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Contribution to the Theory of Critical Inclination of Close Earth Satellites. 2. Case of Asymmetrical Potential*

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The previous theory is extended to the case of an asymmetrical potential of the main body especially in the case of small eccentricity by assuming that J_n, the coefficient of the nth-order zonal harmonics of the potential, is of nth order of magnitude and the eccentricity is of the first order of magnitude. Here there is a peculiar kind of libration which never occurs in the case of moderate eccentricity. This peculiar kind of libration splits into two kinds of libration, depending either on the *antisymmetrical term prevailing case* or on the *symmetrical term prevailing case*, which is a continuous transformation of the type described in the previous theory. Numerical test discloses that for the earth the former peculiar kind of libration occurs. Also it is shown that the fifth coefficient is comparatively large and plays an important role in the asymmetrical theory for the case of small eccentricity. The present paper shows that the antisymmetrical terms cannot be neglected for the earth in the case of small eccentricity.
AUTHOR

I. INTRODUCTION

IN the previous paper by the same author (1962) the motion of a close satellite in the vicinity of the so-called critical inclination was studied. However, in that paper the author omitted the effect of the antisymmetrical terms in the potential of the main bodies. The theory including these terms was given, for example, by Kozai (1961). However, in spite of his comment on the case of small eccentricity, it does not seem very extensive.

Therefore, the author has decided to develop a theory to avoid the difficulty connected with small eccentricity. A preliminary consideration shows that this difficulty arises only in the vicinity of $e=0$, and that there is no trajectory in general extending from the vicinity of $e=0$ to a value of the order of unity. Keeping this in mind, the author has expanded the Hamiltonian into a power series in e, which is assumed to be of the first order of magnitude, assuming that J_n, the coefficient of zonal harmonics of nth order, is of the nth order of magnitude, and that the deviation given by the following formula is assumed to be of the second order:

$$\alpha = 1 - 5H^2(\mu a_0)^{-1}, \tag{1.1}$$

where H, a constant, is the projected angular momentum to the equatorial plane, and a_0 the mean semimajor axis.

The necessary terms up to the sixth order of magnitude in this respect are picked up from the Hamiltonian F^*, which is given by removing the so-called periodic perturbation terms. The terms lower than the sixth order are all constant, therefore the sixth-order terms are considered as the leading terms in our theory.

The author found that the equation of motion can be reduced to the same type as the ones given by

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Andoyer (1903):

$$\frac{dh}{d\tau}=\frac{\partial\tilde{F}}{\partial h}, \quad \frac{dk}{d\tau}=-\frac{\partial\tilde{F}}{\partial k}, \tag{1.2}$$

where

$$\tilde{F}=-\beta h+(\gamma+\beta')h^2+(\gamma-\beta')k^2+(h^2+k^2)^2, \tag{1.3}$$

with a different restriction on the sign of β' than that given by Andoyer.

He assumed a restriction such as $\beta'>0$, which is always confirmed in his theory connected with the librational problem of asteroids. In our theory, however, this restriction should be removed; namely, in some cases we have $\beta'<0$ as well as $\beta'>0$. Especially in the case of the earth, β' is negative. Therefore, some change from his theory must take place. This slight change might easily be overlooked, but as seen later in this paper, some alterations are required afterwards, if we want to have real expressions for the solutions.

The main purpose of this paper is to develop the theory with this difference in mind, and exclusively to give the case for $\beta'<0$. Section II presents a preliminary discussion on obtaining the equations of motion in Andoyer's form. Section III is the same as that of his theory; however, in order to avoid some confusion the present author rewrites the results in a very compact form. In Sec. IV some changes from his own appear in separating the several cases connected with the relation between the quantities $\gamma_*=\gamma/\beta^{\frac{2}{3}}$ and $\beta_*'=\beta'/\beta^{\frac{2}{3}}$. Section V gives a classification under which the solutions connected with the quantity $u_*=u/\beta^{\frac{4}{3}}$, where u is the energy constant of the system, should be written down separately. Section VI gives the whole expressions of the solutions in the real number representation. Section VII is an Appendix which gives the characteristics, which means the trajectory within the plane of $e\cos g$ and $e\sin g$ without any attention to the relation with the time, the independent variable. Several pertinent numerical results are given. Section VIII gives some discussions related with the convergency, the relation to the previous theory and so forth. Finally Sec. IX gives the conclusion.

In order to test the assumption imposed on J_n, the quantities for the earth are listed below—not only in actual values but also in units of proper powers of $a_0\beta^{\frac{1}{3}}=0.0528$, which is assumed to be of the first order of magnitude:

$$J_2=1082.36\times10^{-6}=0.3882\,\beta^{2/3}a_0^2,$$
$$J_3=-2.566\times10^{-6}=-0.0174\,\beta^{3/3}a_0^3,$$
$$J_4=-2.14\times10^{-6}=-0.275\,\beta^{4/3}a_0^4,$$
$$J_5=-0.063\times10^{-6}=-0.154\,\beta^{5/3}a_0^5.$$

Roughly speaking, these numerical values show that J_2, J_4, and J_5 play approximately the same order of role in the vicinity of $e=0$, say $e=0.10$ or 0.05; on the other hand, J_3 does not take any important role here.

II. EQUATIONS OF MOTION

The potential, under which influence a negligible small mass particle moves, is assumed in the following form:

$$V=\frac{\mu}{r}\left[1-\sum J_n\left(\frac{a_e}{r}\right)^n P_n(\sin\delta)\right], \tag{2.1}$$

where $\mu=Mk^2$, k being the Gaussian constant and M the mass of the main body, r is the distance of the particle from the center of the main body, δ is the declination of the particle, J_n represent numerical constants which characterize the spheroidal potential; in this paper the summation $\sum$ extends from 2 to 5, as seen later.

Using a result of the so-called secular and long periodic parts in the original Hamiltonian such as given by Kozai (1962a), we may pick up only the following necessary parts provided that J_n is assumed to be of the nth order of magnitude as well as the eccentricity e to be of the first order:

$$F^*=\frac{3}{16}\frac{\mu^4J_2}{L^6}(e^2-\alpha)^2+\frac{21}{40}\frac{\mu^6J_4}{L^{10}}e^2-\frac{3}{40}\frac{\mu^6J_2^2}{L^{10}}e^2\cos 2g$$
$$-\frac{3}{40}\frac{\mu^6J_4}{L^{10}}e^2\cos 2g+\frac{3}{4(5^{\frac{1}{2}})}\frac{\mu^5J_3}{L^8}e(e^2-\alpha)\sin g$$
$$+\frac{3}{4(5^{\frac{1}{2}})}\frac{\mu^7J_2J_3}{L^{12}}e\sin g+\frac{9}{10(5^{\frac{1}{2}})}\frac{\mu^7J_5}{L^{12}}e\sin g, \tag{2.2}$$

where $\alpha=1-5H^2(\mu a_0)^{-1}=$ a constant which is assumed to be of the second order, and the other notations correspond to Delaunay's. In this expression, the terms of beyond the sixth order of magnitude are neglected. As the Hamiltonian has only sixth order of magnitude and nothing else, neglecting the higher orders, we may take ξ and η as the canonical variables,

$$L^{\frac{1}{2}}e\sin g=\xi, \quad L^{\frac{1}{2}}e\cos g=\eta,$$

in the canonical equations of motion:

$$\frac{d\xi}{dt}=\frac{\partial F^*}{\partial\eta}, \quad \frac{d\eta}{dt}=-\frac{\partial F^*}{\partial\xi}. \tag{2.3}$$

For mathematical simplicity, if the canonical variables and the independent variable are changed:

$$k=e\cos g, \quad h_1=e\sin g,$$
$$\tau=\frac{3}{16}\frac{\mu^4J_2}{L^7}t=\frac{3}{16}\frac{\mu^{\frac{1}{2}}J_2}{a_0^{7/2}}t, \tag{2.4}$$

then the equations of motion will be

$$\frac{dh_1}{d\tau}=\frac{\partial\tilde{F}_1}{\partial k},\quad \frac{dk}{d\tau}=-\frac{\partial\tilde{F}_1}{\partial h_1},\tag{2.5}$$

where by neglecting an unnecessary constant in F^*,

$$\tilde{F}_1=\frac{16}{3}\frac{L^6}{\mu^4J_2}(F^*-\text{const})$$
$$=-\beta_1h+(\gamma_1+\beta_1')h_1^2+(\gamma_1-\beta_1')k^2+\beta_1''(h_1^2+k^2)h_1+(h_1^2+k^2)^2,\tag{2.6}$$

with

$$\beta_1=+\alpha\frac{4}{5^{\frac{1}{2}}}\frac{J_3}{J_2a_0}-\frac{4}{5^{\frac{1}{2}}}\frac{J_3}{a_0^3}-\frac{24}{5(5^{\frac{1}{2}})}\frac{J_5}{J_2a_0^3},$$
$$\gamma_1=\frac{14}{5}\frac{J_4}{J_2a_0^2}-2\alpha,$$
$$\beta_1'=\frac{2}{5}\left(J_2+\frac{J_4}{J_2}\right)\frac{1}{a_0^2},\tag{2.7}$$
$$\beta_1''=\frac{4}{5^{\frac{1}{2}}}\frac{J_3}{J_2a_0}.$$

In order to remove the third-degree terms in h_1 and k, h_1 is changed to h given by

$$h=h_1+\tfrac{1}{4}\beta_1''.\tag{2.8}$$

Further, if the equations are transformed in terms of p and q,

$$p=h+ik,\quad q=h-ik,\tag{2.9}$$

the equations can be written as follows:

$$\frac{dp}{d\tau}=-2i\frac{\partial\tilde{F}}{\partial q},\quad \frac{dq}{d\tau}=+2i\frac{\partial\tilde{F}}{\partial p},\tag{2.10}$$

with

$$\tilde{F}=-\tfrac{1}{2}\beta(p+q)+\gamma pq+\tfrac{1}{2}\beta'(p^2-q^2)+p^2q^2,\tag{2.11}$$

where

$$\tilde{F}-\tilde{F}_1=-\frac{2}{5}\alpha\frac{J_3^2}{J_2^2a_0^2}+\frac{18}{25}\frac{J_3^2}{J_2a_0^4}-\frac{16}{25}\frac{J_3^2J_4}{J_2^3a_0^4}+\frac{24}{25}\frac{J_3J_5}{J_2^2a_0^4}+\frac{3}{25}\frac{J_3^4}{J_2^4a_0^4}=\text{const},$$
$$\beta=\left(-\frac{4}{5}+\frac{8}{5}\frac{J_4}{J_2^2}-\frac{2}{5}\frac{J_3^2}{J_2^3}\right)\frac{4}{5^{\frac{1}{2}}}\frac{J_3}{a_0^3}-\frac{24}{5(5^{\frac{1}{2}})}\frac{J_5}{J_2a_0^3},\tag{2.12}$$
$$\gamma=-2\alpha+\frac{14}{5}\frac{J_4}{J_2a_0^2}-\frac{4}{5}\frac{J_3^2}{J_2^2a_0^2},$$

and

$$\beta'=\frac{2}{5a_0^2}\left(J_2+\frac{J_4}{J_2}-\frac{J_3^2}{J_2^2}\right).$$

The form of the Hamiltonian (2.11) shows that it is the quadratic in p or q, respectively; this was the technique which Andoyer (1903) used for the problem associated with the libration near the commensurability between the mean motion of asteroids and that of the disturbing body, Jupiter. Therefore, we can follow after his development in order to solve the equation of motion. Nevertheless, there is a slight difference between his case and ours concerning the sign of some coefficient. He assumed that the coefficient β' is always positive; but in our case β' can take a negative value as well as a positive one depending on the interrelation of the magnitude of J_n's. Especially for the earth it is negative, as seen later. Therefore, we are restricted to take the negative β' case here unless otherwise mentioned, because for the positive case Andoyer's results are available.

III. GENERAL EXPRESSION OF SOLUTION

The general form of the solutions does not change from Andoyer's results. But, in order to avoid some confusion about the real/complex criterion, we shall make a slight change in the notation. The calculations are omitted and only the results obtained are given here.

The solutions of Eqs. (2.10) are given in the following form:

$$p=\frac{1}{i(a_0')^{\frac{1}{2}}}[\zeta(\tau+\lambda+\tfrac{1}{2}\rho-\tfrac{1}{2}\chi)-\zeta(\tau+\lambda-\tfrac{1}{2}\rho-\tfrac{1}{2}\chi)-\tfrac{1}{2}\zeta(\rho+\chi)-\tfrac{1}{2}\zeta(\rho-\chi)],$$
$$q=\frac{1}{i(a_0')^{\frac{1}{2}}}[\zeta(\tau+\lambda+\tfrac{1}{2}\rho+\tfrac{1}{2}\chi)-\zeta(\tau+\lambda-\tfrac{1}{2}\rho+\tfrac{1}{2}\chi)-\tfrac{1}{2}\zeta(\rho+\chi)-\tfrac{1}{2}\zeta(\rho-\chi)],\tag{3.1}$$

with

$$\wp(\rho)=-(a_1^2+a_0'a_2)(a_0')^{-1}\quad\text{(real)},$$
$$\wp'(\rho)=i(a_0'^2a_3+3a_0'a_1a_2+2a_1^3)(a_0')^{-\frac{3}{2}}\quad\text{(imaginary)},$$
$$\wp(\chi)=\wp(\rho)+\tfrac{1}{4}i(a_0')^{\frac{1}{2}}\beta^{-1}\wp'(\rho)\quad\text{(real)},\tag{3.2}$$
$$\wp'(\chi)=-\beta'\beta^{-1}(a_0')^{\frac{1}{2}}\wp'(\rho)\quad\text{(imaginary)},$$

where

$$a_0'=-8\beta',\quad a_1=-2\beta,$$
$$a_2=\tfrac{1}{3}(8u+2\beta'^2-2\gamma^2),\quad a_3=\beta(\gamma-\beta'),\tag{3.3}$$
$$a_4=8u\beta'-\beta^2;$$

u is the energy constant such that

$$\tilde{F}+u=0.\tag{3.4}$$

$\wp(z)$ is the Weierstrassian elliptic function with the parameters

$$g_2 = -a_0'a_4 - 4a_1a_3 + 3a_2^2, \qquad (3.5)$$
$$g_3 = 2a_1a_2a_3 - a_0'a_2a_4 - a_1^2a_4 + a_0'a_3^2 - a_2^3,$$

such that

$$[\wp'(z)]^2 = 4\wp^3(z) - g_2\wp(z) - g_3, \qquad (3.6)$$

where $\wp'(z)$ denotes the derivative respective to z. $\zeta(z)$ is the associated zeta function defined by

$$\zeta(z) = \frac{1}{z} - \int_0^z \left[\wp(\nu) - \frac{1}{\nu^2}\right] d\nu. \qquad (3.7)$$

In the expression of solution (3.1) λ is an integration constant, which with the energy constant u forms a system of the arbitrary constants in the solution of the equations of motion. u is of course a real value; λ is, however, not necessarily real. λ should be taken such that the expression (3.1) could give real values of h and k,

$$h = \tfrac{1}{2}(p+q), \quad k = \frac{1}{2i}(p-q). \qquad (2.9')$$

IV. DISCRIMINATION AMONG SEVERAL CASES

At the first step it is necessary to know the sign of the discriminant Δ,

$$\Delta = g_2^3 - 27g_3^2. \qquad (4.1)$$

For brevity, let

$$P = \wp(\chi) \equiv \tfrac{1}{3}(4u - \gamma^2 + 6\beta'\gamma - 5\beta'^2),$$

and $\qquad (4.2)$

$$Q = \wp'(\chi)/i \equiv 16u\beta' - 4\beta'(\beta'-\gamma)^2 - 2\beta^2,$$

which are also real values even for our case, then

$$g_2 = 12P^2 - 4(\gamma - 2\beta')Q,$$

and $\qquad (4.3)$

$$g_3 = -8P^3 + Q^2 + 4(\gamma - 2\beta')PQ.$$

It follows accordingly that

$$\Delta = 1024Q^2\Big\{u^3 - \tfrac{1}{2}(\gamma+\beta')^2u^2 + \tfrac{1}{16}(\gamma+\beta')[(\gamma+\beta')^3 + 9\beta^2]u - \tfrac{1}{64}\beta^2\Big[(\gamma+\beta')^3 + \frac{27}{4}\beta^2\Big]\Big\}. \qquad (4.1')$$

For brevity, let

$$u_* = u/\beta^{\frac{4}{3}}, \quad \beta_*' = \beta'/\beta^{\frac{2}{3}},$$
$$\gamma_* = \gamma/\beta^{\frac{2}{3}}, \quad \Delta_* = \Delta/\beta^8, \qquad (4.4)$$
$$Q_* = Q/\beta^2, \quad P_* = P/\beta^{\frac{4}{3}},$$

and so forth, then

$$\Delta_* = 1024Q_*^2\Big\{u_*^3 - \tfrac{1}{2}(\gamma_* + \beta_*')u_*^2 + \tfrac{1}{16}(\gamma_* + \beta_*')[(\gamma_* + \beta_*')^3 + 9]u_* - \frac{1}{64}\Big[(\gamma_* + \beta_*')^3 + \frac{27}{4}\Big]\Big\}. \qquad (4.1'')$$

If the subscript $*$ is omitted in this expression, then every term remains the same as in (4.1′) except for β which is replaced by unity; besides, as is easily seen, β is of the third order of magnitude, and we may choose

$$h_* = h/\beta^{\frac{1}{3}}, \quad k_* = k/\beta^{\frac{1}{3}}. \qquad (4.4')$$

As a result, all quantities are measured in units of proper powers of $\beta^{\frac{1}{3}}$, which is assumed to be of the first order of magnitude, provided that $\beta \neq 0$. Therefore without any restriction we may take $\beta = 1$ hereafter unless otherwise stated. This is an application of the nondimensional analysis.

In any way, there are two cases for $\Delta = 0$:

(i) $$Q = 0. \qquad (4.5)$$

If we consider (4.5) as an equation for u, then we know the value u_0 which satisfies this equation:

$$u_0 = \tfrac{1}{4}(\beta' - \gamma)^2 + \tfrac{1}{8}\beta'^{-1}. \qquad (4.6)$$

(ii) $$f(u) \equiv u^3 - \tfrac{1}{2}(\gamma+\beta')u^2 + \frac{\gamma+\beta'}{16}[(\gamma+\beta')^3 + 9]u - \frac{1}{64}\Big[(\gamma+\beta')^3 - \frac{27}{4}\Big] = 0. \qquad (4.7)$$

Here two cases again are divided according to the sign of the discriminant Δ' of $f(u)$ itself:

$$\Delta' = -\frac{1}{128}\Big\{(\gamma+\beta)^3 + \frac{27}{8}\Big\}^3; \qquad (4.8)$$

namely, if

$$\gamma + \beta' > -\tfrac{3}{2}, \qquad (4.9)$$

then among the three roots u_1, u_2, u_3 of Eq. (4.7) in terms of u, two of them have complex values, while the other, say u_1, remains real. On the other hand, if

$$\gamma + \beta' < -\tfrac{3}{2}, \qquad (4.10)$$

then the three roots are all real,

$$u_3 < u_2 < u_1.$$

Now, it is necessary to know in what sequence the roots $u_i (i = 0,1,2,3)$ appear; in other words, to know the relation between u_0 and $u_i (i = 1,2,3) (\Delta' > 0)$ or $u_1 (\Delta' < 0)$. In the case when (4.9) occurs we can easily determine it in the following way: first of all we have

$$f(u) \gtrless 0, \quad \text{if and only if } u \gtrless u_1; \tag{A}$$

however, since

$$f(u_0)=\frac{9}{64}(\gamma^2+\tfrac{1}{2}\beta'^{-1})P_0^2, \tag{4.11}$$

where P_0 denotes the value of P when $u=u_0$, we have, if $\beta'>0$, the right-hand side of (4.11) is always positive, but if $\beta'<0$, this is not always true. Consequently, we have a new discrimination, which never occurs in the case when $\beta'>0$:

$$\text{if} \quad \gamma^2+\tfrac{1}{2}\beta'^{-1} \lesseqgtr 0 \quad \text{then} \quad u_0 \lesseqgtr u_1. \tag{B}$$

On the other hand, in the case when (4.10) is satisfied, we may only say that

$$\begin{aligned} &\text{if} \quad \gamma^2+\tfrac{1}{2}\beta'^{-1}<0, && \text{then} \quad u_2<u_0<u_1, \\ & && \text{or} \quad u_0<u_3; \\ &\text{and if} \quad \gamma^2+\tfrac{1}{2}\beta'^{-1}>0, && \text{then} \quad u_1<u_0, \\ & && \text{or} \quad u_3<u_0<u_2. \end{aligned} \tag{C}$$

The detailed criterion will be given in a later part.

At the second step it is necessary to divide the cases according to the sign of $(g_3)_i$ $(i=0,1,2,3)$, where $(g_3)_i$ denotes the value of g_3 when $u=u_i$. This is important because, for example, if $(g_3)_i<0$ then the energy constant u_i corresponds to the unstable equilibrium point(s). It is easily confirmed for $(g_3)_0$, since

$$(g_3)_0=-8P_0^3=-\frac{512}{27}\beta'^3(\gamma-\beta'+\tfrac{1}{8}\beta'^{-2})^3; \tag{4.12}$$

when $\beta'<0$,

$$\text{if} \quad \gamma-\beta'+\tfrac{1}{8}\beta'^{-2} \lesseqgtr 0, \quad \text{then} \quad (g_3)_0 \lesseqgtr 0. \tag{D}$$

In order to know the relation between u_0 and u_i $(i=1,2,3)$ in more detail as well as the sign of $(g_3)_i$ it is convenient to divide the γ vs β' plane into several parts so that in each of them the situation never changes. For this purpose, it is sufficient to draw lines where the situation would change. The following four lines serve this purpose:

$$\gamma+\beta'=-\tfrac{3}{2}, \tag{4.13}$$

$$\gamma+(-2\beta')^{-\frac{1}{2}}=0, \tag{4.14}$$

$$\gamma-(-2\beta')^{-\frac{1}{2}}=0, \tag{4.14'}$$

and

$$\gamma-\beta'+\tfrac{1}{8}(\beta')^{-2}=0. \tag{4.15}$$

In effect, $u_0=u_i$ occurs only on the lines (4.14) or (4.14′); $u_1=u_2$ or $u_2=u_3$ only on the line (4.13); and $(g_3)_0=0$ only on (4.15), $(g_3)_i=0$ $(i\neq 0)$ occurs on either (4.14), (4.14′) or (4.13), as is easily confirmed.

Before dividing the plane by the four lines given above, for the sake of simplicity let us consider the division by separating the negative β' half-plane into three parts:

Case I. $\beta'<-2$;

in this case we have

$$-\beta'-\tfrac{3}{2}>+(-2\beta')^{-\frac{1}{2}}>-(-2\beta')^{-\frac{1}{2}}>\beta'-\tfrac{1}{8}(\beta')^{-2}. \tag{4.16}$$

Case II. $-2<\beta'<-\tfrac{1}{2}$,

$$(-2\beta')^{-\frac{1}{2}}>-\beta'-\tfrac{3}{2}>-(-2\beta')^{-\frac{1}{2}}>\beta'-\tfrac{1}{8}(\beta')^{-2}. \tag{4.17}$$

Case III. $-\tfrac{1}{2}<\beta'<0$,

$$(-2\beta')^{-\frac{1}{2}}>-\beta'-\tfrac{3}{2}>-(-2\beta')^{-\frac{1}{2}}>\beta'-\tfrac{1}{8}(\beta')^{-2}. \tag{4.18}$$

The reason why we make a discrimination between case II and case III is that

$$\text{if} \quad \beta'=-\tfrac{1}{2}$$

then

$$-\beta'-\tfrac{3}{2}=-(-2\beta')^{-\frac{1}{2}}=\beta'-\tfrac{1}{8}\beta'^{-2}=-1,$$

so that a domain, $-\beta'-\tfrac{3}{2}>\gamma>-(-2\beta')^{-\frac{1}{2}}$ for example, is cut out at $\beta'=-\tfrac{1}{2}$ and there is no continuous route connecting the regions corresponding to case II and case III without meeting any one of the above four lines.

Now, we subdivide the regions with respect to the four lines stated above; for example, case I_1 means $\beta'<-2$ and $\gamma>-\beta'-\tfrac{3}{2}$, case I_2 $\beta'<-2$ and $-\beta'-\tfrac{3}{2}>\gamma(-2\beta')^{-\frac{1}{2}}$ and so forth. This subdivision is not essential (there is, nevertheless, a practical advantage, because β', or speaking more precisely $\beta'_*=\beta'/\beta^{\frac{3}{2}}$, depends only on the coefficients of the zonal harmonics for the potential and is totally independent of the initial condition) as a whole because cases I_1, II_1, and III_1, for example, have no differences in the sense of continuous deformation. Therefore, it is useful to assemble some of the too subdivided cases into one case, then finally we have the folllowing division:

Case 1	which involves cases	I_1, II_1, III_1;
Case 2′	which involves cases	I_2 ;
Case 2″	which involves cases	II_2, III_2;
Case 3′	which involves cases	I_3, II_3 ;
Case 3″	which involves cases	III_3;
Case 4′	which involves cases	I_4, II_4 ;
Case 4″	which involves cases	III_4;
Case 5	which involves cases	I_5, II_5, III_5.

Figure 1 shows the division in the β' vs γ plane. Here for the sake of comparison, the case where $\beta'>0$ (Andoyer's subcases a', a'', b', and b'') is added.

We evaluate u_i and $(g_3)_i$ in each case either by giving extreme values for β' and γ or by giving special

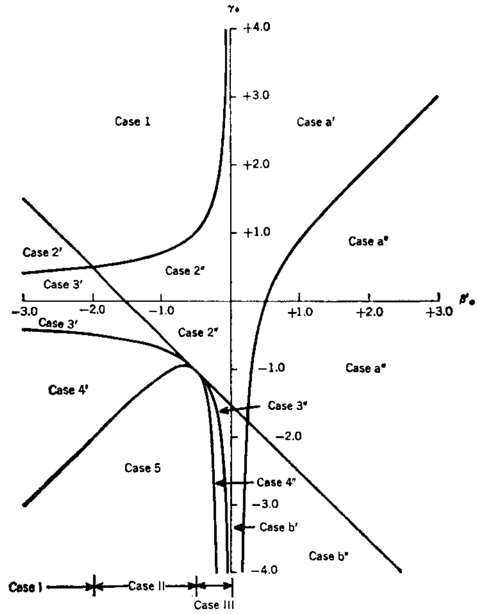

FIG. 1. Division in the β'_* vs γ_* plane. $\beta'_*=\beta'/\beta^{\frac{1}{3}}$ and $\gamma_*=\gamma/\beta^{\frac{2}{3}}$.

numerical values as given in Sec. VII. In any case one set of values for β' and γ is enough to determine the sign of $(g_3)_i$ and the relation of u_0 and u_i $(i=1,2,3)$.

As an example, case 3′ is shown where extreme values are given.

Case 3′ $\quad \beta'=-\epsilon^{-\frac{1}{3}},\quad \gamma=+\epsilon^{\frac{2}{3}},\quad \epsilon\rightarrow+0,$

$$u_0=\tfrac{1}{4}\epsilon^{-\frac{2}{3}}+\tfrac{3}{8}\epsilon^{\frac{1}{3}},$$

$$(g_3)_0=+\frac{512}{27}\epsilon^{-2},$$

$$u_{1,2}=\tfrac{1}{4}\epsilon^{-\frac{2}{3}}\pm\frac{\sqrt{2}}{2}\epsilon^{-1/6},$$

$$(g_3)_{1,2}=+\frac{512}{27}\epsilon^{-2},$$

$$u_3=-\tfrac{1}{4}\epsilon^{\frac{1}{3}},$$

and

$$(g_3)_2=-\frac{7}{28}\epsilon^{-2},$$

where unnecessary higher-order terms are omitted.

Therefore

$$u_3<u_2<u_0<u_1,$$

and

$$(g_3)_0>0,\quad (g_3)_1>0,\quad (g_3)_2>0,\quad \text{and}\quad (g_3)_3<0.$$

Table I shows the results. From this table, combined with (4.7), it is easy to find the sign of $\Delta(u)$ [e.g., in the case 3′, if $u_1<u$ then $\Delta(u)>0$, if $u_0<u<u_1$ or $u_2<u<u_0$ then $\Delta(u)<0$, and so forth].

V. VALUES OF ρ AND χ

For the purpose of giving a real expression of the solutions, it is also necessary to have the range in which ρ or χ falls. As a first step, a comparison is made here between $\wp(\rho)$ or $\wp(\chi)$ and the parameters of $\wp$ function, namely e_1, e_2, and e_3. Let

$$4\wp^3(z)-g_2\wp(z)-g_3 =4[\wp(z)-e_1][\wp(z)-e_2][\wp(z)-e_3], \quad (5.1)$$

then it is well known that if the discriminant $\Delta>0$, the three parameters e_1, e_2, and e_3 are all real; contrary to this, if $\Delta<0$, then only one of them, say e_1, is real.

Since

$$\wp'(\rho)^2=4[\wp(\rho)-e_1][\wp(\rho)-e_2][\wp(\rho)-e_3], \quad (5.2)$$

$$\wp(\rho)=-(a_1^2+a_1^2a_2)a_0' \quad \text{is real,} \quad (3.2)$$

and

$$\wp'(\rho)=i(a_0^2a_3+3a_0'a_1a_2+2a_1^3)(a_0')^{-\frac{1}{2}} \quad \text{is imaginary,}$$

TABLE I. Roots of $\Delta(u)=0$ and sign of g_3 at respective roots.

Case	u_i	$(g_3)_0$	$(g_3)_1$	$(g_3)_2$	$(g_3)_3$
Case 1	$u_1<u_0$	+	+		
Case 2′	$u_3<u_2<u_1<u_0$	+	+	+	−
Case 2″	$u_0<u_1$	+	+		
Case 3′	$u_3<u_2<u_0<u_1$	+	+	+	−
Case 3″	$u_0<u_3<u_2<u_1$	+	+	−	+
Case 4′	$u_3<u_0<u_2<u_1$	+	+	+	−
Case 4″	$u_3<u_0<u_2<u_1$	+	+	−	+
Case 5	$u_3<u_0<u_2<u_1$	−	+	+	+

it follows that, when $\Delta>0$:

$$\wp(\rho)<e_3 \quad \text{or} \quad e_2<\wp(\rho)<e_1, \quad (5.3)$$

where the three arguments are denoted such that

$$e_3<e_2<e_1.$$

Similarly, with

$$\wp'(\chi)=\wp(\rho)+\tfrac{1}{4}i(a_0')^{\frac{1}{2}}\beta^{-1}\wp'(\rho)\equiv P \quad \text{is real}$$

$$\wp'(\chi)=-\beta'\beta^{-1}(a_0')^{\frac{1}{2}}\wp'(\rho)\equiv iQ \quad \text{is imaginary,} \quad (4.2)$$

when $\Delta>0$:

$$\wp(\chi)<e_3 \quad \text{or} \quad e_2<\wp(\chi)<e_1. \quad (5.4)$$

In the case when $\Delta<0$, it follows necessarily that

$$\wp(\rho)<e_1$$

and $\quad (5.5)$

$$\wp(\chi)<e_1.$$

Besides, from (4.2) it follows that

$$\wp(\rho)\lesseqgtr\wp(\chi), \quad \text{if} \quad Q\lesseqgtr 0,$$

that is to say $(\beta'<0)$, $\quad$ (E)

$$u\gtreqless u_0.$$

TABLE II. The values of $P=\wp(\chi)$ and $\wp=\wp(\rho)$ $(\beta'<0)$.

Case	Range of u	Values of P and $\wp$	Sign of $\wp'(\chi)/i$ and $\wp'(\rho)/i$	Class[a]
Case 1	$u_0<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u=u_0$	$P=e_3=e_2=\wp$	0	
	$u_1<u<u_0$	$P<e_3<e_2<\wp<e_1$	$+$	02
	$u<u_1$	$P<\wp<e_1$	$+$	12
Case 2′	$u_0<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u=u_0$	$\wp=e_3=e_2=P$	0	
	$u_1<u<u_0$	$P<e_3<e_2<\wp<e_1$	$+$	02
	$u_2<u<u_1$	$P<\wp<e_1$	$+$	12
	$u_3<u<u_2$	$P<\wp<e_3<e_2<e_1$[b]	$+$	22
	$u<u_3$	$P<\wp<e_1$[b]	$+$	12
Case 2″	$u_1<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u_0<u<u_1$	$\wp<P<e_1$	$-$	11
	$u=u_0$	$\wp=e_3=e_2=P<e_1$	0	
	$u<u_0$	$P<\wp<e_1$	$+$	12
Case 3′	$u_1<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u_0<u<u_1$	$\wp<P<e_1$	$-$	11
	$u=u_0$	$\wp=e_3=e_2=P$	0	
	$u_2<u<u_0$	$P<\wp<e_1$	$+$	12
	$u_3<u<u_2$	$P<\wp<e_3<e_2<e_1$[b]	$+$	22
	$u<u_3$	$P<\wp<e_1$[b]	$+$	12
Case 3″	$u_1<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u_2<u<u_1$	$\wp<P<e_1$[c]	$-$	11
	$u_3<u<u_2$	$\wp<P<e_3<e_2<e_1$[c]	$-$	21
	$u_0<u<u_3$	$\wp<P<e_1$	$-$	11
	$u=u_0$	$\wp=e_2=e_3=P<e_1$	0	
	$u<u_0$	$P<\wp<e_1$	$+$	12
Case 4′	$u_1<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u_2<u<u_1$	$\wp<P<e_1$	$-$	11
	$u_0<u<u_2$	$e_3<e_2<\wp<P<e_1$	$-$	23
	$u=u_0$	$\wp=e_3=e_2=P$	0	
	$u_3<u<u_0$	$P<\wp<e_3<e_2<e_1$[b]	$+$	22
	$u<u_3$	$P<\wp<e_1$[b]	$+$	12
Case 4″	$u_1<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u_2<u<u_1$	$\wp<P<e_1$[c]	$-$	11
	$u_0<u<u_2$	$\wp<P<e_3<e_2<e_1$[c]	$-$	21
	$u=u_0$	$\wp=P=e_3=e_2$	0	
	$u_3<u<u_0$	$e_3<e_2<P<\wp<e_1$	$+$	24
	$u<u_3$	$P<\wp<e_1$	$+$	12
Case 5	$u_1<u$	$\wp<e_3<e_2<P<e_1$	$-$	01
	$u_2<u<u_1$	$\wp<P<e_1$	$-$	11
	$u_0<u<u_2$	$e_3<e_2<\wp<P<e_1$	$-$	23
	$u=u_0$	$e_2=P=\wp=e_1$	0	
	$u_3<u<u_0$	$e_3<e_2<P<\wp<e_1$	$+$	24
	$u<u_3$	$P<\wp<e_1$	$+$	12

[a] See Table III.
[b] e_1 in the lower line is the continuation of e_3 in the upper line as a result of the dropping of e_1 and e_2 in the upper line into imaginary values.
[c] e_1 in the upper line is the continuation of e_3 in the lower line as a result of the dropping of e_1 and e_2 in the lower line into imaginary values.

The above is simply obtained. The next is to decide whether $\wp(\chi)$ and $\wp(\rho)$ drop either in the region between e_1 and e_2 or the region smaller than e_3. This decision can be given by knowing the behavior near $u=u_0$, where generally the situation is changed, and by knowing the behavior in the extreme case such as $u\rightarrow+\infty$. The detailed calculations are omitted here, and only the results thus obtained are listed in Table II. In this table, for brevity, we let:

$$P=\wp(\chi) \quad \text{and} \quad \wp=\wp(\rho). \tag{5.6}$$

Besides, $P'(\chi)/i$ and $\wp'(\rho)/i$ are determined by

$$\wp'(\chi)/i\equiv Q=16\beta'(u-u_0), \tag{5.7}$$

and

$$\wp'(\rho)/i=-16(a_0')^{-\frac{1}{2}}(u-u_0), \tag{5.8}$$

which give us the sign of both values, provided that $(a_0')^{\frac{1}{2}}$, or more rigorously $(a_0')_*^{\frac{1}{2}}\equiv(a_0')^{\frac{1}{2}}/\beta^{\frac{1}{2}}$, is nonnegative without any restriction. In effect, since in the form of the solutions h and k, or more rigorously $h_*\equiv h/\beta^{\frac{1}{2}}$ and $k_*\equiv k/\beta^{\frac{1}{2}}$, have $(a_0')^{-\frac{1}{2}}$, or more rigorously $(a_0')_*^{-\frac{1}{2}}=\beta^{\frac{1}{2}}/2(-2\beta')^{\frac{1}{2}}$, as a factor, the change of sign of $(a_0')^{\frac{1}{2}}$ as a whole does not produce any real difference in the solutions at all. It is to be noted that when $\beta<0$ this provision means $(a_0')^{-\frac{1}{2}}$ itself, which does not mean a nondimensional quantity $(a_0')_*^{-\frac{1}{2}}$, should be taken as negative. By this method we may ignore the difference of sign of β but may unify the procedure as a whole. This is justified directly from the fact that in the equations of motion the sign of β can be changed without any significant alternation except for the change of signs of h and k for the same time.

The next step is to determine the range in which ρ or χ will fall. For this purpose it is convenient to classify the various cases which are associated also with the value of u. This was done in the last column of Table II. The specification of each class is given in Table III. Thus we can determine the range of ρ and χ, if we remember that:

$$\begin{aligned} &\text{when } \Delta<0, \quad \text{if } 0<\nu<\omega_1'', \text{ then} \\ &-\infty<\wp(i\nu)<e_1, \text{ and } \wp'(i\nu)/i<0, \\ &\text{but if } \omega_1''<\nu<2\omega_1'', \text{ then} \\ &e_1>\wp(i\nu)>-\infty, \text{ and } \wp'(i\nu)/i>0, \end{aligned} \tag{F}$$

where $\omega_1''i$ is the purely imaginary semiperiod given by

$$\omega_1''=\int_{-\infty}^{e_1}\frac{dp}{[4(e_1-p)(e_2-p)(e_3-p)]^{\frac{1}{2}}}, \tag{5.9}$$

when e_1 is real.

On the other hand, when $\Delta>0$,

$$\begin{aligned} &\text{if } 0<\nu<\omega'', \text{ then} \\ &e_1>\wp(\omega'+i\nu)>e_2, \quad \wp'(\omega'+i\nu)/i>0, \\ &-\infty<\wp(i\nu)<e_3, \text{ and } \wp'(i\nu)/i<0, \\ &\text{but if } \omega''<\nu<2\omega'', \text{ then} \\ &e_2<\wp(\omega'+i\nu)<e_1, \quad \wp'(\omega'+i\nu)/i<0, \\ &e_3>\wp(i\nu)>-\infty, \text{ and } \wp'(i\nu)/i>0, \end{aligned} \tag{G}$$

where ω' is the real semiperiod and $\omega_1''i$ is the purely imaginary semiperiod given by $(e_3<e_2<e_1)$

$$\omega'=\int_{e_1}^{\infty}\frac{dp}{[4(p-e_1)(p-e_2)(p-e_3)]^{\frac{1}{2}}}, \tag{5.10}$$

and

$$\omega''=\int_{-\infty}^{e_3}\frac{dp}{[4(e_1-p)(e_2-p)(e_3-p)]^{\frac{1}{2}}}. \tag{5.11}$$

TABLE III. Specification of class and range of ρ and χ.

Class	Specification	Sign of $\wp'(\rho)/i$ and of $\wp'(\chi)/i$	Range of ρ and χ		
			$\rho=$	$\chi=$	
01	$\wp<e_3<e_2<P<e_1$	$-$	$i\rho'$,	$\omega'+i\chi'$	$0<\rho'<\omega''<\chi'<2\omega''$
02	$P<e_3<e_2<\wp<e_1$	$+$	$\omega'+i\rho'$,	$i\chi'$	$0<\rho'<\omega''<\chi'<2\omega''$
11	$\wp<P<e_1$	$-$	$i\rho'$,	$i\chi'$	$0<\rho'<\chi'<\omega_1''$
12	$P<\wp<e_1$	$+$	$i\rho'$,	$i\chi'$	$\omega_1''<\rho'<\chi'<2\omega_1''$
21	$\wp<P<e_3<e_2<e_1$	$-$	$i\rho'$,	$i\chi'$	$0<\rho'<\chi'<\omega''$
22	$P<\wp<e_3<e_2<e_1$	$+$	$i\rho'$,	$i\chi'$	$\omega''<\rho'<\chi'<2\omega''$
23	$e_3<e_2<\wp<P<e_1$	$-$	$\omega'+i\rho'$,	$\omega'+i\chi'$	$\omega''<\rho'<\chi'<2\omega''$
24	$e_3<e_2<P<\wp<e_1$	$+$	$\omega'+i\rho'$,	$\omega'+i\chi'$	$0<\rho'<\chi'<\omega''$

Note: Class 0 and Class 2 are corresponding to $\Delta>0$. On the other hand, Class 1 is corresponding to $\Delta<0$.

VI. REAL EXPRESSION OF SOLUTIONS

As seen in the preceding section, the values ρ and χ are uniquely determined within the parallelogram, which is $(2\omega', 2\omega''i)$ for $\Delta>0$ or $(2\omega_1', \omega_1'+\omega_1''i)$ for $\Delta<0$; however, the parameters which are included in the expression of solutions are not ρ and χ, respectively, but $\frac{1}{2}\rho$ and $\frac{1}{2}\chi$. These values can not be uniquely determined in the parallelogram but correspond to the four values, respectively. In effect, ν is uniquely determined by: $\wp(\nu)=$given constant, $\wp'(\nu)$ has given sign, then

$$\tfrac{1}{2}\nu+\omega',\quad \tfrac{1}{2}\nu+\omega''i,\quad \text{and}\quad \tfrac{1}{2}\nu+\omega'+\omega''i,\quad \text{for}\quad \Delta>0,$$

or

$$\tfrac{1}{2}\nu+\omega_1',\quad \tfrac{1}{2}\nu+\tfrac{1}{2}(\omega_1'+\omega_1''i),$$

and

$$\tfrac{1}{2}\nu+\tfrac{3}{2}\omega_1'+\tfrac{1}{2}\omega_1''i,\quad \text{for}\quad \Delta<0,$$

are all required values as well as $\frac{1}{2}\nu$.

This phenomenon, however, does not provide any essential difficulty. In effect, the change from $\frac{1}{2}\nu$ to $\frac{1}{2}\nu+\omega'$ produces only a half-period advance, and this is canceled out by choosing the proper value for the additive constant λ. On the other hand, the change from $\frac{1}{2}\nu$ to $\frac{1}{2}\nu+\omega''i$ or to $\frac{1}{2}\nu+\frac{1}{2}(\omega_1'+\omega_1''i)$ produces another series of solutions, which satisfy the equations of motion as well as the given energy constant u. Reviewing the results in advance, class 2 (which includes classes 21–24) has two series of real solutions; contrary to this, class 1 has only one series of real solutions while the other corresponds to imaginary solutions, and class 0 has no series of real solutions. This change can be also carried out by changing λ to $\lambda'+\omega''i$ or $\lambda'+\frac{1}{2}(\omega_1'+\omega''i)$ without any alteration of $\frac{1}{2}\rho$ and $\frac{1}{2}\chi$, respectively, as was done by Andoyer, since the solutions are periodic *qua* function of λ, $\frac{1}{2}\rho$ or $\frac{1}{2}\chi$, respectively, and the argument is in the form of $\lambda\pm\frac{1}{2}\rho\pm\frac{1}{2}\chi$. Accordingly, we may fix the values for $\frac{1}{2}\rho$ and $\frac{1}{2}\chi$. And from

$$\tfrac{1}{2}(\rho+\chi)=\chi_1,$$

and (6.1)

$$\tfrac{1}{2}(\rho-\chi)=\chi_2,$$

we may construct Table IV.

Thus we have obtained the real expressions of the solutions in each class. From (3.1) the expressions of

TABLE IV. Constants χ_1 and χ_2.

Class	$\chi_1=\frac{1}{2}(\rho+\chi)$		$\chi_2=\frac{1}{2}(\rho-\chi)$	
01	$=\frac{1}{2}\omega'+\chi_1'i$,	$\frac{1}{2}\omega''<\chi_1'<\frac{3}{2}\omega''$	$=-\frac{1}{2}\omega'+\chi_2'i$,	$-\omega''<\chi_2'<0$
02	$\frac{1}{2}\omega'+\chi_1'i$,	$\frac{1}{2}\omega''<\chi_1'<\frac{3}{2}\omega''$	$\frac{1}{2}\omega'+\chi_2'i$,	$-\omega''<\chi_2'<0$
11	$\chi_1'i$,	$0<\chi_1'<\omega_1''$	$\chi_2'i$,	$-\frac{1}{2}\omega_1''<\chi_2'<0$
12	$\chi_1'i$,	$-\omega_1''<\chi_1'<0$	$\chi_2'i$,	$-\frac{1}{2}\omega_1''<\chi_2'<0$
21	$\chi_1'i$,	$0<\chi_1'<\omega''$	$\chi_2'i$,	$-\frac{1}{2}\omega''<\chi_2'<0$
22	$\chi_1'i$,	$\omega''<\chi_1'<2\omega''$	$\chi_2'i$,	$-\frac{1}{2}\omega''<\chi_2'<0$
23	$\omega_1'+\chi_1'i$,	$\omega''<\chi_1'<2\omega''$	$\chi_2'i$,	$-\frac{1}{2}\omega''<\chi_2'<0$
24	$\omega'+\chi_1'i$,	$0<\chi_1'<\omega''$	$\chi_2'i$,	$-\frac{1}{2}\omega''<\chi_2'<0$

TABLE V. Constants included in solutions (6.8, 9). $\chi_1=\chi_1'i$, $\chi_2=\chi_2'i$.

Class	Specification		c_1'	s_1'	c_0'	A	χ_1''	χ_2''	χ_1'''	χ_2'''
11_1	$0<\chi_1'<\frac{\omega_1''}{2}$,	$-\frac{\omega_1''}{2}<\chi_2'<0$	-1	-1	$+1$	0	$\chi_1'-\frac{\omega_1''}{2}$	$\chi_2'+\frac{\omega_1''}{2}$	$2\chi_1'-\frac{\omega_1''}{2}$	$2\chi_2'+\frac{\omega_1''}{2}$
11_2	$\frac{\omega_1''}{2}<\chi_1'<\omega_1''$,	$-\frac{\omega_1''}{2}<\chi_2'<0$	-1	-1	-1	$+\frac{1}{4}$	$\chi_1'-\frac{\omega_1''}{2}$	$\chi_2'+\frac{\omega_1''}{2}$	$2\chi_1'-\frac{3}{2}\omega_1''$	$2\chi_2'+\frac{\omega_1''}{2}$
12_1	$-\frac{\omega_1''}{2}<\chi_1'<0$,	$-\frac{\omega_1''}{2}<\chi_2'<0$	$+1$	$+1$	-1	$+\frac{1}{4}$	$\chi_1'+\frac{\omega_1''}{2}$	$\chi_2'+\frac{\omega_1''}{2}$	$2\chi_1'+\frac{\omega_1''}{2}$	$2\chi_2'+\frac{\omega_1''}{2}$
12_2	$-\omega_1''<\chi_1'<-\frac{\omega_1''}{2}$,	$-\frac{\omega_1''}{2}<\chi_2'<0$	$+1$	$+1$	$+1$	0	$\chi_1'+\frac{\omega_1''}{2}$	$\chi_2'+\frac{\omega_1''}{2}$	$2\chi_1'+\frac{3}{2}\omega_1''$	$2\chi_2'+\frac{\omega_1''}{2}$

Note: The subdivision here is caused by the fact that the restriction $|\mathrm{Im}(\nu)|<\omega_1''$ would not be satisfied especially for $\zeta(2\chi_1)$ in the second subclass; accordingly before expanding into series we have changed the argument keeping in mind the periodicity of h and k *qua* function of χ_1.

h and k are easily obtained:

$$h=\tfrac{1}{2}(p+q)$$
$$=\frac{1}{4i(-2\beta')^{\frac{1}{2}}}[\zeta(\tau+\lambda+\chi_1)-\zeta(\tau+\lambda-\chi_1)$$
$$+\zeta(\tau+\lambda+\chi_2)-\zeta(\tau+\lambda-\chi_2)-\zeta(2\chi_1)-\zeta(2\chi_2)] \quad (6.2)$$

$$k=\frac{1}{2i}(p-q)$$
$$=\frac{1}{4(-2\beta')^{\frac{1}{2}}}[\zeta(\tau+\lambda+\chi_1)+\zeta(\tau+\lambda-\chi_1)$$
$$-\zeta(\tau+\lambda+\chi_2)-\zeta(\tau+\lambda-\chi_2)].$$

When $\Delta>0$, the expression of $\zeta(\nu)$ is given by

$$\zeta(\nu)=\frac{\eta\nu}{\omega'}\mp\frac{\pi i}{2\omega'}+\frac{2\pi}{\omega'}\sum_{n=1}^{\infty}\frac{\mathfrak{q}^n}{1-\mathfrak{q}^{2n}}\sin\frac{n\pi}{\omega'}(\nu\mp\omega''i), \quad (6.3)$$

where the double signs should be taken according to the sign of the imaginary part of ν:

upper sign for $\quad 0<\mathrm{Im}(\nu)<2\omega''$

or

lower sign for $\quad -2\omega''<\mathrm{Im}(\nu)<0, \quad (6.4)$

$$\mathfrak{q}=\exp[-\pi\omega''/\omega'], \quad \eta=\zeta(\omega'),$$

ω' is the real semiperiod, and $\omega''i$ is the purely imaginary semiperiod; when $\Delta<0$,

$$\zeta(\nu)=\frac{\eta_1\nu}{\omega_1'}\mp\frac{\pi}{2\omega_1'}i$$
$$+\frac{2\pi}{\omega_1'}\sum\frac{\mathfrak{q}_1^n}{1-\mathfrak{q}_1^{2n}}\sin\frac{n\pi}{\omega_1'}(\nu\mp\tfrac{1}{2}\omega_1'\mp\tfrac{1}{2}\omega_1''i), \quad (6.5)$$

where the double sign should be taken according to the

TABLE VI. Constants included in solutions (6.10, 11).

Class	Specification		c_1	s_1	A	χ_1''	χ_2''	χ_1'''	χ_2'''
21_1	$\chi_1=\chi_1 i$,	$0<\chi_1'<\omega_1''$	$+1$	$+1$	0	$\chi_1'-\omega''$	$\chi_2'+\omega''$	$2\chi_1'-\omega''$	$2\chi_2'+\omega''$
21_2	$\chi_2=\chi_2'i$,	$-\frac{\omega''}{2}<\chi_2'<0$	$+1$	$+1$	0	χ_1'	χ_2'	$2\chi_1'-\omega''$	$2\chi_2'+\omega''$
22_1	$\chi_1=\chi_1'i$,	$\omega''<\chi_1'<2\omega''$	$+1$	$+1$	$+\frac{1}{4}$	$\chi_1'-\omega''$	$\chi_2'+\omega''$	$2\chi_1'-3\omega''$,	$2\chi_2'+\omega'$
22_2	$\chi_2=\chi_2'i$,	$-\frac{\omega''}{2}<\chi_2'<0$	$+1$	$+1$	$-\frac{1}{4}$	$\chi_1'-2\omega''$	χ_2'	$2\chi_1'-3\omega''$,	$2\chi_2'+\omega''$
23_1	$\chi_1=\omega'+\chi_1'i$,	$\omega''<\chi_1'<2\omega''$	$(-1)^n$	$(-1)^n$	$+\frac{1}{4}$	$\chi_1'-\omega''$	$\chi_2'+\omega''$	$2\chi_1'-3\omega''$,	$2\chi_2'+\omega''$
23_2	$\chi_2=\chi_2'i$,	$-\frac{\omega''}{2}<\chi_2'<0$	$(-1)^n$	$(-1)^n$	$-\frac{1}{4}$	$\chi_1'-2\omega''$	χ_2'	$2\chi_1'-3\omega''$,	$2\chi_2'+\omega''$
24_1	$\chi_1=\omega'+\chi_1'i$,	$0<\chi_1'<\omega''$	$(-1)^n$	$(-1)^n$	0	$\chi_1'-\omega''$	$\chi_2'+\omega''$	$2\chi_1'-\omega''$	$2\chi_2'+\omega''$
24_2	$\chi_2=\chi_2'i$,	$-\frac{\omega''}{2}<\chi_2'<0$	$(-1)^n$	$(-1)^n$	0	χ_1'	χ_2'	$2\chi_1'-\omega''$	$2\chi_2'+\omega''$

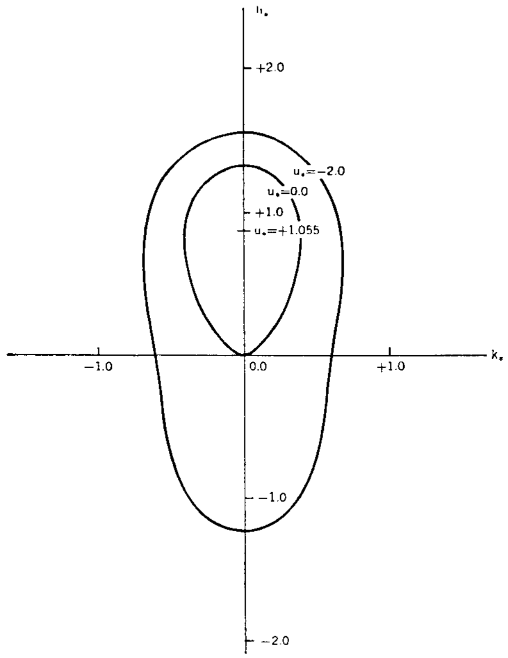

FIG. 2. Characteristics in h_* and k_* plane, $h_*=h/\beta^{\frac{1}{2}}$, $k_*=k/\beta^{\frac{1}{2}}$. $\beta_*'=-3.0$, $\gamma_*=+2.0$. (Case I_1, or case 1.)

sign of the imaginary part of ν:

$$\text{upper sign for} \quad 0<\text{Im}(\nu)<\omega_1''$$

or

$$\text{lower sign for} \quad -\omega_1''<\text{Im}(\nu)<0, \tag{6.6}$$

$$\mathfrak{q}_1=\exp[(\omega_1'+\omega_1''i/2\omega_1')\pi i]$$

$$(=i\mathfrak{q}_2,\ \mathfrak{q}_2 \text{ being a real positive value}),$$

$$\eta_1=\zeta(\omega_1').$$

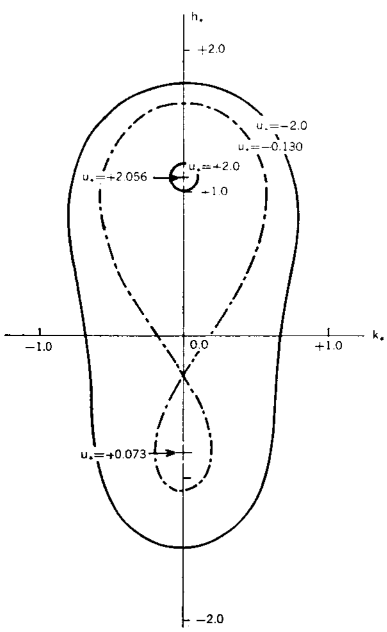

FIG. 3. $\beta'_*=-3.0$, $\gamma_*=+1.0$. (Case I_2, or case 2'.)

The real semiperiod ω_1' is given by

$$\omega_1'=\int_{e_1}^{\infty}\frac{dp}{[4(p-e_1)(p-e_2)(p-e_3)]^{\frac{1}{2}}}, \tag{6.7}$$

and $\omega_1''i$ is purely imaginary semiperiod. [Note: ω_1' and $\frac{1}{2}(\omega_1'+\omega_1''i)$ form a system of the fundamental semiperiod.]

All the necessary preparations have been completed and we are now ready to construct the real expressions

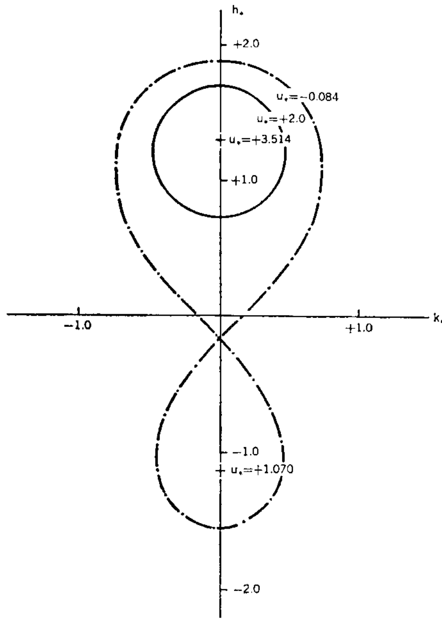

FIG. 4. $\beta'_*=-3.0$, $\gamma_*=0.0$. (Case I_3, or case 3'.)

of the solutions in the following: ($\beta'<0$)

$$\Delta<0 \text{ (Class 1)}$$

$$h=\frac{1}{(-2\beta')^{\frac{1}{2}}}\frac{\pi}{\omega_1'}\Big[\sum{}'\frac{\mathfrak{q}_2^{\,n}}{1+\mathfrak{q}_2^{\,2n}}\Big\{C_n'\cos\frac{n\pi}{\omega_1'}(\tau+\lambda)+C_{n0}'\Big\}$$
$$+\sum{}''\frac{\mathfrak{q}_2^{\,n}}{1-\mathfrak{q}_2^{\,2n}}\Big\{C_n''\cos\frac{n\pi}{\omega_1'}(\tau+\lambda)+C_{n0}''\Big\}+A\Big]$$
$$k=\frac{1}{(-2\beta')^{\frac{1}{2}}}\frac{\pi}{\omega_1'}\Big[\sum{}'\frac{\mathfrak{q}_2^{\,n}}{1+\mathfrak{q}_2^{\,2n}}S_n'\sin\frac{n\pi}{\omega_1'}(\tau+\lambda)$$
$$+\sum{}''\frac{\mathfrak{q}_2^{\,n}}{1-\mathfrak{q}_2^{\,2n}}S_n''\sin\frac{n\pi}{\omega_1'}(\tau+\lambda)\Big], \tag{6.8}$$

where

$$\mathfrak{q}_2=\exp\Big[-\frac{\pi}{2}\frac{\omega_1''}{\omega_1'}\Big], \quad \tau=\tfrac{3}{16}\mu^{\frac{1}{2}}J_2a_0^{-7/2}t,$$

$\sum'$ means the summation extending only over positive odd numbers, and $\sum''$ only over positive even numbers. The coefficients are expressed by

$$\begin{aligned}
C_n' &= c_1' \cosh\frac{n\pi X_1''}{\omega_1'} + \cosh\frac{n\pi X_2''}{\omega_1'},\\
C_n'' &= \sinh\frac{n\pi X_1''}{\omega_1'} + \sinh\frac{n\pi X_2''}{\omega_1'},\\
S_n' &= s_1' \sinh\frac{n\pi X_1''}{\omega_1'} - \sinh\frac{n\pi X_2''}{\omega_1'},\\
S_n'' &= \sinh\frac{n\pi X_1''}{\omega_1'} - \sinh\frac{n\pi X_2''}{\omega_1'},\\
C_{n0}' &= \tfrac{1}{2}c_0' \cosh\frac{n\pi X_1'''}{\omega_1'} - \tfrac{1}{2}\cosh\frac{n\pi X_2'''}{\omega_1'},\\
C_{n0}'' &= -\tfrac{1}{2} \sinh\frac{n\pi X_1'''}{\omega_1'} - \tfrac{1}{2}\sinh\frac{n\pi X_2'''}{\omega_1'}.
\end{aligned} \tag{6.9}$$

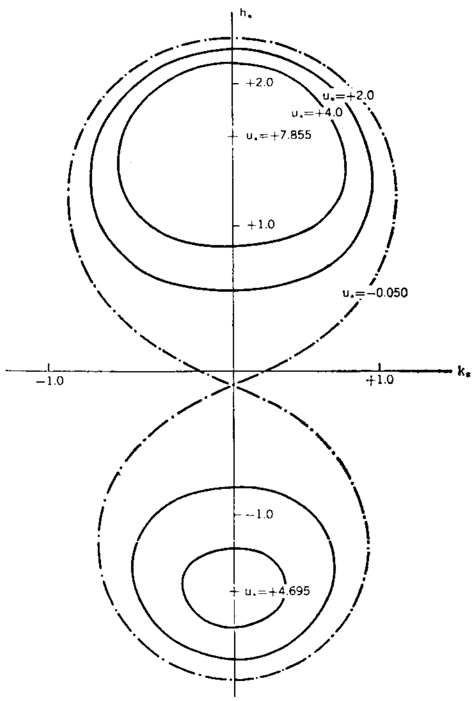

FIG. 5. $\beta'_* = -3.0$, $\gamma_* = -2.0$. (Case I_4, or case 4'.)

The coefficients and the arguments involved in the above expressions are given for each class in Table V, respectively.

$\Delta > 0$ (Class 0 has no real solution; on the other hand class 2 has two series of real solutions; the existence of two subclasses corresponds to this phenomenon. The second subclasses are constructed by making $\lambda = \lambda' + \omega'' i$ in the first subclasses, respectively. After being expanded into series, the unnecessary prime on λ is

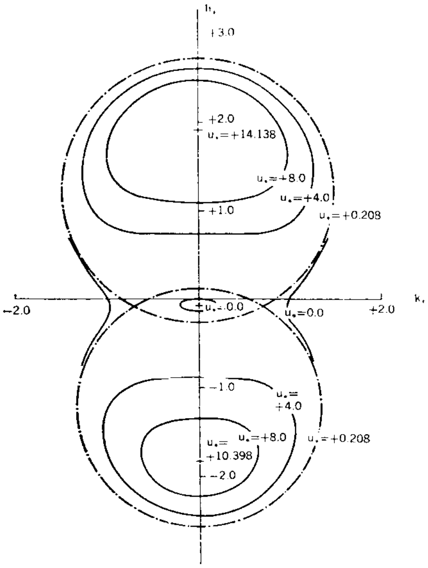

FIG. 6. $\beta'_* = -3.0$, $\gamma_* = -4.0$. (Case I_5, or case 5.)

omitted in the following.)

$$h = \frac{1}{(-2\beta')^{\frac{1}{2}}} \frac{\pi}{\omega'} \times \left[\sum_{n=1}^{\infty} \frac{\mathfrak{q}^n}{1-\mathfrak{q}^{2n}} \left\{ C_n \cos\frac{n\pi}{\omega'}(\tau+\lambda) + C_{n0} \right\} + A \right], \tag{6.10}$$

$$k = \frac{1}{(-2\beta')^{\frac{1}{2}}} \frac{\pi}{\omega'} \sum_{n=1}^{\infty} \frac{\mathfrak{q}^n}{1-\mathfrak{q}^{2n}} \left\{ S_n \sin\frac{n\pi}{\omega'}(\tau+\lambda) \right\},$$

where

$$\mathfrak{q} = \exp\left[-\frac{\pi\omega''}{\omega'} \right], \quad \tau = \tfrac{3}{16}\mu^{\frac{1}{2}} J_2 a_0^{-7/2} t.$$

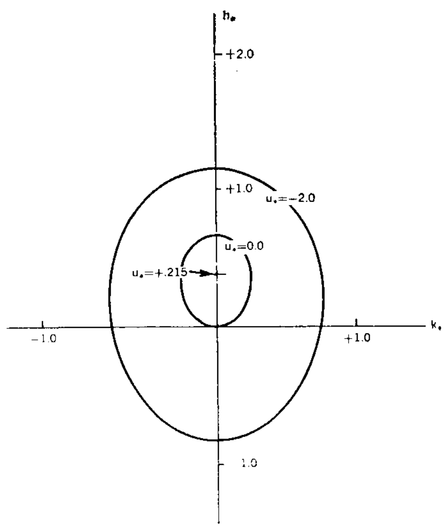

FIG. 7. $\beta'_* = -1.0$, $\gamma_* = +2.0$. (Case II_1, or case 1.)

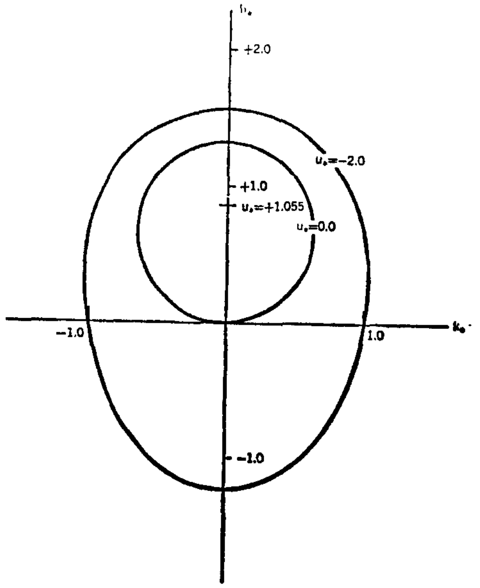

FIG. 8. $\beta'_*=-1.0$, $\gamma_*=0.0$. (Case II_2, or case 2''.)

The coefficients are expressed by

$$C_n=c_1\sinh\frac{n\pi\chi_1''}{\omega'}+\sinh\frac{n\pi\chi_2''}{\omega'},$$

$$S_n=s_1\cosh\frac{n\pi\chi_1''}{\omega'}-\cosh\frac{n\pi\chi_2''}{\omega_1}, \qquad (6.11)$$

$$C_{n0}=-\tfrac{1}{2}\sinh\frac{n\pi\chi_1'''}{\omega'}-\tfrac{1}{2}\sinh\frac{n\pi\chi_2'''}{\omega'}.$$

The coefficients and arguments involved in the above expressions are given in Table VI. These constants are chosen such that all the expansions should be convergent, or in other words, the restriction imposed on (6.3) is always satisfied. If this would not be satisfied, before being expanded into series the argument has been changed keeping in mind the periodicity of h and k *qua* function of χ_1.

It seems necessary to add the dimensionality of ω, χ_1, χ_2, etc. They are all expressed in the following way: $\omega_*=\omega\beta^{\frac{1}{2}}$, and so forth.

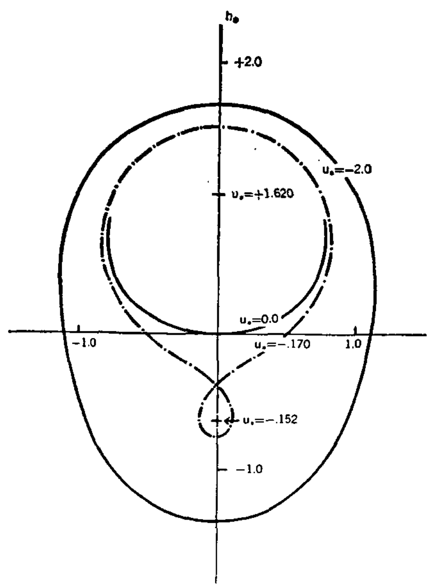

FIG. 9. $\beta'_*=-1.0$, $\gamma_*=-0.6$. (Case II_3, or case 3'.)

VII. CHARACTERISTICS

Some numerical calculations connected with the characteristics are given here. The *characteristics* means the orbit without any regard to the independent variable τ but only the plotted line in h and k space. In this problem it is easy to obtain such a characteristic. In effect,

$$\tilde{F}+u=0 \qquad (7.1)$$

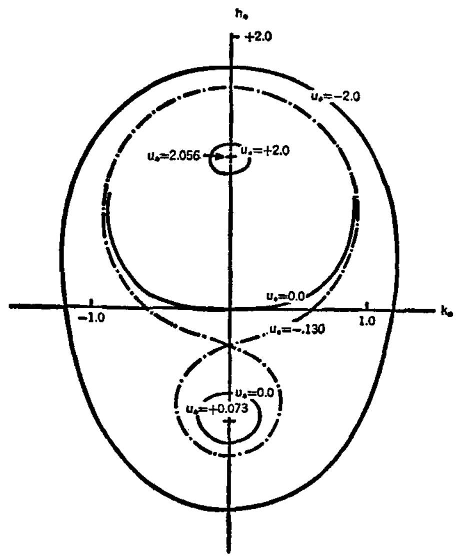

FIG. 10. $\beta'_*=-1.0$, $\gamma_*=-1.0$. (Case II_4, or case 4'.)

gives one; or, in more detail

$$(h^2+k^2)^2+(\gamma+\beta')h^2+(\gamma-\beta')k^2-\beta h+u=0. \qquad (7.2)$$

The same procedure of the nondimensional analysis taken in Sec. V gives the following form (omitting the subscript *):

$$(h^2+k^2)^2+(\gamma+\beta')h^2+(\gamma-\beta')k^2-h+u=0, \qquad (7.2')$$

from which we have simply

$$k^2=-\{h^2+\tfrac{1}{2}(\gamma-\beta')\}$$
$$\pm[-2\beta'h^2+h-u+\tfrac{1}{4}(\gamma-\beta')^2]^{\frac{1}{2}}. \qquad (7.3)$$

Accordingly, by putting

$$U_1=-2\beta'h^2+h+\tfrac{1}{4}(\gamma-\beta')^2,$$

and $\qquad (7.4)$

$$U_2=-h^4-(\gamma-\beta')h^2+h,$$

the following criteria are given:

$$
\begin{array}{llll}
\text{(i)} & \text{if } u>U_1, & \text{then 0 solution for } k; & \\
\text{(ii)}_1 & \text{if } u<U_1, & h^2+\frac{1}{2}(\gamma-\beta')>0, & \\
\text{(ii)}_{11} & \text{and } u>U_2, & \text{then 0 solution,} & \\
\text{(ii)}_{12} & \text{or } u<U_2, & \text{then 2 solutions;} & \text{(H)} \\
\text{(ii)}_2 & \text{if } u<U_1, & h^2+\frac{1}{2}(\gamma-\beta')<0, & \\
\text{(ii)}_{21} & \text{and } u>U_2, & \text{then 4 solutions,} & \\
\text{(ii)}_{22} & \text{or } u<U_2, & \text{then 2 solutions.} &
\end{array}
$$

The critical values for u also come from these criteria as follows:

$$u_0=U_1(h_0), \quad \left(\frac{\partial U_1}{\partial h}\right)_{h=h_0}=0, \tag{7.5}$$

$$u_i=U_2(h_i), \quad \left(\frac{\partial U_2}{\partial h}\right)_{h=h_i}=0 \ (i=1,2,3). \tag{7.6}$$

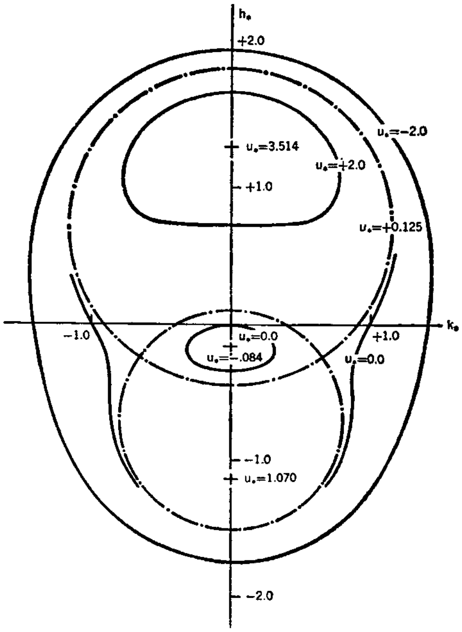

FIG. 11. $\beta'_*=-1.0$, $\gamma_*=-2.0$. (Case II_5, or case 5.)

As an example, one set of β' and γ for each $(I_1,\ldots,III_5)$ is taken and several characteristics corresponding to some values of u are given in Figs. 2–16. Table VII presents the numerical values of u_i $(i=0,\ldots,3)$ etc. Also added is the case E, where $\beta_*'=-0.13$, which is, of course, included in case III, but this is the actual case for the earth. Adopted values of the harmonic coefficients are from Kozai (1962b):

$$
\begin{array}{ll}
J_2=1082.36\times10^{-6}, & J_3=-2.566\times10^{-6}, \\
J_4=\ -2.14\times10^{-6}, & J_5=-0.063\times10^{-6}.
\end{array} \tag{7.7}
$$

Accordingly,

$$
\begin{array}{ll}
\beta=\ \ 0.147\times10^{-3}a_0^{-3}, & \beta^{\frac{1}{3}}=0.0528a_0^{-1}, \\
\beta'=-0.360\times10^{-3}a_0^{-2}, & \\
\gamma=-2\alpha-5.540\times10^{-3}a_0^{-2} & (\alpha=1-5H^2/L^2), \\
\beta_1''=-4.24\ \times10^{-3}a_0^{-1}; &
\end{array} \tag{7.8}
$$

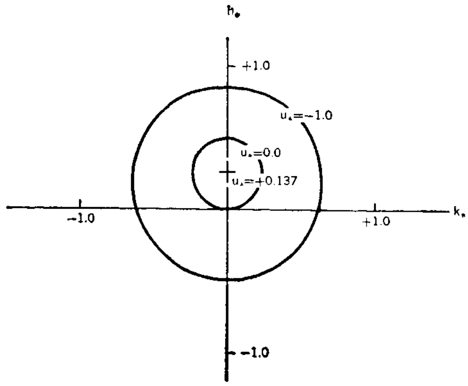

FIG. 12. $\beta'_*=-0.25$, $\gamma_*=+2.0$. (Case III_1, or case 1.)

therefore,

$$
\begin{array}{ll}
\beta_*'=\beta'/\beta^{\frac{2}{3}}=-0.129 & \\
\gamma_*=-71.7a_0^2\alpha-1.99, & \\
h_*=\ \ 18.9a_0h=18.9a_0e\sin g-0.0201 & \\
k_*=\ \ 18.9a_0k=18.9a_0e\cos g, &
\end{array} \tag{7.9}
$$

where a_0 stands for the mean semimajor axis, e for the eccentricity, and g for the argument of perigee.

It is noted that in the case of the earth the asymmetrical part comes mainly from J_5, but not from J_3. The linear shift term (β_1'') which depends only on J_3 in our theory is small. It is also noted that β_*' is not dependent on the mean semimajor axis a_0 but only on J_n's; therefore, if the potential is given, it is an absolute constant.

In concluding this section, we should add some remarks on the critical value u_0. As is seen from Table VII, the critical value u_0 corresponds neither to a stable point nor to an unstable one except for case 5, where of course it corresponds to the unstable points. The sign of $(g_3)_0$ is positive for cases 1–4. Therefore, it might

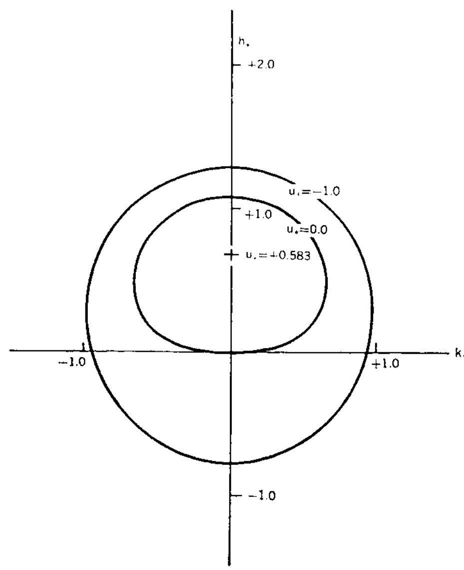

FIG. 13. $\beta'_*=-0.25$, $\gamma_*=0.0$. (Case III_2, or case 2''.)

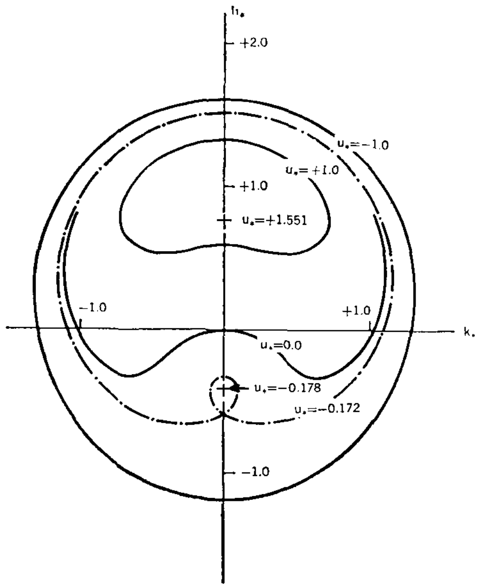

FIG. 14. $\beta'_*=-0.25$, $\gamma_*=-1.3$. (Case III_3, or case 3″.)

be considered that u_0 corresponds to a stable point. In fact, however, it corresponds to imaginary equilibrium points. Or in other words, if we draw the characteristics corresponding to u_0 it is split into two circles given by

$$k_*^2+[h_*\mp\tfrac{1}{2}(-2\beta')^{\frac{1}{2}}]^2=-\frac{\gamma}{2}\pm\frac{1}{2}\frac{1}{(-2\beta')^{\frac{1}{2}}}, \quad (7.10)$$

and for case 5 two circles meet at two points so that the two points become unstable points; on the other hand, for cases 1–4 they cannot meet each other or the radii(us) of one or both circle(s) become(s) imaginary.

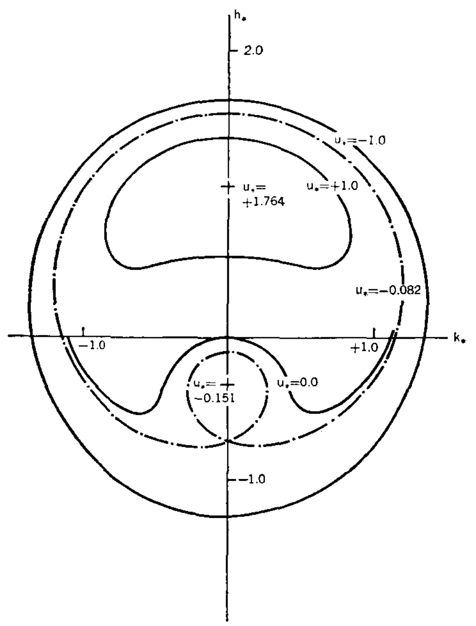

FIG. 15. $\beta'_*=-0.25$, $\gamma_*=-1.5$. (Case III_4, or case 4″.)

VIII. DISCUSSIONS

1. As mentioned before in Sec. VI the expansions of solutions are so arranged that they are always convergent. But it is still a problem to find out the most efficient expansions for the solutions; in other words, to rearrange the series when it is too slowly convergent. This situation will occur when $\mathfrak{q}$ tends to unity. In such a case we have another type of expansion by exchanging the real period and imaginary period with each other, so that for the extreme case of $\mathfrak{q}=1$ we have hyperbolic functions instead of circular functions.

In any case, in order to have suitable expansions of the solutions it is necessary to rearrange them into a different form appropriate to each case, respectively.

Also neglected is a proper method to calculate semiperiods (ω', $\omega''i$, ω_1', or $\omega_1''i$) in terms of given constants

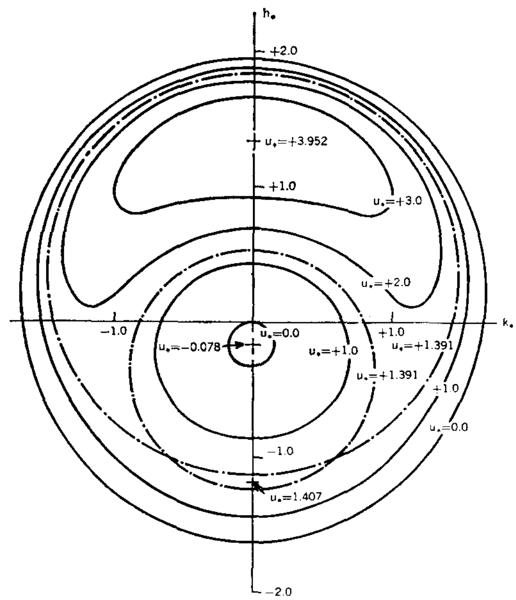

FIG. 16. $\beta'_*=-0.25$, $\gamma_*=-3.0$. (Case III_5, or case 5.)

β', γ and u. These calculations are closely related with the evaluation of $\mathfrak{q}$ (if $\Delta>0$) or $\mathfrak{q}_1$ (if $\Delta<0$). The numerical processes to find ρ and χ are also omitted here.

These practical problems are, of course, important if we wish to obtain the solutions in detail but they are so complicated that their discussion will be postponed.

2. The adoption of nondimensional analysis in units of the proper powers of β is, of course, optional. We can treat the calculation without any nondimensional analysis; or we may have another type of nondimensional analysis in terms of β' or γ for example. However, the process which we treated in this paper has a slight advantage; firstly, in doing so β'_* is an absolute constant depending only on the coefficients of harmonics but independent of the initial condition; secondly, we may treat both the cases $\beta'>0$ or $\beta'<0$, in the same framework, for example, we can draw a diagram such

TABLE VII. Critical values of u_* and corresponding h_* and k_*.

Subcase	γ_*	i	u_{*i}	h_{*i}	k_{*i}	Remark
Case I $\beta_*'=-3.0$						
I_1	+2.0	0	+ 6.2083	...	...	
		1	+ 1.0548	+0.8847	0.0	stable
I_2	+1.0	0	+ 3.9583	...	...	
		1	+ 2.0562	+1.1072	0.0	stable
		2	+ 0.0734	−0.8376	0.0	stable
		3	− 0.1295	−0.2696	0.0	unstable
I_3	0.0	1	+ 3.5139	+1.3008	0.0	stable
		0	+ 2.2083	...	...	
		2	+ 1.0702	−1.1309	0.0	stable
		3	− 0.0840	−0.1699	0.0	unstable
I_4	−2.0	1	+ 7.8553	+1.6290	0.0	stable
		2	+ 4.6947	−1.5286	0.0	stable
		0	+ 0.2083	...	...	
		3	− 0.0500	−0.1004	0.0	unstable
I_5	−4.0	1	+14.1383	+1.9056	0.0	stable
		2	+10.3976	−1.8340	0.0	stable
		0	+ 0.2083	−0.0833	±0.7022	unstable
		3	− 0.0358	−0.0715	0.0	stable
Case II $\beta_*=-1.0$						
II_1	+2.0	0	+ 2.1250	...	...	
		1	+ 0.2148	+0.3855	0.0	stable
II_2	0.0	1	+ 1.0548	+0.8847	0.0	stable
		0	+ 0.1250	...	...	
II_3	−0.6	1	+ 1.6022	+1.0221	0.0	stable
		0	− 0.0850	...	...	
		2	− 0.1524	−0.6396	0.0	stable
		3	− 0.1698	−0.3823	0.0	unstable
II_4	−1.0	1	+ 2.0562	+1.1072	0.0	stable
		2	+ 0.0734	−0.8376	0.0	stable
		0	− 0.1250	...	...	
		3	− 0.1295	−0.2696	0.0	unstable
II_5	−2.0	1	+ 3.5139	+1.3008	0.0	stable
		2	+ 1.0702	−1.1309	0.0	stable
		0	+ 0.1250	−0.2500	±0.6614	unstable
		3	− 0.0840	−0.1699	0.0	stable
Case III $\beta_*'=-0.25$						
III_1	+3.0	0	+ 0.7656	...	...	
		1	+ 0.1371	+0.2645	0.0	stable
III_2	0.0	1	+ 0.5825	+0.6959	0.0	stable
		0	− 0.4844	...	...	
III_3	−1.3	1	+ 1.5506	+1.0111	0.0	stable
		2	− 0.1717	−0.5966	0.0	unstable
		3	− 0.1777	−0.4145	0.0	stable
		0	− 0.2244	...	...	
III_4	−1.5	1	+ 1.7641	+1.0546	0.0	stable
		2	− 0.0815	−0.7296	0.0	unstable
		0	− 0.1094	...	...	
		3	− 0.1512	−0.3249	0.0	stable
III_5	−3.0	1	+ 3.9518	+1.3457	0.0	stable
		2	+ 1.4070	−1.1895	0.0	stable
		0	+ 1.3906	−1.0000	±0.6124	unstable
		3	− 0.0775	−0.1562	0.0	stable
Case E $\beta_*'=-0.13$						
E_1	+3.0	0	+ 1.4877	...	...	
		1	+ 0.0863	+0.1707	0.0	stable
E_2	0.0	1	+ 0.5269	+0.6643	0.0	stable
		0	− 0.9573	...	...	
E_3	−1.5	1	+ 1.6337	+1.0286	0.0	stable
		2	− 0.1397	−0.6608	0.0	unstable
		3	− 0.1655	−0.3679	0.0	stable
		0	− 0.4923	...	...	
E_4	−3.0	1	+ 3.7379	+1.3243	0.0	stable
		2	+ 1.2412	−1.1618	0.0	unstable
		0	+ 1.0977	...	...	
		3	− 0.0806	−0.1625	0.0	stable
E_5	−8.0	1	+18.5553	+2.0463	0.0	stable
		2	+14.5236	−1.9847	0.0	stable
		0	+14.5227	−1.9231	±0.4865	unstable
		3	− 0.0302	−0.0611	0.0	stable

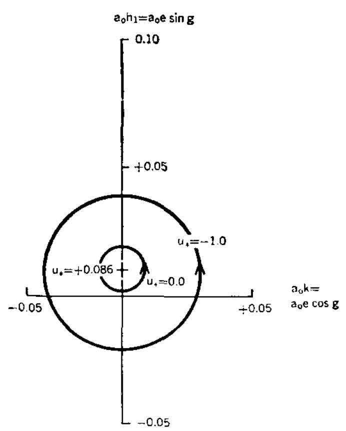

FIG. 17. $\beta'_*=-0.13$, $\gamma_*=+3.0$. (Case E_1, or case 1.) (Figures 17–21 correspond to the case for the earth.)

as Fig. 1 on a single sheet; thirdly, the difference between $\beta>0$ and $\beta<0$ can be neglected automatically by always selecting the sign of $(\beta')^{\frac{1}{2}}$ or $(-\beta')^{\frac{1}{2}}$ such that $(\beta'_*)^{\frac{1}{2}}$, $(-\beta'_*)^{\frac{1}{2}}>0$. This is caused by the fact that the difference between $\beta>0$ and $\beta<0$ remains only on the sign of h and k but there is no difference on the nature of the critical character, etc.

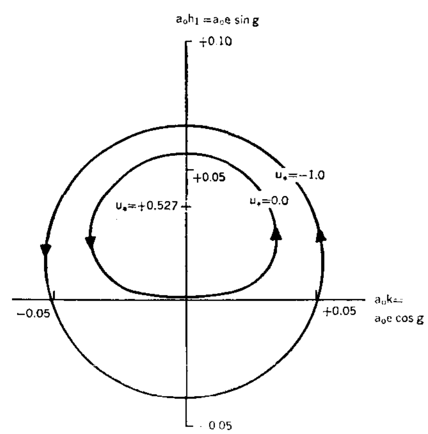

FIG. 18. $\beta'_*=-0.13$, $\gamma_*=0.0$. (Case E_2, or case 2″.)

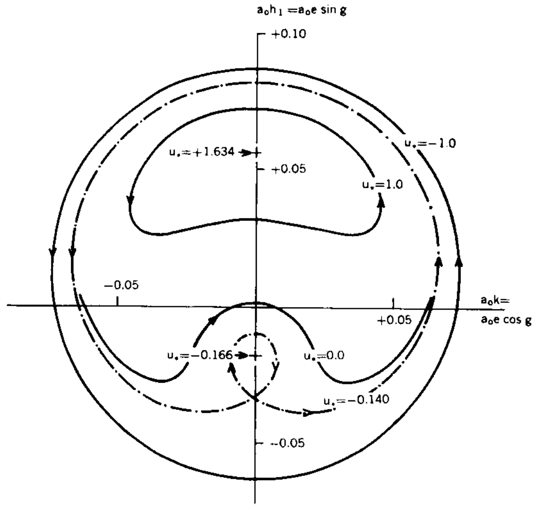

FIG. 19. $\beta'_*=-0.13$, $\gamma_*=-1.5$. (Case E_3, or case 3″.)

3. We have assumed J_n to be of nth order of magnitude. But if those assumptions are broken down, how shall the situation change? We may consider this breakdown as follows within our framework: $\beta\to 0$, in other words a noneffective case of the antisymmetrical terms J_3 and J_5 ($\beta'_*\to\infty$). In this case, the range h_* and k_* for the critical nature (the transition from librational to revolutional) is not restricted in a limited area according as $\gamma_*\to-\infty$, but h and k themselves are limited. Therefore we may let $\beta\to 0$ without any essential difficulty in our theory.

On the other hand if $\beta'\to 0$, keeping β constant the range of h_* and k_* as well as h and k themselves for the critical nature are not restricted as $\gamma_*\to-\infty$. Therefore, another type of theory would be required which

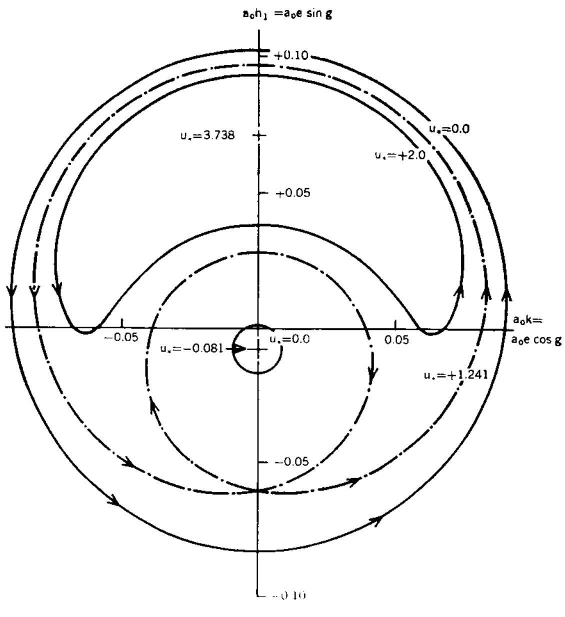

FIG. 20. $\beta'_*=-0.13$, $\gamma_*=-3.0$. (Case E_4, or case 4″.)

could treat not only the small eccentricity case but also the moderate eccentricity case. In this case where the antisymmetrical terms prevail over the symmetrical terms, the general feature, which is constructed here by adding the antisymmetrical terms to the symmetrical terms or at most by considering the same order contributions in some meaning from both sources, would be broken down. Fortunately, since β_*' for the earth is approximately -0.13, it does not seem to necessitate any new theory in this respect.

4. A higher-order theory beyond the one discussed here could be developed along the line which we gave in a previous paper (1962). However we must keep in mind the restriction made there that the leading terms which are, for example, the ones discussed here, should prevail over the remaining terms. In other words, if the leading terms are so small that the neglected terms

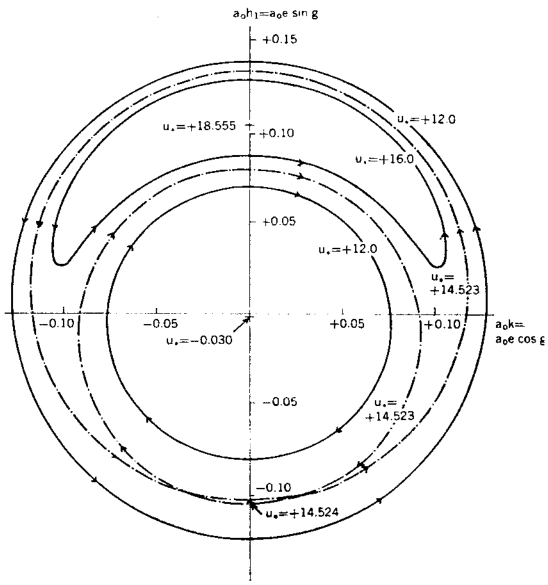

FIG. 21. $\beta'_*=-0.13$, $\gamma_*=-8.0$. (Case E_5, or case 5.)

play an important role in the behavior of the solution, then such an approximation process would be broken down. In addition, in the present paper, we have taken into account two parameters such as β and β', the former is assumed to be third order and the latter is of second order; namely the nondimensional quantity $\beta'_*=\beta'/\beta^{\frac{2}{3}}$ is assumed to be far from zero or from infinity.

In the previous paper, it is assumed that, for example, J_2^3, J_2J_4 and J_6 are all of the third order of J_2+J_4/J_2; this means that, if J_2+J_4/J_2 tends to zero, J_2^3 etc. should tend to zero as well, in order to be able to apply the general treatment described there, otherwise the leading terms are so small that the series obtained along the theory would be divergent. A similar situation occurs in the present theory in subsection 3. This is also true in the case of the higher-order theories here discussed. At any rate it is important to consider first the most significant parts of the terms.

The difference between the previous theory and the present one consists of the fact that here the antisymmetrical terms are given the same importance as the symmetrical terms by assuming that J_n is of the nth order of magnitude in the case of small eccentricity.

IX. CONCLUSION

In this paper we tried to solve the equations of motion of a close satellite near the critical inclination in the case of small eccentricity under the influence of the potential given by

$$V=\frac{\mu}{r}\left[1-J_n\left(\frac{a_e}{r}\right)^n P_n(\sin\delta)\right].$$

Assumed are as follows: the coefficient of zonal harmonics J_n is of the nth order of magnitude, the eccentricity is of the first order, and $\alpha\equiv 1-5H^2/L^2$ is of the 2nd order. The terms which are taken in the Hamiltonian are of the sixth order of magnitude in this respect; namely, the terms which have e^4J_2, e^2J_4, e^3J_3, eJ_2J_3, eJ_5 etc. as factor are included. It is also noted that the lower order terms do not enter the Hamiltonian except for unnecessary constant terms.

In Sec. VI the explicit solutions are given with classification according to the interrelation of β'_*, γ_* and u_* [for the definition of these quantities, see Eqs. (4.4), (2.12) and (3.4)], where β'_* strictly depends on the coefficients of zonal harmonics J_n, γ_* depends on the projected angular momentum to the equatorial plane as well, and u_* is the energy constant in units of $\beta^{\frac{3}{2}}$. This classification is made in order to have real expressions for the solutions. Three parameters involved are expressed by the nondimensional quantities so that one may have the actual solutions by factoring a proper power of $\beta^{\frac{1}{2}}$, which is assumed as of the first order of magnitude.

For the earth $\beta^{\frac{1}{2}}$ is $+0.0528/a_0$; accordingly, if the eccentricity is confined within some range, say 0.05 or 0.10, then the theory developed here is applicable.

In any case it is noted that the transition from the *symmetrical term prevailing case* to the *antisymmetrical prevailing case* within the approximation adopted here occurs at (between case II and case III, where a great change especially in characteristics occurs):

$$\beta'_*=\beta'/\beta^{\frac{3}{2}}=-\tfrac{1}{2}.$$

This quantity for the earth is approximately -0.13; therefore, the earth's potential is in the antisymmetrical term prevailing case in this respect.

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