{c_2}{b_2} b_3 - c_4 d_3$$

$$A_3 = -a_1 c_4 d_3 - b_3 \frac{c_2}{b_2} c_4 d_1$$

There will be stability if $A_1 > 0$, $A_2 > 0$, $A_3 > 0$, and $A_2 > \frac{A_3}{A_1}$

Here again one finds that A_3 alone depends on the sensitivity $d_1 = K/L$ and increases with this sensitivity.

The servo system must become unstable if one increases the sensitivity.

If the displacement z of the controlled axis takes place without a restoring moment proportional to the displacement, one has, moreover, $k = 0$, that is, $d_3 = 0$.

The principles of the linear servo system studied above in a general case, then in a particular case, have been utilized by Mr. Roccard in his "Etude de la stabilité des systèmes accessibles à des mesures" (Study of the stability of systems susceptible to modifications") and have been treated by him for the particular case $L_1 = 0$ and $k = 0$.

This principle has also been used by Mr. Harris in his "The Frequency Response of Automatic Control Systems," with the difference that this author assumes that the intermediary circuit (R, L, i_0) is powered by a source of alternating current.

5. Combination of Automatic Control and Airplane

We can study the reactions of an airplane, a control surface of which is actuated by a servocontrol, by combining the systems of equations relating to the airplane and to the control.

Let us assume that we are dealing with a simple automatic pilot actuating the elevator as a function of the deviation of trim.

The motion of the airplane under the action of a continually variable deflection $\delta\eta$ is given as a function of the aerodynamic time τ by the system

$$\frac{d\delta u}{d\tau} + a_1\delta u + b_1\delta w + c_1\delta q + d_1\delta\theta = 0$$

$$\frac{d\delta w}{d\tau} + a_2\delta u + b_2\delta w + c_2\delta q + d_2\delta\theta = 0$$

$$\frac{d\delta q}{d\tau} + \frac{c}{r^2} a_3\delta u + \frac{c}{r^2} b_3\delta w + \frac{cl}{r^2} c_3\delta q + \frac{cV}{r^2} d_3\delta\theta - \frac{cV}{r^2} h_3\delta\eta = 0$$

$$\frac{d\delta\theta}{d\tau} - \frac{\mu c}{V}\delta q = 0$$

in which the coefficients $a_1 \ldots h_3$ relating to the airplane have been defined in chapter VII. Let us note that $d_3 = 0$.

The motion of the automatic control, in turn, will be defined by an analogous system where the real nature of the input and output signals will be taken into account.

In the case considered, the input is the deviation of trim $\delta\theta$; the output is the deflection $\delta\eta$ of the elevator. The deflections one attempts to realize, and the deviations of trim must have opposite signs; they are not necessarily equal, but simply proportional. One has therefore

$$x = -h_1 \delta\theta$$

$$z = +\delta\eta$$

The resisting moment or hinge moment is, actually, not only a function of the deflection $z = \delta\eta$ and of its derivative $z' = \delta\eta'$, but also of the velocity of the airplane V, of the angle of attack α (defined in the equations by w), of the angular velocity q, and even of the angle of trim if one takes the moment into consideration which is produced by the weight of the control if its center of gravity does not lie on the hinge.

The operation of the automatic control responds to the system

$$\frac{di_0}{dt} + a_1 i_0 + d_1 \delta\eta + h_1 \delta\theta = 0$$

$$\frac{di_1}{dt} + a_2 i_0 + b_2 i_1 + c_2 \delta\eta' = 0$$

$$\frac{d\delta\eta'}{dt} + a_3 i_0 + b_3 i_1 + c_3 \delta\eta' + d_3 \delta\eta + e_3 \delta u + f_3 \delta w + g_3 \delta q + h_3 \delta\theta = 0$$

$$\frac{d\delta\eta}{dt} - \delta\eta' = 0$$

The coefficients $a_1 \ldots d_4$ relating to the automatic control system have been defined in section 2 of the present chapter.

The coefficients e_3, f_3, g_3, h_3 are proportional to the derivatives of the hinge moment, with respect to the variables defining the motion of the airplane.

In order to treat this system simultaneously with the previous one, the aerodynamic time must be adopted as the unit. The transformation will be carried out by multiplying all terms of the four equations, except the derivatives, by $\mu c/V$.

We shall represent each of the coefficients relating to the automatic control system, after they have been multiplied by $\mu c/V$, by $a'_1 \ldots d'_4$, $e'_3 \ldots h'_3$; this will avoid confusions with the factors relating to the airplane.

We thus arrive at eight equations: four equations of motion of the airplane, four equations of motion of the automatic control, and of eight variables.

The first group depends on the four variables δu, δw, δq, $\delta\theta$ defining the motion of the airplane, but the third equation of motion depends, moreover, on $\delta\eta$.

The equations of the second group depend on the four variables i_0, i_1, $\delta\eta'$, and $\delta\eta$ determining the motion of the control; moreover, however, the first equation depends on $\delta\theta$, and the third equation depends on δu, δw, δq, and eventually $\delta\theta$.

The complete system gives rise to a characteristic determinant of eight lines and eight columns.

$$\begin{vmatrix} a_1+\lambda & b_1 & c_1 & d_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2+\lambda & c_2 & d_2 & 0 & 0 & 0 & 0 \\ \frac{c}{r^2}a_3 & \frac{c}{r^2}b_3 & \frac{cl}{r^2}c_3+\lambda & 0 & 0 & 0 & 0 & -\frac{c}{r^2}h_3V \\ 0 & 0 & -\frac{\mu c}{V} & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h'_1 & a'_1+\lambda & 0 & 0 & d'_1 \\ 0 & 0 & 0 & 0 & a'_2 & b'_2+\lambda & c'_2 & 0 \\ e'_3 & f'_3 & g'_3 & h'_3 & 0 & b'_3 & c'_3+\lambda & d'_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mu c}{V} & \lambda \end{vmatrix} = 0$$

The characteristic determinants of the airplane and of the automatic control constitute minors of this new table but the total determinant is not reduced to the simple product of the two.

Even in a simple case - assuming that the resisting moment of the control is independent of the flight conditions $\left(e'_3 = f'_3 = g'_3 = h'_3 = 0\right)$ - the principle of setting up equations prevents one from writing

$$h_3 = 0$$

$$h'_1 = 0$$

and with the method used it is not possible to separate the study of the motion of the airplane from that of the motion of the servocontrol.

Remarks: 1. The argument carried out in the case of a control surface actuated by a servocontrol dependent on the deviations of trim may be extended to a control actuated by deviations in velocity, angle of attack, etc.

Terms with e'_1, f'_1, g'_1, on the fifth line of the determinant would result.

2. An exposition analogous to the preceding one could be given for the study of the lateral motion under the effect of servocontrols actuated by the deviations of the variables defining this motion.

3. It has been assumed in the previous chapter that the detectors of perturbations acted without inertia, and communicated to the amplifying device an input signal rigorously equal to the perturbation.

If that were not so, one would have to write the equations governing the functioning of the detecting instruments, to add them to the system examined, and to raise the degree of the characteristic still higher.

6. Possible Simplifications

The general theory leads to complicated expressions which are almost useless due to the impossibility of making the influence of the different variables apparent.

The foregoing exposition has been given with the purpose of showing that the different problems, treated by a certain number of authors, are actually particular cases of the general problem.

The choice of the simplifications determines in a measure the problem investigated.

The following questions have been treated.

(A) Study of the motion of the airplane under the assumption that the control surface is actuated by a servomotor the moment of which is proportional to the perturbation.

This first investigation amounts to eliminating the study of the amplification stage, and to examining a system of six linear equations.

A report, in accordance with this train of thought, has been written by Weiss.

The setting up of the equations leads necessarily to a characteristic equation of the sixth degree the solution of which is quite laborious. Weiss evades this difficulty by an artifice.

It can actually be admitted that the characteristics of the slow oscillation will not be in any way affected by the moment of inertia of the control.

Let us therefore perform the calculation for the first time under the assumption that the control is without inertia, according to the procedure B below. The characteristic equation is of the fourth degree; it leads to solutions determining a rapid oscillation and a slow oscillation.

In pursuance of the hypothesis made above, the solution defining the slow oscillation in the system of the fourth degree is also a solution of the equation of the sixth degree. Dividing this last equation by the common solution, one arrives again at an equation of the fourth degree which determines the rapid oscillation and the motion of the control.

One can also study the influence of the characteristics of the control: inertia, power of the servomotor, hinge moment, on the motion of the airplane.

Thus one supposes that these characteristics have an influence solely on the rapid oscillation.

It is evident that this manner of reasoning is awkward and indirect.

(B) Study of the motion of the airplane under the assumption that the control is actuated by a servomotor without inertia, producing at every instant a deflection proportional to the perturbation.

Such a hypothesis eradicates the effect of the mechanical characteristics of the automatic control; it retains only - but showing up their full importance - the mechanical characteristics of the motion of the airplane.

Since the automatic control is supposed to produce continually

$$z = x$$

or, in the case where the control is a function of the trim

$$\delta\eta = -h_1\delta\theta$$

it suffices to replace the term in $\delta\eta$ in the equation of equilibrium of moments of the airplane by a term in $\delta\theta$ - which amounts to attributing to d_3 a value different from zero.

If the control is a function of another parameter, it will cause a simple alteration of the corresponding coefficient a_3, b_3, or c_3.

The characteristic equation remains an equation of the fourth degree in the study of the longitudinal motion. It is of the fifth or of the fourth degree according to the law of control adopted in the case of the lateral motion.

The two following chapters are devoted to the study of automatic piloting carried out under this hypothesis.

(C) Study of the motion under the assumption that the control is movable but that there is neither a servomotor nor a detector of perturbation present.

In this case, the control assumes, by itself, the position which ensures the equilibrium of the hinge moment, and the system of equations determines the motion of the airplane flying with free controls.

The fifth and sixth equations of the general system (section 5), likewise the variables i_0 and i_1 must be eliminated.

In the seventh equation one eliminates the terms a_3i and b_3i_1, and writes the rotational equilibrium of the control.

The product of the moment of inertia of the control and the angular acceleration $d\eta'/dt$ must balance the hinge moment at every instant.

The problem is determined by a system of the sixth degree. The lateral motion of the airplane flying with two free controls is determined by a system of the eighth degree.

The possibility of obtaining a usuable solution depends, above all, on the knowledge of the hinge moment and of the derivatives of this moment with respect to the different variables.

The problem of flight with free controls is not the one we have posed.

This problem has already formed the subject of numerous theoretical investigations - which differ especially by the simplifications which have been introduced with the purpose of reducing the degree of the equations.

CHAPTER XV

EFFECT OF THE AUTOMATIC PILOT ON THE LONGITUDINAL MOTION

1. Indirect Effect of the Four Elementary Variables on the Longitudinal Moment

Let us assume that there exist devices producing a deflection $d\eta$ of the elevator, proportional to the deviation:

of velocity δu

of angle of attack $\delta\alpha = -(\delta w/V)$

of angular velocity δq

of trim $\delta\theta$

It will be possible to obtain the effect of these devices on the stability of the airplane motion, on the periods and the damping by introducing into the determinant the desired complementary term:

$$a_3 \text{ becomes } a_3 - \frac{\partial C_M}{\partial \eta}\frac{d\eta}{dV} V = a_3 + a_{3s}$$

$$b_3 \text{ becomes } b_3 + \frac{\partial C_M}{\partial \eta}\frac{d\eta}{d\alpha} = b_3 + b_{3s}$$

$$c_3 \text{ becomes } c_3 - \frac{\partial C_M}{\partial \eta}\frac{d\eta}{dq}\frac{V}{l} = c_3 + c_{3s}$$

$$d_3 \text{ becomes } d_3 - \frac{\partial C_M}{\partial \eta}\frac{d\eta}{d\theta} = d_3 + d_{3s}$$

The quantities a_{3s}, b_{3s}, c_{3s}, d_{3s} characterize the effect of automatic stabilizer.

We have previously (Chapter VII) written the expressions for the coefficients A_1, A_2, A_3, A_4 of the characteristic equation as functions of the four derivatives of the moment a_3, b_3, c_3, d_3.

These expressions lend themselves, without modification, to the prediction of the effects of a stabilizer which is a function of one of the fundamental variables. It suffices to replace a_3 by $a_3 + a_{3s}$, etc. The discussion carried out previously shows that, in general, positive values of a_{3s}, b_{3s}, c_{3s}, d_{3s} lead to stability.

These values are positive when:

$$\frac{\partial C_M}{\partial \eta} \frac{d\eta}{du} < 0$$

(an increase in velocity makes the airplane nose up),

$$\frac{\partial C_M}{\partial \eta} \frac{d\eta}{d\alpha} > 0$$

(an increase in the angle of attack makes it nose down),

$$\frac{\partial C_M}{\partial \eta} \frac{d\eta}{dq} < 0$$

(an angular acceleration, in the nose-down direction, makes the airplane nose up),

$$\frac{\partial C_M}{\partial \eta} \frac{d\eta}{d\theta} < 0$$

(a displacement in trim, in the nose-down direction, makes the airplane nose up).

The linear theory easily permits clearing up the question of the automatic stabilizer, making a preliminary selection among the possible solutions, and research - along general lines - regarding the effect which this or that law of piloting will have on the characteristics of the motion.

Many unfruitful tests could have been avoided, at the cost of a few hours of calculation. It is curious to see how completely the first designers of automatic-control instruments neglected the indications the theory could offer.

2. Numerical Value of the Complementary Terms

Let us show, as an example, to what numerical values the likely laws of deflection will lead.

A. Piloting as a function of the velocity.- Assume a device producing a $\delta\eta$ of -1° when the airplane accelerates by $\delta V = 0.1$ V.

Such a stabilizer will be characterized by

$$a_{3s} = -V \frac{d\eta}{dV} \frac{dC_M}{d\eta} = + 10 \frac{dC_M}{d\eta}$$

A value of $dC_M/d\eta = 0.015$ (angles in degrees) is normal.

The corresponding value of a_{3s} would be +0.15.

In fact, the detectors are generally sensitive to the square of the velocity. It may be seen easily that the apparatus for which the deflection is -1° leads to the same value of a_{3s} when the airplane accelerates by $dV^2 = 0.2\ V^2$

B. Piloting as a function of the angle of attack.- A detector of perturbation in angle of attack set so as to produce a $d\eta = 1^\circ$ when $d\alpha = 2^\circ$, gives

$$b_{3s} = 1/2 \frac{dC_M}{d\eta} \times 57.3$$

if we express, as above, the $dC_M/d\eta$ in degrees.

For $dC_M/d\eta = 0.015$, one obtains $b_{3s} = 0.43$.

C. Piloting as a function of the angular velocity.- An apparatus which would deflect the elevator by an angle $d\eta = +5^\circ$, for an angular velocity $q = -57.3^\circ$ per second, would give

$$c_{3s} = - \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dq} \frac{V}{l}$$

assume for $V/l = 6$ (for instance, $V = 60$ m/sec, $l = 10$ m.)

$$c_{3s} = + 0.015 \times 5/l \times 6 = 0.45$$

D. Piloting as a function of the angle of trim.- A detector of perturbation producing a deflection $d\eta = -0.5\ d\theta$ leads, for the value of $\partial C_M/\partial\eta$ assumed above, to

$$d_{3s} = 0.015 \times 1/2 \times 57.3 = 0.43$$

3. Effect of Piloting Which Is a Function of the Deviations of a Fundamental Variable

A first examination of the possibilities presented by the different laws of piloting may be made by investigating the effect which the introduction of increasing values of a_{3s}, b_{3s}, c_{3s}, d_{3s} exerts on the damping and on the period.

The solutions indicated as interesting by this first examination, may then be investigated in detail later on, after the reactions of the airplane corresponding to certain initial perturbations have been completely calculated.

The diagrams given previously show us, in fact, that sometimes the initial perturbation (this is the case of δw) diminishes rapidly but causes the appearance of secondary perturbations.

If a first examination shows that the period and damping characteristics are satisfactory, it is necessary to make sure that the selected law of piloting will not have the effect of increasing the amplitude of one or the other of the secondary perturbations.

Let us consider hypothetical airplanes, characterized by numerical values chosen arbitrarily for a certain number of given values.

Assume

$$a_1 = +0.125 \qquad Cz = 0.40$$

$$a_2 = -0.80 \qquad \mu = 28.8$$

$$b_1 = +0.345 \qquad c/r = 1.53$$

$$b_2 = +3.75 \qquad l/c = 2.6$$

$$\tau = 1.63 \text{ sec}$$

Let us limit ourselves, on the other hand, to the study of horizontal flight paths traversed at zero angle of attack

$$\alpha = 0 \qquad \tau = 0 \qquad \theta = 0$$

For all airplanes corresponding to these given values, the coefficients A_1, A_2, A_3, A_4 are expressed as follows as functions of the characteristics $\left(a_3 + a_{3s}\right)$, $\left(b_3 + b_{3s}\right)$, $\left(c_3 + c_{3s}\right)$, $\left(d_3 + d_{3s}\right)$:

$$A_1 = 3.8747 + 6.19 \left(c_3 + c_{3s}\right)$$

$$\begin{aligned} A_2 = 0.743 &+ 0 \left(a_3 + a_{3s}\right) \\ &+ 66.4 \left(b_3 + b_{3s}\right) \\ &+ 23.7 \left(c_3 + c_{3s}\right) \\ &+ 66.8 \left(d_3 + d_{3s}\right) \end{aligned}$$

$$\begin{aligned} A_3 = 0 &+ 3.74 \left(a_3 + a_{3s}\right) \\ &+ 8.34 \left(b_3 + b_{3s}\right) \\ &+ 4.55 \left(c_3 + c_{3s}\right) \\ &+ 258.5 \left(d_3 + d_{3s}\right) \end{aligned}$$

$$\begin{aligned} A_4 = 0 &+ 99.8 \left(a_3 + a_{3s}\right) \\ &+ 21.3 \left(b_3 + b_{3s}\right) \\ &+ 42.8 \left(d_3 + d_{3s}\right) \end{aligned}$$

Let us recall that $d_3 = 0$ and suppose, moreover, that the airplane corresponds to

$$a_3 = 0$$

$$c_3 = +1.37$$

In the case where the piloting is a function of the perturbation of the angle of attack, we must take

$$a_{3s} = 0 \qquad c_{3s} = 0 \qquad d_{3s} = 0$$

and give variable values to $\left(b_3 + b_{3s}\right)$.

Let us now investigate the effect of a series of values varying regularly from -0.23 to +0.46.

The roots λ and λ' are then given by the following table:

$(b_3 + b_{3s})$	λ	λ'
0.46	$-6.095 \pm 4.92i$	$-0.0665 \pm 0.388i$
0.344	$-6.092 \pm 4.15i$	$-0.0674 \pm 0.362i$
0.23	$-6.09 \pm 3.03i$	$-0.0713 \pm 0.310i$
0.114	$-6.08 \pm 1.51i$	$-0.0828 \pm 0.230i$
0.057	$-7.45 - 4.65$	$-0.0914 \pm 0.15\ i$
-0.057	$-9.13 - 2.94$	$-0.376 + 0.122$
-0.114	$-9.715 - 2.38$	$-0.473 + 0.207$
-0.172	$-10.11 - 1.79$	$-0.692 + 0.285$
-0.23	$-10.60 - 1.048$	$\pm 0.346i + 0.38$

For the other three laws of piloting considered we shall examine some cases by investigating how the law considered modifies the motion of an airplane possessing a static stability zero $(b_3 = 0)$ and a relatively high static stability

$$b_3 = 0.006 \times 57.3 = 0.3438$$

We shall find as roots:

Variable parameter: a_{3s}

case $b_3 = 0$

$(a_3 + a_{3s})$	λ	λ'
-0.10	$-8.65 - 3.37$	$-0.732 + 0.334$
-0.05	$-8.65 - 3.40$	$-0.569 + 0.301$
0	$-8.65 - 3.43$	$-0.208 \quad 0$
+0.05	$-8.54 - 3.64$	$-0.0712 \pm 0.401i$
+0.10	$-8.50 - 3.81$	$-0.042 \pm 0.555i$

case $b_3 = 0.3438$

-0.10	$-6.074 \pm 4.13i$	$-0.086 \pm 0.2375i$
-0.05	$-6.078 \pm 4.13i$	$-0.0815 \pm 0.191\ i$
0	$-6.092 \pm 4.15i$	$-0.0674 \pm 0.362\ i$
+0.05	$-6.100 \pm 4.20i$	$-0.0600 \pm 0.460\ i$
+0.10	$-6.105 \pm 4.15i$	$-0.0560 \pm 0.56\ i$

Variable parameter: c_{3s}

case $b_3 = 0$

$(c_3 + c_{3s})$	λ	λ'	
0.96	-5.842 - 3.6825	+0.2	0
1.15	-7.01 - 3.71	+0.2	0
1.37	-8.69 - 3.41	+0.202	0
1.63	-9.947 - 3.70	+0.202	0
1.96	-11.834 - 3.834	+0.202	0

case $b_3 = 0.342$

0.96	$-4.313 \pm 4.73i$	$-0.0568 \pm 0.397i$
1.15	$-5.396 \pm 4.45i$	$-0.0661 \pm 0.381i$
1.37	$-6.092 \pm 4.15i$	$-0.0674 \pm 0.362i$
1.63	$-6.85 \pm 3.60i$	$-0.0715 \pm 0.342i$
1.96	$-7.86 \pm 2.23i$	$-0.0756 \pm 0.320i$

Variable parameter: d_{3s}

case $b_3 = 0$

d_{3s}	λ	λ'	
0	-8.60 - 3.43	-0.208	0
0.05	-7.90 - 3.84	-0.4165	-0.1625
0.10	-7.60 - 3.70	-0.865	-0.149
0.20	$-5.35 \pm 0.031i$	-1.455	-0.175
0.30	$-5.05 \pm 2.18\ i$	-2.033	-0.181

case $b_3 = 0.344$

0	$-6.092 \pm 4.15\ i$	$-0.0674 \pm 0.362i$
0.05	$-5.979 \pm 4.36\ i$	$-0.1812 \pm 0.375i$
0.10	$-5.87 \pm 4.475i$	$-0.2895 \pm 0.188i$
0.20	$-5.67 \pm 5.14\ i$	$-0.485 \pm 0.03\ i$
0.30	$-5.55 \pm 5.60\ i$	-0.762 - 0.546

Knowledge of the roots λ permits picking out of the cases of instability (positive roots) and determining the period T and the duration D of the damping of the motions.

Let us recall that, having

$$\lambda = \kappa + \sigma i$$

the T and D, expressed in aerodynamic time, are given by

$$T = \frac{2\pi}{\sigma}$$

$$D = \frac{ln\ 0.5}{\kappa}$$

Although the tables given in the preceding pages, and the conclusions we draw from them, apply to a particular case, they are in good agreement with the results obtained by other authors regarding other particular cases.

Remark: Since it is a matter of indifference whether the variations of a_3, b_3, and c_3 are produced by the particular aerodynamic shapes of the airplane or by the deflection of the elevator, acting without lag or inertia, under the effect of a detector of perturbations, these tables justify the results given qualitatively in section 7, Chapter VII.

4. Practical Effects Obtained

A. Piloting as a function of the velocity.- An airplane which is dynamically unstable due to negative values of a_3 or of b_3 ($A_4 < 0$), could be rendered dynamically stable by an apparatus for piloting as a function of the velocity.

The stability attained by increase of a_{3_s} does not, however, endow the airplane with very good flight characteristics, since the damping of the slow oscillation is, and remains, weak.

The stabilizer sensitive to the velocity will amplify the rotations by which an airplane naturally seeks to maintain a constant velocity. Hence, it amplifies the amplitude of certain secondary perturbations.

One of the current defects of airplanes is the insufficiency of the damping of the slow oscillations. The stabilizer which is a function of the velocity is incapable of improving this situation. One can easily understand the reason.

Let us examine the variation of the V, θ, and q in the course of the slow oscillation.

The stabilizer which is a function of the velocity produces the maximum deflection - that is, applies the maximum pitching moment - when the trim of the airplane is horizontal.

It is clear that it will not contribute to stopping the airplane in this position - which is precisely what is required of it.

The stabilizer which is sensitive to the velocity acts too late; that is the reason why it exerts a detrimental effect on the damping of the slow oscillation.

B. Piloting as a function of the angle of attack.- Any apparatus comprising a wind vane for detecting any perturbation in angle of attack and actuating the elevator in the desired sense, does nothing else but increase the static stability of the airplane, that is, its tendency to maintain a constant value of the angle of attack.

The calculations show that b_3 may vary between rather wide limits without deterioration of the flight characteristics of the airplane.

There does not seem to exist a determined value of b_3 (which could possibly be realized artificially, by producing b_{3s} by means of a mechanical device) which ensures a flight path of clearly superior characteristics.

On the other hand, one can make the coefficient of static stability $b_3 = \partial C_M/\partial\alpha$ vary within wide limits by perfectly natural means: by displacing the center of gravity of the airplane along the axis OX.

This explains the practical failure of the attempts made with the purpose of developing the use of instruments of piloting utilizing the angle-of-attack parameter.

C. Piloting as a function of the angular velocity.- The utilization of the angular-velocity parameter is expressed in the equations by a c_{3s}. It augments A_1 and increases therefore the total available damping.

However, the increment has a bearing solely on the rapid oscillation whereas it is the slow oscillation the damping of which ought to be increased.

The ineffectiveness of the stabilizer which is a function of q for the damping of the slow oscillation can also be explained by examining the figure. The peaks of the curve of the q's correspond to the peaks of the curve of the V's, and the apparatus does not act at the right moment.

The "angular-velocity" parameter used alone does not seem to lead to practical results, but it may be of interest if combined with the "trim"-θ parameter.

It is curious to find that one of the oldest devices studied - the Lucas-Girardville stabilizer, designed in 1911 - used the parameter q.

D. Piloting as a function of the angle of trim θ.- In contrast to instruments sensitive to V and q, the stabilizer which is sensitive to θ assures a maximum deflection at the moment when the airplane is the most inclined and a zero deflection at the moment when the airplane is horizontal.

This situation is favorable for damping the slow oscillations. The calculation confirms this conclusion.

The equation in λ comprises a term in d_3; this term does not increase the value of the coefficient A_1, but it acts effectively on the distribution of the available damping.

The numerical table shows the following facts:

(a) Apparatus mounted on an airplane with neutral stability.- Without stabilizer, the airplane is actuated by a motion the four components of which are aperiodic. The addition of a stabilizer makes the rapid motion oscillatory when the sensitivity of the stabilizer is increased.

The slow motion remains made up of the superposition of two aperiodic motions, within the limits of imaginable sensitivity.

The stabilizer causes a retention of the real part of the root of the rapid motion and transfers this quantity to the root of the slow motion. This transfer of damping increases with the sensitivity of the stabilizer.

(b) Apparatus mounted on a statically stable airplane.- The rapid motion is oscillatory for all hypotheses of sensitivity of the stabilizer.

The slow motion - at first oscillatory - becomes aperiodic for the highest imaginable sensitivity.

The transfer of damping from one root to the other is practically equal to one half of what it is on the statically neutral apparatus.

(c) General conclusion.- The stabilization sensitive to the inclination θ possesses a valuable property which the apparatus studied previously do not possess: the high rate of damping of the slow oscillation.

Any occurring initial perturbation will therefore show a rapidly diminishing amplitude - when it concerns the rapid oscillation as well as when it concerns the slow oscillation.

The same result has been found in the United States where a calculation analogous to the preceding one has been carried out by A. Klemin, P. Pepper, and H. Wittner, of New York University.

The effect of the stabilizer has been investigated for variable sensitivities. The maximum sensitivity corresponds to a deflection $d\eta$ of -3° per degree of inclination θ.

The conclusions of the American reports are:

(a) The damping of the slow oscillation is considerably improved.

(b) If the airplane is stable, the slow oscillation is transformed into two aperiodic motions as soon as $d\eta/d\theta = -0.5$.

(c) After a study of stabilizers up to $d\eta/d\theta = -3$, these American reports state that, if a stabilizer of high sensitivity is placed on the airplane, the characteristics of statically stable and statically neutral airplanes tend to become indistinguishable. In other words, the effect of b_3 disappears before that of d_{3s} when the latter is sufficiently large.

(d) After having combined variations of c_3 with d_{3s}, they state that the influence of the damping factors c_3 on the characteristics of the motion disappears in view of the effect of the factor d_{3s}.

Airplanes which would present unfavorable characteristics, due to insufficiency of c_3, show perfectly admissible characteristics when they are provided with stabilizer sensitive to the inclination θ.

5. Stabilizers as Functions of the Derivatives

The calculations presented in Chapter XI permit predicting the action of a stabilizer which is a function of the derivatives of the fundamental variables.

<u>A. Piloting as a function of U'.</u>- Let us characterize the mechanism by

$$\frac{\partial C_M}{\partial \eta} \frac{d\eta}{du'} = K \quad \text{of dimensions} \quad L^{-1}T^2$$

It suffices to use again the formulas of Chapter XI, Section 7, noting that

$$x_3 = -\frac{\partial C_M}{\partial \eta}\frac{d\eta}{du'} V = -K V$$

$$x_1 = 0$$

$$a_m = -\frac{V}{\mu c} x_3 = +\frac{V^2}{\mu c} K$$

where

a_3	became	$a_3 + a_m a_1$
b_3		$b_3 + a_m b_1$
c_3		$c_3 + a_m c_1 \; 1/l$
Vd_3 (zero)		$Vd_3 + a_m d_1$

The action of the mechanism is the same as that of four equivalent stabilizers, respectively sensitive to the variables u, w, q, θ, and of power

$$\Delta a_3 = K \frac{V^2}{\mu c} a_1$$

$$\Delta b_3 = K \frac{V^2}{\mu c} b_1$$

$$\Delta c_3 = K \frac{V^2}{\mu c}\frac{1}{l} c_1$$

$$\Delta d_3 = \frac{1}{V} K \frac{V^2}{\mu c} d_1$$

On a normal airplane, one has

$a_1 > 0$ (of the order of 1/10)

$b_1 > 0$

$$c_1 = w \frac{\mu c}{V}$$

$$d_1 = -g \cos \theta \frac{\mu c}{V}$$

On the other hand, we take $K < 0$ because it is normal to make the airplane nose up when it accelerates.

Hence

$$\Delta a_3 = K \frac{V^2}{\mu c} a_1 < 0$$

$$\Delta b_3 = K \frac{V^2}{\mu c} b_1 < 0$$

$$\Delta c_3 = K w \frac{V}{l} \quad \text{zero when} \quad w = 0$$

$$\Delta d_3 = -K g \cos \theta > 0$$

Of these four equivalent stabilizers replacing the one which is a function of u', the two first ones exert a negative effect (unfavorable), the third exerts zero effect in the course of a flight at zero angle of attack, the fourth - namely the stabilizer which is a function of the trim - exerts a positive effect.

B. Piloting as a function of w'.- Let us characterize the mechanism by

$$\frac{\partial C_M}{\partial \eta} \frac{d\eta}{dw'} = K \quad \text{dimensions} \quad L^{-1} T^2$$

We refer to Chapter XI

$$z_3 = - \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dw'} V = -K V$$

$$b_3 = - \frac{V}{\mu c} z_3 = \frac{V^2}{\mu c} K$$

where

a_3	became	$a_3 + b_m a_2$
b_3		$b_3 + b_m b_2$
c_3		$c_3 + 1/l \; b_m c_2$
Vd_3		$Vd_3 + b_m d_2$

that is, as if there were four equivalent stabilizers, of the power

$$\Delta a_3 = K \frac{V^2}{\mu c} a_2$$

$$\Delta b_3 = K \frac{V^2}{\mu c} b_2$$

$$\Delta c_3 = K \frac{V^2}{\mu c} \frac{1}{l} c_2$$

$$\Delta d_3 = \frac{1}{V} K \frac{V^2}{\mu c} d_2$$

On a normal airplane

$$a_2 < 0$$

$$b_2 > 0 \quad \text{(of the order of +4)}$$

$$c_2 = -u \frac{\mu c}{V}$$

$$d_2 = -g \sin \theta \frac{\mu c}{V}$$

On the other hand, one will take $K < 0$ because it is normal to make the airplane nose down when α' increases, that is, when a negative w' appears.

Hence

$$\Delta a_3 > 0$$

$$\Delta b_2 < 0$$

$$\Delta d_3 = -K g \sin \theta$$

$$\Delta c_3 = -K \frac{\mu V}{l} > 0$$

The equivalent velocity stabilizer exerts a positive effect. The equivalent angle-of-attack stabilizer exerts a negative effect. The equivalent angular-velocity stabilizer exerts a positive effect. The equivalent trim stabilizer produces zero effect when $\theta = 0$.

<u>C. Piloting as a function of q'</u>.- It is well to present a direct argument:

The equation of rotation is written

$$q' = \frac{1}{B} M + \frac{1}{B} \frac{\partial M}{\partial \eta} \frac{d\eta}{dq'} q'$$

$$q'\left(1 - \frac{1}{B} \frac{\partial M}{\partial \eta} \frac{d\eta}{dq'}\right) = \frac{1}{B} M$$

Let us characterize the stabilizer by

$$\frac{\partial C_M}{\partial \eta} \frac{d\eta}{dq'} = -K \quad K \text{ - of the dimensions - } T^2$$

K will be positive if it causes the airplane to nose up when the angular acceleration is positive.

Hence

$$\frac{\partial M}{\partial \eta} \frac{d\eta}{dq'} = -K\ Sc\ \frac{\rho V^2}{2}$$

Inserting this value into the equation of equilibrium of the rotation, one obtains

$$q'\left(1 + \frac{1}{B} K\ Sc\ \frac{\rho V^2}{2}\right) = q'(1 + C) = \frac{1}{B} M$$

$$q' = \frac{1}{B} M \frac{1}{1 + C} = f_3(\delta u, \delta w, \delta q, \delta\theta) \frac{1}{1 + C}$$

in writing:

$$C = \frac{1}{B} K\ Sc\ \frac{\rho V^2}{2} = \frac{K V^2}{\mu r^2}$$

On an airplane not provided with an automatic pilot, we had

$$\frac{\partial f_3}{\partial u} = \frac{V}{\mu c} \frac{c}{r^2} a_3$$

$$\frac{\partial f_3}{\partial w} = \frac{V}{\mu c} \frac{c}{r^2} b_3$$

$$\frac{\partial f_3}{\partial q} = \frac{V}{\mu c} \frac{c l}{r^2} c_3$$

$$\frac{\partial f_3}{\partial \theta} = 0$$

The addition of a stabilizer sensitive to the angular acceleration will produce the same effect as that which one would obtain by multiplying a_3, b_3, and c_3 by $1/1 + c$ which is < 1 when $K > 0$.

The apparatus considered would reduce the stability.

There is nothing surprising in this conclusion.

We have seen that a stabilizer sensitive to θ, defined by

$$d_{3s} = - \frac{dC_M}{d\eta} \frac{d\eta}{d\theta}$$

increases the stability of the motion.

Now,

$$q' = \frac{dq}{dt} = \frac{d^2\theta}{dt^2}$$

The stabilizer defined by

$$K = - \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dq'}$$

is therefore sensitive to a variable which is opposed to θ.

The action of the visualized instrument will actually be opposed to that of an apparatus recognized to be good. The result reached is unavoidable.

D. Piloting as a function of $d\theta/dt$.- Application of the general formulas makes us state again a fact obvious a priori: piloting as a function of $d\theta/dt$ is nothing else but a piloting which is a function of q, already examined previously.

E. Conclusion.- The foregoing investigation shows that practically only one law of piloting as a function of the derivatives seems of interest, namely the first.

This stems from the fact that the fourth of the effective stabilizers which are equivalent to the apparatus utilizing the acceleration u' is a stabilizer which is a function of the angle of trim acting in the favorable sense.

We see actually that for a sensitivity such as $\delta u' = 5 m/sec^2$ which produces a deflection $\delta\eta = -2^o$, we would have under the assumption that $\partial C_M/\partial\eta = 0.015$ (angles in degrees as before):

$$K' = \frac{\partial C_M}{\partial \eta}\frac{d\eta}{du'} = -0.015 \times 2/5 = 0.006$$

and

$$\Delta d_3 = -g\,K' = +0.06$$

The sensitivity of the equivalent trim stabilizer is not negligible, and the effect of this component can make itself felt.

Obviously one must ask oneself if that effect will not be overcome by the effect of the stabilizer which is a function of the velocity which acts in the unfavorable sense.

We see immediately that this effect will be small.

In fact

$$\Delta a_3 = K\frac{V^2}{\mu c}a_1$$

$$\frac{V^2}{\mu c} = \frac{gV^2}{g\mu c} = g \times \frac{1}{C_z}$$

whence

$$a_3 = \frac{K\,a_1 g}{C_z}$$

for

$$K = +0.006$$
$$a_1 = +0.1$$
$$C_z = +0.4$$

one obtains

$$\Delta a_3 = \frac{-0.006 \times 0.1 \times 10}{0.4} = -0.015$$

an effect which appears relatively small.

It should be remarked that this effect can easily be cancelled by compensating it by a component of piloting as a function of the velocity.

Actually, there does not exist any apparatus directly measuring the derivative du/dt. In order to obtain it one must detect u, then differentiate the obtained result, for instance with the aid of an electric differentiator.

Hence it is possible to extract from the measurement of u that which is necessary to cancel eventually one of the four effects which are functions of u'.

6. Calculation of the Flight Paths After Initial Perturbation

A comparative calculation of the theoretical flight paths, after various initial perturbations, has been performed by Neumark.

We borrow from his report the figures which form the object of plate 39.

These figures describe the behavior of an airplane:

(a) Flying with controls fixed

(b) Piloted by an instrument which is a function of θ

(c) Piloted by an instrument which is a function of u and u' undergoing an initial perturbation

(1) of velocity δu_0 (horizontal gust)

(2) of velocity δw_0 (or angle of attack $\delta\alpha_0$)(ascending gust)

(3) of angle of attack $\delta\alpha_0$ and of trim $-\delta\theta_0$ (actual displacement of the airplane)[9]

The roots of the characteristic equation are characterized by the following numerical values.

For the airplane without automatic pilot

$$\kappa = -3.165 \qquad \kappa' = -0.025$$

$$\sigma = 2.653 \qquad \sigma' = 0.541$$

For the airplane with an automatic pilot which is a function of θ

$$\kappa = -2.621 \qquad \kappa' = -0.569$$

$$\sigma = 3.793 \qquad \sigma' = 0.341$$

[9] The original report of Neumark gives the curves for one single perturbation $\delta\theta$. These curves are without physical significance, and we prefer to add the curves resulting from the perturbations $\delta\alpha = -\delta w/V$ and $-\delta\theta$, in order to obtain an initial perturbation which is more complex but has a physical significance: Displacement of the airplane in space.

For the airplane with an automatic pilot which is a function of V and dV/dt

$$\kappa = -2.610 \qquad \kappa' = -0.580$$

$$\sigma = 2.816 \qquad \sigma' = 0.765$$

The curves relating to the airplane flying with controls fixed are comparable to those we have calculated.

Those relating to the airplane provided with an automatic pilot exhibit a transfer of damping from the rapid oscillation toward the slow oscillation.

The examples calculated by Neumark show that:

(a) In the case of the initial perturbation δu_0 (horizontal gust), the stabilizer which is a function of the trim opposes the nosing up which constitutes the natural reaction of the airplane without stabilizer; hence the perturbation δu diminishes more slowly than it would if the airplane were not provided with an automatic pilot, but the motions of long period are, nevertheless, better damped.

For the same initial perturbation, the stabilizer which is a function of the velocity and of the derivative gives to the secondary perturbation of angle of attack a complicated form which the author has studied in more detail in the original report.

(b) In the case of the initial perturbation $(\delta\alpha_0) = -\delta w_0/V$ (ascending gust) the stabilizers both diminish the rapidity of the decay of $\delta\alpha$.

This is caused by the fact that the decay of $\delta\alpha$ is produced by the exponential term $e^{\kappa t}$; the transfer of a certain quantity of κ toward κ' diminishes therefore the rapidity of the decay of a perturbation.

(c) In the case of a displacement of the airplane in space the stabilizer which is a function of θ adds its effect to that of the static stability.

Remark: The appearance of the diagrams is in good agreement with a fact verified by experience and easily explained by the theory.

The automatic flight control does in no way reduce the vertical accelerations undergone by an airplane flying in bad weather.

These accelerations stem, in fact, from increments in lift, produced chiefly by the modifications in angle of attack $\delta\alpha$ due to the vertical gusts.

No instrument diminishes the maximum $\delta\alpha$ corresponding to an instantaneous gust. All airplanes would be subject to the same accelerations in the theoretical case of the instantaneous gust, whether or not they are provided with an automatic pilot.

If the gust $\delta\alpha$ is established progressively in a fraction of a second (that is, during a time comparable to the duration of disappearance of $\delta\alpha$ by the damping of the rapid oscillation), the graphic construction of Carson shows that the stabilized airplane - for which $\delta\alpha$ diminishes somewhat more slowly - is finally subject to perturbations of the angle of attack (and consequently to accelerations) of a higher degree than a nonstabilized airplane.

This is a consequence of the fact that one has attempted to transfer the damping from the rapid oscillation to the slow oscillation.

7. Stabilizer Acting With Lag

Principle: The detector, sensitive to an arbitrary variable y, actuates the control by the intermediary of a servomotor.

In a simplified calculation one will write that, due to the inertia of the apparatus and the free play, the control occupies at the instant t the position determined by the magnitude the variable y possessed at the instant $t - n$, that is, n seconds earlier.

One may write

$$y_{t-n} = y_t - \frac{dy}{dt} n$$

The motion of rotation of the airplane is determined by

$$\frac{dq}{dt} = \frac{1}{B} M + \frac{1}{B} \frac{\partial M}{\partial \eta} \frac{d\eta}{dy} y_t - \frac{1}{B} \frac{\partial M}{\partial \eta} \frac{d\eta}{dy} \frac{dy}{dt} n$$

The first term of the second member corresponds to the motion of the airplane without stabilizer.

The second term defines the effect of a stabilizer sensitive to the variable y, acting instantaneously.

The third term represents an effect proportional to dy/dt, that is, to the derivative of the variable y.

Writing

$$- \frac{1}{B} \frac{\partial M}{\partial \eta} \frac{d\eta}{dy} y'n = \frac{c}{r^2} \frac{V}{\mu c}\left[-V \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dy} n\right] y' = \frac{c}{r^2} \frac{V}{\mu c}\left[V \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dy'}\right] y'$$

one sees that everything occurs as if the airplane were provided, in addition to the stabilizer sensitive to y, with an instrument sensitive to the derivative of y, and of the sensitivity

$$K V = \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dy'} V = - \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dy} n V$$

Let us examine this effect when y is replaced, successively, by each of the variables.

First case: Lag in the action of a stabilizer sensitive to u, of the power a_{3s}.

This effect is analogous to that of a stabilizer sensitive to the acceleration

$$V \frac{\partial C_M}{\partial \eta} \frac{d\eta}{du'} = K V$$

the sensitivity of which would be equal to

$$-n V = \frac{\partial C_M}{\partial \eta} \frac{d\eta}{du} = +n a_{3s}$$

that is, for which

$$K V = +n a_{3s}$$

$$K = \frac{n a_{3s}}{V}$$

The effect of an apparatus sensitive to u', of the power K, is given to us in section 5.

Replacing K by its value we see that the lag exerts the same effect as four equivalent systems characterized by

$$\Delta a_3 = a_1 a_{3s} n \frac{V}{\mu c}$$

$$\Delta b_3 = b_1 a_{3s} n \frac{V}{\mu c}$$

$$\Delta c_3 = a_{3s} n \frac{w}{l}$$

$$\Delta d_3 = -a_{3s} n \frac{g}{V} \cos \theta$$

The determintal effect of the lag stems, above all, from the fact that Δd_3 has become < 0.

Second case: Lag in the action of a stabilizer sensitive to w, of the power b_{3s} $\left(b_{3s}\right.$ is assumed to be $\left.> 0\right)$.

The same calculation indicates that the lag in the operation of the stabilizer is equal to the presence of four equivalent stabilizers.

$$\Delta a_3 = a_2 b_{3s} n \frac{V}{\mu c}$$

$$\Delta b_3 = b_2 b_{3s} n \frac{V}{\mu c}$$

$$\Delta c_3 = -b_{3s} n \frac{V}{l}$$

$$\Delta d_3 = -b_{3s} n \frac{g}{V} \sin \theta$$

Third case: Lag in the action of a stabilizer sensitive to q.

The calculation shows that, in this case, there is only one single equivalent stabilizer. It is of the type Δc_3. Its action is equal to

$$\Delta_{3s} = -n(g/V)c_{3s}$$

and corresponds necessarily to a reduction in damping.

Fourth case: Lag in the action of a stabilizer sensitive to the angle of trim θ.

This lag is represented by the action of a stabilizer sensitive to the derivative of θ

$$V \frac{\partial C_M}{\partial \eta} \frac{d\eta}{dq} = -n\, V \frac{\partial C_M}{\partial \eta} \frac{d\eta}{d\theta}$$

Then,

$$\frac{\partial \eta}{\partial q} = \frac{d\eta}{d\chi} \frac{l}{V} - \frac{\partial C_M}{\partial \eta} \frac{d\eta}{d\chi} \frac{l}{V} = +n \frac{\partial C_M}{\partial \eta} \frac{d\eta}{d\theta}$$

Everything occurs as if one had a stabilizer which is a function of the angular velocity q, equal to:

$$\Delta c_3 = -n(V/l) d_{3s}$$

The lag in the functioning of a stabilizer sensitive to the inclination θ is equivalent to a reduction of the damping coefficient c_3 of the airplane.

This effect may be numerically evaluated.

For instance, for

$$\begin{aligned} n &= 0.3 \\ d_{3s} &= 0.10 \\ V &= 100 \\ l &= 10 \end{aligned}$$

one would have a reduction in c_3 of 0.3 which a normal airplane can generally stand.

CHAPTER XVI

EFFECT OF THE AUTOMATIC PILOT ON THE LATERAL MOTION

1. Combinations to be Considered

The lateral piloting may be done by actuating the ailerons and the rudder, either as a function of the deviations in the variables

angle of sideslip $\beta = v/u = v/V$

angular velocity of roll p

angular velocity of yaw r

or as a function of the deviations in the angles

φ lateral inclination

ψ course or azimuth

which define the angular position of the airplane in space.

Each of these piloting parameters can actuate one or the other of the two controls, or even govern them both simultaneously.

Each one of the lateral controls exerts, on the other hand, an effect around the two axes OX and OZ.

For an aileron deflection $\Delta\xi$, the rolling moment $dC_L/d\xi \; \Delta\xi$ is the principal effect, the yawing moment $dC_N/d\xi \; \Delta\xi$ is a secondary, nonnegligible effect.

For a rudder deflection $\Delta\zeta$, the yawing moment $dC_N/d\zeta \; \Delta\zeta$ is the principal effect, the rolling moment $dC_N/d\zeta \; \Delta\zeta$ is a secondary effect.

As a result, one has to consider, in the general case, a large number of effects.

When the piloting parameters are β, p, or r, the automatic apparatus modifies effects naturally exerted upon the airplane, according to the aerodynamic characteristics of the plane. The actions exerted are defined by the table:

$$a_{2s} = -\left(\frac{\partial C_L}{\partial \xi}\frac{d\xi}{d\beta} + \frac{\partial C_L}{\partial \zeta}\frac{d\zeta}{d\beta}\right) \quad \text{because} \quad a_2 = -\frac{\partial C_L}{\partial \beta}$$

$$a_{3s} = -\left(\frac{\partial C_N}{\partial \xi}\frac{d\xi}{d\beta} + \frac{\partial C_N}{\partial \zeta}\frac{d\zeta}{d\beta}\right) \quad \text{because} \quad a_3 = -\frac{\partial C_N}{\partial \beta}$$

$$b_{2s} = -\frac{V}{s}\left(\frac{\partial C_L}{\partial \xi}\frac{d\xi}{dp} + \frac{\partial C_L}{\partial \zeta}\frac{d\zeta}{d\beta}\right) \quad \text{because} \quad b_2 = -\frac{\partial C_L}{\partial \tilde{\omega}}$$

$$b_{3s} = -\frac{V}{s}\left(\frac{\partial C_N}{\partial \xi}\frac{d\xi}{dp} + \frac{\partial C_N}{\partial \zeta}\frac{d\zeta}{d\beta}\right) \quad \text{because} \quad b_3 = -\frac{\partial C_N}{\partial \tilde{\omega}}$$

$$c_{2s} = -\frac{V}{s}\left(\frac{\partial C_L}{\partial \xi}\frac{d\xi}{dr} + \frac{\partial C_L}{\partial \zeta}\frac{d\zeta}{d\beta}\right) \quad \text{because} \quad c_2 = -\frac{\partial C_L}{\partial \rho}$$

$$c_{3s} = -\frac{V}{s}\left(\frac{\partial C_N}{\partial \xi}\frac{d\xi}{dr} + \frac{\partial C_N}{\partial \zeta}\frac{d\zeta}{d\beta}\right) \quad \text{because} \quad c_3 = -\frac{\partial C_N}{\partial \rho}$$

When the law of deflection depends on φ or ψ, the automatic stabilizer introduces effects which do not exist on the airplane flying with controls fixed. These effects can be characterized by factors which are written, in the most general case, when all effects add up:

$$d_{2s} = -\left(\frac{\partial C_L}{\partial \xi}\frac{d\xi}{d\varphi} + \frac{\partial C_L}{\partial \zeta}\frac{d\zeta}{d\varphi}\right)$$

$$d_{3s} = -\left(\frac{\partial C_N}{\partial \xi}\frac{d\xi}{d\varphi} + \frac{\partial C_N}{\partial \zeta}\frac{d\zeta}{d\varphi}\right)$$

$$e_{2s} = -\left(\frac{\partial C_L}{\partial \xi}\frac{d\xi}{d\psi} + \frac{\partial C_L}{\partial \zeta}\frac{d\zeta}{d\psi}\right)$$

$$e_{3s} = -\left(\frac{\partial C_N}{\partial \xi}\frac{d\xi}{d\psi} + \frac{\partial C_N}{\partial \zeta}\frac{d\zeta}{d\psi}\right)$$

In the preceding expressions, the first term of each parenthesis represents the effect of the ailerons, the second represents the effect of the rudder.

In each of the products, the first factor characterizes the aerodynamic effect of the control; the second factor characterizes the operation of the automatic pilot.

The action of the different laws is investigated for each of the two groups in the following sections.

2. Stabilizers Which Are Functions of the Variables of the First Group

As an example, the order of magnitude of the complementary terms is indicated below for a particular case.

Let us assume that the effectiveness of the controls is defined by

ailerons $\partial C_L/\partial\xi = 0.003$ (angle ξ in degrees)
$= 0.172$ (angle in radians)

rudder $\partial C_N/\partial\zeta = 0.0015$ (angle ζ in degrees)
$= 0.086$ (angle in radians)

The secondary effects are always a fraction x or z of the principal effects:

$$\frac{\partial C_L}{\partial\xi} = x\,\frac{\partial C_L}{\partial\xi}$$

$$\frac{\partial C_N}{\partial\zeta} = z\,\frac{\partial C_N}{\partial\zeta}$$

we shall assume them to be zero.

If we suppose a displacement of $1/2^\circ$ of the ailerons or of the rudder for 1° of sideslip, we obtain

$$a_{2s} = -0.172 \times 0.5 = -0.086$$

$$a_{3s} = -0.086 \times 0.5 = -0.043$$

A displacement of the ailerons of 5^o for an angular velocity of roll of 57^o 3' per second, and a displacement of the rudder of 5^o for an angular velocity of yaw of 57^o 3' per second give, respectively, for V/s = 10

$$b_{2s} = 0.003 \times 5 \times 10 = 0.15$$

$$c_{3s} = 0.0012 \times 5 \times 10 = 0.06$$

The investigation of the automatic pilot which is a function of the variables β, p, or r will be reduced to a discussion of the roots of the system, for modifications in the magnitude of the parameters

$$a_2 \text{ and } a_3 \qquad b_2 \text{ and } b_3 \qquad c_2 \text{ and } c_3$$

In chapter IX, we have briefly indicated the effect of these parameters. Let us now treat, as an example, the case of a hypothetical airplane which has a certain number of invariable characteristics but for which each of the parameters a_2, a_3, b_2, b_3, c_2, c_3 may vary separately.

Invariable characteristics

$$\mu = 28.2$$

$$C_z = 0.40$$

$$s^2/r^2a = 10$$

$$s^2/r^2c = 4.2$$

$$s/c = 3$$

Combinations investigated

a_2	a_3	b_2	b_3	c_2	c_3
variable	-0.04	+0.24	-0.017	+0.056	+0.045
-0.04	variable	id.	id.	id.	id.
id.	-0.04	variable	id.	id.	id.
id.	id.	+0.24	variable	id.	id.
id.	id.	id.	-0.017	variable	id.
id.	id.	id.	id.	+0.056	variable

We find for the roots λ:

Variable parameter a_2

0	$-0.72 \pm 2.08\,i$	+0.786	-6.28
-0.02	$-0.675 \pm 2.14\,i$	+0.495	-6.28
-0.04	$-0.620 \pm 2.23\,i$	+0.0162	-6.34
-0.06	$-0.577 \pm 2.30\,i$	-0.0114	-6.41
-0.08	$-0.531 \pm 2.357i$	-0.0354	-6.48

Variable parameter a_3

+0.02	$+0.325 \pm 0.413i$	-2.58	-6.34
+0.01	$+0.069 \pm 0.532i$	-1.38	-6.34
+0	$-0.431 \pm 0.908i$	-0.378	-6.34
-0.02	$-0.605 \pm 1.67\,i$	-0.0341	-6.35
-0.04	$-0.627 \pm 2.23\,i$	+0.0162	-6.35
-0.065	$-0.636 \pm 2.75\,i$	+0.0423	-6.36

Variable parameter b_2

0.204	$-0.55 \pm 2.38i$	+0.0182	-5.67
0.234	$-0.629 \pm 2.23i$	+0.0162	-6.33
0.264	$-0.627 \pm 2.22i$	+0.0147	-7.12

Variable parameter b_3

Completely insignificant effect

-0.0108	$-0.617 \pm 2.22i$	+0.0165	-6.35
-0.0168	$-0.620 \pm 2.23i$	+0.0162	-6.33
-0.0228	$-0.622 \pm 2.26i$	+0.0158	-6.32

Variable parameter c_2

0.0413	$-0.600 \pm 2.22i$	-0.00364	-6.35
0.0563	$-0.620 \pm 2.23i$	+0.0162	-6.33
0.0713	$-0.64 \pm 2.25i$	+0.0355	-6.30

Variable parameter c_3

0	$-0.31 \pm 2.21i$	+0.090	-6.31
0.03	$-0.515 \pm 2.25i$	+0.035	-6.33
0.44	$-0.627 \pm 2.23i$	+0.0162	-6.33
0.60	$-0.738 \pm 2.22i$	-0.005	-6.33

The practical effects would be as follows:

(a) The automatic pilot as a function of the angle of sideslip.- This mode of piloting will produce the same effect as an increase in dihedral of the wing or in the power of the vertical tail depending on whether the ailerons or the rudder are actuated.

This mode is little used.

If one had to deal with a badly designed airplane represented by a coordinate a_2 and a_3 situated in a zone of instability or only too closely to the limits, one could visualize a correction of the defects by an appropriate stabilizer a_{2s} or a_{3s}.

However, so far one has always preferred to modify the airplane itself.

(b) The automatic pilot as a function of an angular velocity.- In principle, b_{2s} and c_{3s} increase A_1, that is the total damping available - but whereas b_{2s} increases particularly the root λ_4 which cannot make any use of this increment, the effect of c_{3s} is distributed between the oscillatory motion and the spiral motion, and can exert a useful effect.

The terms b_{3s} and c_{2s} which are not to be found on the diagonal of the determinant and do not affect A_1, are only of little interest.

3. Automatic Pilots Which Are Functions of The Variables of the Second Group

We shall first define the order of magnitude of the complementary terms.

Let us assume that the aerodynamic effectiveness of the controls is the same as in the preceding section.

We shall limit ourselves at first to simple cases:

Ailerons deflected by $1/2^{\circ}$ for 1° of lateral inclination;

Rudder deflected by $1/2^{\circ}$ for 1° change in azimuth

One has

$$d_2 = 0.172 \times 0.5 = 0.086$$

$$e_3 = 0.086 \times 0.5 = 0.043$$

Let us write the characteristic determinant of the system of equations of the lateral motion in its general form, that is, incorporating in it the terms in d_2, d_3, e_2, e_3 which are zero when the airplane flies with controls fixed but which cease to be zero when the lateral controls undergo deflections which are functions of φ and ψ.

Let us recall or set

$$b_1 = -\bar{w}\,\frac{\mu c}{V}$$

$$c_1 = +u\,\frac{\mu c}{V}$$

$$d_1 = g \cos \varphi\,\frac{\mu c}{V} = g\,\frac{\mu c}{V}$$

$$b_4 = -\frac{\mu c}{V}$$

$$c_5 = -\frac{\mu c}{V}$$

The condition

$$\begin{vmatrix} a_1 + \lambda & b_1 & c_1 & d_1 & 0 \\ \frac{b}{r^2{}_a}\,a_2 & \frac{bs}{r^2{}_a}\,b_2 + \lambda & \frac{bs}{r^2{}_a}\,c_2 & \frac{bV}{r^2{}_a}\,d_2 & \frac{bV}{r^2{}_a}\,e_2 \\ \frac{b}{r^2{}_c}\,a_3 & \frac{bs}{r^2{}_c}\,b_3 & \frac{bs}{r^2{}_c}\,c_3 + \lambda & \frac{bV}{r^2{}_c}\,d_3 & \frac{bV}{r^2{}_c}\,e_3 \\ 0 & b_4 & 0 & \lambda & 0 \\ 0 & 0 & c_5 & 0 & \lambda \end{vmatrix} = 0$$

is identical with

$$\lambda^5 + B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5 = 0$$

Comparing with the development carried out in chapter IX, and replacing b by 2s, one obtains

$$B_1 = A_1$$

$$B_2 = A_2 + \mu \frac{2sc}{r^2_a} d_2 + \mu \frac{2sc}{r^2_c} e_3$$

$$B_3 = A_3 + \mu \frac{2sc}{r^2_c}\left(a_1 + \frac{2s^2}{r^2_c} b_3\right) d_2 + \mu \frac{2sc}{r^2_c}\left(a_1 + \frac{2s^2}{r^2_a} b_2\right) e_3 +$$

$$\mu \frac{2sc}{r^2_c}\left(- \frac{2s^2}{r^2_a} c_2\right) d_3 + \mu \frac{2sc}{r^2_a}\left(- \frac{2s^2}{r^2_c} b_3\right) c_2$$

$$B_4 = A_4 + \mu \frac{2sc}{r^2_a}\left(a_1 \frac{2s^2}{r^2_c} c_3 - \frac{2s}{r^2_c} a_3 c_1\right) d_2 +$$

$$\mu \frac{2sc}{r^2_c}\left(\frac{2s}{r^2_a} a_2 c_1 - a_1 \frac{2s^2}{r^2_a} c_2\right) d_3 +$$

$$\mu \frac{2sc}{r^2_a}\left(\frac{2s}{r^2_c} a_3 b_1 - a_1 \frac{2s^2}{r^2_c} b_3\right) e_2 +$$

$$\mu \frac{2sc}{r^2_c}\left(a_1 \frac{2s^2}{r^2_a} b_2 - \frac{2s}{r^2_a} a_2 b_1\right) e_3 +$$

$$\mu^2 \frac{4s^2c^2}{r^2_a r^2_c}\left(d_2 e_3 - e_2 d_3\right)$$

$$B_5 = \mu \frac{2sc}{r^2_a}\left[a_3 \frac{2sc}{r^2_c} \mu C_z\right] e_2 + \mu \frac{2sc}{r^2_c}\left[-a_2 \frac{2sc}{r^2_a} \mu C_z\right] e_3 +$$

$$\mu^2 \frac{4s^2c^2}{r^2_a r^2_c} a_1 \left[d_2 e_3 - e_2 d_3\right]$$

by writing directly

$$c_5 d_1 = - \frac{\mu c}{V} g \frac{\mu c}{V} = - \frac{\mu c}{V^2} g\mu c = -C_z \mu c$$

The presence of the factor c (chord of the wing) gives rise to the same remark as was made already in section 2, chapter IX.

Let us remark immediately that the equation is of the fifth degree only when at least one of the quantities e_2 or e_3 is different from zero, that is, when one of the external actions is a function of ψ.

On the other hand, none of the terms in d_2, d_3, e_2, or e_3 is to be found on the diagonal.

These terms do not contribute to an increase in total damping and can only produce transfers of damping from one root to the other.

Let us retain, as the only variables, the quantities d_2, d_3, e_2, e_3 and write the coefficients B_1, B_2, B_3, B_4, and B_5, giving to the other characteristics the values of the previous example.

We then obtain

$$B_1 = 7.55$$

$$B_2 = 13.15 + 173 d_2 + 104 e_3$$

$$B_3 = 33.41 + 224.5 d_2 - 155 d_3 + 46.4 e_2 + 710.4 e_3$$

$$B_4 = -0.544 + 785.2 d_2 - 787.7 d_3 + 27.8 e_2 + 388 e_3 + 5200 (d_2 e_3 - e_2 d_3)$$

$$B_5 = -288 e_2 + 288 e_3 + 3080 (d_2 e_3 - e_2 d_3)$$

Let us study the characteristic equation in the following cases:

... ...

$d_2 = 0$	0.04	0.08	0.16
$d_3 = 0$			
$e_2 = e_3 = 0$			

... ...

$d_3 = -0.04$	-0.02	0	+0.01
$d_2 = 0$			
$e_2 = e_3 = 0$			

... ...

$e_3 = 0$	+0.05	+0.10	+0.20
$e_2 = 0$			
$d_2 = d_3 = 0$			

... ...

$e_2 = 0$	0.025
$e_3 = 0$	
$d_2 = d_3 = 0$	

... ...

The roots of the characteristic equation are:

Variable parameter d_2

d_2	$\lambda_{1.2}$	λ_3	λ_4
0	$-0.627 \pm 2.23i$	+0.0162	-6.33
0.04	$-0.742 \pm 2.25i$	+0.845	-6.915
0.08	$-0.845 \pm 2.17i$	-2.932	$\pm 1.73i$
0.16	$-0.770 \pm 2.02i$	-3.007	$\pm 4.20i$

Variable parameter d_3

d_3	$\lambda_{1,2}$	λ_3	λ_4
-0.04	$-0.1385 \pm 2.28i$	-0.938	-6.33
-0.02	$-0.3575 \pm 2.21i$	-0.510	-6.33
0	$-0.627 \pm 2.23i$	+0.0162	-6.33
+0.01	$-0.750 \pm 2.25i$	+0.263	-6.32

Variable parameter e_3

e_3	$\lambda_{1,2}$	$\lambda_{3,4}$		λ_5
0	$-0.627 \pm 2.23i$	+0.016	0	-6.33
0.05	$-0.512 \pm 3.02i$	$-0.133 \pm 0.416i$		-6.33
0.10	$-0.434 \pm 3.86i$	$-0.176 \pm 0.517i$		-6.32
0.20	$-0.41 \pm 5.00i$	$-0.212 \pm 0.488i$		-6.32

Variable parameter e_2

e_2	$\lambda_{1,2}$	λ_3	λ_4	λ_5
0	$-0.627 \pm 2.23i$	+0.016	0	-6.33
0.025	$-0.56 \pm 2.29i$	+0.84	-1	-6.32

These roots correspond to the following practical effects:

Effect of d_2: When the stabilizer which is sensitive to φ operates the ailerons and these latter act without secondary effect, we find:

(1) That the oscillation subsists while undergoing progressive modifications.

(2) That the root λ_4 which determines a strongly damped rolling motion and hence is negative begins to increase in absolute value - which is normal because the visualized mode of piloting consists precisely in opposing the rolling motion. But since the sum of the roots is to remain constant, the root λ_3 which characterizes the spiral motion must undergo positive increments, and the pilotage visualized must necessarily produce spiral instability.

For sufficiently large values of d_2, the roots λ_3 and λ_4 are combined into a pair of complex roots, and one then encounters a damped oscillation.

Effect of d_3: The d_3 corresponds in principle to the maneuver of the rudder under the effect of the lateral inclination φ.

$d_3 > 0$ corresponds to a maneuver in the direction tending to make the airplane rotate about the wing which is lowered - which is unfavorable.

The direction in which such a control should act corresponds to $d_3 < 0$. The calculation verifies this fact.

An effect d_3 may also be produced as a secondary effect of the ailerons when these are actuated as a function of the deviation in φ. This d_3 is then > 0 and is unfavorable.

Effect of e_3: This effect is perhaps the most important one that can be produced by the automatic pilot.

In fact, the airplane does not have any sense of azimuth, and the pilot, in flight, must constantly correct the heading.

A stabilizer applying to the rudder a deflection which is a function of the change in azimuth imparts to the airplane a new sense which it does not possess naturally.

One knows that the equation has become an equation of the 5^{th} degree. The calculation shows that the root λ_4 of the airplane without stabilizer appears again in the equation of the 5^{th} degree.

This is an interesting finding which facilitates the calculations, for when this root is known, one can immediately reduce by 1 the degree of the equation of the airplane provided with the automatic pilot.

The root λ_3 which defined the spiral motion is combined with the new root introduced by the stabilizer to give a new oscillatory motion which is rather slightly damped.

In proportion as the power of the stabilizer grows, the damping of the former oscillatory motion decreases.

The characteristics we are setting up here for a particular example have been encountered in other particular cases by other authors, notably by Imlay.

One may assume that for an airplane which presents normal characteristics, the phenomena found above are general.

Effect of e_2: The maneuvering of the ailerons as a function of the change in azimuth would produce and effect e_2.

An effect $e_2 > 0$ produces a detrimental action because there exists a positive root much larger than that found for the same airplane when not provided with a stabilizer.

The reason for this fact is easily found: for an increment $d\psi > 0$, the stabilizer furnishes $\Delta L < 0$.

If the airplane has turned to the left, the effect e_2 would make it incline toward the left. It may easily be seen that this effect necessarily contributes to an increase in spiral instability.

An effect $e_2 > 0$ may be produced as a secondary effect by a rudder controlled by the change in azimuth. This effect exists if the vertical tail is very high.

4. Secondary Effects

We state that the two normal types of automatic pilot, namely $d_2 > 0$ and $e_3 > 0$, can both produce an unfavorable action due to the secondary effect of the controls.

It may be useful to verify that these secondary effects are always less important than the favorable principal effect.

Let us examine the roots corresponding to two combinations:

$d_2 = 0.08$ with (as secondary effect) $d_3 = 0.01$

$e_3 = 0.10$ with (as secondary effect) $e_2 = 0.025$

and let us compare them with the roots corresponding to the same principal effect, without secondary effect.

In the case of the ailerons maneuvered as a function of φ, we find

$$-1.13 \pm 2.225i \qquad -2.647 \pm 1.32i$$

while we had, in the absence of a secondary effect

$$-0.845 \pm 2.17i \qquad -2.932 \pm 1.73i$$

The reduction in damping of the first oscillation is noticeable.

In the case of the azimuth stabilizer we find

$$-0.44 \pm 3.86i \quad -0.175 \pm 0.451i \quad \text{and} \quad -6.32$$

while we had, without secondary effect

$$-0.434 \pm 3.86i \quad -0.176 \pm 0.517i \quad \text{and} \quad -6.32$$

The difference is imperceptible.

These findings show that - at least in the example investigated - the secondary effect of the controls is not of a nature as to modify our conclusions.

5. Stabilizers Which Are Functions of the Derivatives

The problem of the lateral stabilizers sensitive to the derivatives of the variables determining the lateral motion may be treated like that of the longitudinal stabilizers.

One finds again absolutely parallel results: one sees for instance that the stabilizer which is sensitive to the derivative of the angle of sideslip is equivalent to four elementary stabilizers which are functions of the angle of sideslip, of the angular velocities p and r, and of the lateral inclination.

The most interesting result to which this examination leads concerns the effect of the lag of the stabilizer which is a function of the azimuth ψ.

A lag n in the operation of the stabilizer e_3 produces the same effect as if one were adding a stabilizer which is a function of the angular velocity r, and of the power

$$\Delta c_3 = -n \frac{V}{s} e_3$$

It is equivalent to a reduction of the damping coefficient. This effect could be considerable. In fact, for $n = 0.2$, $V = 100$, $s = 10$ $e_3 = 0.10$ we would have:

$$\Delta c_3 = -0.2$$

while we know that the c_3 is of the order of $+0.05$.

There is therefore reason to ask what the effect of a reduction, or even of a change in the sign, of the damping coefficient c_3 will be, in the case of an airplane provided with a stabilizer of the type e_3, sensitive to the azimuth.

We performed the calculation for $e_3 = 0.10$, taking a series of values of $(c_3 + \Delta c_3)$.

The result is as follows:

Coefficients of the equation of the 5th degree

$c_3 + \Delta c_3$	B_1	B_2	B_3	B_4	B_5
+0.044	7.554	23.512	104.28	38.456	28.8
0	6.84	18.742	101.862	36.51	28.8
-0.005	6.40	15.5	100.50	35.3	28.8
-0.05	6.04	13.23	98.75	34.2	28.8
-0.10	5.24	7.78	95.77	32	28.8

One of the roots is necessarily real; its value is practically the same in the five equations. Assume λ_5 to be this root.

Dividing by $(\lambda - \lambda_5)$, one obtains an equation of the 4th degree which leads to the following roots:

Value of $(c_3 + \Delta c_3)$	Pure oscillation	Oscillation arising from the spiral motion	Rolling motion λ_5
0.045	$-0.434 \pm 3.86i$	$-0.164 \pm 0.552i$	-6.30
0	$-0.1 \pm 3.85i$	$-0.176 \pm 0.517i$	-6.32
-0.025	$+0.08 \pm 3.84i$	$-0.165 \pm 0.525i$	-6.32
-0.05	$+0.295 \pm 3.83i$	$-0.164 \pm 0.53\ i$	-6.31
-0.10	$+0.694 \pm 3.75i$	$-0.165 \pm 0.53\ i$	-6.30

The two oscillatory motions are determined by the equation of the 4th degree. These equations are characterized by a coefficient A_1 continually diminishing in proportion as $c_3 + \Delta c_3$ decreases

The total available for the damping of the two oscillations keeps on diminishing, and the pure oscillation may become unstable.

This example shows that a lag in the operation of the directional stabilizer will impart to the airplane a steady yawing motion.

If this instability is of a high degree, it will not suffice to attempt to modify the distribution of the damping between the two oscillations.

After extracting from $-B_1$ the root λ_5, the remaining available amount is not sufficient to damp the two oscillations, whatever the law of distribution.

It would have to be possible either to reduce the root λ_5, or to increase the total available damping.

Unfortunately, it does not seem possible to reduce the root λ_5, and the most effective solution consists in increasing the c_3 by adding a mechanism for controlling the deflection of the rudder as a function of the angular velocity r, that is, by augmenting artificially the damping of the rolling motions.

The characteristics which the calculation indicates for the motion of an airplane provided with an automatic pilot sensitive to the azimuth ψ (whether or not there is a lag in the operation) show that there always exists a serious risk of undulatory motion.

The well-known practical difficulties encountered in creating and operating these instruments constitute a very clear demonstration of the theoretical conclusions.

Let us remark finally that when the instability is not too pronounced and it is sufficient to modify the distribution of damping, several experts suggest utilizing the effect of the component $d_3 > 0$ for this purpose. This exerts a powerful effect on the transfer of damping from the spiral motion toward the oscillatory motion; and this means is usable for stabilizing the latter motion after the spiral instability need no longer be feared - which is the case of an airplane provided with an automatic pilot of the type e_3.

CHAPTER XVII

THE AUTOMATIC PILOT AS A FUNCTION OF THE INTEGRAL OF THE PERTURBATIONS

1. Statement of the Problem

The piloting parameters studied so far were parameters in some way intrinsic to the airplane.

The airplane piloted according to these parameters is insensitive to certain factors which are, however, very important - namely the entrained velocities of the surrounding medium, if these velocities have a constant magnitude.

The airplane which passes abruptly from a zone Z_1 where the atmosphere is motionless to a zone Z_2 where the air has an upward velocity W is, at the instant of this passage, subject to a perturbation δw and reacts abruptly; however, when the transitory period has come to an end, the plane will return to a position of equilibrium characterized by the same relative velocity, the same angle of attack, and the same trim as in the initial state. The entrainment velocity simply adds to the relative velocity, and the trajectory becomes an ascending one without disturbing the power-equilibrium conditions.

If the airplane is provided with instruments of automatic flight control sensitive to the perturbations δu, δw, and $\delta\theta$, this equipment will give rise to reactions during the transitory period but it will be incapable of discerning a difference between the final state of equilibrium and the initial state.

Everything we said here about the lateral motion applies likewise to motions with respect to the surrounding medium. If that medium is possessed of a horizontal entrainment velocity - the wind - the motion of the airplane with respect to the ground is the sum of the motion with respect to the surrounding medium and that of the entrained velocity, and the instruments sensitive to the perturbations examined so far are incapable of detecting the effect exerted on the flight path by a constant wind.

By means of electromagnetic fields, it is possible to set up in space reference lines fixed to the ground. It is possible to detect the deviations with respect to these reference lines and to maneuver the controls accordingly.

Thus one realizes a new class of instruments for automatic flight control.

2. Flight Controlled by Radio Reference Lines

One may visualize:

(a) Flight following a parallel beam

(b) Flight in a beam converging at a point

In the case of parallel reference axes, the airborne receiver gives, by hypothesis, an indication proportional to the distance y between the airplane and the axis.

In the case of a converging beam the center of which is at the distance D from the airplane, the receiver furnishes frequently the angle ϵ; however, since $\tan \epsilon = y/D$, this indication is equivalent to that furnished by a receiver which gives the deviation y but is of a sensitivity varying with the distance D.

We shall not attempt to find out by what means the intensity of an electromagnetic field can be transformed into an input signal x of the servocontrol. If we assume that this part of the operation takes place without lag, the functioning of the servocontrol will be determined by the characteristics of the power relays and of the servomotor used.

The point we are investigating in the present chapter is the effect of the piloting which is a function of references fixed to the ground on the motion of the airplane.

3. Longitudinal Motion

After a line of reference has been set up, the problem posed is to fly along this line, utilizing the indications of an instrument which detects the deviations in height z with respect to the latter. (See fig. 35.)

This deviation z constitutes a new variable, defined, in the case of a horizontal reference line, by

$$dz = V \sin \tau \, dt$$

$$z = \int V \sin \tau \, dt$$

It is proportional to the integral of the angular deviation of the flight path.

Now

$$\tau = -(\theta + \alpha)$$

Approximating the angles by the sines, we have

$$\frac{dz}{dt} = -V(\theta + \alpha)$$

In automatic flight control, the deflection of one of the two longitudinal controls is linked to z by a law of proportionality. If the actuated control is the elevator, the moment M is a function of the new variable z, and one has

$$\frac{dq}{dt} = f_3(u,w,q,\theta,z)$$

On the other hand, the expression

$$\frac{dz}{dt} = -V(\theta + \alpha)$$

becomes a new function f_5

$$\frac{dz}{dt} = f_5(u,w,\theta)$$

and the motion of the airplane is determined by a system of five linear equations.

Since the derivatives

$$\frac{\partial f_3}{\partial z}, \quad \frac{\partial f_5}{\partial u}, \quad \frac{\partial f_5}{\partial w}, \quad \frac{\partial f_5}{\partial \theta}$$

are different from zero, the characteristic equation is of the fifth degree and is written

$$\begin{vmatrix} \frac{\partial f_1}{du} - x & \frac{\partial f_1}{dw} & \frac{\partial f_1}{dq} & \frac{\partial f_1}{d\theta} & 0 \\ \frac{\partial f_2}{du} & \frac{\partial f_2}{dw} - x & \frac{\partial f_2}{dq} & \frac{\partial f_2}{d\theta} & 0 \\ \frac{\partial f_3}{du} & \frac{\partial f_3}{dw} & \frac{\partial f_3}{dq} - x & \frac{\partial f_3}{d\theta} & \frac{\partial f_3}{dz} \\ 0 & 0 & -1 & -x & 0 \\ \frac{\partial f_5}{du} & \frac{\partial f_5}{dw} & 0 & \frac{\partial f_5}{d\theta} & -x \end{vmatrix} = 0$$

The type of piloting which operates by actuating the elevator, as a function of the deviations in altitude, with respect to a reference line, is, in fact, not very logical, for the variable directly controlled in the steady-state condition by the elevator is the velocity along the flight path.

The slope of the flight path is determined by the elevator, in the equilibrium condition, only in an indirect manner, through the effect of the excess of power. The piloting with a view to maintaining the airplane on a flight path defined in altitude should logically take place by acting on the control the effect of which determines directly the upward velocity, that is, the control of the engine power setting.

Such a law of piloting would be necessarily defined by a relation between the thrust T and the deviation z detected; it will produce a modification of the moment M only if the engine power setting exerts a secondary effect on the C_M.

For an airplane piloted in this way, the function f_1 will depend on z,

$$\frac{du}{dt} = f_1(u,w,q,\theta,z)$$

the derivative $\partial f_1/\partial Z$ will always be different from zero; the derivative $\partial f_3/\partial z$ also will be different from zero when the secondary effect is not zero. The characteristic equation likewise will be of the fifth degree.

4. Lateral Motion

The detected deviation y is the horizontal distance with respect to the reference line.

Let us assume that this indication is utilized for the control of the rudder.

In the case of an ideal control mechanism

$$\zeta = k \times y$$

the moment N is a function of y and the equation of equilibrium about the axis OZ is written

$$\frac{dr}{dt} = f_3(v,p,r,\varphi,\psi,y)$$

The derivative dr/dt is a function of a sixth variable, the distance y.

In order to simplify the notation, we assume that the azimuth of the axis of the beam is the origin of the ψ; we have therefore, taking into account the possible sideslip:

$$dy = V \sin(\psi + \beta)\ dt$$

$$y = \int V \sin(\psi + \beta)\ dt$$

and we achieve a piloting which is a function of the integral of the perturbations of the variables ψ and β.

We may write, on the other hand

$$\frac{dy}{dt} = V \sin(\psi + \beta) = f_6(v,p,r,\varphi,\psi,y)$$

and we have a sixth equation connecting the derivative of the variable y with one of the five other variables.

The characteristic determinant becomes

$$\begin{vmatrix}
\frac{\partial f_1}{\partial v} - x & \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial \psi} & 0 \\
\frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial p} - x & \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial \psi} & 0 \\
\frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial p} & \frac{\partial f_3}{\partial r} - x & \frac{\partial f_3}{\partial \varphi} & \frac{\partial f_3}{\partial \psi} & \frac{\partial f_3}{\partial y} \\
\frac{\partial f_4}{\partial v} & \frac{\partial f_4}{\partial p} & \frac{\partial f_4}{\partial r} & \frac{\partial f_4}{\partial \varphi} - x & \frac{\partial f_4}{\partial \psi} & 0 \\
\frac{\partial f_5}{\partial v} & \frac{\partial f_5}{\partial p} & \frac{\partial f_5}{\partial r} & \frac{\partial f_5}{\partial \varphi} & \frac{\partial f_5}{\partial \psi} - x & 0 \\
\frac{\partial f_6}{\partial v} & 0 & 0 & 0 & \frac{\partial f_6}{\partial \psi} & -x
\end{vmatrix} = 0$$

and since $\frac{\partial f_3}{\partial y}$, $\frac{\partial f_6}{\partial v}$ and $\frac{\partial f_6}{\partial \psi}$ are different from zero, the characteristic equation is of the sixth degree.

5. Principal Properties of These Types of Piloting

The piloting as a function of the references fixed to the ground has three essential properties:

1. It is equivalent to the piloting which is a function of the integral of the deviations of one or of several intrinsic variables.

2. It raises the degree of the characteristic equation by one.

3. It permits removing the airplane from the influence of the entrained velocities of the surrounding medium. The flight path may - within certain limits - be rendered independent of these entrained motions.

The study of the properties of the airplanes piloted in such a manner could be made by development of the characteristic equations. However, a superficial examination is sufficient to show the defect of any piloting which is a function of the integral of the deviations: the motions are insufficiently damped and become easily unstable if the sensitivity of the instruments is too high.

In fact, let us examine what takes place in the case of the lateral motion.

If the airplane is at a given instant to the right of the axis ($\Delta y < 0$), the automatic pilot will receive a command ($\Delta N > 0$) proportional to Δy under the effect of which it will turn to the left.

This command will be cancelled only at the moment when the airplane is on the axis. Under the effect of the previously accumulated commands, the airplane will have carried out a considerable rotation, and it will not again contact the axis tangentially with an infinitely small ψ. It will, on the contrary, intersect the axis at a rather large angle ψ, resulting from the integration of the $\delta\psi$, and will pass to the left of the reference line.

After the airplane has passed beyond that position, the same phenomena occur in the opposite direction, and the motion may be amplified.

The experience acquired in the execution of blind landings shows that this is really so. One may consider that the human pilot who attempts to make a blind landing by deflecting the rudder according to the indications of the vertical needle of his ILS receiver, achieves manually the piloting defined above, since the needle indicates the lateral deviation y.

An airplane thus piloted frequently takes up a flight path which becomes more and more undulatory in proportion as the airplane approaches the destination. This stems from the fact that the sensitivity of the receiver grows in proportion as the airplane approaches the transmitter. Equal deviations of the needle correspond to increasingly small deviations δy, and when the pilot endeavers to fit his movements to the indications of the needle, he finally makes the airplane execute an unstable motion as a result of excessive sensitivity.

The theoretical study of such an apparatus could be made by keeping the system of equations in the linear form but giving to the sensitivity factor included in $\partial f_3/\partial y$ a series of increasing values corresponding to different degrees of the progress of the airplane along the landing flight path.

The search for a general expression for the sensitivity factor as a function of time would make $\partial f_3/\partial y$ a function of the independent variable, and would change the type of the equations.

6. Possible Combinations

The type of piloting sensitive to the distance relative to a reference line can be combined with a type of piloting sensitive to the angular deviation of the airplane with respect to the direction of this reference line.

Several cases can be considered for longitudinal piloting: horizontal alinement, angular alinement.

The case of horizontal alinement is rather theoretical; actually, the case of angular alinement is the real case. If the adjustment of the controls and instruments is such that the conditions of equilibrium - when the airplane follows exactly the prescribed flight path - are satisfied by deflections zero, one can visualize the following effects:

Control of elevation dependent on the deviation $\delta\theta$;

Control of power setting dependent on the deviation δz; or else, control of elevation dependent on a combination of the deviations $\delta\theta$ and δz.

For the lateral piloting, the case realized in practice is the one where the rudder is actuated as a function of δy and $\delta\psi$.

Under the assumption that the origin of the ψ corresponds to a particular reference line, and that the y are positive when the airplane is to the left of the reference line, the law of piloting becomes

$$\delta\zeta = -K_1\delta y - K_2\delta\psi$$

The theoretical study shows that one can realize favorable flight paths by a suitable choice of the sensitivities K_1 and K_2. These trajectories improve still more if one adds a piloting component which is a function of r.

However, we see immediately that in case of lateral wind of a velocity W the law of piloting does not permit maintaining the airplane on the reference line. In fact, since W/V is the crab angle, the rectilinear flight along the reference line, at a distance $y = 0$, is not feasible unless the airplane adopts a course equal (in absolute value) to the angle of sideslip W/V, and maintains this course constantly.

This implies the combination

$$d\zeta = 0 \qquad \delta\psi = W/V \qquad \delta y = 0$$

which can be realized only if the pilot - knowing the velocity component W of the wind and the angle of sideslip - modifies the adjustment of the instrument by displacing in some manner the zero point of the ψ.

Without an adjustment of the automatic pilot to take the wind into account, the airplane will necessarily follow a flight path showing constantly deviations $\delta\psi$ and δy.

7. Automatic Landing

Piloting as a function of terrestrial references has the principal aim of permitting the realization of automatic landing.

From the longitudinal viewpoint, the problem consists in following - with deviations in height not exceeding 10 meters - an inclined reference line which one can, in a first solution, assume to have constant slope. Actually, this slope is not necessarily constant, and it would be better to define in space a trajectory identical to the one the pilot imposes on his airplane when he makes a visual landing.

Experience has shown that it is actually not practical to control the slope of the flight path by means of the elevator, because the start of the descent of an airplane can be accompanied by an increase in velocity along the flight path which is absolutely inadmissible in the course of the maneuvers preceding landing.

The devices which have been built avoid these variations in speed by controlling the engine power. This control is indispensable if one prescribes a constant velocity on a flight path of variable slope.

From the lateral viewpoint, the problem consists in guiding the airplane along the axis of a runway, even in the case of a cross wind. Since the width of the runway is of the order of from 80 to 100 meters, the admissible deviation on either side of the axis cannot exceed 20 meters.

Knowledge of the crab angle is indispensable, the heading imposed on the airplane being equal to the azimuth of the runway, corrected by the crab angle. However, this manner of proceeding is admissible only when the correction to be applied does not exceed 6° to 8°.

We reproduce the recording of an automatic landing, effected by means of a Lancaster airplane. This diagram is taken from the report published in 1946 by H. O. Pritchard.

The control apparatus was sensitive to δy and $\delta\psi$. At a distance of 7 miles from the entrance of the runway, the airplane deviated from it by approximately 1 mile, but its axis OY was reasonably parallel to that of the runway. After the automatic flight-control apparatus had been put in action, the airplane effected a change in course of about 45° and approached the axis of the runway.

The curve seems to indicate the existence of an oscillation of very long period (of the order of 2 minutes), which agrees rather well with the concept of piloting as a function of the integral of a perturbation.

CHAPTER XVIII

APPLICATIONS OF SYMBOLIC CALCULUS TO THE PRECEDING PROBLEMS

1. Usefulness of the Symbolic Calculus

The solution of the preceding problems is greatly simplified if one resorts to operational or symbolic calculus.

(a) The integration of the differential equations defining the motion as a function of the given initial conditions by the classical method of Lagrange is simple in theory, but, actually, gives rise to very lengthy numerical calculations. The operational calculus leads to a much speedier method.

(b) In the symbolic calculus, Duhamel's integral is, in general, replaced by an equivalent, but simpler, expression.

(c) In a particular case, for example when a system defined by linear equations is subjected to harmonic, that is, sinusoidal excitation, the solution of the steady motion is found immediately, thanks to the symbolic calculus.

2. Principle of the Symbolic Calculus

Consider a function $f(t)$ of the real variable t.

We put

$$\varphi(p) = p\int_0^\infty e^{-pt}f(t)dt$$

an integral which is found in the second member being assumed convergent.

This formula defines a correspondence between the functions $f(t)$ and $\varphi(p)$ and is symbolically represented by

$$\varphi(p) \subset f(t)$$

It constitutes Carson's transformation.

The function $\varphi(p)$ is called the image of the function $f(t)$.

To Carson's transformation there corresponds an inverse relationship: the formula of Bromwich

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{\varphi(p)}{p} dp$$

where the integral is extended to a line of the complex plane going from $c - \infty i$ to $c + \infty i$.

The function $f(t)$ is called the original of $\varphi(p)$, and the relationship is represented symbolically by

$$f(t) \supset \varphi(p)$$

Any treatment of the symbolic calculus leads to the setting up of tables of correspondence between images and originals.

If one treats a mechanical problem, one must, in general, determine an unknown function of time $f(t)$. The statement of the problem permits the writing down of one or several relations between this unknown function $f(t)$ and the given data of the problem.

The symbolic method permits the solution of such a problem by means of three successive operations.

The first consists in translating the equation which defines the devised function $f(t)$ into symbolic language.

After this translation has taken place, the equation which translates the phenomenon into symbolic notation does no longer contain the variable t. The latter is replaced everywhere by the variable p, and the function to be determined is replaced by a function $\varphi(p)$.

The second stage then consists in determining $\varphi(p)$.

Finally, after the form $\varphi(p)$ has been found, one must proceed to the third phase, that is, to find the original $f(t)$ which corresponds to it which can be done either with the aid of a table of correspondence established once for all, or with the aid of the Bromwich-Mellin formula, if the function $\varphi(p)$ and its original are not indicated in the tables.

Since the purpose of the operational calculus is to simplify the calculations, its employment is justified only if the determination of $\varphi(p)$ is easier than that of $f(t)$.

This is not necessarily the case but does occur always, if the function $f(t)$ is a sum of exponentials.

Since the functions $f(t)$ which are of interest to us are precisely of this form, the operational calculus is a valuable expedient in the investigation of the dynamics of an airplane.

The use of Bromwich's formula requires a thorough acquaintance with the theory of analytical functions and the practice of manipulating complex integrals.

It is generally not necessary to use this formula for the return to the original. This return is achieved either by direct utilization of the tables of correspondence, or - and this happens in the majority of cases - by application of Heaviside's formula which we shall demonstrate further on, without resorting to Bromwich's formula, by the simple use of the table of correspondence.

Important remark: Carson's transformation (with its inverse formula) does not constitute the only transformation one can visualize.

The relationship

$$F(p) = \int_0^\infty f(t)e^{-pt}dt$$

defines a transformation different from the preceding one, namely Laplace's transformation.

Its inverse formula

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}F(p)dp$$

is Cauchy's transformation

One notes that the formulas of Laplace and of Cauchy differ from the formulas of Carson and Bromwich by a factor p.

one has

$$f'(t) \supset \left[p\ \varphi(p) - f(0)\right]$$

In order to show this, let us write

$$p\varphi(p) = p^2 \int_0^\infty e^{-pt} f(t)dt$$

Let us integrate the second member by parts

$$p\varphi(p) = \left[-pe^{-pt} f(t)\right]_0^\infty + p\int_0^\infty e^{-pt} f'(t)dt$$

The term between brackets is equal to $-pf(o)$.

Hence

$$p\varphi(p) - pf(0) = p\int_0^\infty e^{-pt} f'(t)dt$$

The second member is nothing else but Carson's transformation applied to the derivative $f'(t)$. One has therefore exactly

$$f'(t) \supset p\left[\varphi(p) - f(0)\right]$$

This formula expresses the essential property of the transformation of Carson: The operation of the differentiation of $f(t)$ is reduced to the multiplication of the image by p; however, this product is diminished by $pf(o)$.

6th) Integration:

If $f(t) \supset \varphi(p)$

$$\int_0^t f(t)dt = \frac{\varphi(p)}{p}$$

The image of $\int_0^t f(t)dt$ is written by definition

$$p\int_0^\infty e^{-pt}\left(\int_0^t f(t)dt\right)dt$$

or else

$$p\int_0^t f(t)dt \int_0^\infty e^{-pt}dt$$

Let us integrate by parts. We obtain

$$\left[-e^{-pt}\int_0^t f(t)dt\right]_0^\infty + \int_0^t e^{-pt}f(t)dt$$

The first term is zero. The second term is

$$\frac{\varphi(p)}{p}$$

The integration of $f(t)dt$ corresponds to the division of the image by p.

The two preceding properties: differentiation and integration, are the ones in which the entire interest of the operational calculus centers because it permits reducing the integration of a differential equation to an algebraic calculation.

4. Operational Table

The calculations are, in general, remarkably simple.

Let us indicate the following relationships:

Original	Image
t^n	$p^{-n}n!$
e^t	$\frac{p}{p - 1}$
e^{at}	$\frac{p}{p - a}$
$e^{at} - 1$	$\frac{a}{p - a}$
$\cos \omega t$	$\frac{p^2}{p^2 + \omega^2}$
$\sin \omega t$	$\frac{\omega^2}{p^2 + \omega^2}$

5. Value of the Function For a Negative Time

The problem which we pose consists in predicting the behavior of a system as the consequence of the modification of one of the factors of equilibrium.

Generally, this modification starts at the instant $t = 0$ and up to that time the system was in equilibrium.

If the variables of the problem are the perturbations about the position of equilibrium, they are functions of the time, but functions of a particular type - because they are, by hypothesis, zero for $t < 0$ and start varying only at $t = 0$.

This particular characteristic can be expressed in several ways - either by admitting explicitly that all functions of time investigated will be zero for $t < 0$ - or by assuming that the expression describing the variation of the variables contains as a factor Heaviside's unit function; this function has the value

$$0 \quad \text{for} \quad t < 0$$

$$1 \quad \text{for} \quad t > 0$$

We do not intend to study the theoretical consequences of this concept.

It suffices to point out that Carson's transformation applied to the function

$$f(t) = 0 \quad \text{for} \quad t < 0$$

$$f(t) = 1 \quad \text{for} \quad t > 0$$

gives

$$\varphi(p) = 1$$

A function $f(t)$ which satisfies the preceding condition (value zero for $t = 0$) possesses an important property if one displaces the origin of the time.

Let us find the image of $f(t - s)$, that is, the image of the function displaced by the quantity s with respect to t.

Assume

$$\varphi(p) = p \int_0^{\infty} e^{-pt} f(t)dt$$

We want to calculate

$$\varphi_1(p) = p \int_0^{\infty} e^{-pt} f(t - s)dt$$

Let us put

$$\varphi_1(p) = pe^{-ps} \int_{-s}^{\infty} e^{-pt_1} f(t_1) dt_1$$

Since $f(t)$ is zero for the negative values of t:

$$\int_{-s}^{\infty} = \int_{0}^{\infty}$$

whence

$$\varphi_1(p) = e^{-ps}\varphi(p)$$

or else

$$e^{-ps}\varphi(p) \subset f(t - s)$$

The operator e^{-ps}, multiplying a function $\varphi(p)$, is equivalent to a displacement of the variable t by the quantity s.

6. Application of the Operational Calculus to the Solution of Linear Differential Equations

The operational calculus permits rapid solution of systems of linear differential equations.

Assume a system of four equations, in four dependent variables x, y, z, s, and the independent variable t.

$$\frac{dx}{dt} + a_1 x + b_1 y + c_1 z + d_1 s = h_1$$

$$\frac{dy}{dt} + a_2x + b_2y + c_2z + d_2s = h_2$$

$$\frac{dz}{dt} + a_3x + b_3y + c_3z + d_3s = h_3$$

$$\frac{ds}{dt} + a_4x + b_4y + c_4z + d_4s = h_4$$

The quantities $h_1 \dots h_4$ appearing in the second member are constants (case of the equations encountered previously).

The integration of the system has the purpose of determining four unknown functions of time

$$x = F_1(t)$$

$$y = F_2(t)$$

$$z = F_3(t)$$

$$s = F_4(t)$$

Let us write that

$F_1(t)$ is the original of an unknown function $\xi(p)$

$F_2(t)$ is the original of an unknown function $\eta(p)$

$F_3(t)$ is the original of an unknown function $\zeta(p)$

$F_4(t)$ is the original of an unknown function $\sigma(p)$

One has therefore

$$x \supset \xi$$
$$y \supset \eta$$
$$z \supset \zeta$$
$$s \supset \sigma$$

according to what has been said previously.

$$\frac{dx}{dt} \supset p\xi - px_0$$

$$\frac{dy}{dt} \supset p\eta - py_0$$

$$\frac{dz}{dt} \supset p\zeta - pz_0$$

$$\frac{ds}{dt} \supset p\sigma - ps_0$$

where x_0, y_0, z_0, s_0 are the values of the functions x, y, z, s at the time $t = 0$, that is, the initial conditions.

Let us write the equations to be integrated by going to the images

$$\begin{array}{lllll}
(a_1 + p)\xi & + b_1\eta & + c_1\zeta & + d_1\sigma & = h_1 + px_0 \\
a_2\xi & + (b_2 + p)\eta & + c_2\zeta & + d_2\sigma & = h_2 + py_0 \\
a_3\xi & + b_3\eta & + (c_3 + p)\zeta & + d_3\sigma & = h_3 + pz_0 \\
a_4\xi & + b_4\eta & + c_4\zeta & + (d_4 + p)\sigma & = h_4 + ps_0
\end{array}$$

It suffices to solve the algebraic equations for finding the images ξ, η, ζ, and σ as function of the initial values x_0, y_0, z_0, s_0 and of the constants h_1, h_2, h_3, and h_4.

The introduction of the initial values x_0, y_0, z_0, s_0 is equivalent to the determination of the constants of integration because the latter are - in the conventional methods - determined by introducing into the solutions the initial values corresponding to the time $t = 0$.

The solution of the algebraic equations by determinants gives expressions such as

$$\xi = \frac{\begin{vmatrix} h_1 + px_0 & b_1 & c_1 & d_1 \\ h_2 + py_0 & b_2 + p & c_2 & d_2 \\ h_3 + pz_0 & b_3 & c_3 + p & d_3 \\ h_4 + ps_0 & b_4 & c_4 & d_4 + p \end{vmatrix}}{\begin{vmatrix} a_1 + p & b_1 & c_1 & d_1 \\ a_2 & b_2 + p & c_2 & d_2 \\ a_3 & b_3 & c_3 + p & d_3 \\ a_4 & b_4 & c_4 & d_4 + p \end{vmatrix}}$$

Let us call $Z(p)$ the denominator. This polynomial in p is identical to the characteristic equations in λ written in chapters VII and IX; it is sufficient to replace λ by p.

Let us designate the minors of the numerator by

$$H_{1,1}(p) \qquad H_{2,1}(p) \qquad H_{3,1}(p) \qquad H_{4,1}(p)$$

with the first subscript designating the suppressed line; with the second subscript designating the suppressed column.

We obtain the following symbolic expression

$$\xi = (h_1 + px_0)\,\frac{H_{1,1}(p)}{Z(p)} + (h_2 + py_0)\,\frac{H_{2,1}(p)}{Z(p)} + (h_3 + pz_0)\,\frac{H_{3,1}(p)}{Z(p)} + (h_4 + ps_0)\,\frac{H_{4,1}(p)}{Z(p)}$$

it remains to find the original of this expression.

7. Heaviside's Formula

The theoretical treatises solve the question by application of the formula of Bromwich-Mellin. However, there exists a method which permits finding directly the expression

$$F_1(t) \supset \xi(p)$$

when the numerators are polynomials of a degree lower than or equal to the polynomial of the denominator.

The expressions we have to deal with satisfy precisely this restriction.

Let us limit ourselves to the study of one single term

$$\xi = h_1 \frac{H_{1,1}(p)}{Z(p)} + x_0 \frac{pH_{1,1}(p)}{Z(p)}$$

and let us, in order to simplify the notation, provisionally drop the double subscript of H.

When the equation $Z(p) = 0$ does not possess multiple roots (the usual case in practice), the rational fraction may be decomposed by purely algebraic means, in a sum

$$\frac{H(p)}{Z(p)} = \sum_1^n \frac{H(\lambda)}{Z'(\lambda)} \frac{1}{(p - \lambda)}$$

with λ designating the roots of the equation $Z(p) = 0$.

The image ξ may consequently be represented by

$$\xi = h_1 \sum_1^n \frac{H(\lambda)}{\lambda Z'(\lambda)} \frac{\lambda}{p - \lambda} + x_0 \sum_1^n \frac{H(\lambda)}{Z'(\lambda)} \frac{p}{p - \lambda}$$

Each of the terms which constitute the second member is a product of which only the factors

$$\frac{\lambda}{p - \lambda} \text{ and } \frac{p}{p - \lambda}$$

are functions of p.

These two quantities appear in the table of correspondence

$$\frac{\lambda}{p - \lambda} \subset e^{\lambda t} - 1$$

$$\frac{p}{p - \lambda} \subset e^{\lambda t}$$

which permits to immediately come back to the original x

$$x = F_1(t) = h_1 \sum_1^n \frac{H(\lambda)}{\lambda Z'(\lambda)} \left(e^{\lambda t} - 1\right) + x_0 \sum_1^n \frac{H(\lambda)}{Z'(\lambda)} e^{\lambda t}$$

We can transform the solution, noting that, if we make $p = 0$ in the expression $H(p)/Z(p)$, we obtain

$$\frac{H(0)}{Z(0)} = -\sum_1^n \frac{H(\lambda)}{\lambda Z'(\lambda)}$$

This permits, in fact, writing, again making use of the subscript for H:

$$x = F_1(t) = h_1 \left[\sum_1^n \frac{H_{1,1}(\lambda)}{\lambda Z'(\lambda)} e^{\lambda t} + \frac{H_{1,1}(0)}{Z(0)}\right] + x_0 \sum_1^n \frac{H_{1,1}(\lambda)}{Z'(\lambda)} e^{\lambda t}$$

the expression known as "Heaviside's formula." It is generally written in the particular case where $x_0 = 0$.

The Σ, from 1 to 4, affects the four roots λ but does not apply to the subscripts of the minors.

The complete solution x will contain also analogous terms in h_2 and y_0, with the minor to be used $H_{2,1}$

h_3	z_0	$H_{3,1}$
h_4	s_0	$H_{4,1}$

The solutions in y, z, and s will be obtained in the same manner; the second subscript of the minors H will become, respectively, 2, 3, or 4.

The differential system the solution of which we have studied just now is identical with the systems we had established previously for investigating:

(a) The response of the airplane to an initial perturbation which occurs suddenly at the time $t = 0$, under the assumption that the airplane is flying with controls fixed

(b) The response of the airplane to an abrupt displacement of a control surface carried out at the time $t = 0$

(c) The response of an automatic control system to command, of constant amplitude, applied at $t = 0$

In the first problem, the h are zero, but at least one of the quantities x_0, y_0, z_0 or s_0 is different from zero.

In the second problem, at least one of the expressions h is different from zero. The x_0, y_0, z_0 and s_0 are zero unless one superposes the first and the second problem.

In the third problem, only the expression h_1 is different from zero.

It is clear that the application of Heaviside's formula works rapidly when the n roots of $Z(p) = 0$ are real.

8. Trigonometric Transformation of Heaviside's Formula

When one has to deal with imaginary roots, the calculations are longer and it will be of advantage to use graphical constructions.

A complex root may be written

$$\lambda = \kappa + i\sigma = Re^{ir} = R \cos r + iR \sin r$$

hence

$$\frac{H(\lambda)}{\lambda Z'(\lambda)} = \frac{H\left(Re^{ir}\right)}{Re^{ir}Z'\left(Re^{ir}\right)}$$

the polynominal $H\,Re^{ir}$ is a complex quantity which can easily be represented vectorially. We shall denote this quantity by He^{ih} where H is the modulus, h the argument.

Likewise, $Z'\,Re^{ir}$ will be designated by $Z'e^{iz'}$ where Z' is the modulus and z' the argument.

We obtain therefore

$$\frac{H(\lambda)}{\lambda Z'(\lambda)} = \frac{H}{RZ'}\, e^{ir(h-r-z')}$$

We shall put

$$\frac{H}{RZ'} = M$$

$$h - r - z' = \theta$$

Then

$$\frac{H(\lambda)}{\lambda Z'(\lambda)}\, e^{\lambda t} = Me^{i\theta}e^{Re^{ir}t}$$

and, for the pair of complex roots

$$\sum_1^2 \frac{H(\lambda)}{\lambda Z'(\lambda)}\, e^{\lambda t} = M\left[e^{\left(i\theta + Re^{ir}t\right)} + e^{\left(-i\theta + Re^{-ir}t\right)}\right]$$

which leads, after a few calculations, to the simple expression

$$\sum_1^2 \frac{H(\lambda)}{\lambda Z'(\lambda)} e^{\lambda t} = 2Me^{\kappa t} \cos(\sigma t + \theta)$$

Let us write, in order to recapitulate this calculation, the response $x = \delta u$ to a deflection of the elevator, producing $h_1 = h_2 = h_4 = 0$ and $h_3 = 1$, applied at an instant $t = 0$ where no other perturbation pertains.

This response is, when one has at the same time real roots and complex roots

$$x = \frac{H_{3,1}(0)}{Z(0)} + \Sigma \frac{H_{3,1}(\lambda)}{\lambda Z'(\lambda)} e^{\lambda t} + \Sigma 2Me^{\kappa t} \cos(\sigma t + \theta)$$

$\dfrac{H_{3,1}(\lambda)}{Z'(\lambda)}$ is the transient damped response corresponding to the real roots of $Z(p) = 0$

$\dfrac{H_{3,1}(0)}{Z'(0)}$ is the response at infinity, that is, the change in steady state imposed by the manipulation of the control

$2Me^{\kappa t}\cos(\sigma t + \theta)$ is the transient oscillatory response which corresponds to the imaginary roots

The sign Σ is applied to pairs of conjugate roots: for one pair of conjugate roots, the Σ is not needed.

Let us note, as a conclusion, that the method of solution we have just described does not contribute a single element not already contained in the general method described in chapter VI; however, the execution of the method above is infinitely more rapid.

9. Theorem of the Product

Assume $\varphi_1(p)$ and $\varphi_2(p)$ to be the images of the functions $f_1(s)$ and $f_2(t)$

$$\varphi_1(p) = p \int_0^{\infty} e^{-ps} f_1(s) ds$$

$$\varphi_2(p) = p \int_0^{\infty} e^{-pt} f_2(t) dt$$

There then exist between the products of the functions of s $f_1(s)$ and $f_2(t - s)$ and the products of the images $\varphi_1(p)$ and $\varphi_2(p)$ the relationships

$$\frac{1}{p} \varphi_1(p)\varphi_2(p) \subset \int_0^t f_1(s) f_2(t - s) ds$$

and

$$\frac{1}{p} \varphi_1(p)\varphi_2(p) \subset \int_0^t f_1(t - s) f_2(s) ds$$

From

$$\varphi_2(p) \subset f_2(t)$$

we extract, for $t > s$ (by displacement of the origin)

$$e^{-ps}\varphi_2(p) \subset f_2(t - s)$$

Let us multiply the two members by $f_1(s)$

$$e^{-ps} f_1(s)\varphi_2(p) \subset f_1(s) f_2(t - s)$$

Let us integrate for the variable s

$$\int_0^\infty e^{-ps} f_1(s)\varphi_2(p)ds \subset \int_0^\infty f_1(s)f_2(t - s)ds$$

$$\frac{1}{p}\,\varphi_1(p)\varphi_2(p) \subset \int_0^\infty f_1(s)f_2(t - s)ds$$

When s is larger than t, the quantity $(t - s)$ represents a negative time, and the function f_2 is zero for these values of the variable. Hence there results that

$$\int_0^\infty f_1(s)f_2(t - s)ds = \int_0^t f_1(s)f_2(t - s)ds$$

and the first formula is therewith demonstrated.

The second will be demonstrated in the same manner.

10. Image of Duhamel's Integral

Let us write the preceding relationship, replacing f_2 by F_1 and designating by Φ_2 the image of the following function

$$\phi_2(p) \subset F(t)$$

Let us write, moreover, that the function $f_1(s)$ is the derivative of a function $f(s)$

$$f_1(s) = \frac{df(s)}{ds} = f'(s)$$

Let us put

$$\phi_1(p) \subset f(s)$$

We have, by definition

$$\varphi_1(p) \subset f'(s)$$

on the other hand, by virtue of the rule of differentaiation

$$p\left[\phi_1(p) - f(0)\right] \subset f'(s)$$

whence

$$\varphi_1(p) = p\left[\phi\ (p) - f(0)\right]$$

Substituting this into the first formula of the product:

$$\left[\phi_1(p) - f(0)\right]\phi_2(p) \subset \int_0^t f'(s)F(t - s)ds$$

however

$$f(0)\phi_2(p) \subset f(0)F(t)$$

Let us add, term by term

$$\phi_1(p)\phi_2(p) \subset f(0)F(t) + \int_0^t f'(s)F(t - s)ds$$

The second term is nothing else but Duhamel's integral which gives us the response of a system to an arbitrary perturbation $f(s)$ which is variable as a function of the response F to a constant perturbation equal to unity.

Thus we see that the image of the response to an arbitrary perturbation is equal to the product of the image of the perturbation and the image of the response to the unit perturbation.

The symbolic representation of Duhamel's integral considerably simplifies this expression.

11. Application of Duhamel's Integral to the Study of Automatic Flight Control

Let us apply the methods of operational calculus to the solution of the following problem.

An airplane is provided with an apparatus which detects the perturbations of any arbitrary variable, for instance δu, and produces instantly a deflection of the elevator proportional to this deviation.

One has therefore

$$\delta\eta = +k\delta u$$

In practice, k will be negative in the case considered because one would choose such a direction for the maneuver that the control surface would tend to make the airplane nose down when the speed diminishes.

What is then the behavior of the variations δu, δw, δq, $\delta\theta$ following an arbitrary initial perturbation $(\delta n)_0$?

Solution of this problem is possible if one knows:

(a) The reaction of the airplane - which is supposed to fly with controls fixed - under the action of the initial perturbation

(b) The response of the airplane, under the action of a constant deflection $\delta\eta$ equal to unity

(c) The law $\delta\eta = k\delta u$ characterizing the automatic apparatus.

Let us write the equations, denoting, in a general manner, the time by t.

(a) The response of the airplane which is flying with controls fixed to the initial perturbation is

$$\begin{aligned} \delta u &= G_1(t) \\ \delta w &= G_2(t) \\ \delta q &= G_3(t) \\ \delta\theta &= G_4(t) \end{aligned}$$

These functions give rise to the images

$$\begin{aligned} \gamma_1(p) &\subset G_1(t) \\ \gamma_2(p) &\subset G_2(t) \\ \gamma_3(p) &\subset G_3(t) \\ \gamma_4(p) &\subset G_4(t) \end{aligned}$$

(b) The response of the airplane under the effect of a deflection $\eta = 1$ is

$$\begin{aligned} \delta u &= F_1(t) \\ \delta w &= F_2(t) \\ \delta q &= F_3(t) \\ \delta \theta &= F_4(t) \end{aligned}$$

These funtions give rise to the images

$$\begin{aligned} \varphi_1(p) &\subset F_1(t) \\ \varphi_2(p) &\subset F_2(t) \\ \varphi_3(p) &\subset F_3(t) \\ \varphi_4(p) &\subset F_4(t) \end{aligned}$$

The unknown motion

$$\begin{aligned} \delta u &= x(t) \\ \delta w &= z(t) \\ \delta q &= q(t) \\ \delta \theta &= \theta(t) \end{aligned}$$

is the superposition of the normal return motion of the airplane, after the initial perturbation, and of the response of the airplane under the action of a deflection which is at any instant proportional to the deviation δu.

The functions $x(t)$, $z(t)$, etc. have as derivatives $x'(t)$, $z'(t)$, etc., and as images[10]

$$\begin{aligned}\xi(p) &\subset x(t)\\ \zeta(p) &\subset z(t)\\ \chi(p) &\subset q(t)\\ \theta(p) &\subset \theta(t)\end{aligned}$$

The deflection η is, at any instant, given by

$$\eta = kx(t)$$

under the assumption that the piloting is a function of δu.

Duhamel's formula permits determination of the component of the motion which is due to the action of this variable deflection. If one superimposes the return motion of the airplane with controls fixed and the response under the action of the variable deflection, one obtains, at the instant t_b

$$x(t_b) = G_1(t_b) + F_1(t_b)x(0) + \int_0^{t_b} kF_1(t_b - t)\,x'(t)dt$$

$$z(t_b) = G_2(t_b) + F_2(t_b)x(0) + \int_0^{t_b} kF_2(t_b - t)\,x'(t)dt$$

$$q(t_b) = G_3(t_b) + F_3(t_b)x(0) + \int_0^{t_b} kF_3(t_b - t)\,x'(t)dt$$

If one assumes the integration of the second term to have been carried out, one may eliminate everywhere the subscripts b.

---

[10]One may represent the original and the image by the same letter if one indicates that the one is the function of t, and the other of p.

Going over to the images, taking into account the symbolic transposition of Duhamel's integral, one has

$$\begin{aligned}
\xi(p) &= \gamma_1(p) + k\varphi_1(p)\xi(p) \\
\zeta(p) &= \gamma_2(p) + k\varphi_2(p)\xi(p) \\
\chi(p) &= \gamma_3(p) + k\varphi_3(p)\xi(p) \\
\theta(p) &= \gamma_4(p) + k\varphi_4(p)\xi(p)
\end{aligned}$$

Hence there results that the desired functions are

$$\xi(p) = \frac{1 - k\gamma_1(p)}{\varphi_1(p)}$$

$$\zeta(p) = \frac{\gamma_2(p) + k\left[\varphi_2(p)\gamma_1(p) - \varphi_1(p)\gamma_2(p)\right]}{1 - k\varphi_1(p)}$$

The expressions of $\chi(p)$ and $\theta(p)$ will be obtained by substitution of the subscripts 3 or 4 for the subscripts 2.

If the automatic pilot were a function of the deviation $\delta\theta$, instead of the deviation δu, one would have

$$\eta = k\delta\theta$$

and the symbolic expression of the resultant motion would be

$$\begin{aligned}
\xi(p) &= \varphi_1(p) + k\varphi_1(p)\theta(p) \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
\theta(p) &= \varphi_4(p) + k\varphi_4(p)\theta(p)
\end{aligned}$$

which would give expressions such as

$$\zeta(p) = \frac{\gamma_1(p) + k\left[\varphi_1(p)\gamma_4(p) - \varphi_4(p)\gamma_1(p)\right]}{1 - k\varphi_4(p)}$$

....

....

$$\theta(p) = \frac{\varphi_4(p)}{1 - k\varphi_4(p)}$$

12. Usefulness of These Formulas

It seems at first sight that the employment of these symbolic expressions would necessitate the return to the original and therefore lead to long calculations.

We shall show in the following chapter that this is not the case if one examines sinusoidal perturbations.

Thus it is interesting to develop these formulas.

Let us take the general equations of the preceding chapter, replacing x, y, z, s by δu, δw, δq, $\delta\theta$, and writing the coefficients of the third equation in the form

$$\frac{c}{r^2}\, a_3 \qquad \frac{c}{r^2}\, b_3 \qquad \frac{cl}{r^2}\, c_3 \qquad \frac{cV}{r^2}\, d_3$$

and the independent term in the form

$$\frac{c}{r^2}\, Vh_3$$

In the most general case the initial perturbation which we assume to be arbitrary can be considered as a sum

$$\delta n = (\delta u)_0 + (\delta w)_0 + (\delta q)_0 + (\delta\theta)_0$$

Hence, representing the principal determinant by $Z(p)$

$$Z(p) = \begin{vmatrix} a_1 + p & b_1 & c_1 & d_1 \\ a_2 & b_2 + p & c_2 & d_2 \\ \frac{c}{r^2}\, a_3 & \frac{c}{r^2}\, b_3 & \frac{cl}{r^2}\, c_3 + p & \frac{cV}{r^2}\, d_3 \\ a_4 & b_4 & c_4 & d_4 + p \end{vmatrix}$$

the images of the response to an initial perturbation are

$$\gamma_1(p) = \begin{vmatrix} p(\delta u)_0 & b_1 & c_1 & d_1 \\ p(\delta w)_0 & b_2 + p & c_2 & d_2 \\ p(\delta q)_0 & \frac{c}{r^2}\, b_3 & \frac{cl}{r^2}\, c_3 + p & \frac{cV}{r^2}\, d_3 \\ p(\delta \theta)_0 & b_4 & c_4 & d_4 + p \end{vmatrix} \times \frac{1}{Z(p)}$$

$$\gamma_2(p) = \begin{vmatrix} a_1 + p & p(\delta u)_0 & c_1 & d_1 \\ a_2 & p(\delta w)_0 & c_2 & d_2 \\ \frac{c}{r^2}\, a_3 & p(\delta q)_0 & \frac{cl}{r^2}\, c_3 + p & \frac{cV}{r^2}\, d_3 \\ a_4 & p(\delta \theta)_0 & c_4 & d_4 + p \end{vmatrix} \times \frac{1}{Z(p)}$$

and so forth in the same manner.

The symbolic response to the deflection, $\eta = 1$, will be given by the following expressions in which the term $h_3 = dC_M/d\eta$.

$$\varphi_1(p) = \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ 0 & b_2 + p & c_2 & d_2 \\ \frac{c}{r^2}\, Vh_3 & \frac{c}{r^2}\, b_3 & \frac{cl}{r^2}\, c_3 + p & \frac{cV}{r^2}\, d_3 \\ 0 & b_4 & c_4 & d_4 + p \end{vmatrix} \times \frac{1}{Z(p)}$$

$$\varphi_2(p) = \begin{vmatrix} a_1 + p & 0 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ \frac{c}{r^2}\, a_3 & \frac{c}{r^2}\, Vh_3 & \frac{cl}{r^2}\, c_3 + p & \frac{cV}{r^2}\, d_3 \\ a_4 & 0 & c_4 & d_4 + p \end{vmatrix} \times \frac{1}{Z(p)}$$

and so forth in the same manner.

In all these determinants we have:

$d_3 = 0$

$a_4 = b_4 = d_4 = 0$ (see the reasons above)

For simplification, we shall content ourselves to search for the response to an initial perturbation $(\delta w)_0$.

We assume therefore: $(\delta u)_0 = (\delta q)_0 = (\delta\theta)_0 = 0$

Let us again denote by H the minors of the determinant Z, with the first subscript designating the suppressed line, the second designating the suppressed column.

The functions γ are of the form

$$\gamma_1(p) = \frac{p(\delta w)_0 H_{2,1}(p)}{Z(p)}$$

$$\gamma_2(p) = \frac{p(\delta w)_0 H_{2,2}(p)}{Z(p)}$$

The functions φ are of the form

$$\varphi_1(p) = \frac{\frac{cV}{r^2} h_3H_{3,1}(p)}{Z(p)}$$

$$\varphi_2(p) = \frac{\frac{cV}{r^2} h_3H_{3,2}(p)}{Z(p)}$$

Let us assume that the automatic pilot is sensitive to the perturbation in trim $\delta\theta$.

One obtains expressions such as

$$\xi(p) = p(\delta w)_0 \frac{H_{2,1}(p) + k \frac{cV}{r^2} h_3 \left[H_{3,1}(p)H_{2,4}(p) - H_{3,4}(p)H_{2,1}(p)\right] \frac{1}{Z(p)}}{Z(p) - k \frac{cV}{r^2} H_3H_{3,4}(p)}$$

....

....

$$\theta(p) = p(\delta w)_C \frac{H_{2,4}(p)}{Z(p) - k \frac{cV}{r^2} h_3H_{3,4}(p)}$$

All these expressions will have the same denominator.

This denominator is nothing else but the characteristic determinant of the motion of the airplane flying with controls fixed, plus the term

$$-k \frac{cV}{r^2} h_3H_{3,4}(p)$$

If we now examine the development of the determinant $Z(p)$ as a function of the terms of the third line, we find that this development contains a term

$$\frac{cV}{r^2} d_3H_{3,4}(p)$$

We see therefore that the periods and the damping of the motion investigated will be determined by the roots in λ of the characteristic determinant in which one will have replaced

$$d_3 \text{ by } d_3 - kh_3$$

Since

$$h_3 = \partial C_M / \partial \eta$$

$$k = d\eta / d\theta$$

everything occurs as if the term in d_3 of the characteristic determinant became

$$d_3 - \frac{\partial C_M}{\partial \eta} \frac{\partial \eta}{\partial \theta}$$

with the mechanical connections imposing the sign of $\partial\eta/\partial\theta$ which will, in fact, be negative.

The application of the operational calculus justifies the method of calculation which we have used in chapter XV for the calculation of the period and of the damping of the motion of an airplane provided with an automatic pilot operating without lag.

CHAPTER XIX

FREQUENCY RESPONSE

1. Definition

Let us examine the general properties of the oscillating systems.

Let us assume two quantities x and z, connected with one another by a mechanical system. These quantities are functions of time.

The quantity x is the command or input signal.

The quantity z is the response or output signal.

The command x varies as a function of time, following a law of input

$$x = f_1(t)$$

The response z is determined by the law of output

$$z = f_2(t)$$

An airplane constitutes a particular case of such a mechanical system. The deflection of a control (for instance of the elevator η) constitutes the input signal x. Each one of the quantities which define the motion of the airplane - especially the variables u, w, θ - constitutes an output signal. The entrained motions of the surrounding medium also constitute input signals; the vertical gusts are identical to initial perturbations δw, the horizontal gusts are identical to initial perturbations δu.

Another special case is presented to us by the servocontrols. In such an arrangement the displacement of a control constitutes the output signal; it is a function of an input signal which can be either a command given by the pilot; or the indication given by a detector of perturbations.

In this last case, if one chooses x as input signal, the difference δn existing between the instantaneous value n of a variable and its steady-state value $\bar{n}$ will be

$$\delta n = n - \bar{n}$$

and if one adopts such a direction of deflection that the control imparts to the airplane a moment which tends to reduce δn, the arrangement realized becomes an automatic pilot.

In the previous chapters we investigated the methods which give us the response of the system for an input signal changing abruptly from zero to a constant value. If the system is stable, this response is a transient motion: namely the motion by which the system passes from the position of initial equilibrium to the position of final equilibrium.

It will be of advantage to study the effect of an input signal equal to unity.

We have remarked on the existence of Duhamel's integral which permits calculation of the motion of the system caused by an input signal constituting an arbitrary function of time, if one knows the response to the abruptly applied unit signal.

There exists another means for studying the systems considered. This means consists in determining the effect of an input signal, assumed to be zero for $t < 0$ but constituting a sinusoidal function of time for $t > 0$:

$$x = x_m \sin \omega t$$

When a system is subjected to such an excitation, a transient motion is established at the beginning of the phenomenon, but it disappears gradually, and the motion tends toward a steady state which is constituted by a sinusoidal motion of the same period but different amplitude and phase.

The system undergoes a forced oscillation, of the same period as the excitation. This oscillation is the frequency response.

2. Calculation of the Frequency Response

Duhamel's integral is a general formula. Solved for a sufficiently large time t_b, it furnishes the characteristics of the steady motion.

Let us apply the method to the calculation of the frequency response of an airplane subjected to a deflection η varying according to a sinusoidal law

$$\eta = \eta_m \sin \omega t = f(t)$$

where η_m represents the amplitude of the deflection, ω the circular frequency of the excitation.

Let us write the response δu to the displacement $\Delta\eta = 1$ in its sinusoidal form

$$F(t) = \Delta u + e^{kt}A_u \sin(st + \varphi) + e^{k't}A'_u \sin(s't + \varphi')$$

and Duhamel's integral in its form

$$\delta u_b = f(0)F(t_b) + \int_0^{t_b} f'(t_b - t)F(t)dt$$

Since $f(0) = 0$, we obtain

$$\delta u_b = \eta_m \int_0^{t_b} - \omega \cos(t_b - t)\left[\Delta u + \Sigma A_u e^{kt} \sin(st + \varphi)\right]dt$$

The only part of the integral of interest to us is the one corresponding to a very large t_b.

The solution of this integral appears in the appendix.

The calculations lead to an expression of the form

$$\delta u_b = -\left(C \sin \omega t_b + D \cos \omega t_b\right)\eta_m$$

with

$$C = \Delta u + \Sigma A_u \frac{\omega^2 \cos \varphi_u}{k^2 + (s + \omega)^2}$$

$$D = \Sigma A_u \frac{s\omega \sin \varphi_u - k\omega \cos \varphi_u}{k^2 + (s + \omega)^2}$$

The factors A and B which define the frequency response are functions:

(a) Of the circular frequency ω of the excitation

(b) Of the characteristics of the response of the airplane under the effect of the deflection $\Delta\eta = 1$, applied abruptly.

3. Complex Expression of the Frequency Response

The preceding method supposes that the characteristics A_u, φ_u, s, and t of the response to the unit signal have been determined. This calculation is lengthy and it requires especially the numerical solution of the characteristic equation.

It is possible to determine the steady-state part of the response in a more rapid manner, aside from Duhamel's integral.

Let be:

R_b the response at the time t_b

$F(t)$ the response to the unit perturbation

$f(t)$ a sinusoidal excitation

$$f(t) = \eta_m \sin \omega t = \eta_m \frac{e^{i\omega t} - e^{-i\omega t}}{2}$$

The conventions usually agreed upon for the representation of sinusoidal motions by rotating vectors permit writing symbolically:

$$\sin \omega t = e^{i\omega t}$$

an expression which we shall utilize below. Besides, it would be sufficient to make the calculation complete by using the two exponentials in order to find the complete analytical expression of the result at which we shall arrive.

Let us this time write Duhamel's integral as follows:

$$R_b = f(0)F\ t_b + \int_0^{t_b} f'(t)F(t_b -)dt$$

There follows necessarily

$$f(0) = 0$$

Let us temporarily omit the factor η_m, that is to say, let us suppose an exciting motion of unit amplitude.

$$R_b = \int_0^{t_b} i\omega e^{i\omega t} F(t_b - t)dt$$

Let us put $t = t_b - s$

whence

$$dt = -ds$$

$$t_b - t = s$$

Let us replace t by s and take the new limits into account. We obtain

$$R_b = \int_{t_b}^{0} - i\omega e^{i\omega(t_b - s)} F(s)ds$$

$$= i\omega e^{i\omega t_b} \int_0^{t_b} e^{-i\omega s} F(s)ds$$

The integral

$$\int_0^{t_b} = \int_0^{\infty} - \int_{t_b}^{\infty}$$

and $\int_{t_b}^{\infty}$ is zero because the transient motions are regarded as having ended within t_b. (We concern ourselves only with the steady-state response.)

One has therefore

$$R_b = i\omega e^{i\omega t_b} \int_0^{\infty} e^{-\omega s} F(s)ds$$

F(s) is, by definition, the response to the unit excitation. Its Carson transformation is

$$\varphi(p) = p \int_0^{\infty} e^{-ps} F(s)ds$$

Let us replace p by $i\omega$

$$\varphi(i\omega) = i\omega \int_0^{\infty} e^{-i\omega s} F(s)ds$$

As a result,

$$R_b = \varphi(i\omega)e^{i\omega t_b}$$

and we see that the original of the steady-state part of the response to a sinusoidal perturbation is obtained from the image of the response to a unit perturbation, substituting $i\omega$ for the operator p, and multiplying by $e^{i\omega t}$.

This expression represents a sinusoidal motion of the frequency ω and the complex amplitude $\varphi(i\omega)$.

When the phenomenon investigated is represented by linear equations, the image of the response is given by a quotient of polynomials in p. By substitution of $i\omega$ for p one will find the complex amplitude, that is to say, the magnitude and the phase displacement of the response.

This result is obtained by elementary calculations or constructions which no longer necessitate the finding of the roots of the characteristic equation.

Assume that one has to find the response

$$\delta u = x(t)$$

of an airplane (not provided with an automatic pilot) to a sinusoidal excitation consisting:

of a motion, of amplitude η, of the elevator

or of variations, of amplitude n, in the entrained velocities of the surrounding medium (atmospheric swell).

The response to an abrupt perturbation, in symbolic notation, is of the form

$$\xi(p) = \eta \frac{H(p)}{Z(p)}$$

or

$$\xi(p) = np \frac{H(p)}{Z(p)}$$

according to whether it is a matter of the one or the other case.

The minor H must be affected by the desired subscripts (see section 12 of the previous chapter) and incorporates, if it takes place, the constant factors.

In view of what was said above, the complex amplitude of the original of the response is obtained by replacement of p by $i\omega$. It is written

$$\eta \frac{H(i\omega)}{Z(i\omega)} \quad \text{or} \quad n \frac{i\omega H(i\omega)}{Z(i\omega)}$$

is always of the form

$$\frac{X_n + iY_n}{X_d + iY_d}$$

where X_n, Y_n, X_d, Y_d are polynomials in ω.

The amplitude or the modulus M of the response is

$$M = \frac{\sqrt{X_n^2 + Y_n^2}}{\sqrt{X_d^2 + Y_d^2}} = f_1(\omega)$$

The phase displacement of the response with respect to the excitation is

$$\varphi = \frac{1}{tg\left(\frac{Y_n}{X_n}\right)} - \frac{1}{tg\ \frac{Y_d}{X_d}} = f_2(\omega)$$

An identical reasoning permits the calculation of the frequency response of a linear automatic control system.

For command $x = 1$, the symbolic response in z is

$$\zeta = \frac{H(p)}{Z(p)}$$

where $H(p)$ is the minor

$$\begin{vmatrix} a_2 & b_2 + p & C_2 \\ 0 & b_3 & C_3 + p \\ 0 & 0 & -1 \end{vmatrix}$$

corresponding to the fourth term of the first line (fourth variable calculated - unit action applied in the first equation only) in such a manner that

$$\zeta = \frac{a_2 b_3}{Z(p)}$$

and the complex amplitude of the response is nothing else but $Z(i\omega)$.

4. The Transfer Function

The expression $H(p)/Z(p) = Y(p)$ in which one substitutes afterwards $p = i\omega$ characterizes an oscillatory system which transforms an input signal - the deflection of a control - into an output signal: perturbation of one of the variables defining the motion of the airplane.

The expression $Y(p)$ is called: "transfer function" by the American authors, "admittance" by the French authors.

If the input signal were a perturbation of the surrounding medium, one would have

$$Y(p) = \frac{pH(p)}{Z(p)}$$

The concept of transfer function is extended to the case of automatic control mechanisms.

The inverse of the admittance

$$I(p) = \frac{1}{Y(p)}$$

is the "impedance" of the system.

5. Graphical Representation of the Frequency Response

The characteristics of the system subjected to sinusoidal excitation can be represented graphically, either by Cartesian diagrams giving the amplitude and the phase displacement as functions of the excitation ω, or by a polar diagram.

In this last case, the locus of the frequency response is the locus of the extremity of a vector the length of which is the ratio of the amplitudes of the response to the command, and which forms with the axis OX an angle representing the difference in phase. This locus is graduated according to the values of the frequency.

In order to trace it, it suffices to plot, on the plane of the complex variable, the admittance function $Y(i\omega)$.

The curve of admittance, traced on the plane of the complex variable, is actually nothing else but the curve of the frequency response, in polar representation.

Any curve of this nature presents the following characteristics:

For $\omega = 0$, the angle φ is zero. There is no phase displacement; the period of the input signal is so long that the system may be considered as being in a static state.

For $\omega = \infty$, the curve passes through the origin which means that, for an infinite frequency of excitation, the amplitude of the response of the system is zero.

For intermediate frequencies, the response vector lags with respect to the excitation.

One may visualize an inverse curve, called frequency-demand curve.

The locus of the frequency demand is the locus of the extremity of a vector the length of which is the ratio of the amplitudes of the command to the response, and which forms with OX an angle which is the difference in phase between the command and the response.

This locus is nothing else but the curve representing the impedance in the plane of the complex variable.

The frequency-demand curve is deduced from the curve of response by an inversion of the modulus 1 with respect to the origin, and a symmetry of the angles with respect to OX, with the demand leading the response.

When the demand vector is large, the system requires a large excitation in order to furnish a prescribed response.

Figure 45 represents the demand curve and the corresponding curve of response.

The natural frequencies of the system are those where resonant phenomena are produced which amplify the response. They correspond to the parts of the curve for which the amplitude of the vector goes through a maximum in a diagram of response, or through a minimum in a diagram of demand.

6. Equality of the Two Concepts

Knowledge of the transient motion of a system subjected to an abrupt unit excitation, and knowledge of the frequency response of the system realized in steady state by a sinusoidal excitation are equivalent.

The frequency response may be deduced from the response to the unit excitation, by means of Duhamel's integral. Inversely, the transient motion corresponding to the response to the unit excitation may be deduced from the frequency response by a Fourier series.

Let us replace the step function, that is

$$f(t) = 0 \quad \text{for} \quad t < 0$$

$$f(t) = 1 \quad \text{for} \quad t > 0$$

by a periodic function formed by a series of impulses $f(t) = 1$, of a duration $T/2$ each, separated by equal intervals $T/2$ during which $f(t) = 0$.

The period T will be chosen sufficiently large that after a time of application of the perturbation equal to $T/2$ the response to the impulse differs from its final value only by a negligible quantity.

One replaces therefore the continuous impulse by successive impluses each of which is applied for a sufficient length of time.

Any series of successive impulses may be represented by a series of the type

$$y = \frac{4k}{\pi}\left(\sin \omega t + \frac{1}{3} \sin 3 \omega t + \frac{1}{5} \sin 5 \omega t + . . .\right)$$

when a symmetrical function with respect to the t-axis is involved.

A change of the origin gives immediately the Fourier series, representing steps the successive values of which are +1 and 0.

$$v = \frac{1}{2} + \frac{2}{\pi}\left(\sin \omega t + \frac{1}{3} \sin 3 \omega t + \frac{1}{5} \sin 5 \omega t + . . .\right)$$

The frequency ω of the fundamental harmonic is linked to the period T (that is, to the duration of application $T/2$ of the excitation) by

$$\omega = 2\pi/T$$

The function represented by the series approaches the alternative unit function the more closely, the more one augments the number of harmonics.

The response to the unit excitation will be obtained by picking out, on the frequency-response diagram, the characteristics of the response to each of the harmonics and by adding these partial responses, taking the respective amplitudes into consideration.

7. Milliken's Experiments

The preceding material contains all the information which permits calculation of the frequency response curves of airplanes subjected to a sinusoidal excitation. If the input signal is a motion of the elevator, one can calculate the frequency response curve for each of the variables α, θ, V as well as for the functions of these variables, acceleration J_Z or angular velocity q.

The given data to be utilized are those we have encountered in the calculation of the dynamic lateral stability, but the solution of the characteristic equation is not necessary.

Tests have been made in the United States, at the Cornell Laboratory, for determining these curves in flight. Milliken applied, by means of a convenient modification of the automatic pilot, an alternating deflection to the elevator and recorded the effectively applied deflection η, as a function of time, and also the variation of the θ, α, J_Z considered as output signals.

We have plotted in a polar diagram the result obtained by Milliken for the ratio θ_m/η_m. The measurements were made by means of a twin-engined B-25J light bomber. Certain tests have been made using the same automatic pilot: the Sperry A12 at different flight velocities for the airplane.

The curves found vary according to the speed of the airplane, that is to say, according to its angle of attack. This is normal since the $a_1 \ldots c_3$ which determine the response of the airplane depend on the angle of attack.

Other tests have been made with successive use of different automatic pilots (the Honeywell C_1, Honeywell C_1A, and Sperry A12) but using the airplane at the same condition of flight velocity.

These tests have led to curves differing among themselves; since the pertinent known variables are the angle of trim θ and the angle of deflection η actually applied, one may be surprised, at a first glance, that the experimental results differ when the manner varies in which the displacement of the control surface has been produced.

This fact is explained, however, when one examines the automatic pilots used. Although one may hope that the Sperry will cause a more or less sinusoidal deflection when one of the elements of adjustment varies according to a specific law, this is not the case for the Honeywell. This apparatus acts in an on-and-off manner and is not linear. The deflection is controlled by a coupling which intermittently connects the control surface with a driving motor rotating at constant speed. It is obvious that such an apparatus is absolutely incapable of producing a sinusoidal motion of the control surface. Besides, the recording diagram of η shows that the curve of variation in deflection is more nearly saw-tooth than sinusoidal. Thus the motion of excitation contains numerous harmonics, and the curve of the response is, under these conditions, the response to a motion much more complex than a purely sinusoidal motion.

We do not pursue here the theoretical developments which Milliken has given in his publications.

The calculation methods he used seem more primitive than those which we recommend.

Milliken introduced in his developments especially the hypothesis of a constant V.

This means that he excluded systematically the influence of the long-period oscillation.

This is compatible with the experiments made (since the tests were carried out for an ω between 1 and 7 radians per second), but reduces the generality of the conclusions.

The concept of frequency response applies also to the lateral motion of the airplane, and tests aiming at the measurement of this characteristic have been performed.

However, the results have not been published.

8. Automatic Control System Subjected to Sinusoidal Excitation

Let us again take up the investigation of automatic control systems and seek for their response to a sinusoidal excitation.

The arrangements for automatic control always contain an element of return or of feedback, reproducing the response in z before the indicator communicating the command x to the apparatus.

For the apparatus studied in chapter XIV, the equality of the response to the command (ultimately defined, except for a factor of proportionality) is obtained because the slider 3 catching up with the slider 2 stops the control motor.

In fact, one subtracts the output z from the input command x; the amplification circuit and the servomotor are actuated by the difference $x - z$.

Any automatic control system containing a means of return:

(a) Constitutes a closed-loop system

(b) May be regarded as a system with negative feedback.

Let us note explicitly that such a device exists in the Sperry A_3; the displacement of the blocking device which is governed by a displacement of the control surface reestablishes the equality of pressure on the two faces of the membrane when the deflection η has taken on the value imposed as a function of θ. In short, it is the kinematic linkages making up the connecting elements between the control surface and the instrument case which determine the magnitude of $\delta\eta/\delta\theta = k$.

We shall show in what follows that one can study the frequency response and determine the transfer function of an automatic control system which constitutes a closed-loop system in two different ways.

The first procedure consists in utilizing the system of equations set up in chapter XIV, to send a sinusoidal input signal x, and to find the output signal z, either by calculating the transfer function of the closed-loop system as it actually is, or by making experiments with the system.

In the course of such an operation the amplifying circuit and the servomotor should constantly function under the action of the difference

$$\epsilon = x - z$$

Hence there results that there exists a second method of examination, consisting in cutting off the feedback path by immobilizing the slider 3, in sending to the mecahnism - by the displacement of the slider 2 - the totality of an independent sinusoidal signal ϵ, and in calculating or observing the response z of the system thus simplified.

This amounts to putting

$$d_1 = 0$$

into the system of equations and calculating z as a function of an input signal ϵ.

The system with the feedback path cut off will henceforward be called open system.

Study of the closed-loop system with negative feedback and study of the open system are equivalent since well-defined relations exist between the curve of frequency response z/x of the closed-loop system and the curve of response z/ϵ of the open system.

9. Relations Between the Curves of Response for the Open and for the Closed-Loop Systems

The definition

$$\epsilon = x - z$$

leads to

$$\frac{z}{x} = \frac{z/\epsilon}{1 + z/\epsilon}$$

or

$$\frac{x}{z} = \frac{\epsilon}{z} + 1$$

In many cases the transfer function of the open system can be determined more easily than that of the original closed-loop system.

Once one has plotted the curve of frequency response or transfer function of the open system, one can deduce from it the principal properties of the transfer function of the corresponding closed-loop system.

When the transfer function of the open system assumed to be known:

$$\frac{z}{\epsilon} = \frac{X_N + iY_N}{X_D + iY_D}$$

where X_N, Y_N, X_D and Y_D are functions of ω - has been plotted on the plane of the complex variable, one determines immediately the loci of the points representing the same modulus z/x or the same phase displacement φ. These loci are, by definition, concentric circles around the origin, or radii issuing from the origin.

The simplicity of the relations existing between z/ϵ and z/x permits the prediction that it will be possible to deduce, from the curve of the transfer function of the open system, certain properties of the corresponding closed-loop system.

One sees immediately that, if there exists a frequency for which the curve of response of the open system passes through the point -1, one has

$$z/\epsilon = -1$$

which entails

$$z/x = \infty$$

$$z/x = 0$$

An input signal of this frequency and of the amplitude zero excites the closed-loop system and induces a response of finite amplitude. This means that the closed-loop system becomes capable of free oscillation, at a particular frequency, and indicates to us the possibility of finding a criterion of stability of the closed-loop system according to the position of the curve of response of the open system, with respect to the point -1.

On the other hand, one may put into the complex plane serving for the representation of the response z/ϵ of the open system graduations useful for the evaluation of the properties of the response z/x of the corresponding closed-loop system.

In order to determine the loci of the same modulus z/x, let us consider a vector $V = z/\epsilon$ issuing from the origin, and a vector $V_1 = z/\epsilon + 1$ issuing from the point 1.

The loci where these vectors terminate when their lengths are in constant ratios $M = V/V_1$, make up series of circles the centers of which lie on the axis of the abscissas. This grid indicates, by its intersections with the transfer function of the open system, the modulus of the function of the closed-loop system.

Analogously, the phase displacement of the curve of response of the closed-loop system is given by the loci of equal phase displacement transferred to figure 50.

10. Relations Between the Curves of Requirement for the Open and for the Closed-Loop Systems

The demand curves have particular properties.

From

$$\frac{x}{z} = \frac{\epsilon}{z} + 1$$

one deduces that the vector expressing the demand of the closed-loop system is equal to the demand vector of the open system (for the same frequency) increased by 1.

This means that the demand vector of the closed-loop system starts from the point -1, to end at the point of the demand curve of the open system corresponding to the frequency considered.

The point -1 then is the center of the circles of the same modulus, and the origin of radii of the same phase displacement.

The stability of the closed-loop system becomes critical when there exists a frequency for which the demand vector has zero length, that is, when the demand curve of the open system passes through -1.

This confirms the previous results, because it results from the definitions of the curves of demand and of response that — if one of them passes through the point -1 — the other one does the same.

CHAPTER XX

1. Combination of Several Oscillating Systems by the Method of Frequency Response

The methods used in chapters XV and XVI permit the study of the effect exerted on the motion of an airplane by an apparatus for automatic flight control acting without lag or inertia. These methods make it possible to determine the type of stabilizer one can advantageously use if one wants to control the flight path; they show to what extent the law of piloting to be achieved by the automatic apparatus depends on the characteristics of the airplane.

If one wants to take the real operational characteristics of the automatic control system into account, as they are defined in chapter XVI, one must combine the system of equations of the automatic control with that of the pilot.

Determination of the motions corresponding to abrupt perturbations, that is, to transient conditions, by the methods of classical mechanics becomes impossible because of the complexity of the calculations.

In contrast, investigation of the frequency response remains possible since it is easy to combine the conclusions of the study of the frequency response of the airplane with the conclusions of the study of the frequency response of the automatic control and to determine the frequency response of the combination - without having to combine, in the course of the calculations, the two sets of parameters among themselves.

2. Oscillating Systems Placed In Series

Let us imagine a chain formed of several open systems, placed in series and controlling one another, and set up in such a manner that the functioning of any one system does not affect the functioning of the preceding system.

The transfer function of the total system will be the product of the transfer functions of the partial systems.

One can plot the curve of response of the total system by finding points at equal frequency ω on the curves of response of the partial systems, by multiplying the corresponding vectors according to the rule of multiplication of imaginary quantities and then joining the points thus obtained.

The demand curve of a system formed by several systems in series will be obtained by multiplication of the vectors corresponding to points of equal frequency.

These properties are obvious and do not require any demonstration.

The only restriction results from the rule: the functioning of any one system must not have any effect on the functioning of the preceding system.

3. The Criterion of Nyquist

The considerations of section 9 of the preceding chapter show that the frequency-response characteristics of a closed-loop system are related to the response characteristics of the same system considered as an open system.

The frequency response of a system containing several elementary systems in series, on the other hand, is obtained by forming the product of the responses of the elementary systems.

Thus one observes that here appears a possibility of treating complex systems by relatively simple methods.

It is, especially, possible to determine whether or not a closed-loop system is stable or not by examining the position of the response curve of the corresponding open system, with respect to the point -1 if the output signal is subtracted from the input signal, with respect to the point +1 if the output signal is added to the input signal.

In the case of negative feedback (output signal subtracted from the input signal) the passing through the point -1 of the curve characterizing the open system indicates a finite response for an input zero. It constitutes therefore the boundary between stability and instability. The only question which remains to be determined is, on which side are found stability and instability.

The criterion of Nyquist answers this question.

Let $R(p)$ be the transfer function connecting an output signal z with the input signal x:

$$z = R(p)x$$

where p is a complex variable.

To a value zero of x there corresponds a nonzero value of z when $R(p)$ is infinite.

The corresponding values of p are the poles of the analytical function R.

Instability of the system is produced if the real part of one or several poles is positive; in contrast, stability is attained when none of the poles of the function has a positive real part - in other words, when not any one value of p represented on the plane of the complex variable by a point situated on the right half-plane constitutes a pole of the function R.

In a system called a system with localized (lumped) constants we have

$$R(p) = \frac{H(p)}{Z(p)}$$

where H and Z are polynomials in p.

The stability condition is that $Z(p)$ must not have a zero situated to the right of the imaginary axis.

A theorem of Cauchy gives us an indication regarding the number of zeros and poles of an analytical function, contained in a particular region of the plane.

Let us consider, on the plane of the variable p, a contour C. Let $(F(p))$ be an analytical function which does not present either zeros or poles on the contour C.

Let us carry out a conformal transformation defined by

$$P = F(p)$$

This transformation permits us to plot in the plane of the variable P a curve Γ which is the transformation of C.

Let N be the number of revolutions of the curve Γ around the origin of the plane P when the point p describes the contour C.

The theorem of Cauchy states that this number N is equal to the difference $K - Q$ between the number K of zeros and the number Q of poles of the function $F(p)$ inside the contour C:

$$N = K - Q$$

As a result, we can - if we know the number of poles K of the function $F(p)$ within the contour C - determine the number Q of zeros inside this contour by inspection of the curve Γ.

Let us apply this theorem to the problem we are interested in.

Let us take as the contour C, in the plane p, the contour separating the entire right part of the half-plane, that is to say, the imaginary axis from $-\infty i$ to $+\infty i$ and a circular arc of infinite radius joining, through the right part of the plane, the two extremities of the imaginary axis.

The transformation of this contour by the function

$$P = F(p)$$

gives us:

(a) For the upper part of the imaginary axis, the curve $F(i\omega)$ for ω variable between 0 and ∞;

(b) For the lower part of the imaginary axis the curve symmetrical to the preceding one;

(c) For the circle passing through infinity, circles situated at infinity.

Let us apply this theorem to a system the transfer function of which is

$$z = \frac{R_1}{1 + R_1} x$$

that is, to a closed-loop system, with negative feedback where R_1 is the transfer function of the corresponding open system.

Let us examine the function $R_1 + 1$. Let K be its number of zeros, and Q the number of poles.

The number of poles of the function $R_1 + 1$ situated in a closed contour is necessarily the same as the number of poles of the function R_1. Q represents therefore likewise the number of poles of the function R_1 inside the contour considered.

If the system R_1 is stable, $Q = 0$ for the visualized contour enclosing the right half of the p-plane.

The number of zeros of the function $1 + R_1$ is then equal to the number of turns about the origin of the transformed Γ defined by

$$P = 1 + R_1(p)$$

or also to the number of turns around the point -1 of the transformation defined by

$$P = R_1(p)$$

Therefore one must draw on the complex plane of the variable P this transformation of the contour enclosing the right half of the p-plane defined above. This transformation is a closed curve the number of turns of which around the point -1 is counted.

Since the system R_1 (open) is supposed to be stable, the corresponding closed-loop system is stable if the curve Γ does not encircle the point -1.

This closed curve Γ comprises a branch which is nothing else but the transfer function $R\,(i\omega)$, a branch symmetrical with respect to the axis and, ultimately, circles, or parts of a circle of infinite radius.

If the open system is made up of several oscillators placed in series, R_1 is the product of the transfer functions of the elements. It may occur that one of these elements is unstable, that is, presents a pole in the right part of the plane. The condition of stability of the corresponding closed-loop system will then be that the transformation of R_1 must encircle once the point -1.

The relation which makes the stability of a closed-loop system depend on the position of the point -1 with respect to a closed curve plotted, starting from the transfer function of the open system, constitutes Nyquist's criterion.

4. Application to Systems of Automatic Pilot and Airplane

The systems formed by an airplane and an automatic pilot constitute systems with negative feedback. Their study constitutes a simple extension of that of a simple automatic control system.

Let us examine a series of cases of increasing complexity.

A. Simple automatic control system with negative feedback.- Let us take up again the linear automatic control studied in chapter XIX.

Let x be the input
z the output

We have shown before that the apparatus is actually sensitive to the difference $\epsilon = x - z$ owing to a device which subtracts the output signal from the input signal.

The airplane constitutes a first oscillating system. When it is subjected to a sinusoidal deflection, it transforms this input signal and furnishes four output signals δu, δw, δq, $\delta\theta$.

The automatic pilot is a second oscillator. It transforms any input signal received into a deflection η. It is fed by the discriminator which adds:

(a) The independent input signals x_1, produced by the pilot by means of the device 3

(b) One of the output signals of the airplane, with the sign changed, for instance

$$x_\theta = -\theta$$

Under these conditions, the input signal to the oscillator 2 is

$$x = x_i - \theta$$

The airplane (oscillator 1) carries out the operations

$$\theta = R_1\eta \qquad \eta = D_1\theta$$

The automatic pilot (oscillator 2) gives

$$\eta = R_2(x_i - \theta) \qquad (x_i - \theta) = D_2\eta$$

The total motion for the characteristics of which we are actually seeking is that produced by the independent excitation x_i

$$\theta = R_t x_i \qquad x_i = D_t\theta$$

The relation

$$\eta = R_2(x_i - \theta)$$

may be written

$$D_1\theta = R_2 D_t\theta - R_2\theta$$

which gives, by means of several transformations

$$R_2 D_t = D_1 + R_2$$

$$R_2/R_t = D_1 + R_2$$

$$R_t = \frac{R_2}{D_1 + R_2} = \frac{R_1 R_2}{1 + R_1 R_2}$$

and likewise

$$D_t = D_1 D_2 + 1$$

Let us examine the principles of operation of the system. In fact, we have simply formed - with respect to the quantities x_i and θ - a closed-loop control system with negative sensitivity but where two elements - the systems 2 and 1 - are placed in series.

We again find the expression of the characteristics of the closed-loop system as a function of those of the open system which one would obtain by cutting the feedback path, the response of the said open system being determined by the product of the responses R_1R_2 of two elements placed in series.

The graphical criterion of stability will consist in tracing the curve of the product R_1R_2 and in examining the position of the point -1 with respect to this curve.

In figure 55 the curves R_1 and R_2 have been plotted. The critical frequency corresponding to the appearance of steady oscillations is the one for which $R_1R_2 = -1$ or $D_1D_2 = -1$.

We state here the well-founded practical rule which permits estimation of the stability of closed-loop systems. This rule consists in cutting the loop at any point whatsoever and in constructing, by forming the product of the two functions R_1R_2, the transfer function of the open system containing the two elements placed in series.

<u>C. Airplane equipped with an automatic pilot and an independent control which acts after the output of the automatic pilot.</u>- Let there be a combination comprising:

(1) An airplane

(2) An automatic pilot

(3) An independent control system acting after the output of the automatic pilot

The schematic diagram is given by figure 56.

The airplane plays the same role as in the preceding case.

The automatic pilot receives only one single excitation: one of the output signals of the airplane; it transforms this excitation into a motion of deflection which we shall call η_p if the variable of reference is the angle of trim θ.

The independent control acts between the output of the automatic pilot and the control surface. Let us represent it by x_i.

If the control is accomplished by a connecting rod, one may for instance suppose an eccentric inserted into the rod, linking the automatic pilot to the control mechanism.

$$\eta = \eta_p + x_i$$

The automatic pilot produces

$$\eta_p = R_2(-\theta) \qquad -\theta = D_2\eta_p$$

The airplane produces

$$\theta = R_1\eta \qquad \eta = D_1\theta$$

and the response of the combination with respect to the independent control is

$$\theta = R_t x_i \qquad x_i = D_t\theta$$

A calculation analogous to the preceding one gives

$$R_t = \frac{R_1}{1 + R_1R_2}$$

and

$$D_t = D_1 + R_2 = \frac{D_1D_2 + 1}{D_2}$$

The expression of the response to an independent external control is different from the preceding one, but the stability condition of the system (flight of the airplane under action of the automatic pilot alone, without any action exerted on the independent control, is the same as in the previous case.

D. Airplane equipped with an automatic pilot which is a function of two variables and an independent control.- Likewise, we shall distinguish two cases, according to whether the independent control acts before the automatic pilot or after it.

The airplane constitutes always an oscillating system under the action of a sinusoidal deflection of the control surface, but two of the output signals, for instance θ and V, are utilized by the automatic pilot after having been changed in sign.

Figure 57 represents the schemetic diagram; the airplane forms the oscillator system No. 1, the automatic pilot constitutes the oscillator No. 2.

(a) Let us suppose that the independent control is applied before the automatic pilot.

The automatic pilot receives:

x_i and transforms this signal into $\eta = R_x x_i$

θ and transforms this signal into $\eta = -R_2\theta$

V and transforms this signal into $\eta = -R'_2 V$

so that $\eta = R_x x_i - R_2\theta - R'_2 V$

Since we have for the system 1

$$\theta = R_1\eta \qquad \eta = D_1\theta$$

$$V = R'_1\eta \qquad \eta = D_1 V$$

and for the total system

$$\theta = R_t x_i \qquad x_i = D_t\theta$$

$$V = R'_t x_i \qquad x_i = D'_t V$$

we may write

$$D_1 = D_t R_x - R_2 - R'_2 \frac{R'_t}{R_t}$$

$$D'_1 = D'_t R_x - R_2 \frac{R'_t}{R_t} - R'_2$$

which leads us, after all calculations have been made, to

$$R_t = \frac{R_1 R_x}{1 + R_1 R_2 + R'_1 R'_2}$$

$$R'_t = \frac{R'_1 R_x}{1 + R_1 R_2 + R'_1 R'_2}$$

When the denominator is zero, there is a frequency at which the airplane can oscillate freely. Hence one sees that the passing through the point -1 of the curve $(R_1 R_2 + R'_1 R'_2)$ determines the stability.

To express the demand curves, let us agree to write

$$x_i = D_x \eta$$

$$\theta = -D_2 \eta$$

$$V = -D'_2 \eta$$

One then has

$$D_t = D_x D_1 \left(\frac{1 + D_1 D_2 + D'_1 D'_2}{D_1 D_2 + D'_1 D'_2} \right)$$

$$D'_1 = D_x D'_1 \left(\frac{1 + D_1 D_2 + D'_1 D'_2}{D_1 D_2 + D'_1 D'_2} \right)$$

The output signals of the airplane are therefore

$$u = R_A \eta + R_B w_e$$

$$\theta = R_C \eta + R_D w_e$$

(b) If the independent control acts after the automatic pilot (fig. 58), everything occurs as though D_x and R_x are equal to unity.

Thus it is sufficient to make the substitution in the preceding formulas.

Remark: We have examined a general case; we have supposed that the stabilizer treats the two signals received θ and V differently and transforms them with ratios R_2 and R'_2 differing by the modulus and the phase difference.

If the two signals are added up, before being applied to the automatic pilot, they will both be transformed in the same ratio R_2. One may achieve such an operation by introducing a purely numerical factor k which takes account of the fact that the scales with which one measures the V and the θ are different, and by replacing $R_1R_2 + R'_1R'_2$ by $(R_1 + kR'_1)R_2$.

The corresponding graphical construction is evident.

E. Response of an airplane equipped with an automatic pilot to external perturbations.- So far, we have studied the response of the airplane to the action of an independent control.

This concept is somewhat artificial. It had the advantage of leading to cases which get progressively farther away from the simplest case - elementary automatic-control mechanism - and of facilitating the argument.

The essential point does not consist, in general, in an investigation of how the airplane responds to a sinusoidal excitation of an independent control but, on the contrary, in making sure that, in the absence of any independent control, the airplane's motion will be stable.

The stability condition follows readily from the argument. In the determination of this condition, the manner in which the control acts loses all significance: the stability condition is the same whatever may be the location of the chain where the action of the control is applied.

We shall show that the same arguments permit study of the reactions of an airplane equipped with an automatic pilot under the effect of external perturbation, for instance, gusts, assumed to be applied according to a sinusoidal law.

Let us imagine an airplane subjected simultaneously to two excitations:

An excitation due to the sinusoidal displacement η of the elevator

An excitation due to sinusoidal perturbations, of amplitude w_e, of the entrained velocity of the surrounding medium.

By hypothesis, the airplane is provided with a device for automatic piloting sensitive to the angle of trim θ. Figure 59 gives the basic diagram of the system considered.

The airplane constitutes the system 1. It reacts differently under the action of the input signals η and w_e.

Under the effect of each of these input signals, the system furnishes four output signals u, w or α, θ, and q. It is therefore characterized by eight transfer functions which we can calculate by the previously established methods.

Of these eight transfer functions, there are four of interest to us, in the problem studied. We should represent the responses by R with a first subscript 1 to indicate that it applies to the system 1 airplane, and with supplementary subscripts indicating which is the variable of interest for the response and which the variable considered for the excitation.

We shall use a simplification by adopting a less systematic notation and agreeing that:

R_A constitutes the response in u to an excitation in η

R_B constitutes the response in u to an excitation in w_e

R_C constitutes the response in θ to an excitation in η

R_D constitutes the response in θ to an excitation in w_e

The automatic pilot constitutes the system 2 and is determined by

$$\eta = -R_2\theta \quad \text{or} \quad \theta = -D_2\eta$$

Let us replace θ in the preceding equation, and obtain

$$-D_2\eta = R_C\eta + R_D w_e$$

Let us eliminate η, taking u into account; we obtain

$$u = \left[\frac{R_B + \left(R_B R_C - R_A R_D\right)R_2}{1 + R_C R_2}\right] w_e$$

which constitutes the expression of the total response in u to an external excitation w_e.

Since R_C is the same quantity which we have called R_1 in the problems B and C, we state that we find again the same stability condition which was to be foreseen: the stability condition, deduced from the expression of the response to an external perturbation must be independent of the variable, the variation of which constitutes the response, and of the perturbation considered.

The total transfer function sought, response in u to an excitation in w_e, will be calculable if one knows the curve of the frequency response of the automatic pilot and the frequency responses $R_A \ldots R_D$ of the airplane. The latter are calculable and given by

$$R_A = \frac{cV}{r^2} h_3 \frac{H_{3,1}(i\omega)}{Z(i\omega)}$$

$$R_B = \frac{(i\omega)H_{2,1}(i\omega)}{Z(i\omega)}$$

$$R_C = \frac{cV}{r^2} h_3 \frac{H_{3,4}(i\omega)}{Z(i\omega)}$$

$$R_D = \frac{(i\omega)H_{2,4}(i\omega)}{Z(i\omega)}$$

The curve representing the frequency response of the complete system can therefore be plotted, each point being obtained by calculating, for the value of ω considered, the different polynomials, and graphically carrying out the construction.

Remark: In the theoretical case of an automatic pilot which does not introduce any phase displacement, R_2 is reduced to the constant $-k$.

Inserting this value $R_2 = -k$ into

$$R_x = \frac{R_B + R_2\left(R_B R_C - R_A R_D\right)}{1 + R_C R_2}$$

and replacing $R_A \ldots R_D$ by the expressions given above, one obtains

$$R = (i\omega)w_e \frac{H_{2,1}(i\omega) + k\frac{cV}{r^2}h_3\left[H_{3,1}(i\omega)H_{2,4}(i\omega) - H_{2,1}(i\omega)H_{3,4}(i\omega)\right]\frac{1}{Z^2(i\omega)}}{1 - \left[k\frac{cV}{r^2}h_3H_{3,4}(i\omega)\right]\frac{1}{Z(i\omega)}}$$

This formula corresponds exactly to the expressions found in section 12 of chapter XVIII, with p being replaced by $i\omega$.

In the case of the motion δu, produced by a perturbation δw, with an automatic pilot senstive to $\delta\theta$ but acting without inertia or lag, the symbolic response is in fact

$$\xi(p) = \frac{\gamma_1(p) + k\left[\varphi_1(p)\gamma_4(p) - \gamma_1(p)\varphi_4(p)\right]}{1 - k\varphi_4(p)}$$

and since

$$\varphi_1 = \frac{cV}{r^2}h_3\frac{H_{3,1}(p)}{Z(p)} \qquad \gamma_1 = \frac{p(\delta w)_0H_{2,1}(p)}{Z(p)}$$

$$\varphi_4 = \frac{cV}{r^2}h_3\frac{H_{3,1}(p)}{Z(p)} \qquad \gamma_4 = p(\delta w)_0\frac{H_{3,4}(p)}{Z(p)}$$

one would have

$$\xi(p) = p(\delta w)_0 \frac{H_{2,1}(p) + k\frac{cV}{r^2}h_3\left[H_{3,1}(p)H_{2,4}(p) - H_{2,1}(p)H_{3,4}(p)\right]\frac{1}{Z^2(p)}}{1 - \left[k\frac{cV}{r^2}h_3H_{3,4}(p)\right]\frac{1}{Z(p)}}$$

a symbolic expression which permits passing to the frequency response by substitution of $i\omega$ for p.

5. Recapitulation of the Principles

For any transient state, the properties of a system are, in fact, completely described if one analyzes the phenomena for the sinusoidal steady states at all frequencies contained between zero and infinity, and the method of the frequency response leads thus to the investigation of the motion of airplanes in the cases that are insolvable by the classical methods.

There exists equivalence between the curve of frequency response and the curve giving the response under a unit action. We have seen how one can go from one to the other.

If one is content with finding out whether a system is stable, the criterion of Nyquist furnishes quickly an answer to the question.

The method of frequency response offers advantages which the classical method does not possess, because it permits:

(a) Analysis of the action of every parameter of the system separately, on the resultant behavior of the whole

(b) Determination of the performances of the mechanism, even if it is very complicated

(c) Eventual experimental study of certain elements of the mechanism defying calculation, and introduction of the experimental result into the theoretical calculation of the combined system by graphical method

(d) Guidance in the search for the modifications which would be recognized as necessary for improvement of an existing system thanks to the possibility of representing the effect of the elementary mechanisms by graphical methods.

6. Use of the Oscillating Table

If an element of a chain seems to defy calculation, one may consider to determine from it experimentally the transfer function.

This procedure is especially suitable for elements the functioning of which presents numerous causes of nonlinearity.

The Sperry A3, for instance, is in this case. It is not certain that the difference in pressure acting on the membrane controlling the distributor will be rigorously proportional to the angular displacement of the gyroscope.

It is doubtful that the displacement of the slide valve constituting the distributor will be rigorously proportional to the pressure difference acting on the membrane.

Finally, it is almost certain that the force applied to the piston of the servomotor will not be proportional to the displacement of the distributor. As soon as one port is uncovered by the distributor, there is actually a tendency for a pressure on the piston to establish itself which is equal to the pressure given by the oil pump, reduced by the pressure loss in traversing the passageway; the resulting pressure is certainly not proportional to the displacement of the slide valve.

But if it is not possible, due to the nonlinearity of the system, to calculate with certainty its curve of frequency response, there exists always a possibility of determining this curve experimentally.

In practice, one may actually subject the detecting organ, in the laboratory, to a sinusoidal excitation, apply to the output of the servomotor a return moment varying according to the same law as the hinge moment, and record simultaneously the excitation and the response for an entire range of frequencies.

For an apparatus sensitive to the angle of trim, such as Sperry A3, it is sufficient to place the detector on a table the inclination of which may vary sinusoidally at increasing frequency.

For a linear system, the curve of response is independent of the amplitude of the excitations because an essential characteristic of the linearity is a rigorous proportionality between the excitation and the response.

Repeating the experiment for different amplitudes of the excitation, one will see from the spread of the curves of unit response what is the effect of the nonlinearity.

We have not yet the possibility of performing such tests but we hope to have it shortly.

7. Interposition of Filters or of Amplifiers in the Case of Electric Controls

Since the transfer function of a chain of elements is equal to the product of the transfer functions of the elements when none of the latter reacts on any of the preceding ones, there exist possibilities of transforming the curve of response by introducing into the chain filters or amplifiers which act on certain frequencies.

Let us take up again the diagram of the linear control of chapter XIV. Let us suppose the feedback path to be cut, so that

$$d_1 = 0$$

The transfer function z/x may be written with separate consideration of the component systems

$$\frac{z}{x} = \frac{z}{i_1} \cdot \frac{i_1}{i} \frac{i}{x}$$

It is clear that the interposition of filters or amplifiers permits, for harmonic state, modification of the transfer function i_1/i, and improvement of the curve of total response if the latter possesses undesirable characteristics for certain values of the frequency.

8. The Frequencies

At a first glance, one may object that the natural frequencies of the automatic pilot and of the airplane are sufficiently distant from each other that the combination of the characteristics of the two systems is not necessary.

We believe this objection to be unfounded because it is always useful to verify, at the price of a few hours of calculation, that a theoretically possible resonance is not produced.

There are so many examples of accidents that have happened in all branches of mechanics, because a possible resonance had been neglected, that one cannot possibly say that application of the calculation methods developed above is not useful.

We have indicated that the oscillatory characteristics of servo-controls can be modified by electric filters. The combinations of condensers, inductances, etc. generally produce an effect on frequencies which are very high from the standpoint of the designer and hardly of interest to him.

However, instruments have been created which modify the response of electric circuits with large lags - that is, which act efficiently at very low frequencies. They are the "chronotrons."

Such an apparatus is a Wheatstone bridge where the resistances placed in the arms are subjected to the action of heating resistances through which run the currents to be manipulated.

Due to the thermic inertia, the effects of the currents sent through the heating resistances are manifested, in the circuits constituting the bridge, with lags to be expressed in seconds, and it is thus possible to add effects of small period to the input currents.

In view of such possibilities, knowledge of complete investigation methods seems indispensable.

9. Various Consequences of the Preceding Considerations

A. Theoretical case.- In chapters XV and XVI we made the assumption that the deflection of the control surface took place by means of mechanisms without inertia. This amounts to supposing that the transformation R_2, achieved by the automatic pilot takes place, in the case of harmonic excitation, with a phase displacement continuously zero whatever the pulsation frequency ω may be. One may again plot the curve R_1R_2, but this curve is nothing else but the curve R_1 all vectors of which are multiplied by a constant.

This leads immediately to an interesting conclusion.

The airplane possesses as many curves of response or transfer functions as the number of output variables considered.

If one of these curves looks as indicated in b on figure 60, we are certain that, by addition of a device for automatic piloting which is a function of the variable considered and has negative feedback, we shall arrive at instability, if we choose a sufficiently high sensitivity factor.

In fact, since the phase angle of the product R_1R_2 is never changed, there will always arrive a moment where the curve R_1R_2 will pass through -1 when the constant factor R_2 attains a sufficient magnitude. This does not occur if R_1 corresponds to a curve such as a.

By subtraction of output or negative feedback we stipulate a device such that:

A positive $\Delta\theta$ (nose-down) makes the airplane nose up

A positive Δq (nose-down acceleration) makes the airplane nose up

A positive Δu makes the airplane nose up

A positive $\Delta\alpha$, that is, a negative Δw, makes the airplane nose down.

The directions of action described above are those we have visualized, in chapter XIV, as standard directions.

If we have a piloting apparatus the action of which is inversed, that is, if we operate by adding or reinjecting at the input a part of the output, we have a system of positive feedback.

The stability condition will be no longer determined by the position of the curve R_1R_2 with respect to the point -1, but with respect to the point +1. One can easily verify that mechanisms of this type will lead to instability when their sensitivity becomes sufficient.

B. Case of a constant lag in the functioning of the stabilizer.- If we suppose that the automatic pilot acts with a constant lag, this lag corresponds to an angular increment of the response R_2, increasing with the pulsation frequency.

In the application of the graphical criterion of stability one sees that this operation could lead the extremity of the vector produced to describe a curve surrounding the point -1, if, at high frequencies, the modulus of the vector R_2 is not very small.

C. Airplane equipped with an automatic control which is a function of the derivative of one of the output signals of the airplane.- It is possible to study the harmonic motion of an airplane equipped with an automatic pilot that is a function of the derivative of one of the output signals of the airplane.

Let R_1 be the function defining the fundamental output signal, for instance V, under the action of the input signal η. We suppose that the automatic pilot is excited by the derivative dV/dt. R_2 is the function defining the output η under the action of dV/dt.

The response R_1 in V can be calculated and plotted. There corresponds a vector to each frequency ω. However, the transformation defined by R_2 is to be applied not to this vector but to its derivative.

One must therefore proceed, first, to carry out the differentiation of this vector - an operation which is carried out by multiplying the modulus of the vector by ω and by displacing it forward in phase by $\pi/2$. This operation must be carried out on each of the vectors representing V, before performing the multiplication by each of the vectors R_2 of the same frequency.

One has therefore plotted the locus of the product

$$\frac{dR_1}{dt} R_2$$

and the stability of the system will be indicated by the position of this curve with respect to the point -1.

One sees immediately that this operation can contribute to deviating the curve from the region occupied by the point -1.

We investigated in section 4, D, the functioning of the control actuated by the indication of two variables. It is evident that one of the two may be the derivative of the other and that, instead of detecting it, one may produce it by means of a differentiator.

D. Airplane equipped with an automatic control that is a function of the integral of one of the output signals of the airplane.- Let R_1, R'_1, . . . be the functions defining the various output signals of the airplane, for instance, in the case of the lateral motion, the angles ψ and β.

We suppose that the automatic pilot is sensitive to the distance y between the actual flight path and a required flight path. R_2 is the function defining the output ζ under the action of y. When the required trajectory is obtained through radio alinements, the deviation with respect to the latter can be measured on board of the airplane by means of receivers.

On the other hand, we can calculate this deviation as a function of the intrinsic variables of the motion.

Since

$$dy = V(\psi + \beta)dt$$

$$y = \int V(\psi + \beta)dt$$

where ψ and β are sinusoidal functions of the pulsation ω. Thus we find that for any pulsation ω the output vector y is nothing else but the integral of the sum of the vectors ψ and β, multiplied by V.

The integral of a rotating vector is obtained by multiplying the modulus by $1/\omega$ and by shifting its phase backward by $\pi/2$.

We can therefore construct the vector y for any pulsation ω.

Plotting the curve R_1R_2 consists in finding the locus of the product of this vector y and the vector defining the response R_2.

One finds immediately that this construction always approaches this locus in the region of the plane occupied by the point -1.

The automatic landing depends, above all, on the possibility of producing and of detecting, on board the airplane, beams which occupy a position invariable with respect to the ground.

It is known that these beams undergo distortions which up to now have been incompletely explained.

The first point to be improved is the stability of the beam, and this problem depends solely on the technique of radio communications.

The airborne detection seems more certain than the production of the beam; here also the problem depends on radio technique.

However, once these problems have been solved, the investigation of the motion of the airplane must be made by combining all preceding factors. Complete knowledge of the reactions of the airplane under all circumstances is, of course, indispensable.

We want to stress a remark made in chapter XVII. In proportion as the airplane approaches the transmitter, the effect is as though the sensitivity of the automatic pilot increased.

But in setting up the equations, one introduces this sensitivity by a factor which must remain constant.

As a result it will be necessary to investigate the reactions of the airplane for a series of different values of the sensitivity factor, corresponding to different distances of the airplane with respect to the transmitter.

One may represent on the same diagram the transfer function of the open system for different values of this parameter. One then finds that the corresponding curves approach the point -1 in proportion as the airplane approaches the transmitter.

11. Conclusions

The investigation of the automatic piloting of airplanes by the method of sinusoidal oscillations is a particular application of the investigation methods of servo-mechanisms, established in the United States during the war and developed very rapidly to a degree of high perfection.

This method is much more powerful than the classical method; it permits the study of complicated schemes of operation without leading to inextricable calculations, but it presents its conclusions in the form of curves called, according to the authors:

curves of frequency response

transfer functions

locus of admittance or locus of impedance

the interpretation of which is not always immediately possible.

We have shown that, fundamentally, the transfer function contains all elements necessary for knowledge of the motions of the system since the elements which it defines, introduced in a Fourier series, permit construction of the response to a unit impulse.

We have indicated that knowledge of the transfer function permits the utilization of a criterion of stability in a simple application. We have also attempted to show in the present report that these new methods could be deduced from the classical theory by considerations which constitute a transposition into the mechanical domain of calculation methods used by electrical engineers.

The present report does not yet contain any application of these methods to the solution of particular problems, to the study of frequencies of resonance, to the study of better combinations, etc. We hope, however, to have convinced the reader that these new methods make a complete investigation of the automatic flight control of airplanes possible, and to have communicated to him our certainty that, thanks to them, all arising problems will be solved.

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APPENDIX I

In sections 9 and 11 of chapter III, we introduced a factor n' which permits expressing $d\gamma/dT$ when the rotational speed of a propeller of constant pitch varies with the speed of the airplane as a result of changes in the resisting moment.

One has, by definition

$$Q = k_Q \rho n^2 D^5$$

whence

$$k_Q = \frac{Q}{\rho D^3 n^2 D^4} = \frac{Q\gamma^2}{\rho D^3 V^2} = K\,\frac{\gamma^2}{V^2}$$

Passing to logarithms:

$$\log k_Q = \log K + 2 \log \gamma - 2 \log V$$

$$\log \gamma = \frac{1}{2} \log k_Q + \log V + C$$

$$d \log \gamma = \frac{1}{2}\, d \log k_Q + d \log V$$

or

$$\frac{d \log V}{d \log \gamma} + \frac{1}{2}\,\frac{d \log k_Q}{d \log \gamma} = 1$$

$$\frac{dV}{V}\,\frac{\gamma}{d\gamma} + \frac{1}{2}\,\frac{d \log k_Q}{d \log \gamma} = 1$$

$$\frac{d\gamma}{dV} = \frac{\gamma}{V} \times \frac{1}{1 + \frac{1}{2}\,\frac{d \log k_Q}{d \log \gamma}}$$

Representing, by means of logarithmic scales, the curve of the k_Q as a function of γ, one can find the absolute slope of the curve and thus determine

$$\frac{d \log k_Q}{d \log \gamma} = s$$

The curve of the $\log k_Q$ is - except for one constant - identical with the curve of the $\log k_s$.

One may therefore utilize the well-known logarithmic power curves for the determination of s.

One obtains

$$\frac{d\gamma}{dV} = \frac{\gamma}{V} \frac{2}{2 + s} = n' \frac{\gamma}{V}$$

whence

$$n' = \frac{2}{2 + s}$$

It suffices to measure the absolute slope of the logarithmic characteristic of the propeller to find n'.

APPENDIX II

The solution

$$\delta u = C_1 e^{x1t} + C_2 e^{x2t} + C_3 e^{x3t} + C_4 e^{x4t}$$

$$\delta w = l_1 C_1 e^{x1t} + l_2 C_2 e^{x2t} + l_3 C_3 e^{x3t} + l_4 C_4 e^{x4t}$$

etc., can be transformed into

$$\delta u = e^{kt} A_u \sin(st + \varphi_u) + e^{k't} A'_u \sin(s't + \varphi'_u)$$

$$\delta w = e^{kt} A_w \sin(st + \varphi_w) + e^{k't} A'_w \sin(s't + \varphi'_w)$$

etc.

Since the roots $x_{1,2}$ are conjugate imaginaries, the factors l_1 and l_2, m_1 and m_2, etc., also are conjugate imaginaries.

Let us write

$$l_1 = L_1 + L_2 i \qquad m_1 = M_1 + M_2 i$$

$$l_2 = L_1 - L_2 i \qquad m_2 = M_1 - M_2 i$$

The integration constants which one determines by identifying the δu for $t = 0$, with the initial conditions determining the kind of case studied, also are conjugate imaginaries.

Let us write

$$C_1 = A + Bi$$

$$C_2 = A - Bi$$

The transformation of the solution in exponential form to the solution in sinusoidal form then takes place by means of the following transformations:

$$l_1 C_1 e^{x1t} + l_2 C_2 e^{x2t} = e^{kt}\left[\left(l_1 C_1 + l_2 C_2\right)\cos st + i\left(l_1 C_1 + l_2 C_2\right)\sin st\right]$$

$$= e^{kt}\left[\left(2AL_1 - 2BL_2\right)\cos st - i\left(2AL_2 + 2BL_1\right)\sin st\right]$$

$$= e^{kt} A_w \sin\left(st + \varphi_w\right)$$

with

$$A_w = 2\sqrt{\left(AL_1 - BL_2\right)^2 + \left(AL_2 + BL_1\right)^2}$$

$$\sin \varphi_w = \frac{2\left(AL_1 - BL_2\right)}{A_w}$$

$$\cos \varphi_w = \frac{2\left(AL_2 + BL_1\right)}{A_w}$$

For the solution in δu where the constants $C_1 \ldots C_4$ are not multiplied by any of the factors $l_1 \ldots n_4$, one obtains quite simply

$$A_u = 2\sqrt{A^2 + B^2}$$

$$\sin \varphi_u = \frac{2A}{A_u}$$

$$\cos \varphi_u = \frac{2B}{Au}$$

APPENDIX III

One calculates

$$\delta u = + \int_0^{t_b} \eta_m \omega \cos \omega(t_b - t)\left[\Delta u + \Sigma e^{kt} A_u \sin(st + \varphi)\right] dt$$

for very large t_b and negative k, according to hypothesis.

Let us calculate successively

$$+\Delta u \eta_m \omega \int_0^{t_b} \cos \omega(t_b - t) dt$$

$$et \eta_m \omega \int_0^{t_b} \Sigma e^{kt} \cos \omega(t_b - t) \sin(st - \varphi) dt$$

For the first integral

$$\int_0^{t_b} \cos \omega(t_b - t) dt = \cos \omega t_b \int_0^{t_b} \cos \omega t \, dt + \sin \omega t_b \int_0^{t_b} \sin \omega t \, dt$$

$$= \cos \omega t_b \frac{1}{\omega} \left[\sin \omega t\right]_0^{t_b} + \sin \omega t_b \frac{1}{\omega} \left[-\cos \omega t\right]_0^{t_b}$$

$$= \frac{1}{\omega} \cos \omega t_b \sin \omega t_b - \frac{1}{\omega} \sin \omega t_b \cos \omega t_b + \frac{1}{\omega} \sin \omega t_b$$

$$= \frac{1}{\omega} \sin \omega t_b$$

The first integral gives therefore

$$+\eta_m \Delta u \sin \omega t_b$$

For the second integral, we shall calculate one of the terms of Σ

$$\int_0^{t_b} e^{kt} \sin(st + \varphi) \cos(\omega t_b - \omega t) dt$$

which is equal to

$$\int_0^{t_b} e^{kt}[\sin st \sin \varphi + \cos st \cos \varphi] [\cos \omega t_b \cos \omega t + \sin \omega t_b \sin \omega t] dt$$

or to the sum of

$$\sin \varphi \cos \omega t_b \int_0^{t_b} e^{kt} \sin st \cos \omega t \, dt +$$

$$\sin \varphi \sin \omega t_b \int_0^{t_b} e^{kt} \sin st \sin \omega t \, dt +$$

$$\cos \varphi \sin \omega t_b \int_0^{t_b} e^{kt} \cos st \cos \omega t \, dt +$$

$$\cos \varphi \cos \omega t_b \int_0^{t_b} e^{kt} \cos st \sin \omega t \, dt$$

For the solution of these integrals we shall make use of the formula given on page 115 of the table of indefinite integrals published by the Service de Documentation et d'Information Aéronautique (Trad. no. 4221).

$$\int e^{ax} \sin bx \cos cx \, dx = \frac{1}{2} e^{ax} \left\{ \frac{a \sin(b + c)x - (b + c)\cos(b + c)x}{a^2 + (b + c)^2} + \frac{a \sin(b - c)x - (b - c)\cos(b - c)x}{a^2 + (b + c)^2} \right\} + C^{te}$$

This integral must be taken as definite integral with very large t_b as upper limit, 0 as lower limit.

Since a is negative by hypothesis, the exponential factor = 0 when t_b is sufficiently large and the integral is zero at the upper limit. We have to concern ourselves only with the lower limit $x = 0$, and we obtain

$$\int_0^{t_b} e^{ax} \sin bx \cos cx \, dx = -\frac{1}{2} \frac{-(b + c) - (b - c)}{a^2 + (b + c)^2} = \frac{b}{a^2 + (b + c)^2}$$

In order to find

$$\int_0^{t_b} e^{ax}\sin bx \sin cx \, dx$$

$$\int_0^{t_b} e^{ax}\cos bx \cos c^x \, dx$$

we must transform these expressions.

Let us integrate by parts. We obtain

$$\int e^{ax}\sin bx \sin cx \, dx = -\frac{1}{c} e^{ax}\sin bx \cos cx + \frac{a}{c}\int e^{ax}\sin bx \cos cx \, dx +$$

$$\frac{b}{c}\int e^{ax}\cos bx \cos cx \, dx$$

$$\int e^{ax}\cos bx \cos cx \, dx = \frac{1}{c} e^{ax}\cos bx \sin cx - \frac{a}{c}\int e^{ax}\cos bx \sin cx \, dx +$$

$$\frac{b}{c}\int e^{ax}\sin bx \sin cx \, dx$$

Let us introduce the limits of integration. We then have

$$\Big[e^{ax}\sin bx \cos cx\Big]_0^{t_b} = \Big[e^{ax}\cos bx \sin cx\Big]_0^{t_b} = 0$$

$$\int_0^{t_b} e^{ax}\sin bx \sin cx \, dx = \frac{a}{c}\,\frac{+b}{a^2 + (b + c)^2} + \frac{b}{c}\int_0^{t_b} e^{ax}\cos bx \cos cx \, dx$$

$$\int_0^{t_b} e^{ax}\cos bx \cos cx \, dx = -\frac{a}{c}\,\frac{+c}{a^2 + (b + c)^2} + \frac{b}{c}\int_0^{t_b} e^{ax}\sin bx \sin cx \, dx$$

whence one obtains immediately

$$\int_0^{t_b} e^{ax}\sin bx \sin cx \, dx = 0$$

$$\int_0^{t_b} e^{ax}\cos bx \cos cx \, dx = \frac{-a}{a^2 + (b + c)^2}$$

If one substitutes in these expressions

$$a = k, \quad b = s, \quad c = \omega$$

the sum of the four integrals becomes

$$\frac{1}{k^2 + (s + \omega)^2}\left[+s \sin\varphi \cos\omega t_b - k \cos\varphi \cos\omega t_b + \omega \cos\varphi \sin\omega t_b\right]$$

Taking into account the factors placed before the $\int$ sign, one obtains as the total of the expression to be calculated

$$+\eta_m \,\Delta u \sin\omega t_b$$

$$+\eta_m \sin\omega t_b \Sigma A_u \frac{\omega^2 \cos\varphi}{k^2 + (s + \omega)^2}$$

$$+\eta_m \cos\omega t_b \Sigma A_u \frac{s\omega \sin\varphi - k\omega\cos\varphi}{k^2 + (s + \omega)^2}$$

The Σ indicates that one must take account of the rapid oscillation and of the slow oscillation:

$(s, \ k, \ \varphi$ in one case, $s', \ k', \ \varphi'$ in the other)

If one defines

$$\delta u_b = \left(C \sin\omega t_b + D \cos\omega t\right)\eta_m$$

one obtains

$$C = \Delta u + A_u \frac{\omega^2 \cos\varphi}{k^2 + (s + \omega)^2} + A'_u \frac{\omega^2 \cos\varphi'}{k'^2 + (s' + \omega)^2}$$

$$D = \frac{s\omega \sin\varphi - k\omega\cos\varphi}{k^2 + (s + \omega)^2} + A'_u \frac{s'\omega \sin\varphi' - k'\omega\cos\varphi}{k'^2 + (s' + \omega)^2}$$

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We divide the bibliography into four parts which we treat in a very different manner.

I. Stability of airplanes.- The investigation of the stability of airplanes, flying with controls fixed, does not constitute the ultimate goal of our report, but on the contrary, the starting point for our calculations.

We have, therefore, not attempted to give a complete bibliography regarding this question and are content to point out some classical books and several reports on particular points of a special interest for the aim we strive for.

II. Stabilizers or instruments for automatic piloting of airplanes.- Here we have tried to give, on the contrary, a bibliography complete up to October 1, 1948, and have eliminated only a few popularized articles.

III. Servo-mechanisms in general.- The theoretical investigation of these instruments has made considerable progress during these last years.

We have to limit ourselves to pointing out a few general books.

IV. Symbolic or operational calculus.- There exist already complete bibliographies pertaining to this question. We refer only to the reports of which we made use.

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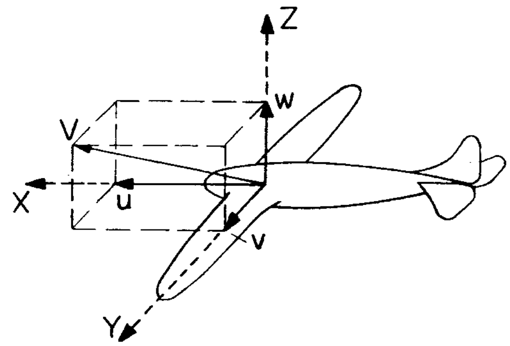


Figure 3.- Projections of the linear velocity.

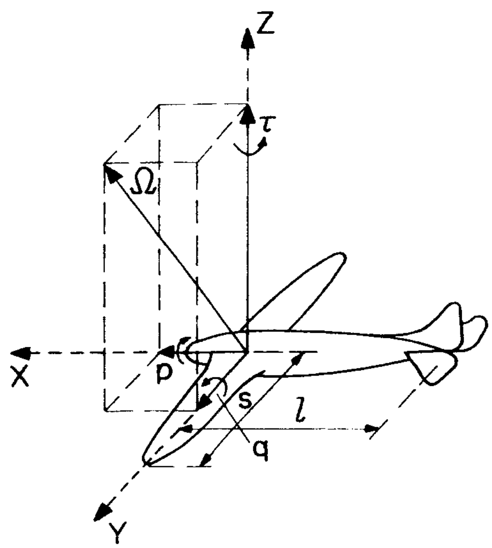


Figure 4.- Projections of the angular velocity.

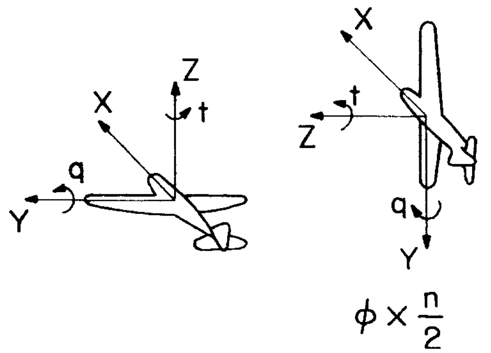


Figure 5.- Relationship between the motions of the airplane and its displacement with respect to the horizon.

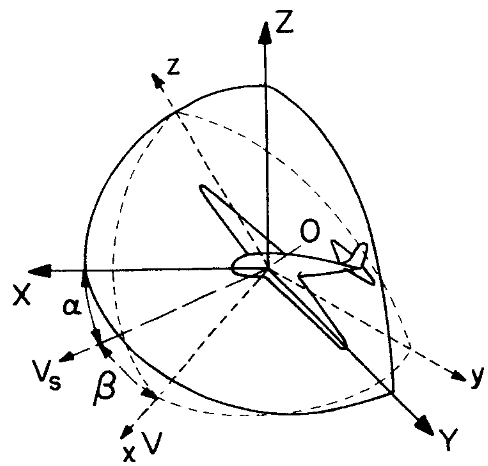


Figure 6.- Aerodynamic orthogonal system of axes.

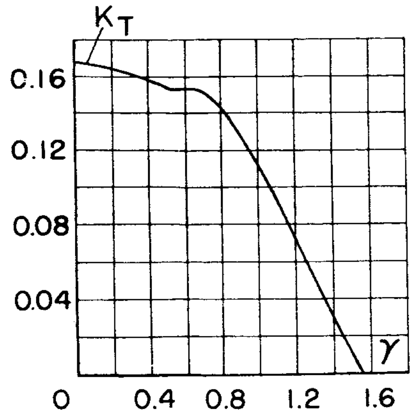


Figure 7.- Thrust coefficient of a propeller.

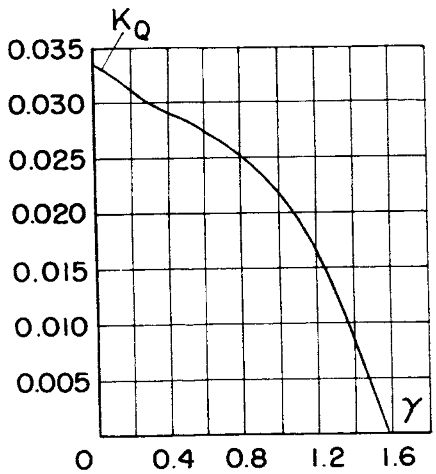


Figure 8.- Moment coefficient of a propeller.

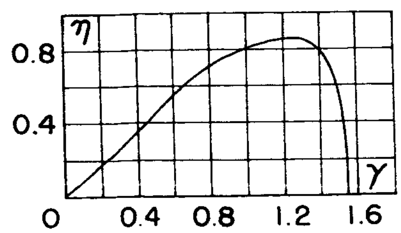


Figure 9.- Efficiency of a propeller.

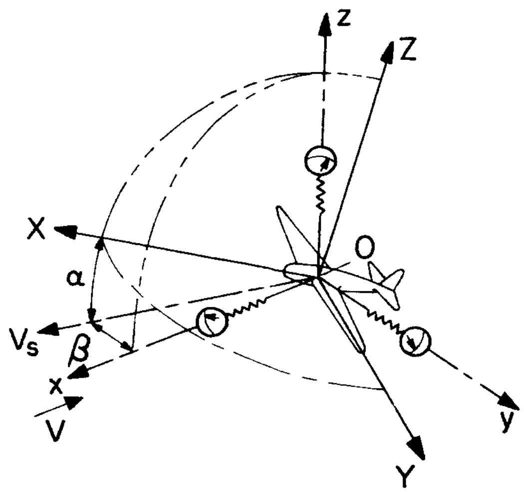


Figure 10.- Orientation of a model in the tunnel. Components measured.

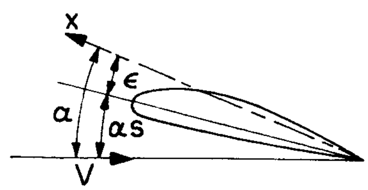


Figure 11.- Definition of the angle of attack in the tunnel.

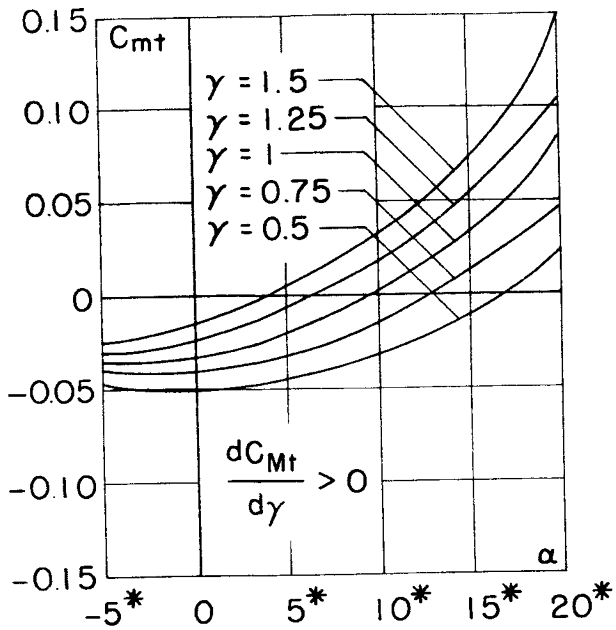


Figure 12.- Curve of the total-moment coefficient.

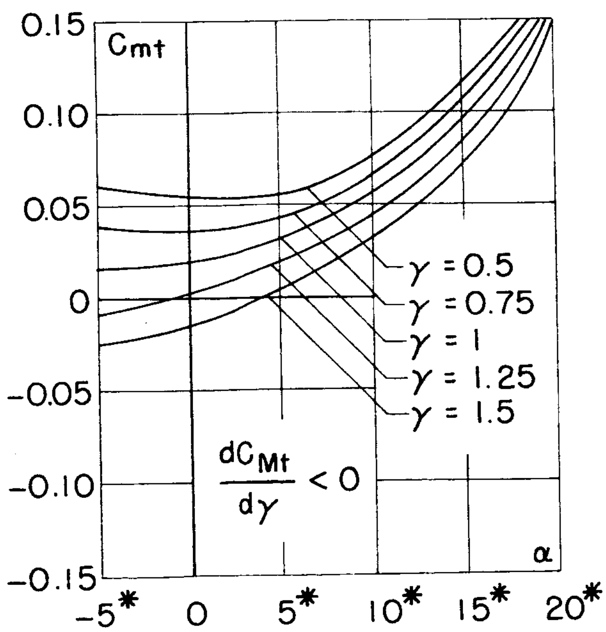


Figure 13.- Curve of the total-moment coefficient.

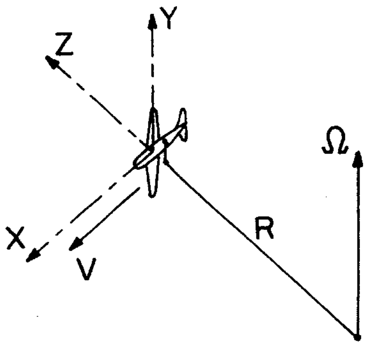


Figure 14.- Whirling-arm model test.

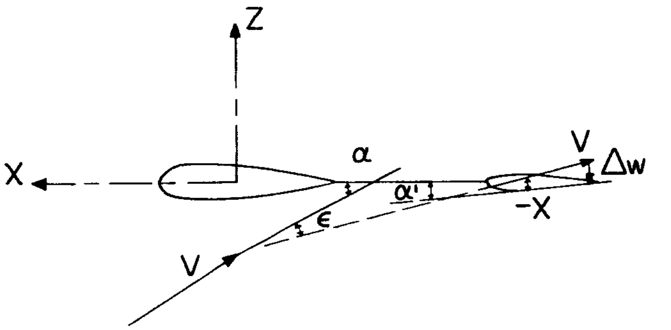


Figure 15.- Combination of the velocities acting on the tail surfaces.

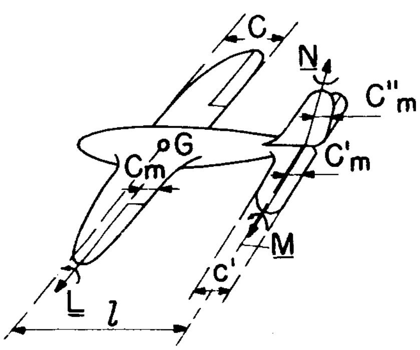


Figure 16.- Schematic diagram of the three principal controls.

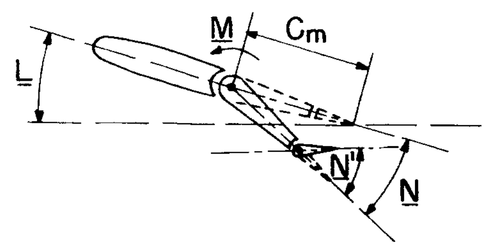


Figure 17.- Reaction compensator.

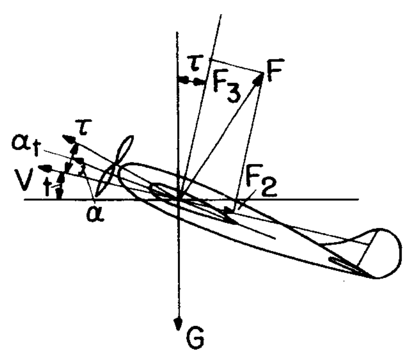


Figure 18.- Forces acting on the airplane.

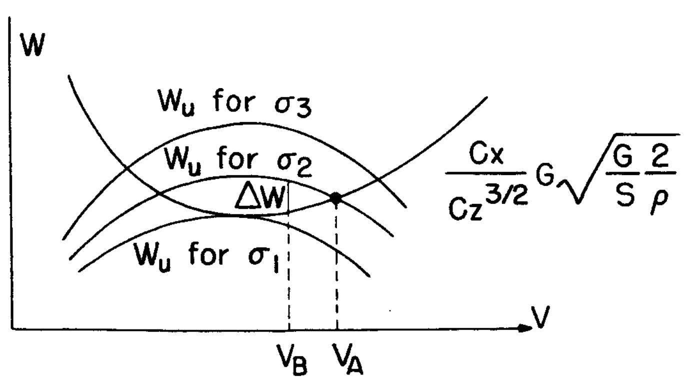


Figure 19.- Curve of the necessary and available power.

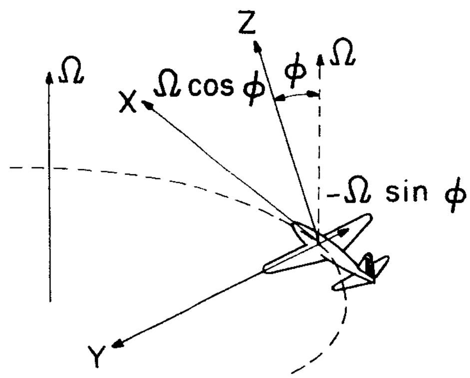


Figure 20.- Airplane turning about a vertical axis.

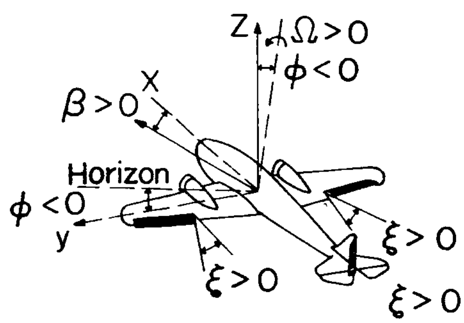


Figure 21.- Deflection of the controls and sideslip.

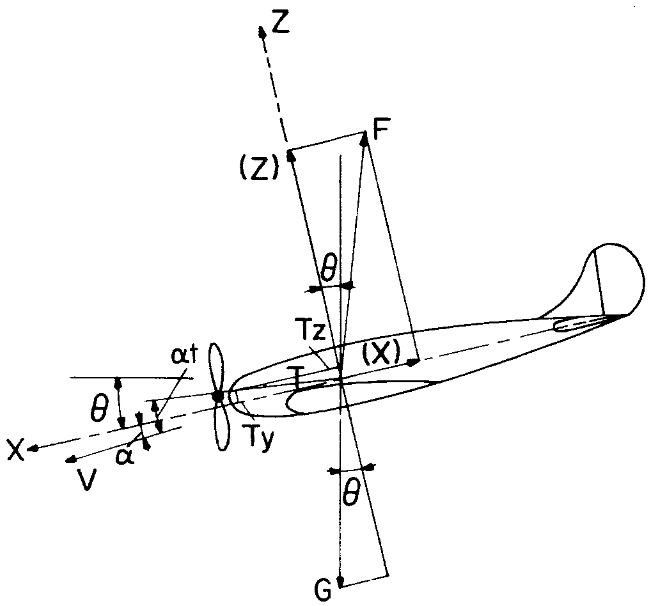


Figure 22.- Forces acting on the airplane.

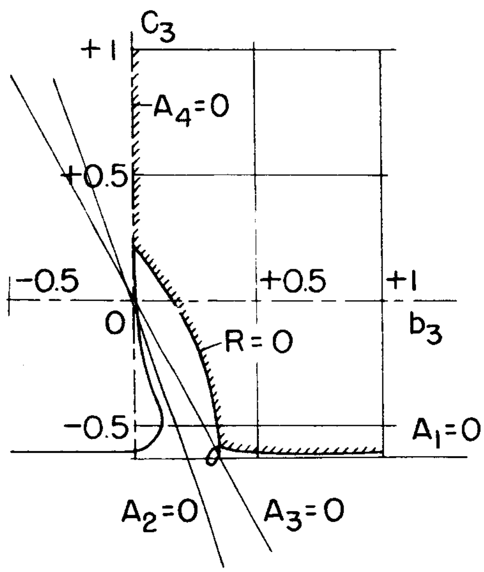


Figure 23.- Stability domain of the longitudinal motion.

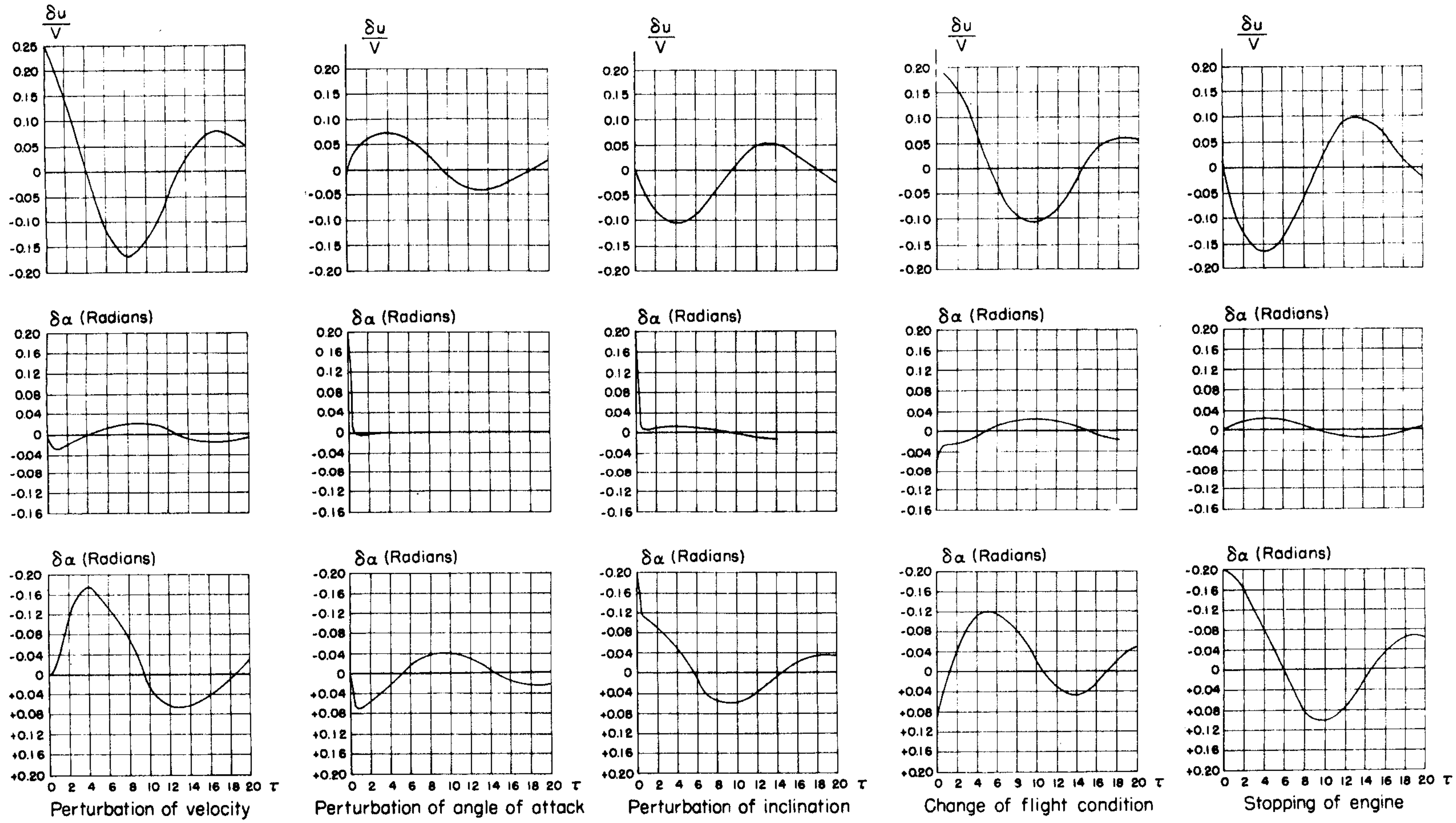


Figure 24.- Longitudinal motion following different initial perturbations. (Note that the negative θ corresponding to the nose-up motion are plotted in an upward direction.)

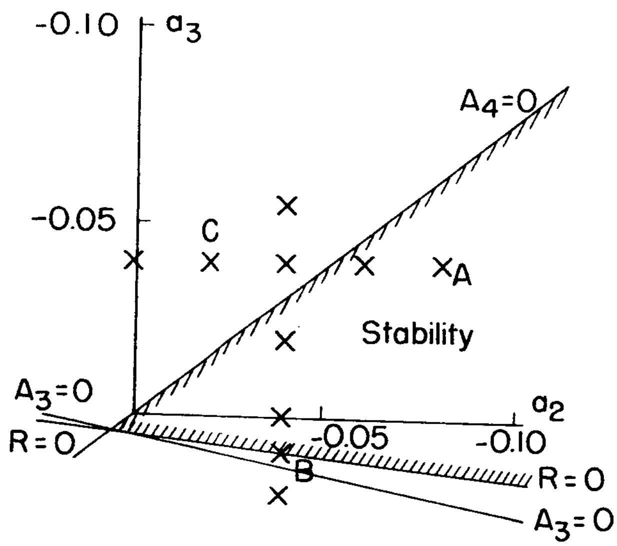


Figure 25.- Stability domain of the lateral motion.

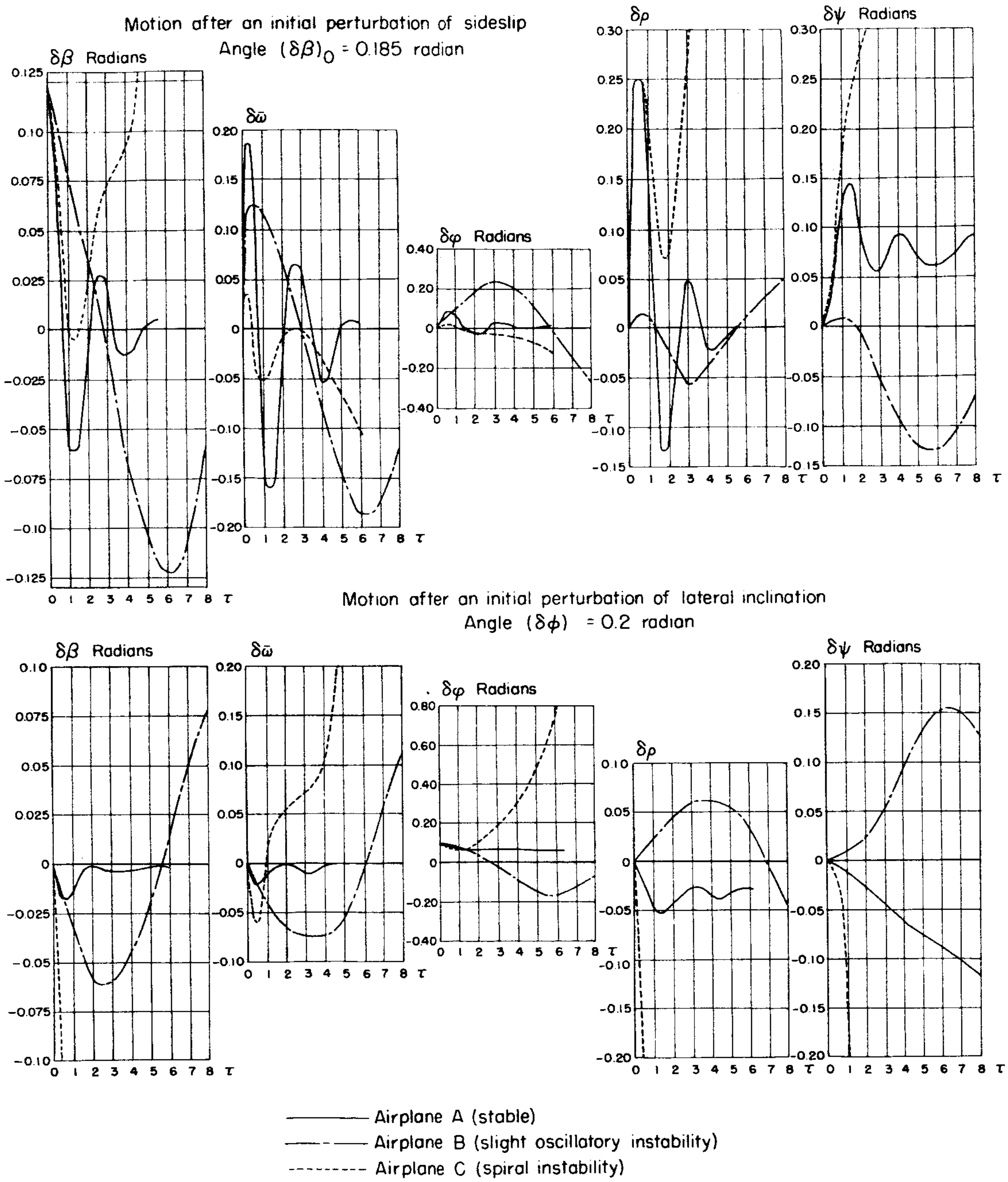


Figure 26.- Lateral motion following different initial perturbations.

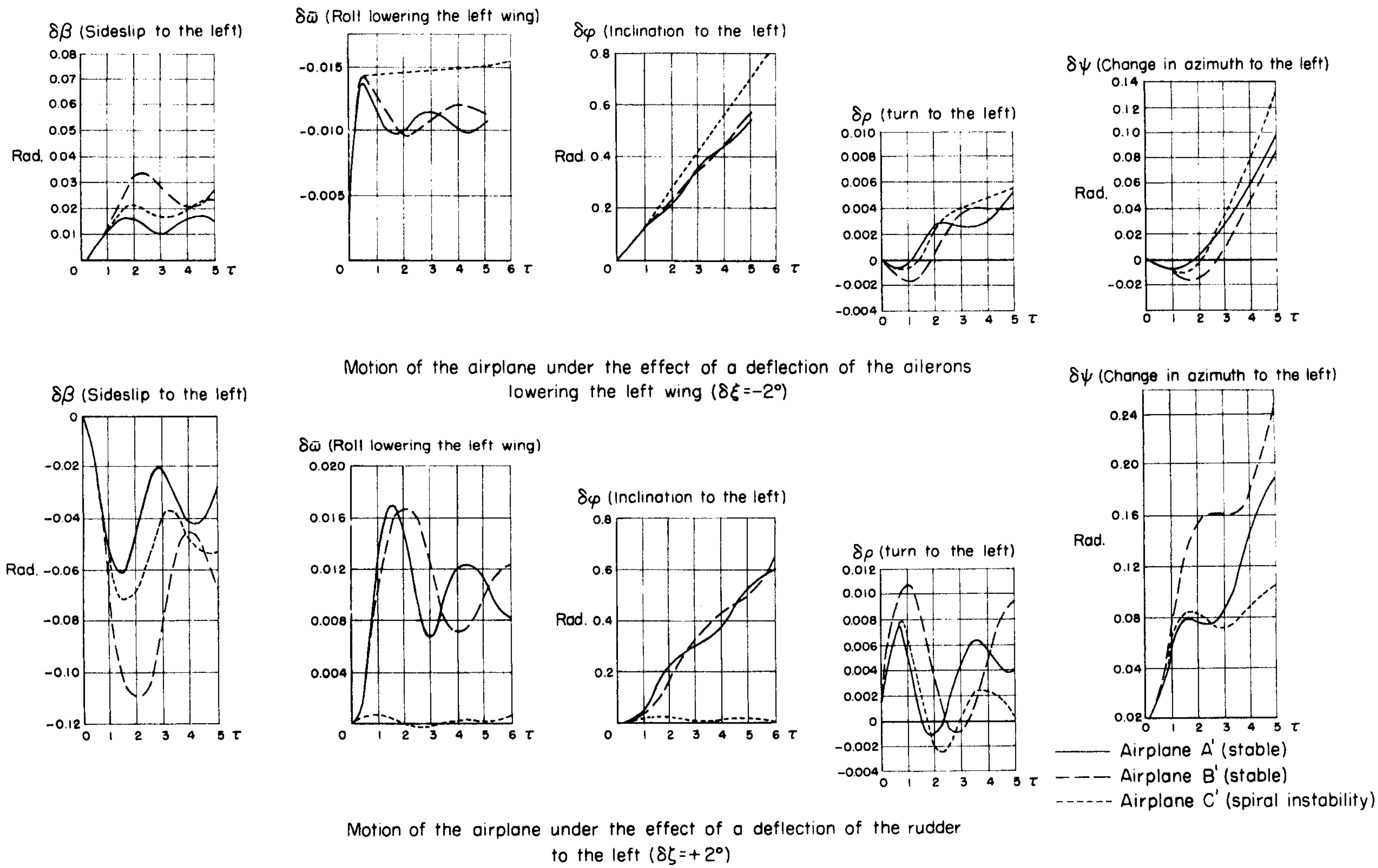


Figure 27.- Lateral motion produced by the deflection of the ailerons and of the rudder. (Note the direction of the motions indicated on the figure.)

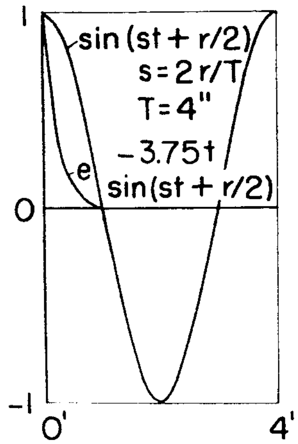


Figure 28.- Characteristics of an oscillatory motion when D is small compared to T.

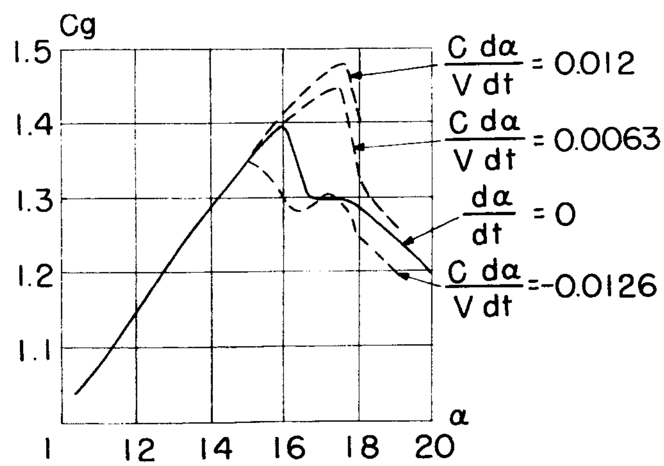


Figure 29.- Curves of Cz as a function of the angle of attack for different rates of variation in angle of attack.

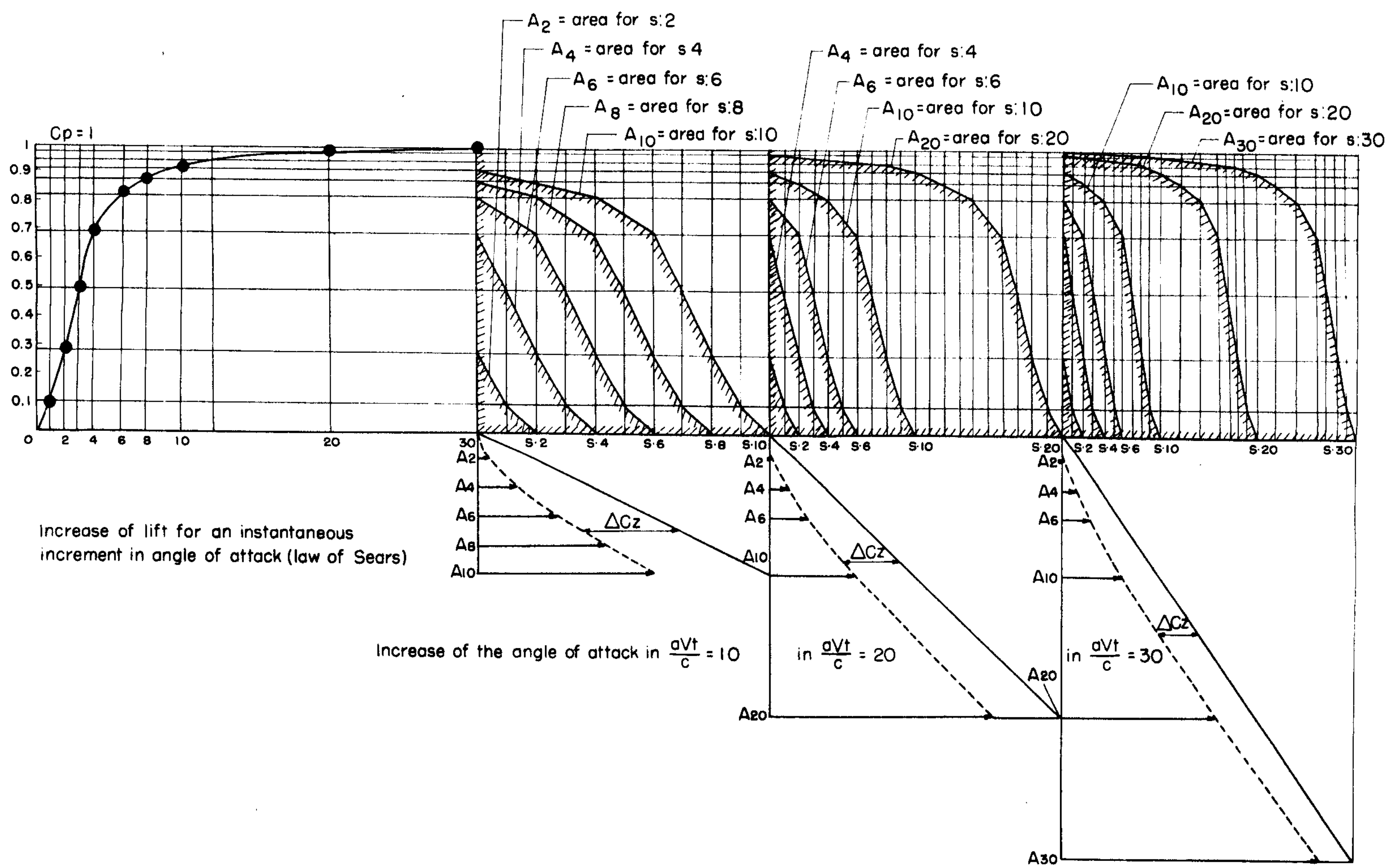


Figure 30.- Graphical calculation relative to the effect of lag in the establishment of the lift.

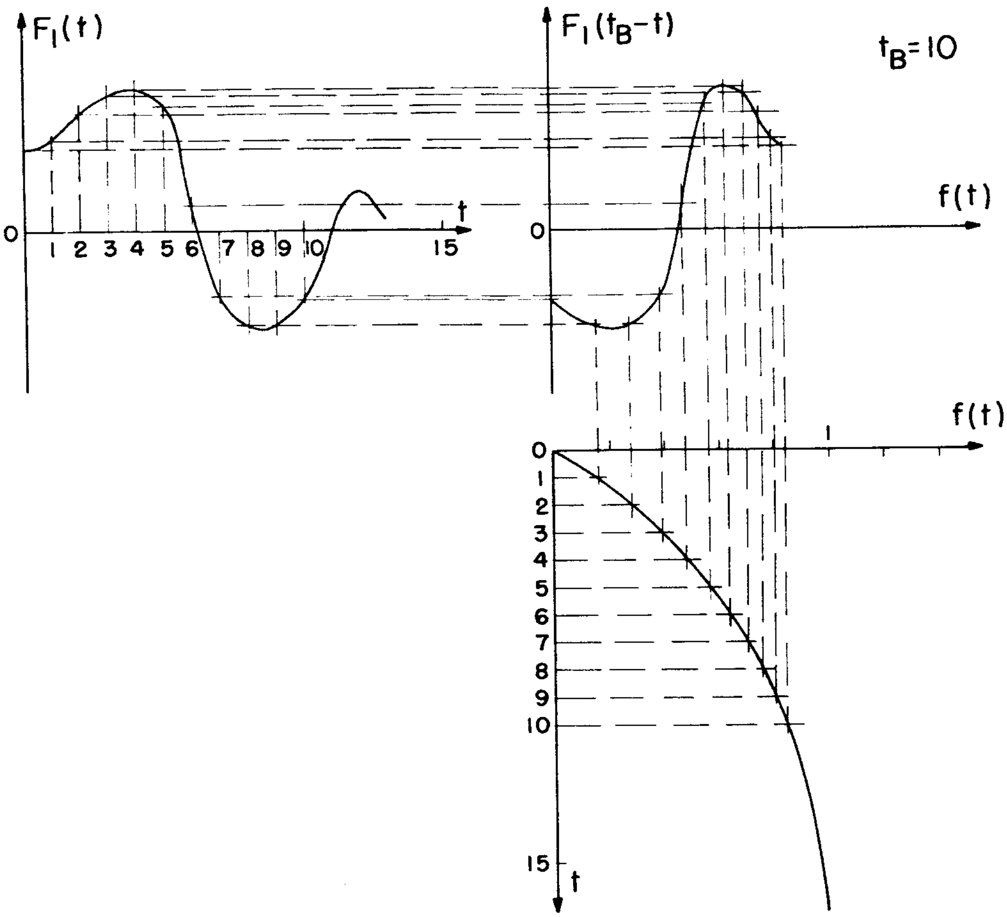


Figure 31.- Graphical method for solution of Duhamel's integral.

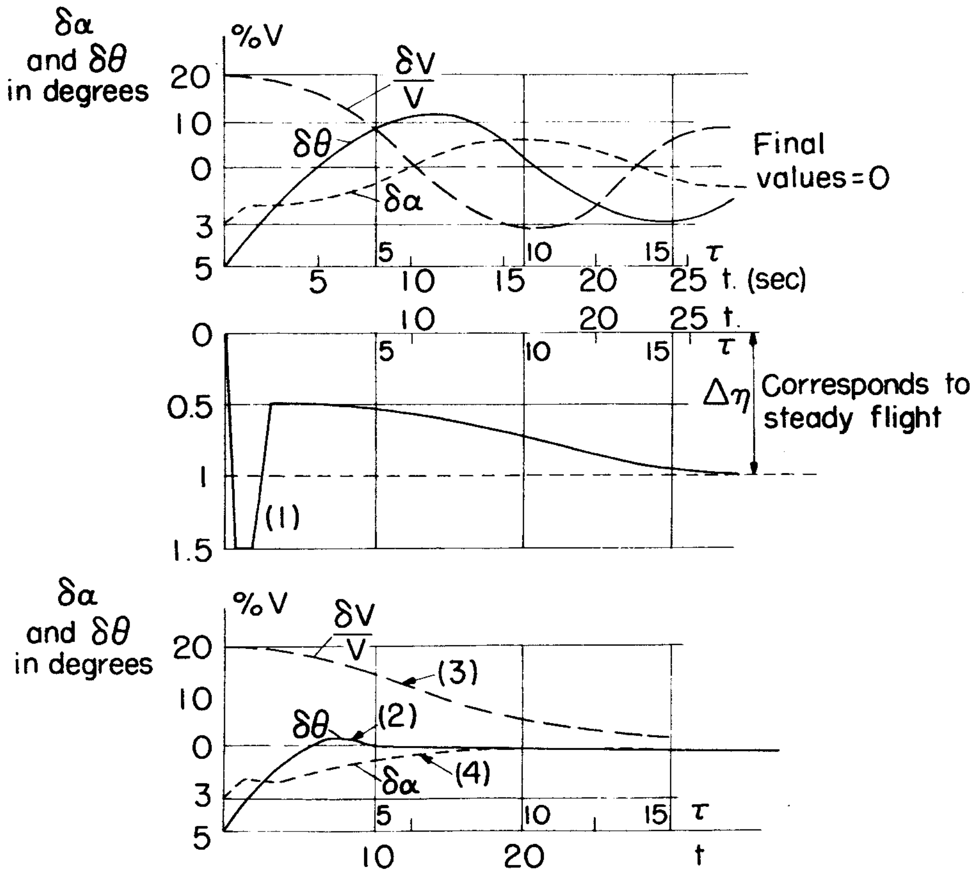


Figure 32.- Longitudinal motion produced by two different laws of displacement of the elevator.

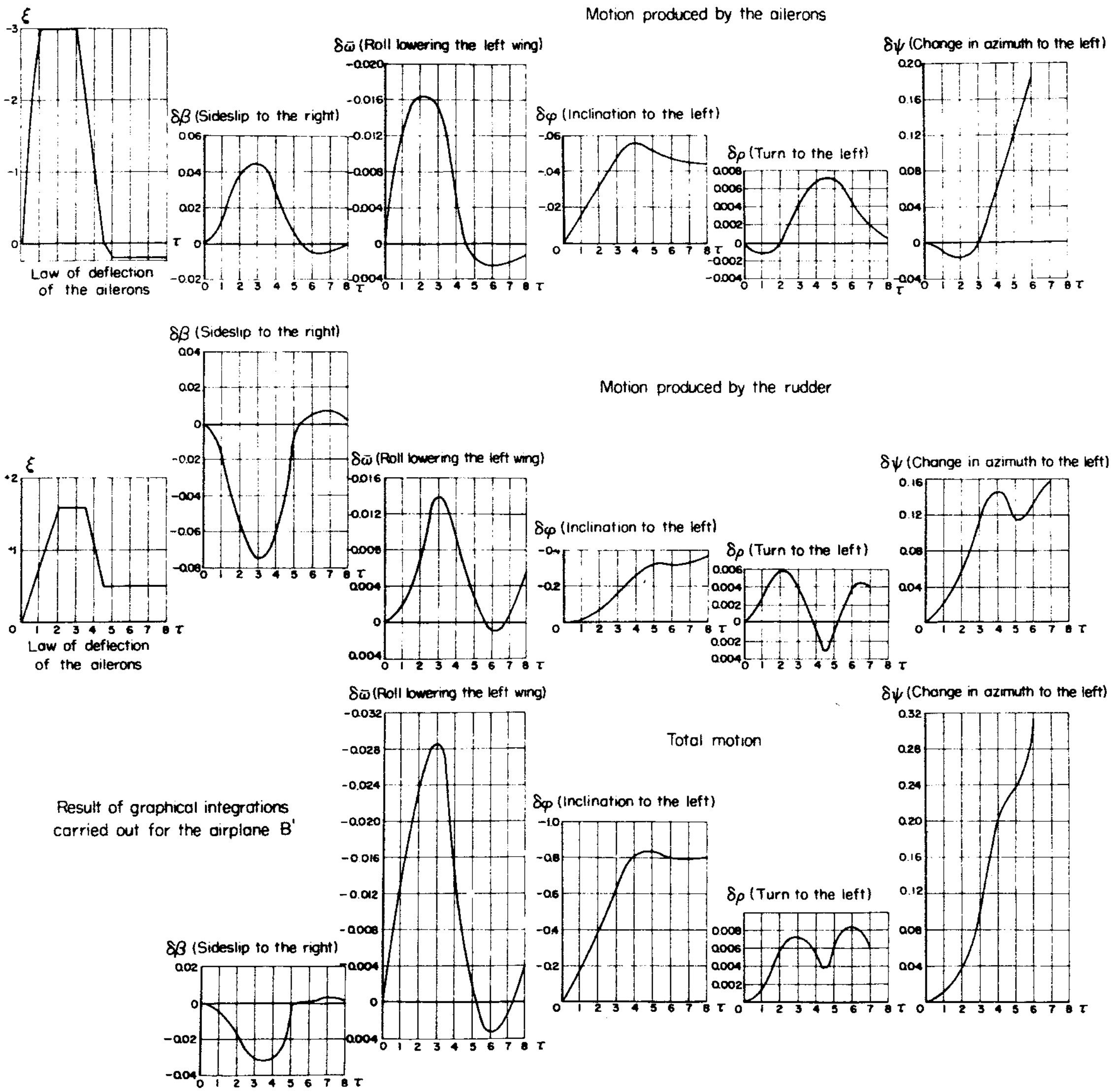


Figure 33.- Initiation of turn of an airplane. Calculation of the effect of displacement of any of the control surfaces.

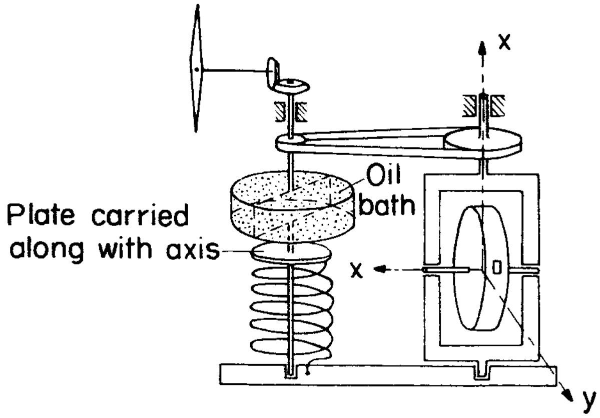


Figure 34.- Principle of a gyrometer developed by Mr. Bouny. The angular velocity q causes a moment about OZ. The magnitude of this moment is measured by the deformation of a spiral spring. An oil damper makes the instrument aperiodic.

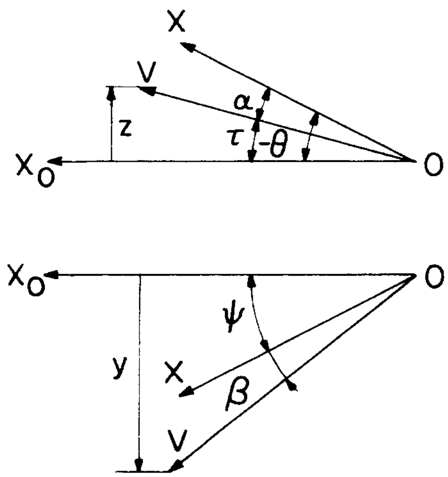


Figure 35.- Relation between the distances z and x, with respect to a reference line, and the angles $\theta + \alpha$ and $\psi + \beta$.

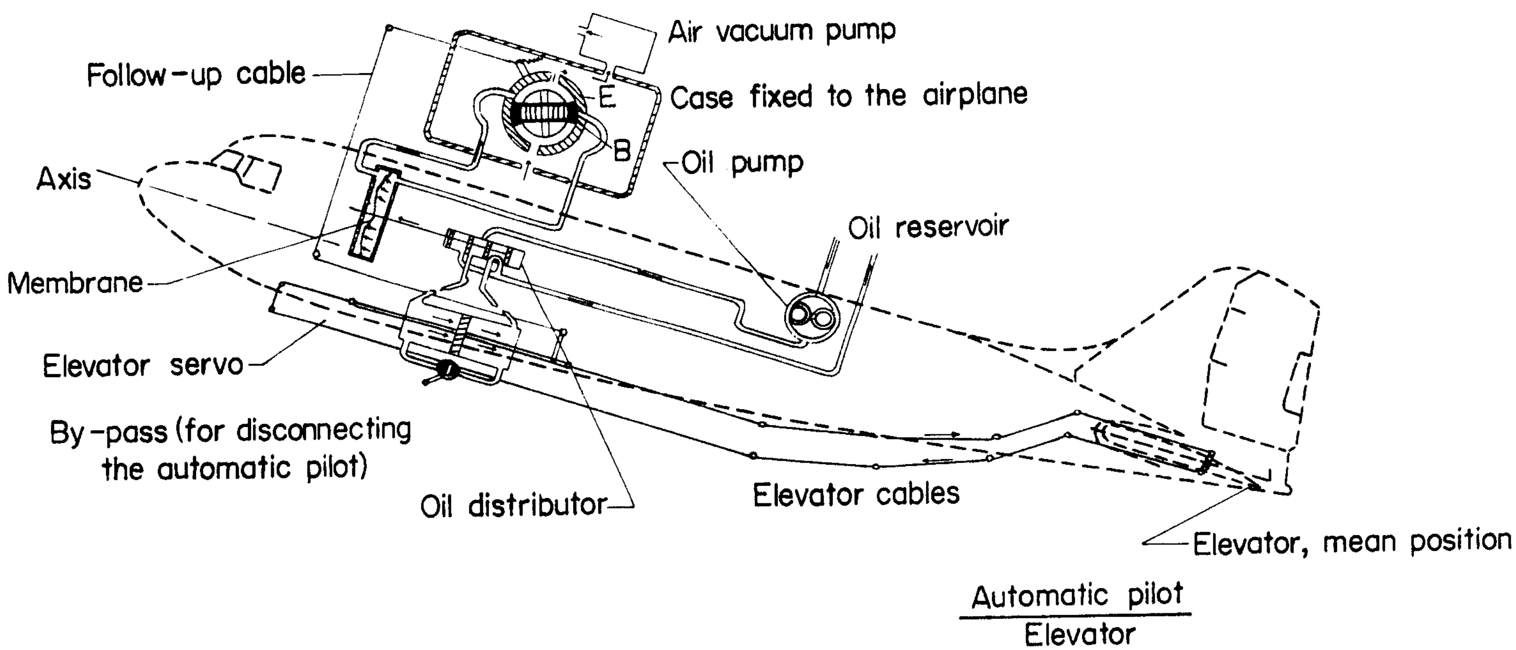


Figure 36.- Schematic diagram of the principle of the Sperry A_3.

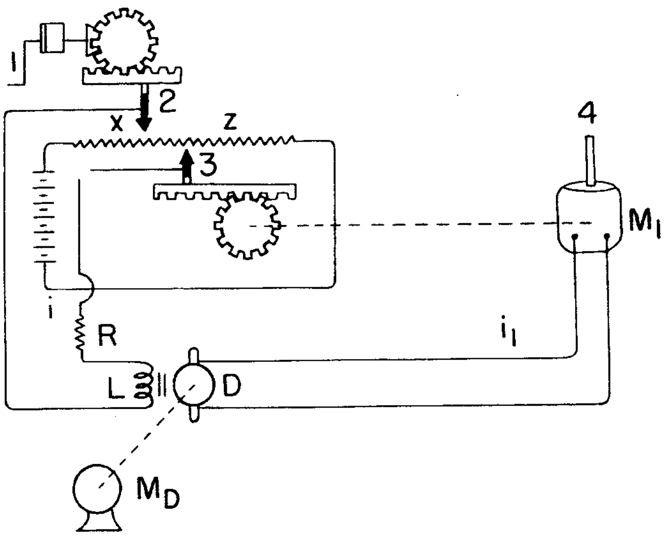


Figure 37.- Principle of a linear servocontrol.

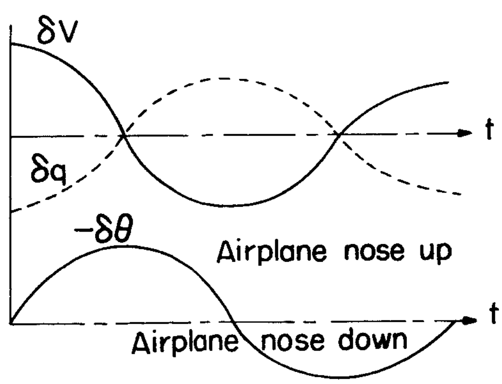


Figure 38.- Phase difference between the V, θ, and q pertaining to the slow oscillation.

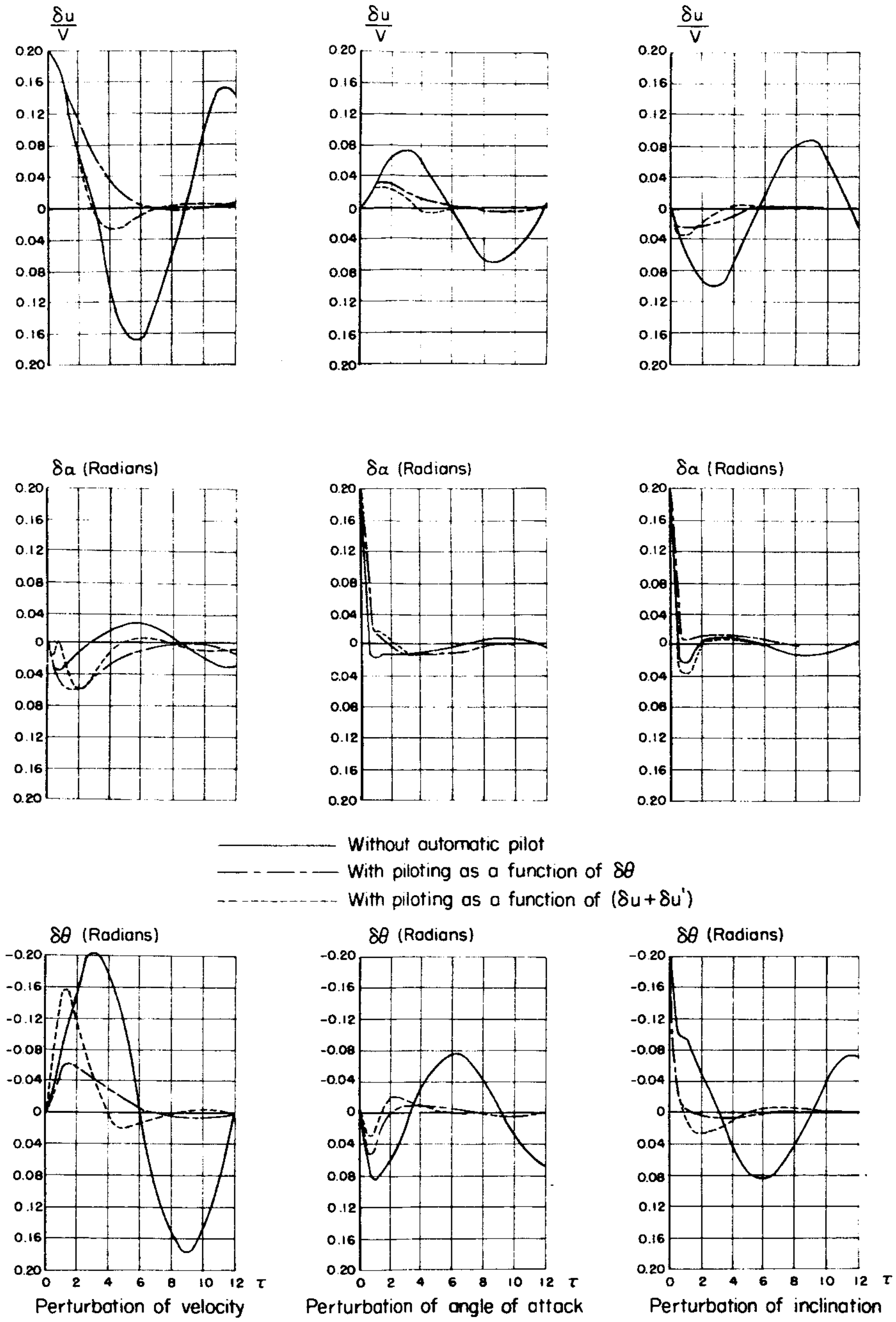


Figure 39.- Behavior following three different initial perturbations for three different laws of piloting: (a) controls fixed, (b) airplane piloted as a function of $\delta\theta$, and (c) airplane piloted as a function of $\delta u + \delta u'$.

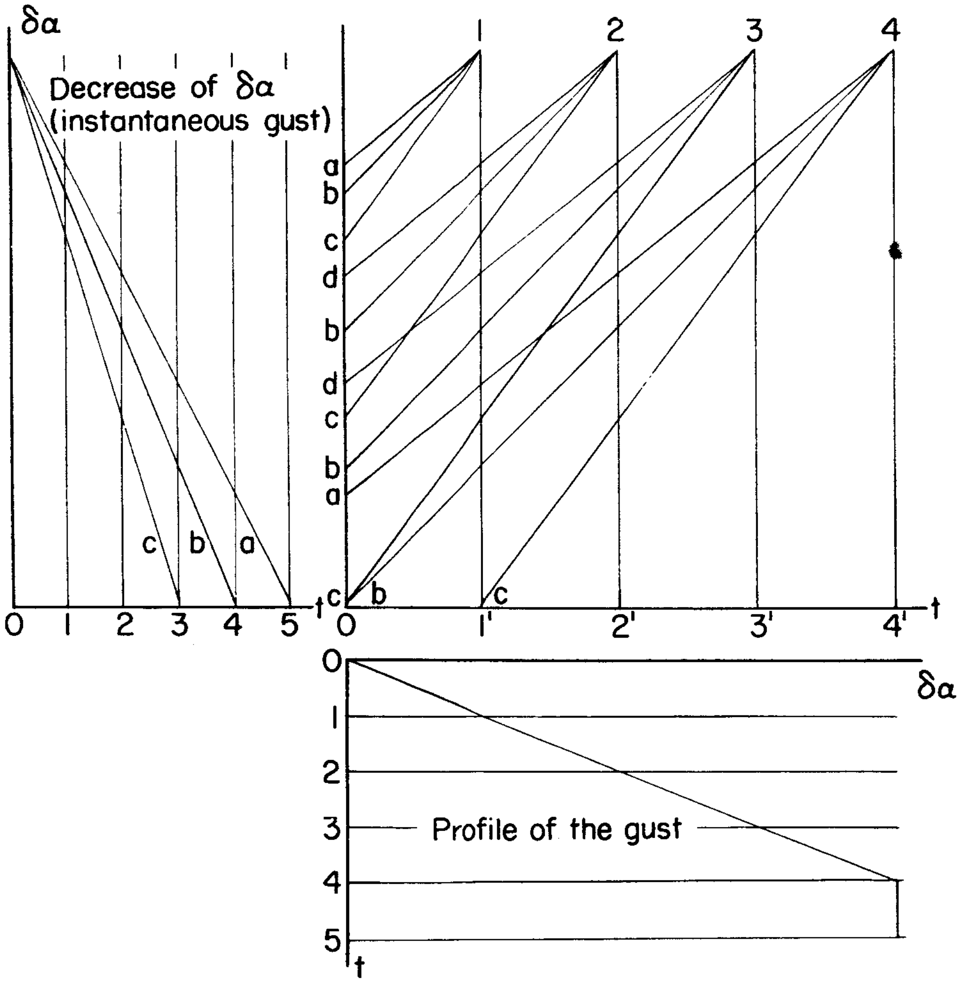


Figure 40.- Construction leading to the graphical calculation of the increments in angle of attack produced at the times 1, 2, 3, 4 by a gust the duration of buildup of which is 4 time units. The construction is made for three airplanes, characterized respectively by the laws of reduction a, b, c of the perturbation in angle of attack by an instantaneous gust. One will note that the duration of the decrease of this perturbation is of the same order of magnitude as the duration of buildup of the progressive gust visualized here. The increments in angle of attack at the times 1, 2, 3, 4 are proportional to the areas 0, 1', 1, a, b, or c; 0, 2', 2, a, b, or c. One sees that this area is always larger for the polygon characterized by a than for the corresponding polygons designated by b or c.

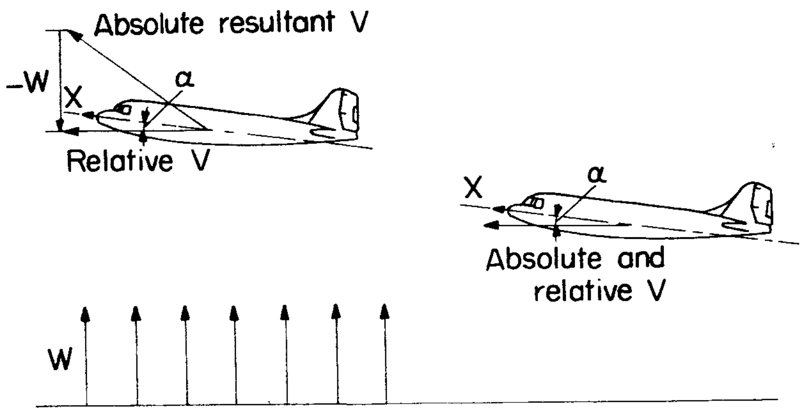


Figure 41.- Flight of the airplane in zones of air having different vertical velocities.

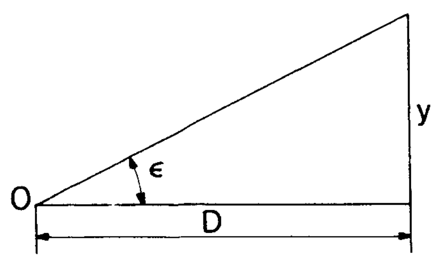


Figure 42.- Geometrical relation between y and ϵ.

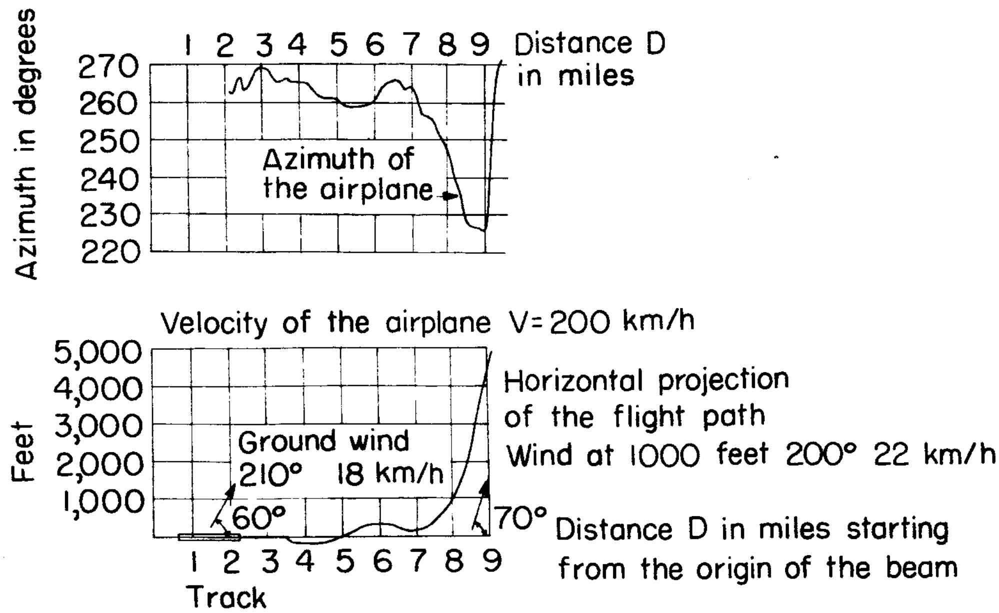


Figure 43.- Flight path actually attained in the course of an automatically controlled approach.

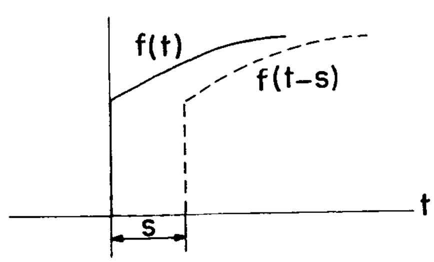


Figure 44.- Function shifted by s.

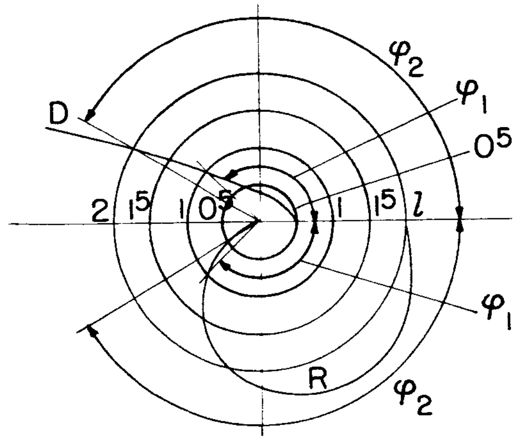


Figure 45.- Curve R: Frequency response. Curve D: Frequency demand.

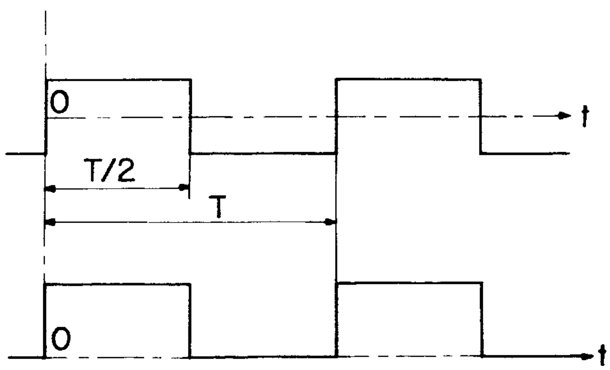


Figure 46.- Replacement of the steady unit-impulse function by a periodic function of a sufficiently large period T.

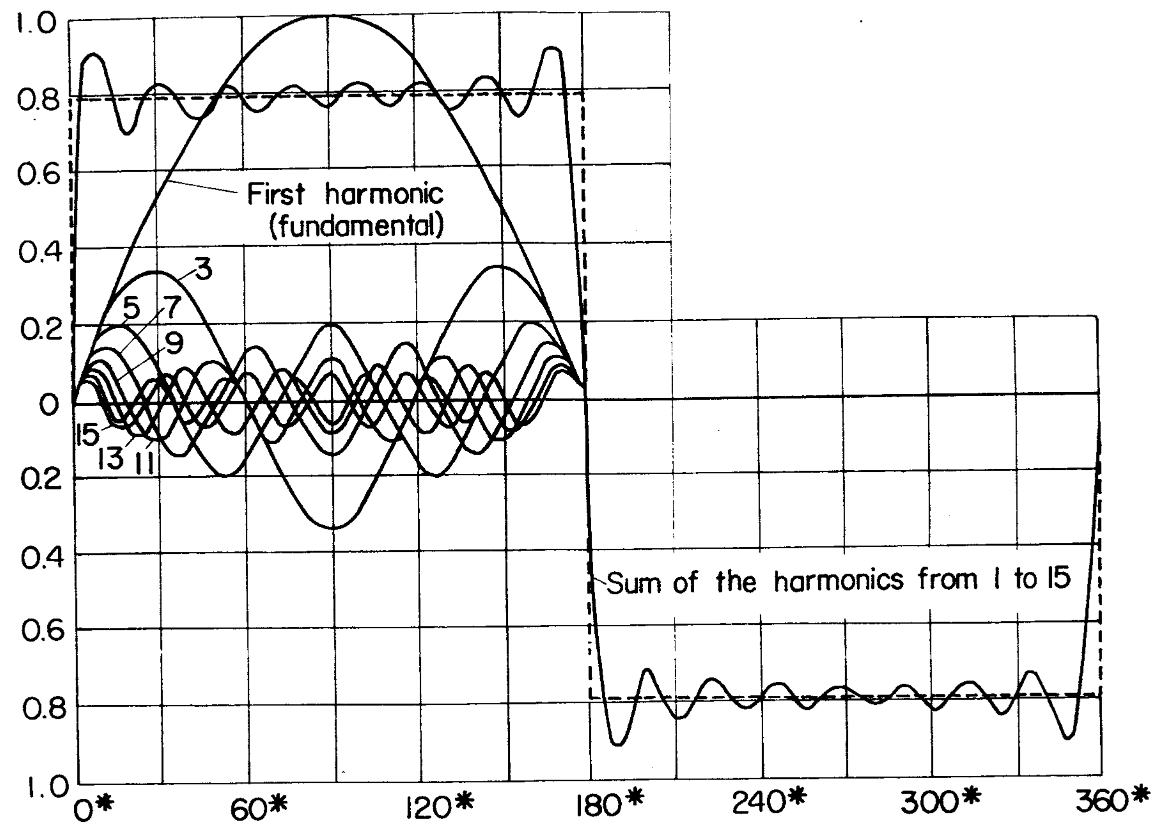
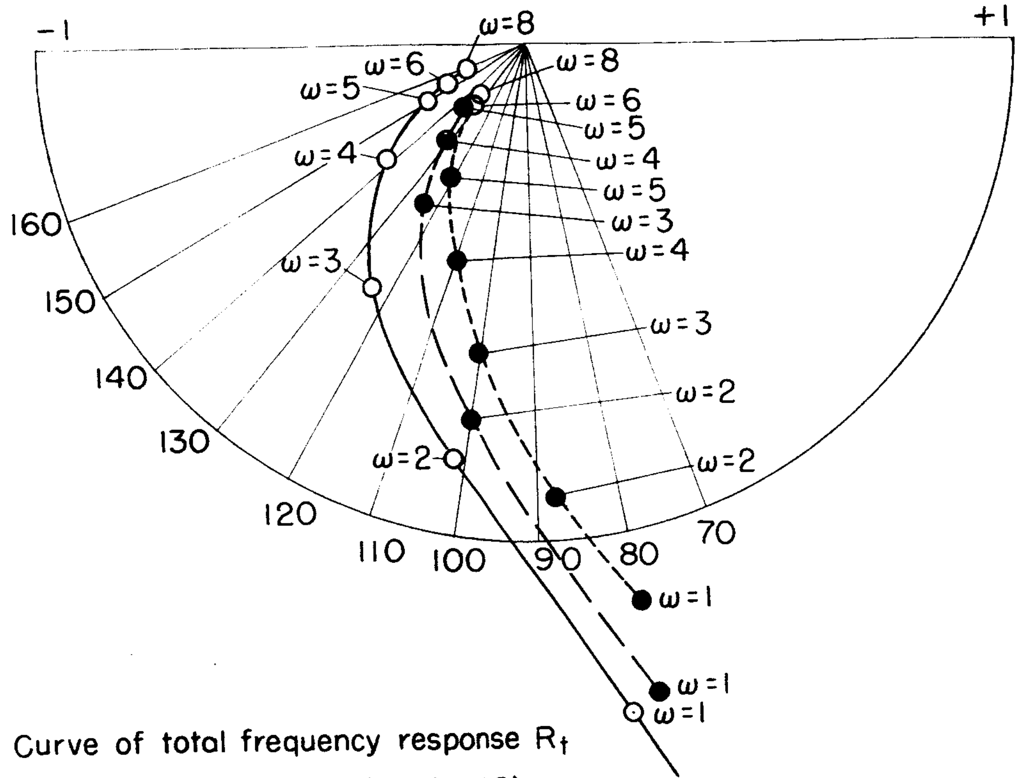


Figure 47.- Graph of the harmonics 1 to 15 and of the resultant.



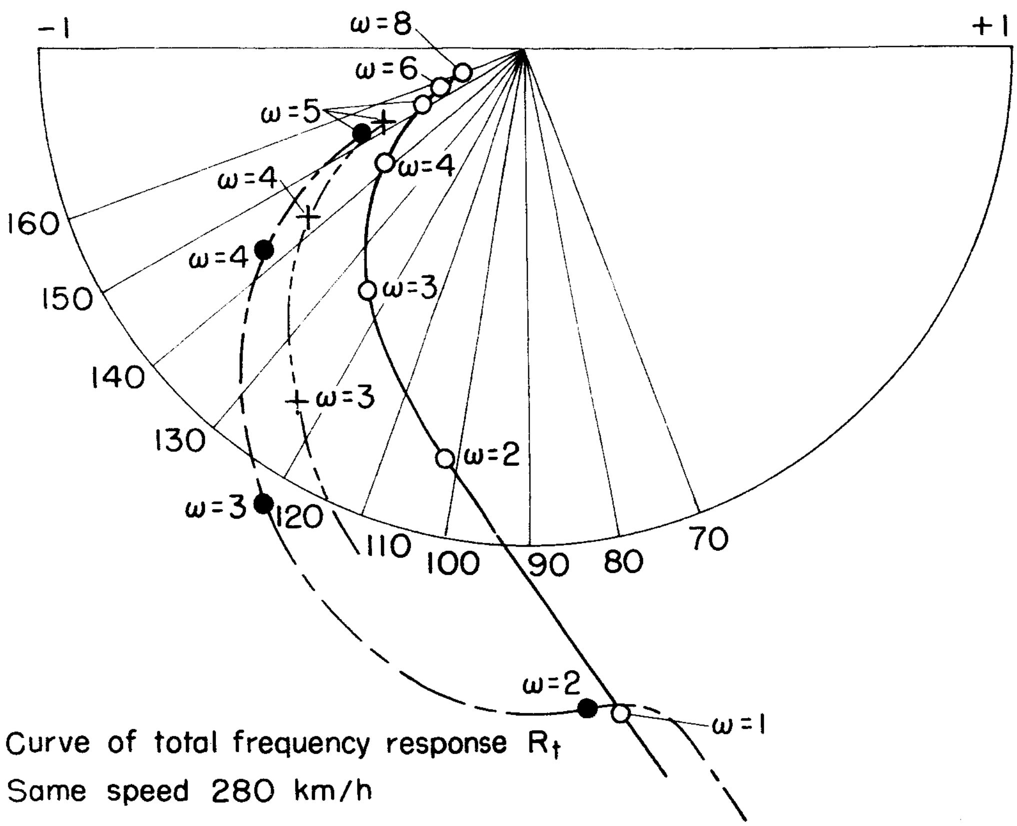
Curve of total frequency response R_t

Same with automatic pilot (A 12)

Speed of the airplane:

- - - - - -	200 mph	360 km/h
————	175 mph	280 km/h
—— ——	135 mph	216 km/h

Figure 48.- Curves of frequency response found experimentally at three different velocities.



Curve of total frequency response R_t

Same speed 280 km/h

——— automatic pilot A 12

– – – — automatic pilot C 1

—— – — automatic pilot C 1A

Figure 49.- Curves of frequency response found experimentally with three different automatic pilots.

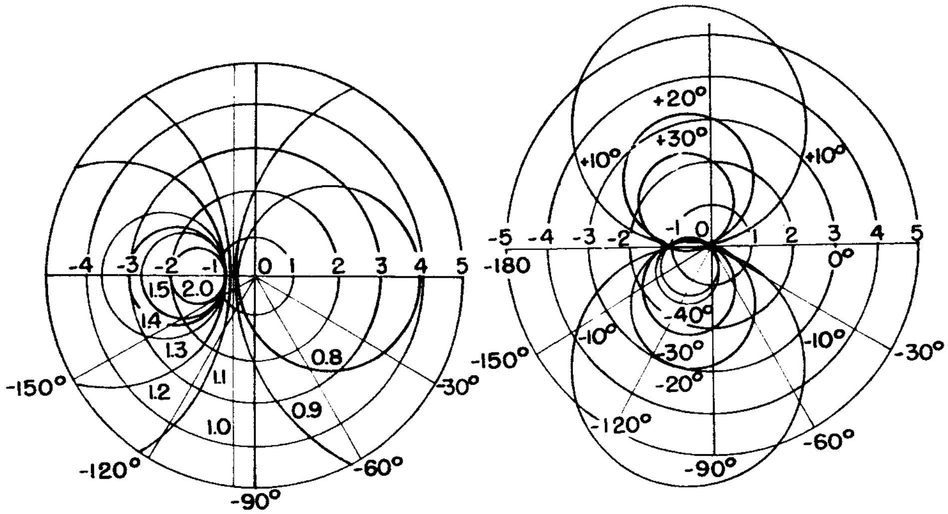


Figure 50.- Locus of equal modulus of $\frac{Z}{X}$ and locus of equal phase displacement of $\frac{Z}{X}$ of the closed-loop system, transferred to the diagram designed for outlining the curve of response $\frac{Z}{\epsilon}$ of the open system.

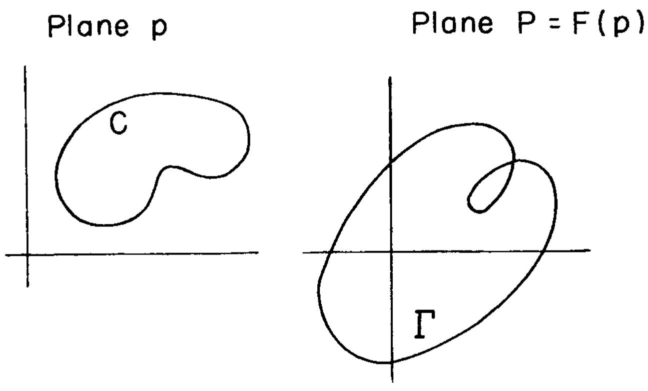


Figure 51.- Curve in the plane p and its transformation in the plane P.

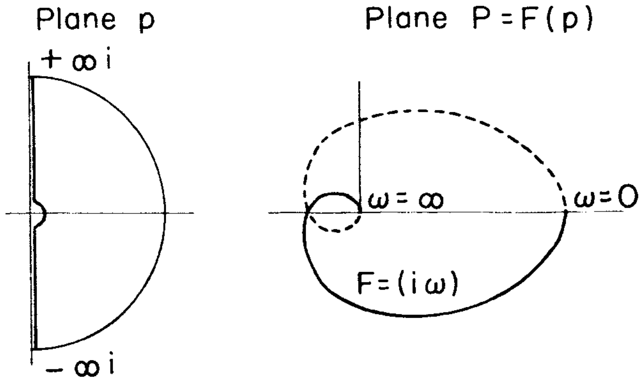


Figure 52.- Transformation in the plane P of the contour enclosing the right half of the p-plane.

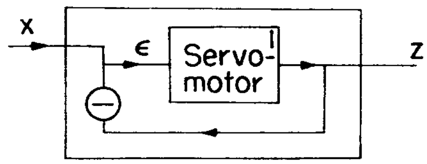


Figure 53.- Schematic diagram of a simple automatic control.

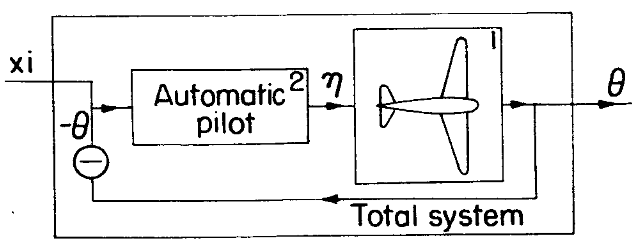


Figure 54.- Schematic diagram of an airplane equipped with an automatic pilot and an independent control acting ahead of the automatic pilot.

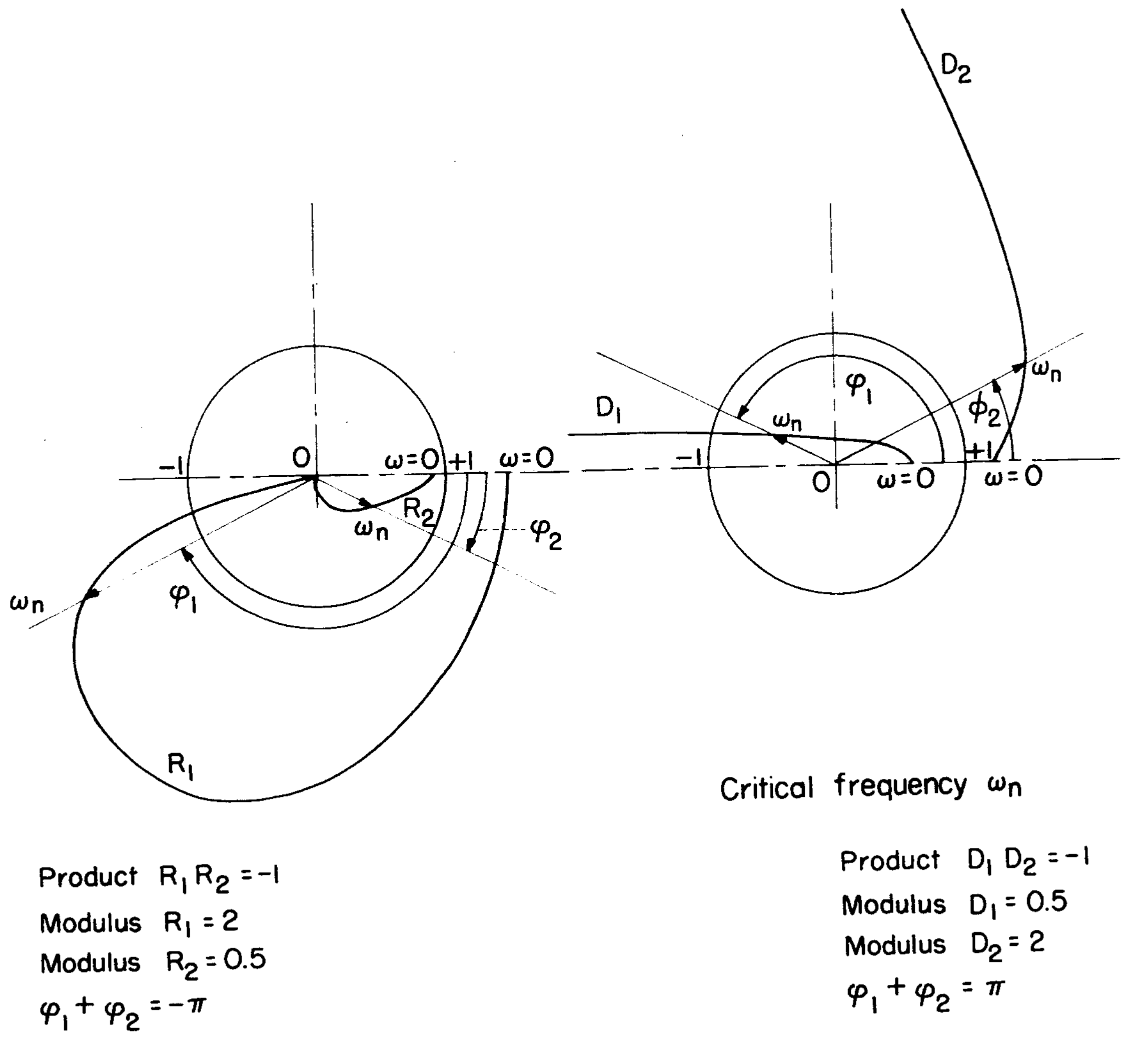


Figure 55.- Curves of response R_1 and R_2 and corresponding curves of demand D_1 and D_2 showing a critical frequency ω_n.

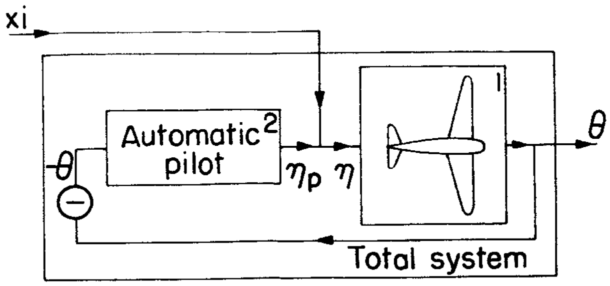


Figure 56.- Schematic diagram airplane, automatic pilot, and independent control acting after the automatic pilot.

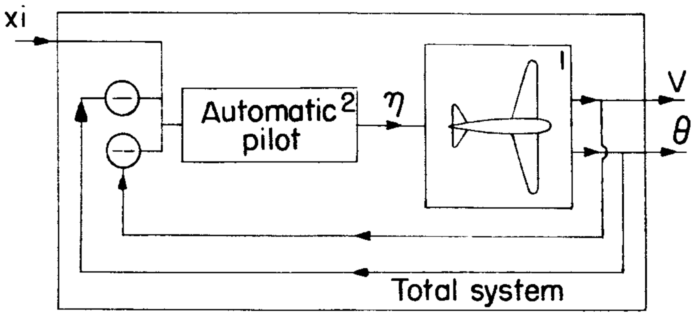


Figure 57.- Airplane and automatic pilot as a function of two variables, with independent control acting ahead of the automatic pilot.

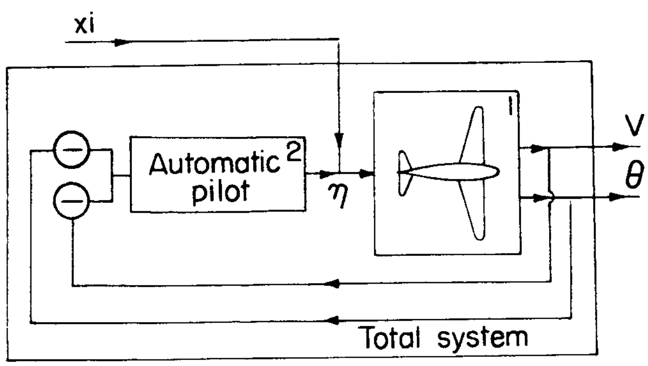


Figure 58.- Airplane, automatic pilot as a function of two variables and independent control acting after the automatic pilot.

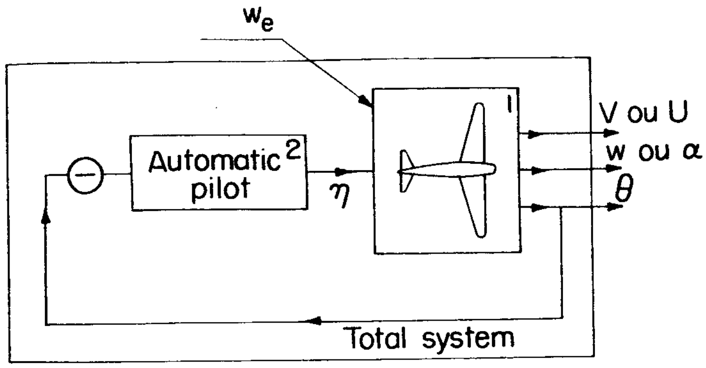


Figure 59.- Schematic diagram of an airplane equipped with an automatic pilot sensitive to the angle θ and subjected to perturbations of the entrained velocity w_e of the surrounding medium.

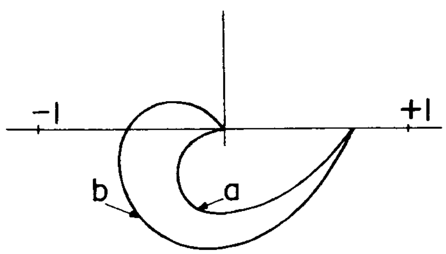


Figure 60.- Forms of the transfer function leading to instability.

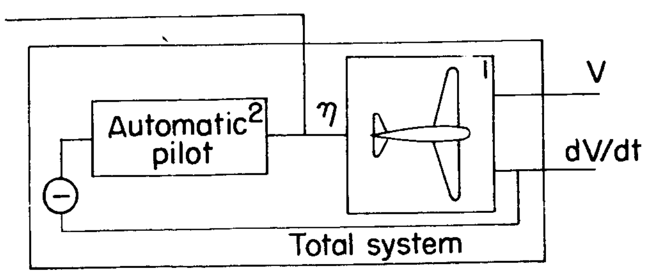


Figure 61.- Airplane and automatic pilot sensitive to the derivative of a perturbation.

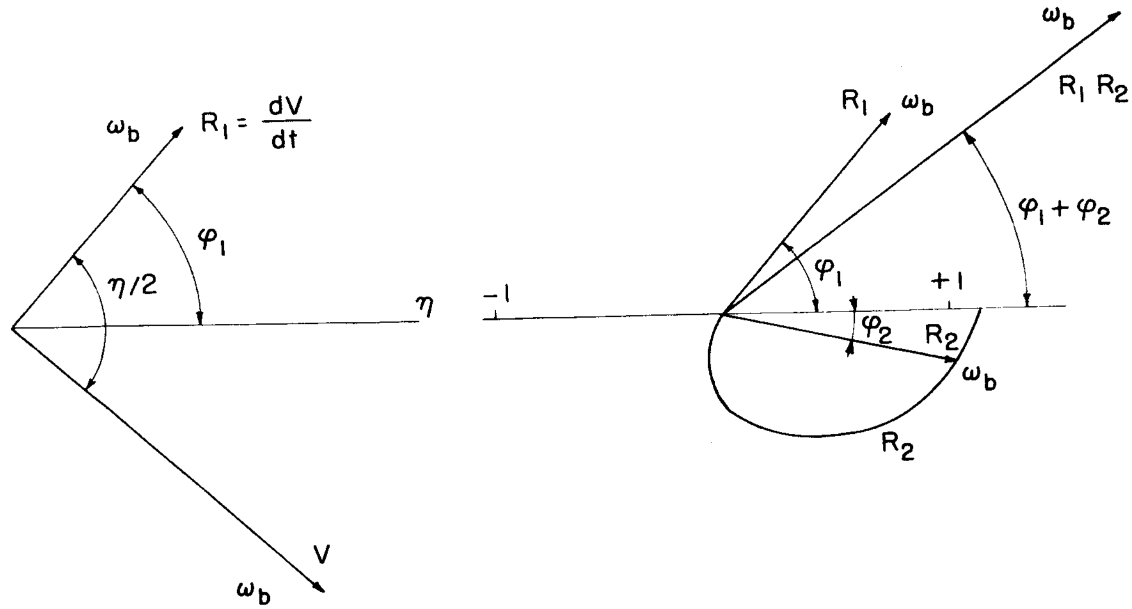


Figure 62.- Vectorial diagram showing that the presence of a derivative deviates the vector product R_1R_2 from the point -1.

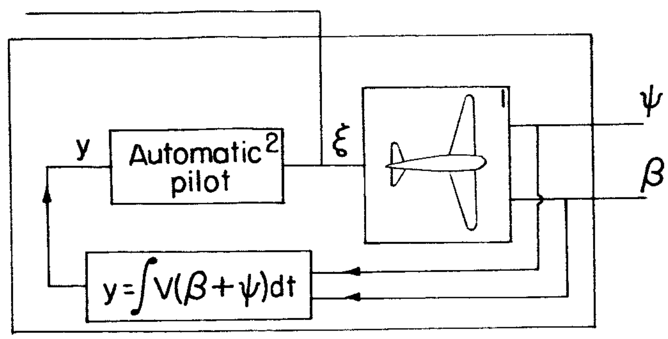


Figure 63.- Airplane and automatic pilot sensitive to the integral of a perturbation.

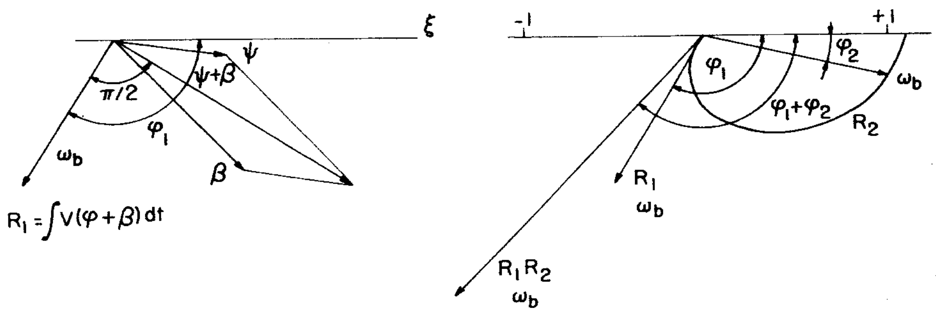


Figure 64.- Vectorial diagram showing that the presence of an integral makes the vector product R_1R_2 approach the point -1.