NASA TECHNICAL REPORT

NASA TR R-188

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ASTRORELATIVITY

by Helmut G. L. Krause
George C. Marshall Space Flight Center
Huntsville, Alabama

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1964

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Office of Technical Services, Department of Commerce,
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ABSTRACT

In this paper the special theory of relativity is extended to include the vectorial and scalar transformations of kinematical and dynamical quantities associated with bodies whose proper or rest mass is variable with time (i. e. rockets). Besides the known transformation formulas for force (or momentum flow rate) the general transformation formulas for power (or total energy flow rate) are given for the first time. All text books on the theory of relativity introduce the classical definition for the power which is an overspecification. It would be correct only when the rest mass is not changeable with time. This assumption is always fulfilled when applying the theory of special relativity to fast moving electrons, atoms, or nuclear particles, but it is not true for fast moving rockets. Generalized relativistic conservation laws of momentum and total energy (mass) are derived.

An application of relativistic dynamics to rocket propulsion gives the data of an arbitrarily accelerated rocket in free space (without external forces) in the system of a stationary earth observer, and in the rest system of an astronaut centered in the moving rocket itself.

Two special cases of rectilinear motion of a rocket with constant exhaust velocity are treated:

(1) constant thrust acceleration (hyperbolic motion)
(2) constant mass flow rate or constant thrust

Tables with numerical values for dimensionless flight parameters will be given for both cases.

In this report Einstein's general theory of relativity (gravitational theory) is applied to the motion of an artificial satellite revolving in an arbitrary orbit around a central body and the time dilatation effect for this satellite is given. This relativistic perturbation theory is based on Einstein's general field theory, differential geometry of non-Euclidean spaces, potential theory, and celestial mechanics. The short periodic perturbations are excluded by using time average values over a revolution. The secular and long-periodic (non-relativistic as well as relativistic) perturbations of the osculating orbital elements, which represent deviations from the elliptic orbit, are presented here for the case of a rotating, non-homogeneous, oblated spheroidal central body. This is an extension of the work of Einstein (1915) who considered motion around a mass point as well as the work of deSitter (1916) and, independently, of Lense and Thirring (1918), who treated the relativistic motion around a rotating, homogeneous, spherical central body, omitting the terms due to the square of the angular velocity.

A formula for the relative difference of the time rates of a satellite clock, compared against a standard earth clock (time dilatation effect) is derived for orbits of any eccentricity and equatorial inclinations, thus extending the paper of Winterberg (1955), Singer (1956) and Hoffmann (1957).

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SUMMARY

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I. RELATIVISTIC ROCKET KINEMATICS

A. General Lorentz Transformations

The special theory of relativity is based on two principles:

1. The postulate of relativity. It is impossible to measure or detect unaccelerated translatory motion of a system through free space.

2. The postulate of the constancy of the velocity of light. The velocity of light in free space is the same for all observers, independent of the relative velocity of the source of light and the observer.

Using these postulates, A. Einstein (Ref. 1) derived the Lorentz transformations (transformation formulae between two reference frames in relative uniform motion to each other) previously derived by H. A. Lorentz (Ref. 2) and H. Poincare (Ref. 3) on the basis of the electron theory.

Two coordinate systems S and S′ moved against each other with a constant system velocity, (where $\mathbf{v}$ is the velocity vector of the system S′ relative to S and $\mathbf{v}' = -\mathbf{v}$ is the velocity vector of the system S relative to S′) have origins 0 and 0′, respectively, coinciding at the time $t=t'=0$. The transformation of a position vector $\mathbf{r} = (x, y, z)$ and the time t in the system S to the corresponding quantities $\mathbf{r}' = (x', y', z')$ and t′ in the system S′ is given by the general Lorentz transformation (without rotation) in a paper by G. Herglotz (Ref. 4), namely

$$\mathbf{r}' = \mathbf{r} + \mathbf{v}\left[\left(\frac{1}{a} - 1\right)\frac{\mathbf{r}\cdot\mathbf{v}}{v^2} - \frac{t}{a}\right] ; \quad t' = \frac{1}{a}\left(t - \frac{\mathbf{r}\cdot\mathbf{v}}{c^2}\right) \tag{1}$$

$$\mathbf{r}' + \mathbf{v}t' = \mathbf{r} - \mathbf{v}\left[(1-a)\frac{\mathbf{r}\cdot\mathbf{v}}{v^2}\right]$$

or

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}\left[\left(\frac{1}{a} - 1\right)\frac{\mathbf{r}'\cdot\mathbf{v}}{v^2} + \frac{t'}{a}\right] ; \quad t = \frac{1}{a}\left(t' + \frac{\mathbf{r}'\cdot\mathbf{v}}{c^2}\right) \tag{2}$$

$$\mathbf{r} - \mathbf{v}t = \mathbf{r}' - \mathbf{v}\left[(1-a)\frac{\mathbf{r}'\cdot\mathbf{v}}{v^2}\right]$$

where c is the light velocity and

$$a = \sqrt{1-(v/c)^2} < 1 \quad ; \quad 1 - a^2 = (v/c)^2 \tag{3}$$

All these transformations satisfy the equation

$$s^2 = (\mathbf{r}\cdot\mathbf{r}) - c^2 t^2 = (\mathbf{r}'\cdot\mathbf{r}') - c^2 t'^2 \text{ (invariant)} \tag{4}$$

The inverse equations (2) follow from equations (1) by interchanging $(\mathbf{r}', t')$ and $(\mathbf{r}, t)$

and replacing $\mathbf{v}$ by $\mathbf{v}' = -\mathbf{v}$.

The general Lorentz transformation with rotation (when the Cartesian axes in S and S′ do not have the same orientation) are

$$D\mathbf{r}' = \mathbf{r} + \mathbf{v}\left[\left(\frac{1}{a} - 1\right)\frac{\mathbf{r}\cdot\mathbf{v}}{v^2} - \frac{t}{a}\right] \tag{5}$$

$$t' = \frac{1}{a}\left(t - \frac{\mathbf{r}\cdot\mathbf{v}}{c^2}\right)$$

where D is the rotation operator (a tensor or matrix). Due to

$$D\mathbf{v}' = -\mathbf{v} \quad ; \quad \mathbf{v}' = -D^{-1}\mathbf{v} \tag{6}$$

there is

$$\mathbf{r}' = D^{-1}\mathbf{r} - \mathbf{v}'\left[\left(\frac{1}{a} - 1\right)\frac{\mathbf{r}\cdot\mathbf{v}}{v^2} - \frac{t}{a}\right] \tag{7}$$

$$t' = \frac{1}{a}\left(t - \frac{\mathbf{r}\cdot\mathbf{v}}{c^2}\right)$$

and the inverse relations

$$\mathbf{r} = D\mathbf{r}' - \mathbf{v}\left[\left(\frac{1}{a} - 1\right)\frac{\mathbf{r}'\cdot\mathbf{v}'}{v^2} - \frac{t'}{a}\right] \tag{8}$$

$$t = \frac{1}{a}\left(t' - \frac{\mathbf{r}'\cdot\mathbf{v}'}{c^2}\right)$$

because of the identity

$$\left(\frac{1}{a} - 1\right)\frac{\mathbf{r}\cdot\mathbf{v}}{v^2} - \frac{t}{a} = \left(\frac{1}{a} - 1\right)\frac{\mathbf{r}'\cdot\mathbf{v}'}{v^2} - \frac{t'}{a} \tag{9}$$

These equations also satisfy equation (4). For inhomogeneous Lorentz transformations (where the origins 0 and 0′ do not coincide at the time $t = t' = 0$) the distance s in the space-time world will no longer be invariant. The invariant quantity for these general transformations is now the square of the four-dimensional line-element $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$.

Special Lorentz transformations (where the constant velocity v is in the direction of the positive x-axis, thus $\mathbf{v} = [v, o, o]$) are

$$x' = \frac{x - vt}{a} = x\cosh\psi - ct\sinh\psi$$

$$y' = y \;;\; z' = z$$

$$ct' = \frac{ct - xv/c}{a} = ct\cosh\psi - x\sinh\psi$$

$$x = \frac{x' + vt'}{a} = x'\cosh\psi + ct'\sinh\psi$$

$$y = y' \;,\; z = z'$$

$$ct = \frac{ct' + x'v/c}{a} = ct'\cosh\psi + x'\sinh\psi$$

where

$$\psi = \cosh^{-1}(1/a) = \cosh^{-1}\frac{1}{\sqrt{1-(v/c)^2}} = \tanh^{-1}(v/c) \tag{10}$$

B. Transformation of Particle Velocities

The motion of an arbitrarily moving particle will be given in the system S by its instantaneous position $x = x(t)$, $y = y(t)$, $z = z(t)$ or $\mathbf{r}(t)$. The particle velocity vector is $\mathbf{u} = d\mathbf{r}/dt = (u_x, u_y, u_z) = (\dot{x}, \dot{y}, \dot{z})$ and the velocity itself is given by $u = (u_x^2 + u_y^2 + u_z^2)^{1/2}$. Corresponding quantities in the System S' are marked by a prime.

By differentiation of eqs. (1) and (2) one obtains the transformation formulae for the time elements and for the Lorentz contraction factor α:

$$\frac{dt'}{dt} = \frac{1}{\alpha}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \; ; \quad \frac{dt}{dt'} = \frac{1}{\alpha}\left(1 + \frac{\mathbf{u}'\cdot\mathbf{v}}{c^2}\right) \tag{11}$$

$$\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)\left(1 + \frac{\mathbf{u}'\cdot\mathbf{v}}{c^2}\right) = \alpha^2 \equiv 1 - \frac{v^2}{c^2} \tag{12}$$

Transformation formulae also follow for the local velocity vector of a particle in the reference system S and in the system S' in relative uniform motion with constant velocity $\mathbf{v}$ to S, namely

$$\mathbf{u}' = \frac{d\mathbf{r}'}{dt'} = \frac{d\mathbf{r}'/dt}{dt'/dt} = \frac{\alpha\mathbf{u} + \mathbf{v}\;\{(1-\alpha)(\mathbf{u}\cdot\mathbf{v})/v^2 - 1\}}{1 - (\mathbf{u}\cdot\mathbf{v})/c^2} \tag{13}$$

or

$$\mathbf{u} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}/dt'}{dt/dt'} = \frac{\alpha\,\mathbf{u}' + \mathbf{v}\;\{(1-\alpha)(\mathbf{u}'\cdot\mathbf{v})/v^2 + 1\}}{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2} \tag{14}$$

These formulae are the basis for the relativistic kinematics. By squaring eqs. (13) and (14) there results

$$\left(\frac{u'}{c}\right)^2 = 1 - \alpha^2 \frac{1 - (u/c)^2}{\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^2} \leqslant 1 \;\; ; \;\; u' \leqslant c \tag{15}$$

$$\left(\frac{u}{c}\right)^2 = 1 - \alpha^2 \frac{1 - (u'/c)^2}{\left(1 + \frac{\mathbf{u}'\cdot\mathbf{v}}{c^2}\right)^2} \leqslant 1 \;\; ; \;\; u \leqslant c \tag{16}$$

combining eqs. (11), (15) and (16) yields

$$\frac{dt'}{dt} = \frac{1 - (\mathbf{u}\cdot\mathbf{v})/c^2}{\alpha} = \frac{\alpha}{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2} = \sqrt{\frac{1 - (u/c)^2}{1 - (u'/c)^2}} \tag{17}$$

and thus

$$dt\sqrt{1 - (u/c)^2} = dt'\sqrt{1 - (u'/c)^2} = d\tau \;\; \text{(invariant)} \tag{18}$$

where τ is called the proper time measured in a system centered in the moving particle.

For $\mathbf{u} \parallel \mathbf{v}$ respectively $\mathbf{u}' \parallel \mathbf{v}$ eqs. (13) and (14) yield the usually quoted addition theorem of particle velocities

$$\mathbf{u}' = \frac{\mathbf{u} - \mathbf{v}}{1 - (\mathbf{u}\cdot\mathbf{v})/c^2} \quad ; \quad \mathbf{u} = \frac{\mathbf{u}' + \mathbf{v}}{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2} \tag{19}$$

For $\mathbf{u} \perp \mathbf{v}$, respectively, $\mathbf{u}' \perp \mathbf{v}$ eqs. (13) and (14) yield

$$\mathbf{u}' = \alpha\mathbf{u} - \mathbf{v} \;\; ; \;\; \mathbf{u} = \alpha\mathbf{u}' + \mathbf{v} \tag{20}$$

For $|\mathbf{u}| = c$ (photons) there is $\mathbf{u}' = -\mathbf{u}$, while for $\mathbf{u} = \mathbf{v}$ there follows $\mathbf{u}' = 0$ (transformation to rest) meaning that the coordinate system S' is centered in the particle itself (rest system S_o).

For the special case where $\mathbf{v}$ is in the direction of the positive x - axis, eqs. (13) and (14) give

$$u_x' = \frac{u_x - v}{1 - (u_x v/c^2)} \;\; ; \;\; u_y' = \frac{\alpha u_y}{1 - (u_x v/c^2)} \;\; ; \;\; u_z' = \frac{\alpha u_z}{1 - (u_x v/c^2)} \tag{21}$$

or

$$u_x = \frac{u_x' + v}{1 + (u_x' v/c^2)} \;\; ; \;\; u_y = \frac{\alpha u_y'}{1 + (u_x' v/c^2)} \;\; ; \;\; u_z = \frac{\alpha u_z'}{1 + (u_x' v/c^2)} \tag{22}$$

Taking scalar and vector products of eq. (13) with $\mathbf{v}$ produces

$$\mathbf{u}' \cdot \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v} - v^2}{1 - (\mathbf{u} \cdot \mathbf{v})/c^2} \; ; \;\; |\mathbf{u}' \times \mathbf{v}| = \frac{\alpha\, |\mathbf{u} \times \mathbf{v}|}{1 - (\mathbf{u} \cdot \mathbf{v})/c^2} \tag{23}$$

Introducing the angles $\theta = \sphericalangle(\mathbf{u}, \mathbf{v})$ and $\theta' = \sphericalangle(\mathbf{u}', \mathbf{v})$ the above equations can be written after division by v, in the following form

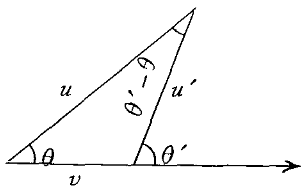
$$u' \cos\theta' = \frac{u \cos\theta - v}{1 - \frac{u v \cos\theta}{c^2}} \;\; ; \;\; u' \sin\theta' = \frac{\alpha u \sin\theta}{1 - \frac{u v \cos\theta}{c^2}} \tag{24}$$

and thus

$$u' = \frac{[(u\cos\theta - v)^2 + (\alpha u \sin\theta)^2]^{1/2}}{1 - \frac{u v \cos\theta}{c^2}} = u\,\frac{[1 - 2(v/u)\cos\theta + (v/u)^2 - (v/c)^2 \sin^2\theta]^{1/2}}{1 - \frac{u v \cos\theta}{c^2}} \tag{25}$$

and

$$\tan\theta' = \frac{\alpha \sin\theta}{\cos\theta - (v/u)} \tag{26}$$



The inverse equations follow at once by interchanging the primed and unprimed quantities and replacing v by $-v$, thus

$$u\cos\theta = \frac{u'\cos\theta' + v}{1 + \frac{u' v \cos\theta'}{c^2}} \;\; ; \;\; u \sin\theta = \frac{\alpha u' \sin\theta'}{1 + \frac{u' v \cos\theta'}{c^2}} \tag{27}$$

$$u = \frac{[(u'\cos\theta' + v)^2 + (\alpha u' \sin\theta')^2]^{1/2}}{1 + \frac{u' v \cos\theta'}{c^2}} = u'\,\frac{[1 + 2(v/u')\cos\theta' + (v/u')^2 - (v/c)^2 \sin^2\theta']^{1/2}}{1 + \frac{u' v \cos\theta}{c^2}} \tag{28}$$

$$\tan\theta = \frac{\alpha \sin\theta'}{\cos\theta' + (v/u')} \tag{29}$$

Applying eqs. (24) and (26) respectively eqs. (27) and (29) to photons $(u = u' = c)$ the relativistic formulae for the aberration of light follow:

$$\cos\theta' = \frac{\cos\theta - (v/c)}{1 - (v/c)\cos\theta} \quad ; \quad \sin\theta' = \frac{a \sin\theta}{1 - (v/c)\cos\theta} \quad ; \quad \tan\theta' = \frac{a\sin\theta}{\cos\theta - (v/c)} \tag{30}$$

or

$$\cos\theta = \frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'} \quad ; \quad \sin\theta = \frac{a\sin\theta'}{1 + (v/c)\cos\theta'} \quad ; \quad \tan\theta = \frac{a\sin\theta'}{\cos\theta' + (v/c)} \tag{31}$$

A more rigorous proof can be derived from the invariance of the phase of an electromagnetic wave. This principle also gives the relativistic formula for the Doppler effect. If ν is the frequency, w the phase velocity, $\mathbf{n}$ the wave normal or the unit vector in the direction of the ray, and [$\mathbf{k} = (\nu/w)\,\mathbf{n}$] the wave propagation vector, the invariance of the phase can be expressed by

$$\nu' t' - (\mathbf{k}' \cdot \mathbf{r}') = \nu t - (\mathbf{k} \cdot \mathbf{r}) \quad \text{or} \quad \nu'\left(t' - \frac{\mathbf{n}' \cdot \mathbf{r}'}{w'}\right) = \nu\left(t - \frac{\mathbf{n} \cdot \mathbf{r}}{w}\right) \tag{32}$$

Eliminating $\mathbf{r}'$ and t' by means of eq. (2) and comparing space-dependent and time-dependent terms respectively on both sides of the above equation yields the transformation formulae

$$\mathbf{k} = \mathbf{k}' + \mathbf{v}\left[\left(\frac{1}{a} - 1\right)\frac{\mathbf{k}' \cdot \mathbf{v}}{v^2} + \frac{\nu'}{a c^2}\right] \; ; \quad \nu = \frac{\nu' + (\mathbf{k}' \cdot \mathbf{v})}{a} \tag{33}$$

The elimination of $\mathbf{r}$ and t by means of eq. (11.2) leads to the inverse relations

$$\mathbf{k}' = \mathbf{k} + \mathbf{v}\left[\left(\frac{1}{a} - 1\right)\frac{\mathbf{k} \cdot \mathbf{v}}{v^2} - \frac{\nu}{a c^2}\right] \; ; \quad \nu' = \frac{\nu - \mathbf{k} \cdot \mathbf{v}}{a} \tag{34}$$

Introducing the direction angle of the wave normal $\theta = \sphericalangle(\mathbf{n}, \mathbf{v})$ the transformation formula for the frequency can be written

$$\nu' = \frac{\nu - (\mathbf{k} \cdot \mathbf{v})}{a} = \nu\,\frac{1 - (\mathbf{n} \cdot \mathbf{v})/w}{a} = \nu\,\frac{1 - (v/w)\cos\theta}{a} \tag{35}$$

Taking the scalar and vector products of $\mathbf{k}'$ with $\mathbf{v}$ produces

$$\mathbf{k}' \cdot \mathbf{v} = \frac{(\mathbf{k} \cdot \mathbf{v}) - \nu\,(v/c)^2}{a} \quad \text{or} \quad \frac{\nu'}{w'}\,(\mathbf{n}' \cdot \mathbf{v}) = \frac{\nu}{w}\,\frac{(\mathbf{n} \cdot \mathbf{v}) - w\,(v/c)^2}{a}$$

and

$$|\mathbf{k}' \times \mathbf{v}| = |\mathbf{k} \times \mathbf{v}| \quad \text{or} \quad \frac{\nu'}{w'}\,|\mathbf{n}' \times \mathbf{v}| = \frac{\nu}{w}\,|\mathbf{n} \times \mathbf{v}|$$

These equations can be written

$$\frac{\nu'}{w'}\cos\theta' = \frac{\nu}{w}\,\frac{\cos\theta - (v w)/c^2}{a} \; ; \quad \frac{\nu'}{w'}\sin\theta' = \frac{\nu}{w}\sin\theta \tag{36}$$

Dividing both equations gives the transformation formula for the direction of the wave normal

$$\tan\theta' = \frac{a\sin\theta}{\cos\theta - (v w)/c^2} \tag{37}$$

Squaring and then adding both equations produces

$$\frac{\nu'}{w'} = \frac{\nu}{w} \left[\sin^2\theta + \{\cos\theta - (vw)/c^2\}^2/a^2 \right]^{1/2}$$

Eliminating ν'/ν yields the transformation formula for the phase velocity

$$w' = w \frac{1 - (v/w)\cos\theta}{\left[a^2\sin^2\theta + \{\cos\theta - (vw)/c^2\}^2\right]^{1/2}}$$

$$= \frac{w - v\cos\theta}{\left[1 - \left(\frac{v}{c}\sin\theta\right)^2 - 2\frac{vw}{c^2}\cos\theta + \left(\frac{vw}{c^2}\right)^2\right]^{1/2}} \tag{38}$$

The inverse transformation formulae for the wave characteristics are

$$\nu = \nu' \frac{1 + (v/w')\cos\theta'}{a} \; ; \; \tan\theta = \frac{a\sin\theta'}{\cos\theta' + (vw')/c^2} \tag{39}$$

$$w = w' \frac{1 + (v/w')\cos\theta'}{\left[a^2\sin^2\theta' + \{\cos\theta' + (vw')/c^2\}^2\right]^{1/2}}$$

$$= \frac{w' + v\cos\theta'}{\left[1 - \left(\frac{v}{c}\sin\theta'\right)^2 + 2\frac{vw'}{c^2}\cos\theta' + \left(\frac{vw'}{c^2}\right)^2\right]^{1/2}} \tag{40}$$

A comparison of eq. (26) with eq. (37) and eq. (38) with eq. (25) shows that $u = c^2/w$ and $u' = c^2/w'$. Therefore velocity and direction of a particle are transformed in the same way as corresponding quantities for a wave with phase velocity $w = c^2/u$. DeBroglie used this result in his wave theory of elementary particles.

Applying the above mentioned formulae to a light source *in vacuo* $(w = w' = c)$ the following transformation formulae result

$$\nu' = \nu \frac{1 - \frac{v}{c}\cos\theta}{a} \; ; \; \nu = \nu' \frac{1 + \frac{v}{c}\cos\theta'}{a} \tag{41}$$

$$\cos\theta' = \frac{\cos\theta - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta} \; ; \; \sin\theta' = \frac{a\sin\theta}{1 - \frac{v}{c}\cos\theta} \; ; \; \tan\theta' = \frac{a\sin\theta}{\cos\theta - \frac{v}{c}} \tag{42}$$

$$\cos\theta = \frac{\cos\theta' + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta'} \; ; \; \sin\theta = \frac{a\sin\theta'}{1 + \frac{v}{c}\cos\theta'} \; ; \; \tan\theta = \frac{a\sin\theta'}{\cos\theta' + \frac{v}{c}} \tag{43}$$

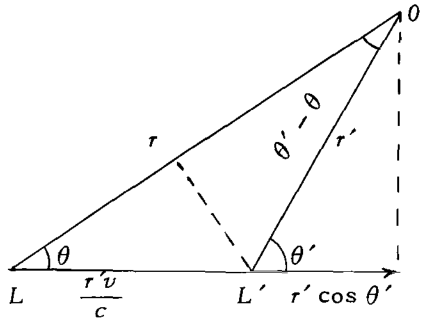
$$\tan\frac{\theta'}{2} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\tan\frac{\theta}{2} \; ; \; 1 + \frac{v}{c}\cos\theta' = \frac{a^2}{1 - \frac{v}{c}\cos\theta} \tag{44}$$

It is easy to derive the classical aberration formula from these equations, namely

$$\sin(\theta' - \theta) = \sin\theta' \cos\theta - \cos\theta' \sin\theta$$

$$= \sin\theta \frac{\frac{v}{c} - (1-a)\cos\theta}{1 - \frac{v}{c}\cos\theta}$$

$$= \sin\theta' \frac{(1-a)\cos\theta' + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta'}$$



$$\theta' - \theta = \frac{(r'v/c)\sin\theta}{r'} = \frac{v}{c}\sin\theta$$

For $\frac{v}{c} \ll 1$ $(a \approx 1)$ the above mentioned formulae yield

$$\theta' - \theta \approx \frac{v}{c}\sin\theta' \approx \frac{v}{c}\sin\theta \quad , \tag{45}$$

the same equation which follows directly from the figure.

In discussing the Doppler effect there are two special cases

a) the usual longitudinal Doppler effect: $\mathbf{n} \parallel \mathbf{v}$ $(\theta = \theta' = 0)$

$$\nu' = \nu \frac{1 - \frac{v}{c}}{a} = \nu \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} < \nu \tag{46}$$

b) the transverse Doppler effect: $\mathbf{n} \perp \mathbf{v}$ $(\theta = \frac{\pi}{2} :\ \theta' = \pi - \cos^{-1}\frac{v}{c} = \pi - \sin^{-1} a)$

$$\nu' = \frac{\nu}{a} = \frac{\nu}{\sqrt{1 - (v/c)^2}} > \nu \tag{47}$$

The decrease of the frequency, ν', of the emitted light corresponds to a shift towards the red in accordance with the time dilatation of a moving clock.

It is useful to have available the transformation formulae of certain other quantities. Using eq. (44) the transformation formula for the solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$

$$\frac{d\Omega'}{d\Omega} \approx \frac{d(\cos\theta')}{d(\cos\theta)} = \frac{a^2}{(1 - \frac{v}{c}\cos\theta)^2} = \frac{(1 + \frac{v}{c}\cos\theta')^2}{a} \tag{48}$$

The transformation formulae for the amplitude A, the volume V of a laterally bounded, finite wave and the total Energy $E = \frac{1}{2} A^2 V$ of the wave are

$$\frac{A'}{A} = \frac{V}{V'} = \frac{E'}{E} = \frac{\nu'}{\nu} = \frac{1 - \frac{v}{c}\cos\theta}{a} = \frac{a}{1 + \frac{v}{c}\cos\theta'} \tag{49}$$

The total energy density $u = E/V = \frac{1}{2} A^2$ will be transformed according to

$$\frac{u'}{u} = \left(\frac{A'}{A}\right)^2 = \left(\frac{1 - \frac{v}{c}\cos\theta}{a}\right)^2 = \left(\frac{a}{1 + \frac{v}{c}\cos\theta'}\right)^2 = \frac{d\Omega}{d\Omega'} \tag{50}$$

thus

$$u'\, d\Omega' = u\, d\Omega$$

Due to the transformation formula for the number density of photons $D = u/h\nu$:

$$\frac{D'}{D} = \left(\frac{u'}{u}\right)\left(\frac{\nu}{\nu'}\right) = \frac{A'}{A} = \frac{V}{V'} \tag{51}$$

the number of photons $N = D \cdot V = E/h\nu$ *is a relativistic invariant:* $N' = N$. Thus the relativistic transformation from one to another coordinate system is not connected with a creation or destruction of photons.

For practical applications of the relativistic Doppler effect and aberration formulae, it is more advantageous to replace the angle θ between the velocity vector $\mathbf{v}$ of the light source (relative to the observer) and the actual direction from the light source *to the observer* by the angle $\Theta = \theta + \pi$ between $\mathbf{v}$ and the actual direction from the observer *to the light source.* Likewise the angle θ' has now to be replaced by $\Theta' = \theta' + \pi$ for the corresponding apparent angles. Furthermore, ν' should be the proper frequency ν_o of a light source or a transmitter while ν is the frequency measured by an observer or a receiver. The eqs. (41) to (44) therefore yield ($\beta = v/c;\ a = \sqrt{1-\beta^2}$)

$$\nu_o = \nu\, \frac{1 + \beta\cos\Theta}{\sqrt{1-\beta^2}}\ ; \quad \nu = \nu_o\, \frac{1 - \beta\cos\Theta'}{\sqrt{1-\beta^2}} \tag{52}$$

$$\cos\Theta' = \frac{\cos\Theta + \beta}{1 + \beta\cos\Theta}\ ; \quad \sin\Theta' = \frac{\sqrt{1-\beta^2}\,\sin\Theta}{1 + \beta\cos\Theta}\ ; \quad \tan\Theta' = \frac{\sqrt{1-\beta^2}\,\sin\Theta}{\cos\Theta + \beta} \tag{53}$$

$$\cos\Theta = \frac{\cos\Theta' - \beta}{1 - \beta\cos\Theta'}\ ; \quad \sin\Theta = \frac{\sqrt{1-\beta^2}\,\sin\Theta'}{1 - \beta\cos\Theta'}\ ; \quad \tan\Theta = \frac{\sqrt{1-\beta^2}\,\sin\Theta'}{\cos\Theta' - \beta} \tag{54}$$

$$\tan\frac{\Theta'}{2} = \sqrt{\frac{1-\beta}{1+\beta}}\,\tan\frac{\Theta}{2}\ ; \quad 1 - \beta\cos\Theta' = \frac{1-\beta^2}{1+\beta\cos\Theta} \tag{55}$$

$\Theta = \Theta' = 0$ (receding space vehicle):

$$\nu/\nu_o = \frac{1-\beta}{a} = \frac{a}{1+\beta} = \sqrt{\frac{1-\beta}{1+\beta}} < 1 \quad \text{(shift to the red)}$$

$\Theta = \frac{\pi}{2}\ ;\ \Theta' = \cos^{-1}\beta$:

$$\nu/\nu_o = \sqrt{1-\beta^2} = a < 1 \quad \text{(shift to the red)}$$

$\Theta' = \frac{\pi}{2}\ ;\ \Theta = \pi - \cos^{-1}\beta$:

$$\nu/\nu_o = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{a} > 1 \quad \text{(shift to the violet)}$$

$\Theta = \Theta' = \pi$ (approaching space vehicle):

$$\nu/\nu = \frac{1+\beta}{a} = \frac{a}{1-\beta} = \frac{1+\beta}{1-\beta} > 1 \quad \text{(shift to the violet)}$$

The above mentioned formulae also apply to the case where the (earth) observer is moving with respect to the fixed stars and the light source (fixed stars) is at rest. It is only necessary to interchange primed and unprimed quantities and to replace $\mathbf{v}$ by $-\mathbf{v}$.

An illustration of the before mentioned formulae is shown in the following table:

Table 1. Aberration and Doppler Effect

θ	$\theta' = \cos^{-1}\frac{\beta + \cos\theta}{1+\beta\cos\theta}$			$\nu/\nu_o = \frac{(1-\beta^2)^{1/2}}{1+\beta\cos\theta}$		
	$\beta = v/c$			$\beta = v/c$		
	0	0.5	0.995	0	0.5	0.995
0°	0°	0°	0°	1	0.5773	0.0501
30	30	17.588	1.537	1	0.6043	0.0537
60	60	36.869	3.311	1	0.6928	0.0668
90	90	60.000	5.732	1	0.8660	0.1000
120	120	90.000	9.912	1	1.1547	0.1990
150	150	139.792	21.166	1	1.5274	0.7230
180	180	180.000	180.000	1	1.7321	20.0000

C. Lorentz Contraction and Time Dilatation

Two important applications of the Lorentz transformations can be made:

1. A measuring rod at rest ($\mathbf{u} = 0$) relative to the system S has the length $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. In the system S' moving with the velocity $\mathbf{v}$ relative to S the two end-points of the rod have a relative velocity $\mathbf{u}' = -\mathbf{v}$ and simultaneously

$$[\Delta t' = t_2' - t_1' = 0 \quad \text{or} \quad \Delta t = (\Delta\mathbf{r}\cdot\mathbf{v})/c^2 = \{(\Delta\mathbf{r}\cdot\mathbf{v})/v^2\}\ (1-a^2)]$$

have a distance

$$\Delta\mathbf{r}' \equiv \mathbf{r}_2' - \mathbf{r}_1' = \Delta\mathbf{r} + \mathbf{v}\left[\left(\frac{1}{a} - 1\right)\frac{\Delta\mathbf{r}\cdot\mathbf{v}}{v^2} - \frac{\Delta t}{a}\right] = \Delta\mathbf{r} - \mathbf{v}\left[(1-a)\frac{\Delta\mathbf{r}\cdot\mathbf{v}}{v^2}\right] \tag{56}$$

For $\Delta\mathbf{r} \parallel \mathbf{v}$ there is $\Delta\mathbf{r}' = a\Delta\mathbf{r}$ (Lorentz contraction) while for $\Delta\mathbf{r} \perp \mathbf{v}$ there is no contraction:

$$\Delta\mathbf{r}' = \Delta\mathbf{r}$$

2. Two events observed in the same point ($\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = 0$) of the system S in the time interval $\Delta t = t_2 - t_1$, appear in the system S' in the time interval $\Delta t' = \Delta t/a$ (time dilatation), however, now in the distance $\Delta\mathbf{r}' = -\mathbf{v}\Delta t/a = -\mathbf{v}\Delta t'$.

Thus, if l_o is the rest length of a rod, the length of the moved rod is

$$l = l_o\sqrt{1-(v/c)^2} \tag{57}$$

and, if τ is the proper time of a clock which is at rest, the time of the moved clock is

$$t = \frac{\tau}{\sqrt{1-(v/c)^2}} \tag{58}$$

so that the moving clock will lag behind one at rest.

D. Transformation of Particle Accelerations

Differentiation of eqs. (13) and (14) gives the corresponding transformation equations for the acceleration:

$$\mathbf{a}' = \frac{d\mathbf{u}'}{dt'} = \frac{d\mathbf{u}'/dt}{dt'/dt} = \alpha^2 \frac{\mathbf{a}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) + \frac{\mathbf{a}\cdot\mathbf{v}}{c^2}\left(\mathbf{u} - \frac{\mathbf{v}}{1+\alpha}\right)}{\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^3} \tag{59}$$

or

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}/dt'}{dt/dt'} = \alpha^2 \frac{\mathbf{a}'\left(1 + \frac{\mathbf{u}'\cdot\mathbf{v}}{c^2}\right) - \frac{\mathbf{a}'\cdot\mathbf{v}}{c^2}\left(\mathbf{u}' + \frac{\mathbf{v}}{1+\alpha}\right)}{\left(1 + \frac{\mathbf{u}'\cdot\mathbf{v}}{c^2}\right)^3} \tag{60}$$

Some special cases will be considered now. For $\mathbf{a}\parallel\mathbf{u}$ there follows

$$\mathbf{a}' = \alpha^2 \frac{\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{v}}{c^2}\frac{\mathbf{v}}{1+\alpha}}{\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^3}$$

and for $\mathbf{a}\parallel\mathbf{u}\parallel\mathbf{v}$ there is

$$\mathbf{a}' = \frac{\alpha^3\,\mathbf{a}}{\left(1 - \frac{uv}{c^2}\right)^3} = \left(\frac{\alpha}{1 - uv/c^2}\right)^3 \mathbf{a}$$

Setting $\mathbf{a}\parallel\mathbf{v}$ yields

$$\mathbf{a}' = \alpha^2 \frac{\mathbf{a}\left(\alpha - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) + \mathbf{u}\,\frac{\mathbf{a}\cdot\mathbf{v}}{c^2}}{\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^3}$$

and setting $\mathbf{a}\perp\mathbf{v}$ $(\mathbf{a}\cdot\mathbf{v} = 0)$ gives

$$\mathbf{a}' = \frac{\alpha^2\,\mathbf{a}}{[1 - (\mathbf{u}\cdot\mathbf{v})/c^2]^2} = \left(\frac{\alpha}{1 - (\mathbf{u}\cdot\mathbf{v})/c^2}\right)^2 \mathbf{a}$$

Another case, $\mathbf{u}\perp\mathbf{v}$ $(\mathbf{u}\cdot\mathbf{v} = 0)$ provides

$$\mathbf{a}' = \alpha^2 \; \mathbf{a} + \left[\frac{\mathbf{a}\cdot\mathbf{v}}{c^2}\left(\mathbf{u} - \frac{\mathbf{v}}{1+\alpha}\right)\right]$$

For the special case where $\mathbf{v} = (v, o, o)$ is in the direction of the positive x-axis the eqs. (59) and (60) yield

$$a_x' \equiv \ddot{x}' = \frac{\alpha^3 a_x}{(1 - u_x v/c^2)^3}\;;\quad a_x \equiv \ddot{x} = \frac{\alpha^3 a_x'}{(1 + u_x' v/c^2)^3}$$

$$a_y' \equiv \ddot{y}' = \frac{\alpha^2 a_y}{(1 - u_x v/c^2)^2} + \frac{\alpha^2 (a_x v/c^2) u_y}{(1 - u_x v/c^2)^3}\;;\quad a_y \equiv \ddot{y} = \frac{\alpha^2 a_y'}{(1 + u_x' v/c^2)^2} - \frac{\alpha^2 (a_x' v/c^2) u_y'}{(1 + u_x' v/c^2)^3}$$

$$a_z' \equiv \ddot{z}' = \frac{\alpha^2 a_z}{(1 - u_x v/c^2)^2} + \frac{\alpha^2 (a_x v/c^2) u_z}{(1 - u_x v/c^2)^3}\;;\quad a_z \equiv \ddot{z} = \frac{\alpha^2 a_z'}{(1 + u_x' v/c^2)^2} - \frac{\alpha^2 (a_x' v/c^2) u_z'}{(1 + u_x' v/c^2)^3}$$

A very important case is obtained by putting $\mathbf{u} = \mathbf{v}$ or $\mathbf{u}' = 0$ (transformation to a rest system). This means that the primed system is centered in the moving vehicle itself. The corresponding time t' is called the proper time τ. The quantities $v = u$ and $\alpha = \sqrt{1-(v/c)^2} = \sqrt{1-(u/c)^2}$ are now functions of the time t. The eqs. (59) and (60) are now

$$\mathbf{a}' = \frac{\alpha\mathbf{a} + \dfrac{\mathbf{a}\cdot\mathbf{u}}{c^2}\dfrac{\mathbf{u}}{1+\alpha}}{\alpha^3} = \frac{\alpha\mathbf{a} + \dfrac{\mathbf{a}\cdot\mathbf{u}}{u^2}(1-\alpha)\,\mathbf{u}}{\alpha^3} \tag{61}$$

$$\mathbf{a} = \alpha^2\left[\mathbf{a}' - \frac{\mathbf{a}'\cdot\mathbf{u}}{c^2}\frac{\mathbf{u}}{1+\alpha}\right] = \alpha^2\left[\mathbf{a}' - \frac{\mathbf{a}'\cdot\mathbf{u}}{u^2}(1-\alpha)\,\mathbf{u}\right] \tag{62}$$

and thus

$$\mathbf{a}'\cdot\mathbf{u} = \frac{\mathbf{a}\cdot\mathbf{u}}{\alpha^3} \tag{63}$$

There are two special cases. Namely, for $\mathbf{a}\parallel\mathbf{u}$ or $\mathbf{a}'\parallel\mathbf{u}$ there is

$$\mathbf{a}' = \mathbf{a}/\alpha^3 \tag{64}$$

and for $\mathbf{a}\ \ \mathbf{u}$ or $\mathbf{a}'\ \ \mathbf{u}$ there is

$$\mathbf{a}' = \mathbf{a}/\alpha^2 \tag{65}$$

In the section on dynamics there appears the function

$$\frac{d}{dt}\left(\frac{\mathbf{u}}{\sqrt{1-(u/c)^2}}\right) = \frac{\mathbf{a}}{(1-u^2/c^2)^{1/2}} + \frac{\mathbf{u}}{(1-u^2/c^2)^{3/2}}\,\frac{\mathbf{a}\cdot\mathbf{u}}{c^2} =$$

$$= \frac{\alpha^2\mathbf{a} + \dfrac{\mathbf{a}\cdot\mathbf{u}}{c^2}\,\mathbf{u}}{\alpha^3} = \frac{\alpha^2\mathbf{a} + \dfrac{\mathbf{a}\cdot\mathbf{u}}{u^2}(1-\alpha^2)\,\mathbf{u}}{\alpha^3} \tag{66}$$

where $\mathbf{a} = d\mathbf{u}/dt$ and $u\,du/dt = \mathbf{u}\cdot d\mathbf{u}/dt = \mathbf{a}\cdot\mathbf{u}$ has been used. Using eqs. (62) and (63) eq. (66) can be written

$$\frac{d}{dt}\left(\frac{\mathbf{u}}{\sqrt{1-(u/c)^2}}\right) = \frac{\mathbf{a}}{\alpha} + \frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\,\frac{\mathbf{u}}{\alpha^3} = \alpha\,\mathbf{a}' + \frac{\mathbf{a}'\cdot\mathbf{u}}{c^2}\,\frac{\mathbf{u}}{1+\alpha} = \tag{67}$$

$$= \alpha\,\mathbf{a}' + (\mathbf{a}' - \mathbf{a}/\alpha^2) = (1+\alpha)\,\mathbf{a}' - \frac{\mathbf{a}}{\alpha^2}$$

For $\mathbf{a}\parallel\mathbf{u}$ or $\mathbf{a}'\parallel\mathbf{u}$ there is

$$\frac{d}{dt}\left(\frac{\mathbf{u}}{\sqrt{1-(u/c)^2}}\right) = \mathbf{a}/\alpha^3 = \mathbf{a}' \tag{68}$$

while for $\mathbf{a}\perp\mathbf{u}$ or $\mathbf{a}'\perp\mathbf{u}$ there is

$$\frac{d}{dt}\left(\frac{\mathbf{u}}{\sqrt{1-(u/c)^2}}\right) = \mathbf{a}/\alpha = \alpha\,\mathbf{a}' \tag{69}$$

2. RELATIVISTIC ROCKET DYNAMICS

A. Definitions

In accordance with the principles of relativity that the theorems of conservation of mass and momentum hold in all sets of coordinate systems in uniform relative motion (using the more complicated Lorentz transformations instead of the Galilean transformation equations), it is necessary to modify the older Newtonian mechanics by assuming that the mass of the particle depends on its velocity.

If m_o is the proper mass or rest mass of a particle moving with the velocity $\mathbf{u}$ the following definition equations hold. The mass is given by

$$m = \frac{m_o}{\sqrt{1 - u^2/c^2}} \tag{70}$$

This yields, for the momentum vector,

$$\mathbf{p} = m\mathbf{u} = \frac{m_o \mathbf{u}}{\sqrt{1 - u^2/c^2}} \tag{71}$$

The total energy is introduced by

$$E = m c^2 = \frac{m_o c^2}{\sqrt{1 - u^2/c^2}} = \frac{E_o}{\sqrt{1 - u^2/c^2}} \tag{72}$$

where $E_o = m_o c^2$ is the rest energy. The difference between E and E_o is the kinetic energy of the particle

$$T = E - E_o = c^2 (m - m_o) = m_o c^2 \left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right) = \frac{m_o}{2} u^2 \left(1 + 3/4\, u^2/c^2 + \cdots\right)$$

The force is defined in relativistic mechanics as the momentum flow rate

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{u}) = \frac{d}{dt}\left(\frac{m_o \mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) \tag{73}$$

Tolman (Ref. 5, p. 46) says that "the inclusion of m_o inside the bracket makes the expression applicable also in cases where the proper mass of the particle varies, as it might, for example, from an inflow of heat." In 1934, he probably did not have in mind an application to fast-moving rockets expelling proper mass (exhaust gases). As in the author's paper of 1955 (Ref. 8) where the Special Theory of Relativity was extended to systems with timely changeable rest masses (rockets), the proper or rest mass m_o will be treated as a function of the time t (but not of the flight velocity u) in this paper. Thus the last equation yields

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} + \frac{dm}{dt}\mathbf{u} = m\mathbf{a} + \frac{dm}{dt}\mathbf{u} = \frac{m_o}{\sqrt{1 - u^2/c^2}}\frac{d\mathbf{u}}{dt} + \frac{d}{dt}\left(\frac{m_o}{\sqrt{1 - u^2/c^2}}\right)\mathbf{u} \tag{74}$$

showing that in relativistic mechanics the force $\mathbf{F}$ and acceleration $\mathbf{a}$ will generally not be in the same direction, as it was in Newtonian mechanics. The power or work done on a particle per unit time will be defined here as the total energy flow rate

$$P = \frac{dE}{dt} = c^2\frac{dm}{dt} = \frac{d}{dt}\left(\frac{m_o c^2}{\sqrt{1 - u^2/c^2}}\right) = \frac{d}{dt}\left(\frac{E_o}{\sqrt{1 - u^2/c^2}}\right) \tag{75}$$

B. Transformations

The general Lorentz transformation (eqs. 1 and 2) provided the relations between the space and time coordinates of system S to those of system S'. The corresponding transformation equations for velocity and acceleration were derived from them. It is very important to possess the transformation relations for certain other quantities of particle dynamics. From eq. (70) there follows

$$m\sqrt{1-u^2/c^2} = m'\sqrt{1-u'^2/c^2} = m_o \text{ (invariant)} \tag{76}$$

The proper time element ($d\tau$ or dt_o) is given by

$$dt\sqrt{1-u^2/c^2} = dt'\sqrt{1-u'^2/c^2} = dt_o \text{ (invariant)} \tag{77}$$

A division of both equations shows the following simple relations

$$\frac{dt}{m} = \frac{dt'}{m'} = \frac{dt_o}{m_o} \quad \text{(invariant)} \tag{78}$$

By means of eq. (17) the eqs. (76) and (77) immediately give

$$\frac{m}{m'} = \frac{dt}{dt'} = \sqrt{\frac{1-u'^2/c^2}{1-u^2/c^2}} = \frac{a}{1-(\mathbf{u}\cdot\mathbf{v})/c^2} = \frac{1+(\mathbf{u}'\cdot\mathbf{v})/c^2}{a} = \sqrt{\frac{1+(\mathbf{u}'\cdot\mathbf{v})/c^2}{1-(\mathbf{u}\cdot\mathbf{v})/c^2}} \tag{79}$$

where

$$a = \sqrt{1-v^2/c^2}$$

The transformation equations for these quantities of particle dynamics are,

(1) for the mass:

$$m = m'\frac{1+(\mathbf{u}'\cdot\mathbf{v})/c^2}{a} = \frac{m'+(\mathbf{p}'\cdot\mathbf{v})/c^2}{a}\ ;\ m' = m\frac{1-(\mathbf{u}\cdot\mathbf{v})/c^2}{a} = \frac{m-(\mathbf{p}\cdot\mathbf{v})/c^2}{a} \tag{80}$$

(2) for the mass flow rates

$$\frac{dm}{dt} = \frac{dm'}{dt}\cdot\frac{1+(\mathbf{u}'\cdot\mathbf{v})/c^2}{a} + \frac{m'}{a}\left(\frac{d\mathbf{u}'}{dt}\cdot\frac{\mathbf{v}}{c^2}\right) = \frac{dm'}{dt'} + m'\frac{(\mathbf{a}'\cdot\mathbf{v})/c^2}{1+(\mathbf{u}'\cdot\mathbf{v})/c^2} \tag{81}$$

or

$$\frac{dm'}{dt'} = \frac{dm}{dt} - m\frac{(\mathbf{a}\cdot\mathbf{v})/c^2}{1-(\mathbf{u}\cdot\mathbf{v})/c^2} \tag{82}$$

thus

$$m'\frac{(\mathbf{a}'\cdot\mathbf{v})/c^2}{1+(\mathbf{u}'\cdot\mathbf{v})/c^2} = m\frac{(\mathbf{a}\cdot\mathbf{v})/c^2}{1-(\mathbf{u}\cdot\mathbf{v})/c^2}\ ;\ \frac{\mathbf{a}'\cdot\mathbf{v}}{\mathbf{a}\cdot\mathbf{v}} = \left(\frac{m}{m'}\right)^3 \tag{83}$$

(3) for the momentum vector:

$$\mathbf{p} = m\mathbf{u} = \frac{m\,a\,\mathbf{u}' + m\,\mathbf{v}\,\{(1-a)(\mathbf{u}'\cdot\mathbf{v})/v^2+1\}}{1+(\mathbf{u}'\cdot\mathbf{v})/c^2} = m'\mathbf{u}' + m'\frac{\mathbf{v}}{a}\left[(1-a)\frac{\mathbf{u}'\cdot\mathbf{v}}{v^2}+1\right]$$

thus

$$\mathbf{p} = \mathbf{p}' + \frac{\mathbf{v}}{a}\left[(1-a)\frac{\mathbf{p}'\cdot\mathbf{v}}{v^2} + \frac{E'}{c^2}\right] = \mathbf{p}' + \frac{\mathbf{v}}{c^2}\,\frac{(\mathbf{p}'\cdot\mathbf{v})/(1+a)+E'}{a} \tag{84}$$

or

$$\mathbf{p}' = \mathbf{p} + \frac{\mathbf{v}}{a}\left[(1-a)\frac{\mathbf{p}\cdot\mathbf{v}}{v^2} - \frac{E}{c^2}\right] = \mathbf{p} + \frac{\mathbf{v}}{c^2}\,\frac{(\mathbf{p}\cdot\mathbf{v})/(1+a)-E}{a} \tag{85}$$

(4) for the total energy:

$$E = mc^2 = m'c^2 \frac{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2}{\alpha} = \frac{E' + (\mathbf{p}'\cdot\mathbf{v})}{\alpha}; \quad E' = \frac{E - (\mathbf{p}\cdot\mathbf{v})}{\alpha} \tag{86}$$

(5) for the force (momentum flow rate):

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d\mathbf{p}/dt'}{dt/dt'} = \frac{\alpha\mathbf{F}' + \mathbf{v}\,\{(1-\alpha)(\mathbf{F}'\cdot\mathbf{v})/v^2 + (dE'/dt')/c^2\}}{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2} = \frac{\alpha\mathbf{F}' + \frac{\mathbf{v}}{c^2}\left[\frac{\mathbf{F}'\cdot\mathbf{v}}{1+\alpha} + \frac{dE'}{dt'}\right]}{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2} \tag{87}$$

or

$$\mathbf{F}' = \frac{\alpha\mathbf{F} + \mathbf{v}\,\{(1-\alpha)(\mathbf{F}\cdot\mathbf{v})/v^2 - (dE/dt)/c^2\}}{1 - (\mathbf{u}\cdot\mathbf{v})/c^2} = \frac{\alpha\mathbf{F} + \frac{\mathbf{v}}{c^2}\left[\frac{\mathbf{F}\cdot\mathbf{v}}{1+\alpha} - \frac{dE}{dt}\right]}{1 - (\mathbf{u}\cdot\mathbf{v})/c^2} \tag{88}$$

(6) for the power (total energy flow rate):

$$\frac{dE}{dt} = \frac{dE/dt'}{dt/dt'} = \frac{dE'/dt' + (\mathbf{F}'\cdot\mathbf{v})}{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2}; \quad \frac{dE'}{dt'} = \frac{dE/dt - (\mathbf{F}\cdot\mathbf{v})}{1 - (\mathbf{u}\cdot\mathbf{v})/c^2} \tag{89}$$

Instead of differentiating eq. (86) it is also possible to multiply eqs. (81) or (82) with c^2 yielding

$$\frac{dE}{dt} = \frac{dE'}{dt'} + \frac{m'(\mathbf{a}'\cdot\mathbf{v})}{1 + (\mathbf{u}'\cdot\mathbf{v})/c^2}; \quad \frac{dE'}{dt'} = \frac{dE}{dt} - \frac{m(\mathbf{a}\cdot\mathbf{v})}{1 - (\mathbf{u}\cdot\mathbf{v})/c^2} \tag{90}$$

It is easy to see that the eqs. (89) and (90) are the same since

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{u}) = m\mathbf{a} + \frac{dm}{dt}\mathbf{u}; \quad \mathbf{F}' = \frac{d}{dt}(m'\mathbf{u}') = m'\mathbf{a}' + \frac{dm'}{dt'}\mathbf{u}' \tag{91}$$

The transformation eqs. (89) are not given in any standard book on Theory of Relativity, although they follow from the definition $\mathbf{F} = d\mathbf{p}/dt$ or $\mathbf{F}' = d\mathbf{p}'/dt'$. See, for example, the books of Tolman (Ref. 5) and Moller (Ref. 6) on Theory of Relativity. Instead of eq. (89) a further definition is introduced, namely

$$dE/dt = (\mathbf{F}\cdot\mathbf{u}) \quad \text{or} \quad dE'/dt' = (\mathbf{F}'\cdot\mathbf{u}')$$

That would mean that in any system the change of kinetic or total energy per unit time is equal to the work done by the force per unit time. However, this is an overspecification.

Identifying the system S' with a rest system S_0 ($\mathbf{u}' = 0$; $\mathbf{u} = \mathbf{v}$) the definitions $dE/dt = (\mathbf{F}\cdot\mathbf{u})$ and $dE'/dt = (\mathbf{F}'\cdot\mathbf{u}')$ are correct only when the rest mass is not changeable with time. This assumption is always fulfilled when applying the Theory of Special Relativity to fast moving electrons, atoms or nuclear particles. However, this is not the case when applying it to fast moving rockets, expelling rest mass (exhaust gases). In the general case, these definitions are wrong and have to be replaced by the correct transformation formula eq. (89) for the power.

C. Transformation to a Rest System

Putting $\mathbf{u}' = 0$ ($\mathbf{u} = \mathbf{v}$) in the before-mentioned transformation equations the following relations will result when the prime is replaced by the subscript 0 (to designate that quantities belong to the rest system):

$$\frac{m}{m_o} = \frac{E}{E_o} = \frac{dt}{dt_o} = \frac{1}{\alpha} = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{92}$$

and for the mass flow rate

$$\frac{dm}{dt} = \frac{dm_o}{dt_o} + m_o \left(\frac{\mathbf{a}_o \cdot \mathbf{u}}{c^2}\right) ; \quad \frac{dm_o}{dt_o} = \frac{dm}{dt} - \frac{m}{a^2}\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right) \tag{93}$$

thus

$$\mathbf{a}_0 \cdot \mathbf{u} = \frac{\mathbf{a}\cdot\mathbf{u}}{a^3}$$

The force is transformed according to

$$\mathbf{F} = a\,\mathbf{F}_o + \mathbf{u}\left[(1-a)\frac{\mathbf{F}_o \cdot \mathbf{u}}{u^2} + \frac{1}{c^2}\frac{dE_o}{dt_o}\right] = a\,\mathbf{F}_o + \frac{\mathbf{u}}{c^2}\left[\frac{\mathbf{F}_o\cdot\mathbf{u}}{1+a} + \frac{dE_o}{dt_o}\right] \tag{94}$$

$$\mathbf{F}_o = \frac{\mathbf{F}}{a} + \frac{\mathbf{u}}{a^2}\left[(1-a)\frac{\mathbf{F}\cdot\mathbf{u}}{u^2} - \frac{1}{c^2}\frac{dE}{dt}\right] = \frac{\mathbf{F}}{a} + \frac{u}{a^2 c^2}\left[\frac{\mathbf{F}\cdot\mathbf{u}}{1+a} - \frac{dE}{dt}\right] \tag{95}$$

thus

$$(\mathbf{F}\cdot\mathbf{u}) = (\mathbf{F}_o\cdot\mathbf{u}) + (1-a^2)\frac{dE_o}{dt_o} = (\mathbf{F}_o\cdot\mathbf{u}) + u^2\frac{dm_o}{dt_o} \tag{96}$$

The transformation law for the power is given by

$$\frac{dE}{dt} = \frac{dE_o}{dt_o} + m_o(\mathbf{a}_o\cdot\mathbf{u}) = \frac{dE_o}{dt_o} + (\mathbf{F}_o\cdot\mathbf{u}) \tag{97}$$

$$\frac{dE_o}{dt_o} = \frac{dE}{dt} - \frac{m}{a^2}(\mathbf{a}\cdot\mathbf{u}) = \frac{1}{a^2}\left[\frac{dE}{dt} - (\mathbf{F}\cdot\mathbf{u})\right] \tag{98}$$

The eqs. (96), (97), (98) were already given by the author in 1955 (Ref. 8). Writing eqs. (61) and (62) as

$$\mathbf{a} = a^2\left[\mathbf{a}_o - \frac{\mathbf{u}}{1+a}\left(\frac{\mathbf{a}_o\cdot\mathbf{u}}{c^2}\right)\right] = a^2\left[\mathbf{a}_o - (1-a)\left(\frac{\mathbf{a}_o\cdot\mathbf{u}}{u^2}\right)\mathbf{u}\right] \tag{99}$$

$$\mathbf{a}_o = \frac{1}{a^3}\left[a\,\mathbf{a} + \frac{\mathbf{u}}{1+a}\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)\right] = \frac{1}{a^3}\left[a\,\mathbf{a} + (1-a)\left(\frac{\mathbf{a}\cdot\mathbf{u}}{u^2}\right)\mathbf{u}\right] \tag{100}$$

and using these transformation laws for the acceleration or the proper acceleration, respectively, together with eqs. (92) and (93) in the definitive equations for the forces

$$\mathbf{F} = \frac{d}{dt}(m\,\mathbf{u}) = m\mathbf{a} + \frac{dm}{dt}\,\mathbf{u} \;;\quad \mathbf{F}_o = m_o\mathbf{a}_o \tag{101}$$

then the above-given transformation equations (94) and (95) are obtained. Dividing eqs. (97) and (98) by c^2 yields

$$\frac{dm}{dt} = \frac{dm_o}{dt_o} + \frac{\mathbf{F}_o\cdot\mathbf{u}}{c^2} \;;\quad \frac{dm_o}{dt_o} = \frac{1}{a^2}\left[\frac{dm}{dt} - \frac{\mathbf{F}\cdot\mathbf{u}}{c^2}\right] \tag{102}$$

The first of these equations can be obtained directly without using the transformation equations by differentiation:

$$\frac{dm}{dt} = \frac{d}{dt}\left(\frac{m_o}{a}\right) = \frac{m_o}{a^3}\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right) + \frac{1}{a}\frac{dm_o}{dt} = m_o\left(\frac{\mathbf{a}_o\cdot\mathbf{u}}{c^2}\right) + \frac{dm_o}{dt_o} = \left(\frac{\mathbf{F}_o\cdot\mathbf{u}}{c^2}\right) + \frac{dm_o}{dt_o}$$

Instead of splitting up the force, $\mathbf{F}$, in one part proportional to the acceleration $\mathbf{a}$ and another part proportional to the velocity $\mathbf{u}$ (see eq. 101), $\mathbf{F}$ can be subdivided into one term due to the change of the rest mass with time and the rest term $\mathbf{F}_*$ identical with $\mathbf{F}$ for particles with constant rest mass.

$$\mathbf{F} = \frac{d}{dt}\left(\frac{m_o \mathbf{u}}{\alpha}\right) = m_o \frac{d}{dt}\left(\frac{\mathbf{u}}{\alpha}\right) + \frac{\mathbf{u}}{\alpha}\frac{dm_o}{dt} = \mathbf{F}_* + \frac{dm_o}{dt_o}\mathbf{u} \tag{103}$$

Due to eq. (67)

$$\frac{d}{dt}\left(\frac{\mathbf{u}}{\alpha}\right) = \frac{\mathbf{a}}{\alpha} + \left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)\frac{\mathbf{u}}{\alpha^3} = \frac{\mathbf{a}}{\alpha} + \left(\frac{\mathbf{a}_o\cdot\mathbf{u}}{c^2}\right)\mathbf{u} = \alpha\mathbf{a}_o + \frac{\mathbf{a}_o\cdot\mathbf{u}}{c^2}\left(\frac{\mathbf{u}}{1+\alpha}\right)$$

thus

$$\mathbf{F}_* = m_o \frac{d}{dt}\left(\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}\right) = m_o \frac{d}{dt}\left(\frac{\mathbf{u}}{\alpha}\right) = m\mathbf{a} + \left(\frac{\mathbf{F}_o\cdot\mathbf{u}}{c^2}\right)\mathbf{u} = m\mathbf{a} + \left(\frac{dm}{dt} - \frac{dm_o}{dt_o}\right)\mathbf{u} \tag{104}$$

and

$$\mathbf{F}_* = \alpha\mathbf{F}_o + \left(\frac{\mathbf{F}_o\cdot\mathbf{u}}{c^2}\right)\frac{\mathbf{u}}{1+\alpha} = \alpha\mathbf{F}_o + (1-\alpha)\left(\frac{\mathbf{F}_o\cdot\mathbf{u}}{u^2}\right)\mathbf{u} \tag{105}$$

Multiplying the last equation with $\mathbf{u}$ yields

$$(\mathbf{F}_*\cdot\mathbf{u}) = (\mathbf{F}_o\cdot\mathbf{u}) = \frac{dE}{dt} - \frac{dE_o}{dt_o} \tag{106}$$

Putting eq. (106) into eq. (104) gives the equation of motion in the form.

$$m\frac{d\mathbf{u}}{dt} = \mathbf{F}_* - \left(\frac{\mathbf{F}_*\cdot\mathbf{u}}{c^2}\right)\mathbf{u} \tag{107}$$

Compared with the corresponding equation in the book of Møller (Ref. 6) eq. (107) has the additional term $-\alpha^2 (dm_o/dt_o)\mathbf{u}$ on the right side and the advantage that it is also applicable to systems with timely changeable rest masses (rockets). Introducing the force $\mathbf{F}$, or the Newtonian force $\mathbf{F}_o$, into equation (107) yields [by means of eqs. (102), (105) and (106)]:

$$m\frac{d\mathbf{u}}{dt} = \mathbf{F} - \frac{dm}{dt}\mathbf{u} = \mathbf{F} - \left[\left(\frac{\mathbf{F}\cdot\mathbf{u}}{c^2}\right) + \alpha^2\frac{dm_o}{dt_o}\right]\mathbf{u} \tag{108}$$

and

$$m\frac{d\mathbf{u}}{dt} = \alpha\left[\mathbf{F}_o - \left(\frac{\mathbf{F}_o\cdot\mathbf{u}}{c^2}\right)\frac{\mathbf{u}}{1+\alpha}\right] = \alpha\left[\mathbf{F}_o - (1-\alpha)\left(\frac{\mathbf{F}_o\cdot\mathbf{u}}{u^2}\right)\mathbf{u}\right] \tag{109}$$

In the special case $\mathbf{F}_o \parallel \mathbf{u}$ there is

$$\mathbf{F}_* = \mathbf{F}_o;\quad \mathbf{F} = \mathbf{F}_o + \frac{dm_o}{dt_o}\mathbf{u};\quad m\mathbf{a} = \alpha^2\mathbf{F}_o \quad \text{or} \quad \mathbf{a}/\alpha^3 = \mathbf{a}_o \tag{110}$$

D. Application of Relativistic Dynamics to Rocket Propulsion

In the following, the data for a rocket and its exhausted gases will be marked without a subscript in the system S of a stationary earth observer and with the subscript 0 in the rest system S_o of an astronaut centered in the moving rocket itself. Applying the transformation eqs. (13) and (14) for particle velocities to the exhaust velocities $\mathbf{v}_{eo}$ in the system S_o where the thrust is generated and $\mathbf{v}_e$ in the system S of an earth observer the following formulas are obtained:

$$\mathbf{v}_e = \frac{a\,\mathbf{v}_{eo} + \mathbf{u}\,\{(1-a)(\mathbf{u}\cdot\mathbf{v}_{eo})/u^2 - 1\}}{1-(\mathbf{u}\cdot\mathbf{v}_{eo})/c^2} = \frac{a\,\mathbf{v}_{eo} + \mathbf{u}\{(\mathbf{u}\cdot\mathbf{v}_{eo}/c^2)/(1+a) - 1\}}{1-(\mathbf{u}\cdot\mathbf{v}_{eo})/c^2} \tag{111}$$

$$\mathbf{v}_{eo} = \frac{a\,\mathbf{v}_e + \mathbf{u}\,\{(1-a)(\mathbf{u}\cdot\mathbf{v}_e)/u^2 + 1\}}{1+(\mathbf{u}\cdot\mathbf{v}_e)/c^2} = \frac{a\mathbf{v}_e + \mathbf{u}\,\{(\mathbf{u}\cdot\mathbf{v}_e/c^2)/(1+a) + 1\}}{1+(\mathbf{u}\cdot\mathbf{v}_e)/c^2} \tag{112}$$

where the relative velocity between the two systems is $\mathbf{u}$ and $a = \sqrt{1-u^2/c^2}$. Eq. (79) can be written, for this case,

$$\sqrt{\frac{1-v_e^2/c^2}{1-v_{eo}^2/c^2}} = \frac{a}{1-(\mathbf{u}\cdot\mathbf{v}_{eo})/c^2} = \frac{1+(\mathbf{u}\cdot\mathbf{v}_e)/c^2}{a} = \sqrt{\frac{1+(\mathbf{u}\cdot\mathbf{v}_e)/c^2}{1-(\mathbf{u}\cdot\mathbf{v}_{eo})/c^2}} = \frac{dm_o}{dm} \tag{113}$$

where dm and dm_o are the elements of mass flow after ejection in the two systems S and S_o, respectively. The element of mass flow before ejection

$$dm\sqrt{1-v_e^2/c^2} = dm_o\sqrt{1-v_{eo}^2/c^2} = dm_o^* \tag{114}$$

is invariant. The mass flow rate in the system S_o of the astronaut is

$$\mu_o \equiv -\frac{dm_o}{dt_o} = \frac{\mu_o^*}{\sqrt{1-v_{eo}^2/c^2}} \tag{115}$$

where $\mu_o^* = -dm_o^*/dt_o$ is the mass flow rate in a system in rest relative to the exhaust gases of the rocket. The corresponding mass flow rate in the system S of the earth observer is

$$\mu \equiv -\frac{dm}{dt} = \frac{a\mu_o^*}{\sqrt{1-v_e^2/c^2}} = \mu_o^*\frac{1-(\mathbf{u}\cdot\mathbf{v}_{eo})/c^2}{\sqrt{1-v_{eo}^2/c^2}} = \mu_o\left(1-\frac{\mathbf{u}\cdot\mathbf{v}_{eo}}{c^2}\right) = \mu_o - \left(\frac{\mathbf{F}_o\cdot\mathbf{u}}{c^2}\right) \tag{116}$$

in accordance to eq. (102). The quantity

$$\mathbf{F}_o = \mu_o\,\mathbf{v}_{eo} = \frac{\mu^*\,\mathbf{v}_{eo}}{\sqrt{1-v_{eo}^2/c^2}} \tag{117}$$

is the primary thrust force of the rocket in the rest system S_o where the thrust is generated. The corresponding thrust in system S which the earth observer would measure is

$$\mathbf{F} = \mu\,\mathbf{v}_e = \frac{a\,\mu_o^*\,\mathbf{v}_e}{\sqrt{1-v_e^2/c^2}} = \mu_o^*\,\frac{1-(\mathbf{u}\cdot\mathbf{v}_{eo})/c^2}{\sqrt{1-v_{eo}^2/c^2}}\,\mathbf{v}_e \tag{118}$$

or using eqs. (111) and (117)

$$\mathbf{F} = \mu_o^*\,\frac{a\,\mathbf{v}_{eo} + \mathbf{u}\,\{(1-a)(\mathbf{u}\cdot\mathbf{v}_{eo})/u^2 - 1\}}{\sqrt{1-v_{eo}^2/c^2}} = a\,\mathbf{F}_o + \mathbf{u}\left[(1-a)\frac{\mathbf{F}_o\cdot\mathbf{u}}{u^2} - \mu_o\right] \tag{119}$$

in accordance with eq. (94). The results can be summarized in table 2.

An important quantity is that part of the total energy of the exhaust gases which can be converted into useful work (kinetic energy). In the rest system S of the astronaut this conversion factor, first introduced by E. Saenger (Ref. 9), is given by

$$\gamma = \frac{(dE_o/dt_o) - (dE_o^*/dt_o)}{dE_o/dt_o} = \frac{\mu_o - \mu_o^*}{\mu_o} = 1-\sqrt{1-v_{eo}^2/c^2} \tag{120}$$

and thus

$$v_{eo}/c = \sqrt{1-(1-\gamma)^2} = \sqrt{\gamma(2-\gamma)} \tag{121}$$

Putting $a = m_o/m$ into eq. (113) gives

$$\frac{dm}{m} = \frac{dm_o}{m_o}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}_{eo}}{c^2}\right) \tag{122}$$

The equation of motion is

$$\frac{d}{dt}(m\mathbf{u}) \equiv m\,\frac{d\mathbf{u}}{dt} + \frac{dm}{dt}\,\mathbf{u} = \mathbf{F} \equiv -\,\frac{dm}{dt}\,\mathbf{v}_e \tag{123}$$

or

$$m\,\frac{d\mathbf{u}}{dt} = -(\mathbf{v}_e + \mathbf{u})\,\frac{dm}{dt}\ ;$$

thus, using eq. (122),

$$d\mathbf{u} = -(\mathbf{v}_e + \mathbf{u})\,\frac{dm}{m} = -(\mathbf{v}_e + \mathbf{u})\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}_{eo}}{c^2}\right)\frac{dm_o}{m_o}$$

Applying eq. (111) then yields

$$d\mathbf{u} = -a\left[\mathbf{v}_{eo} - (1-a)\left(\frac{\mathbf{u}\cdot\mathbf{v}_{eo}}{u^2}\right)\mathbf{u}\right]\frac{dm_o}{m_o} \tag{124}$$

Considering the special case $\mathbf{v}_{eo} \parallel \mathbf{u}$ equation (124) reduces to

$$\frac{dm_o}{m_o} = -\frac{1}{v_{eo}}\,\frac{du}{a^2} = -\frac{1}{v_{eo}}\,\frac{du}{1-u^2/c^2} \tag{125}$$

which can be easily integrated for constant exhaust velocity. Let M_o be the rest mass of the rocket for $t = 0\,(u = 0)$; the integration yields

$$\ln\frac{m_o}{M_o} = -\frac{c}{v_{eo}}\int_0^u \frac{(1/c)\,du}{1-u^2/c^2} = -\frac{c}{2\,v_{eo}}\ln\frac{1+u/c}{1-u/c}$$

thus

$$\frac{1}{r} \equiv \frac{m_o}{M_o} = \left(\frac{1-u/c}{1+u/c}\right)^{\frac{c}{2v_{eo}}}\ ;\ r \equiv \frac{M_o}{m_o} = \left(\frac{1+u/c}{1-u/c}\right)^{\frac{c}{2v_{eo}}} \tag{126}$$

where r is the mass ratio of the rocket. The inverted formulas are

$$\frac{u}{c} = \frac{1-(1/r)^{2v_{eo}/c}}{1+(1/r)^{2v_{eo}/c}} = \frac{r^{2v_{eo}/c}-1}{r^{2v_{eo}/c}+1} \tag{127}$$

The eqs. (126) and (127) represent the fundamental relativistic rocket equation derived by J. Ackeret (Ref. 10) in 1946 from the conservation law of momentum. Using $a = \sqrt{1-u^2/c^2}$; $u/c = \sqrt{1-a^2}$ eq. (126) can be transformed into

$$\frac{1}{r} \equiv \frac{m_o}{M_o} = \left(\frac{a}{1+\sqrt{1-a^2}}\right)^{\frac{c}{v_{eo}}} \tag{128}$$

This formula was found by R. Esnault-Pelterie (Ref. 11) in 1928 from a relativistic treatment of the special case of a rocket with constant proper acceleration a_o. He noted that this equation is valid for any law of mass consumption since it is independent of a_o. In classical physics ($c \to \infty$ or $u/c \to 0$) the basic relativistic rocket equation becomes the well-known formula

$$r \equiv \frac{M_o}{m_o} = e^{u/v_{eo}} \quad \text{or} \ \ln r = \frac{u}{v_{eo}}$$

since

$$\lim_{u/c\to 0} \ln r = \lim_{u/c\to 0} \frac{u}{v_{eo}} \frac{\ln(1+u/c) - \ln(1-u/c)}{2u/c} = \lim_{u/c\to 0} \frac{u}{v_{eo}} \frac{1/(1+u/c) + 1/(1-u/c)}{2} = \frac{u}{v_{eo}}$$

Representing

$$y \equiv \frac{1}{r} = \left[\frac{1-x}{1+x}\right]^{c/2v_{eo}}, \quad (0 \leqslant y \leqslant 1)$$

as function of $x \equiv u/c$ $(0 \leqslant x \leqslant 1)$, a double differentiation shows that an inflection point $(y'' = 0)$ appears for

$$x_i = \frac{c}{2v_{eo}} < 1 \quad \text{and} \quad y_i = \left[\frac{1-x_i}{1+x_i}\right]^{x_i}$$

The most favorable case $v_{eo} = c$ (photons) yields $x_i = ½$ and $y_i = 1/\sqrt{3}$; that is, the inflection point appears when $u = c/2$ and $r = \sqrt{3}$

Before closing this part, it should be mentioned that in the special case $\mathbf{F}_o \parallel \mathbf{u}$

$$F = F_o - \mu_o u = F_o \left(1 - \frac{u}{v_{eo}}\right) \tag{129}$$

If $u = v_{eo}$ is attained, then v_e will vanish and also $F = 0$. From the technical point of view the thrust force F_o of the rocket in the rest system S_o is naturally the matter of primary interest.

E. Motion and Mass Consumption of a Rocket Under a Constant Proper Acceleration

The motion of a body under the action of a constant proper acceleration a_o in the direction of the velocity ($\mathbf{a}_o \parallel \mathbf{u}$), that is rectilinear uniformly accelerated motion, is known as hyperbolic motion. It was first considered by Minkowski (Ref. 12) in 1908, but was discussed more fully by M. Born (Ref. 13) in 1909 and A. Sommerfeld (Ref. 14) in 1910. It was R. Esnault-Pelterie (Ref. 1) in 1928 who applied this kind of motion to rockets with constant thrust acceleration and discovered the basic relativistic rocket equation for any law of mass consumption. Contributions to this problem were also made by Shepherd (Ref. 15), W. L. Bade (Ref. 16), E. Saenger (Ref. 17) and others. This presentation follows closely the author's treatment (Ref. 8).

For a rocket in rectilinear mótion with a constant proper acceleration in a system moving with the rocket (rest system) and thus a different one at each instant-eq. (67) can be written as

$$\frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) = \frac{a}{(1-u^2/c^2)^{3/2}} = a_o \ (\text{const.}) \tag{130}$$

or, integrated,

$$\frac{u}{\sqrt{1 - u^2/c^2}} = a_o t \qquad (\text{with } u = 0 \text{ for } t = 0) \tag{131}$$

This gives for the velocity

$$u = \frac{dx}{dt} = c\,\frac{a_o t/c}{\sqrt{1 + (a_o t/c)^2}} = a_o t \left[1 - \frac{1}{2}\left(\frac{a_o}{c}t\right)^2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{a_o}{c}t\right)^4 \mp \cdots\right] \tag{132}$$

and, thus, for α

$$\alpha = \frac{dt_o}{dt} = \frac{m_o}{m} = \sqrt{1 - u^2/c^2} = \frac{u}{a_o t} = \frac{1}{\sqrt{1 + (a_o t/c)^2}} = 1 - \frac{1}{2}\left(\frac{a_o}{c}t\right)^2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{a_o}{c}t\right)^4 \mp \cdots \tag{133}$$

Eqs. (130) and (133) yield for the acceleration

$$a = a_o\alpha^3 = \frac{a_o}{[1 + (a_o t/c)^2]^{3/2}} = a_o\left[1 - \frac{1\cdot 3}{1}\,\frac{1}{2}\left(\frac{a_o}{c}t\right)^2 + \frac{1\cdot 3\cdot 5}{1\cdot 2}\,\frac{1}{2^2}\left(\frac{a_o}{c}t\right)^4 \mp \cdots\right] \tag{134}$$

The distance is obtained by integration of eq. (132) with

$$x = \frac{c^2}{a_o}\left[\sqrt{1 + (a_o t/c)^2} - 1\right] = \frac{a_o}{2}t^2\left[1 - \frac{1}{4}\left(\frac{a_o}{c}t\right)^2 + \frac{1\cdot 3}{4\cdot 6}\left(\frac{a_o}{c}t\right)^4 \mp \cdots\right] \tag{135}$$

thus

$$1 + \frac{a_o}{c^2}x = \quad 1 + \frac{a_o}{c}t^{\;2} = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\alpha} \tag{136}$$

Eqs. (135) or (136) give immediately

$$\left(\frac{a_o}{c^2}x + 1\right)^2 - \left(\frac{a_o t}{c}\right)^2 = 1 \quad \text{or} \quad \left(x + \frac{c^2}{a_o}\right)^2 - (c\,t)^2 = \left(\frac{c^2}{a_o}\right)^2 \tag{137}$$

Therefore, the world lines are hyperbolas in an $x - t$ diagram and the motion is called hyperbolic motion in comparison to parabolic motion $(x = \frac{1}{2}a_o t^2)$ in Newtonian mechanics. The inversion of the above-mentioned formulas yields for the time

$$\frac{a_o t}{c} = \frac{u/c}{\sqrt{1 - u^2/c^2}} = \frac{\sqrt{1 - \alpha^2}}{\alpha} = \sqrt{\left(1 + \frac{a_o}{c^2}x\right)^2 - 1} = \sqrt{\frac{a_o}{c^2}x\left(2 + \frac{a_o}{c^2}x\right)} \tag{138}$$

The proper time of the astronaut follows from integration of eq. (133):

$$t_o = \frac{c}{a_o}\ln\left[\frac{a_o}{c}t + \sqrt{1 + \left(\frac{a_o}{c}t\right)^2}\right] = \frac{c}{a_o}\sinh^{-1}\left(\frac{a_o}{c}t\right)$$

$$= t\left[1 - \frac{1}{2\cdot 3}\left(\frac{a_o}{c}t\right)^2 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\left(\frac{a_o}{c}t\right)^4 \mp \cdots\right] \tag{139}$$

The inverted formula reads

$$t = \frac{c}{a_o}\sinh\left(\frac{a_o}{c}t_o\right) = \left[\frac{c}{2a_o}\quad e^{a_o t_o/c} - e^{-a_o t_o/c}\right] =$$

$$= t_o\left[1 + \frac{1}{3!}\;\frac{a_o}{c}t_o \;+ \frac{1}{5!}\;\frac{a_o}{c}t_o^{\;4} + \cdots\right] \tag{140}$$

Inserting this relation into eqs. (133), (134), (132) and (135) yields the acceleration, velocity and distance of the rocket as a function of the proper or local time of the astronaut.

$$a = \frac{1}{\cosh(a_o t_o/c)} = 1 - \frac{1}{2!}\left(\frac{a_o}{c} t_o\right)^2 + \frac{5}{4!}\left(\frac{a_o}{c} t_o\right)^4 \mp \cdots \tag{141}$$

$$a = \frac{a_o}{\cosh^3(a_o t_o/c)} = a_o \left[1 - \frac{3}{2}\left(\frac{a_o}{c} t_o\right)^2 + \frac{11}{8}\left(\frac{a_o}{c} t_o\right)^2 \mp \cdots\right] \tag{142}$$

$$u = c \tanh\left(\frac{a_o}{c} t_o\right) = a_o t_o \left[1 - \frac{1}{3}\left(\frac{a_o}{c} t_o\right)^2 + \frac{2}{15}\left(\frac{a_o}{c} t\right)^4 \mp \cdots\right] \tag{143}$$

$$x = \frac{c^2}{a}\left[\cosh\left(\frac{a_o}{c} t_o\right) - 1\right] = \frac{a_o}{2} t_o^2 \left[1 + \frac{1}{3\cdot4}\left(\frac{a_o}{c} t_o\right)^2 + \frac{1}{3\cdot4\cdot5\cdot6}\left(\frac{a_o}{c} t_o\right)^4 + \cdots\right] \tag{144}$$

In addition to eq. (139) for the proper time t_o the inversion of the preceding formulas yields

$$\begin{aligned}\frac{a_o t_o}{c} &= \ln\left(\frac{1 + \sqrt{1 - a^2}}{a}\right) = \cosh^{-1}\left(\frac{1}{a}\right) = \ln\sqrt{\frac{1 + u/c}{1 - u/c}} = \tanh^{-1}\left(\frac{u}{c}\right) \\ &= \ln\left[1 + \frac{a_o}{c^2} x + \sqrt{\left(1 + \frac{a_o}{c^2} x\right)^2 - 1}\right] = \cosh^{-1}\left(1 + \frac{a_o}{c^2} x\right)\end{aligned} \tag{145}$$

A rocket travelling with constant proper acceleration a_o requires, in order to attain a given distance X, a time

$$T = \sqrt{\frac{2X}{a_o}} \tag{146}$$

in the system of classical physics;

$$\begin{aligned} t &= \frac{c}{a_o}\sqrt{\frac{a_o}{c^2} X \left(2 + \frac{a_o}{c^2} X\right)} = T\sqrt{1 + \frac{a_o X}{2c^2}} \\ &= T\left[1 + \frac{1}{2}\left(\frac{a_o X}{2c^2}\right) - \frac{1}{2\cdot4}\left(\frac{a_o X}{2c^2}\right)^2 + \frac{1\cdot3}{2\cdot4\cdot6}\left(\frac{a_o X}{2c^2}\right)^3 \mp \cdots\right]\end{aligned} \tag{147}$$

in the system of the stationary earth observer;

$$\begin{aligned} t_o &= \frac{c}{a_o}\cosh^{-1}\left(1 + \frac{a_o X}{c^2}\right) = T\,\frac{\cosh^{-1}(1 + a_o X/c^2)}{\sqrt{2 a_o X/c^2}} = T\,\frac{\sinh^{-1}\sqrt{a_o X/(2c^2)}}{\sqrt{a_o X/(2c^2)}} \\ &= T\left[1 - \frac{1}{2\cdot3}\left(\frac{a_o X}{2c^2}\right) + \frac{1\cdot3}{2\cdot4\cdot5}\left(\frac{a_o X}{2c^2}\right)^2 - \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7}\left(\frac{a_o X}{2c^2}\right)^3 \pm \cdots\right]\end{aligned} \tag{148}$$

in the system of the astronaut traveling with the rocket. In general $t_o < T < t$. Thus the astronauts undergo an immense gain in time from the relativistic principle.

The preceding relations contained only kinematical data. The equation of motion in the system of the astronaut

$$m_o a_o = F_o \equiv -v_{eo}\frac{dm_o}{dt_o} \quad \text{or} \quad \frac{dm_o}{m_o} = -\frac{a_o}{v_{eo}} dt_o \tag{149}$$

yields, after integration, the mass ratio of the rocket:

$$r \equiv \frac{M_o}{m} = e^{a_o t_o / v_{eo}} = \left[\frac{a_o}{c} t + \sqrt{1 + \left(\frac{a_o}{c} t\right)^2}\right]^{c/v_{eo}} = \left[\frac{1 + \sqrt{1 - a^2}}{a}\right]^{c/v_{eo}}$$

$$= \left[\frac{1 + u/c}{1 - u/c}\right]^{c/(2v_{eo})} = \left[1 + \frac{a_o}{c} x + \sqrt{\left(1 + \frac{a_o}{c} x\right)^2 - 1}\right]^{c/v_{eo}} \tag{150}$$

The thrust is given by

$$F_o = m_o a_o = M_o a_o / r = M_o a_o e^{-a_o t_o / v_{eo}} \quad ; \quad \frac{a_o}{F_o / M_o} = r \tag{151}$$

The other kinematical data, expressed as functions of r, are:

$$\frac{u}{c} = \frac{r^{2v_{eo}/c} - 1}{r^{2v_{eo}/c} + 1} \quad ; \quad a = \sqrt{1 - u^2/c^2} = \frac{2r^{v_{eo}/c}}{r^{2v_{eo}/c} + 1} \tag{152}$$

$$x = \frac{c^2}{a_o}\left(\frac{1}{a} - 1\right) = \frac{c^2}{2a_o} \frac{(r^{v_{eo}/c} - 1)^2}{r^{v_{eo}/c}} \tag{153}$$

$$t_o = \frac{v_{eo}}{a_o} \ln r \; ; \; t = \frac{c}{a_o}\left(\frac{u/c}{a}\right) = \frac{c}{2a_o} \frac{r^{2v_{eo}/c} - 1}{r^{v_{eo}/c}} \tag{154}$$

All relations in this paragraph can be written in a dimensionless form by selecting c/a_o as the unit of time, c as the unit of velocity and c^2/a_o as the unit of length. With $c = 3 \cdot 10^{10}$ cm/sec and $a_o = g_o = 981$ cm/sec^2

$$\frac{c^2}{g_o} = \frac{9 \cdot 10^{20}}{981} = 9.18 \cdot 10^{17} \text{ cm} = 9.18 \cdot 10^{12} \text{km} = 6.14 \cdot 10^4 \text{ A.U.} = 0.97 \text{ light years}$$

$$\frac{c}{g_o} = \frac{3 \cdot 10^{10}}{981} = 3.06 \cdot 10^7 \text{ sec} = 354.2 \ d = 0.97 \text{ years}$$

F. Motion and Mass Consumption of a Rocket Under a Constant Thrust (Constant Mass Flow Rate)

In this case, first treated by the author (Ref. 8) in 1955, and also by Kooy (Ref. 18) the investigation again proceeds from the equation of motion of the rocket in the system of the astronaut traveling with the rocket, namely

$$\frac{m_o a}{(1 - u^2/c^2)^{3/2}} = m_o a_o = F_o = v_{eo} \mu_o \tag{155}$$

Taking a constant mass flow rate, μ_o, the mass then decreases linearly with time t_o according to the law

$$m_o = M_o - \mu_o t_o = M_o \left(1 - \frac{\mu_o}{M_o} t_o\right) \tag{156}$$

Again taking the exhaust velocity v_{eo} of the gases to be a constant, then the thrust, F_o, also must be constant. On the other hand, $F = F_o \ (1 - u/v_{eo}) = \mu_o \ (v_{eo} - u)$ is variable. Combining the two eqs. (155) and (156) the proper acceleration is

$$\frac{a}{(1 - u^2/c^2)^{3/2}} = a_o = \frac{F_o}{m} = v_{eo} \frac{\mu_o / M_o}{1 - (\mu_o / M_o) t_o} \tag{157}$$

or (since $a = du/dt = a\ du/dt_o$)

$$\frac{du}{1 - u^2/c^2} = v_{eo} \frac{\mu_o/M_o}{1 - (\mu_o/M_o)t_o} dt_o$$

An integration yields

$$\frac{c}{2} \ln \frac{1 + u/c}{1 - u/c} = - v_{eo} \ln \left(1 - \frac{\mu_o}{M} t_o\right)$$

Thus, the reciprocal mass ratio is

$$\frac{1}{r} \equiv \frac{m_o}{M_o} = 1 - \frac{\mu_o}{M_o} t_o = \left[\frac{1 - u/c}{1 + u/c}\right]^{c/2v_{eo}} \tag{158}$$

This is again the basic relativistic rocket equation which is valid for any law of mass consumption. The inversion of eq. (158) gives the velocity

$$u = c \frac{1 - (1/r)^{2v_{eo}/c}}{1 + (1/r)^{2v_{eo}/c}} = c \frac{r^{2v_{eo}/c} - 1}{r^{2v_{eo}/c} + 1} \tag{159}$$

therefore

$$a = \frac{dt_o}{dt} = \sqrt{1 - u^2/c^2} = \frac{2(1/r)^{v_{eo}/c}}{1 + (1/r)^{2v_{eo}/c}} = \frac{2r^{v_{eo}/c}}{r^{2v_{eo}/c} + 1} \tag{160}$$

In the two preceding equations, $1/r$ can also be replaced by m_o/M_o or $1 - (\mu_o/M_o)\, t_o$. The acceleration is easily found from $a = a_o a^3$ using eq. (160). The time and the distance can be determined in the following manner, using eqs. (158), (159) and (160):

$$t = \int_o^{t_o} \frac{dt_o}{a} = \frac{M_o}{\mu_o} \int_1^r \frac{dr}{r^2 a} = \begin{cases} \dfrac{M_o}{2\mu_o} \left[\dfrac{1 - (1/r)^{1-(v_{eo}/c)}}{1 - (v_{eo}/c)} + \dfrac{1 - (1/r)^{1+(v_{eo}/c)}}{1 + (v_{eo}/c)}\right] \text{ for } v_{eo} \neq c & (161) \\ \\ \dfrac{M_o}{2\mu_o} \left[\ln r + \dfrac{1 - (1/r)^2}{2}\right] = \dfrac{M_o}{4\mu_o} \ln\left[r^2 + (1 - 1/r^2)\right] \text{ for } v_{eo} = c & (162) \end{cases}$$

and

$$x = c \int_o^{t_o} \frac{u/c}{a} dt_o = c \frac{M_o}{\mu_o} \int_1^r \frac{u/c}{a r^2} dr =$$

$$\begin{cases} \dfrac{c}{2} \dfrac{M_o}{\mu_o} \left[\dfrac{1 - (1/r)^{1-(v_{eo}/c)}}{1 - (v_{eo}/c)} - \dfrac{1 - (1/r)^{1+(v_{eo}/c)}}{1 + (v_{eo}/c)}\right] \text{ for } v_{eo} \neq c & (163) \\ \dfrac{c}{2} \dfrac{M_o}{\mu_o} \left[\ln r - \dfrac{1 - (1/r)^2}{2}\right] = \dfrac{c}{4} \dfrac{M_o}{\mu_o} \left[\ln r^2 - (1 - 1/r^2)\right] \text{ for } v_{eo} = c & (164) \end{cases}$$

In these last equations, $1/r$ can be substituted by m_o, t_o or u according to eq. (158).

In classical physics $(c \to \infty\ ;\ \epsilon \equiv v_{eo}/c \to 0)$ $a = 1$, $a = a_o$ and $t = t_o$. For eqs. (159) and (163) the limit must be found by means of the Bernoulli-de l'Hospital formula:

$$\frac{u}{v_{eo}} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{r^{2\epsilon} - 1}{r^{2\epsilon} + 1} = \frac{1}{2} \lim_{\epsilon \to 0} \frac{r^{2\epsilon} - 1}{\epsilon} = \frac{1}{2} \lim_{\epsilon \to 0} (2r^{2\epsilon} \ln r) = \ln r$$

and

$$\frac{x}{v_{eo}\, M_o/\mu_o} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\left[\frac{1-(1/r)^{1-\epsilon}}{1-\epsilon} - \frac{1-(1/r)^{1+\epsilon}}{1+\epsilon}\right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\left[\frac{2\epsilon - (1+\epsilon)(1/r)^{1-\epsilon} + (1-\epsilon)(1/r)^{1+\epsilon}}{1-\epsilon^2}\right]$$

$$= \lim_{\epsilon \to 0}\left[1 - \frac{(1+\epsilon)(1/r)^{1-\epsilon} - (1-\epsilon)(1/r)^{1+\epsilon}}{2\epsilon}\right]$$

$$= \lim_{\epsilon \to 0}\left\{1 - \frac{1}{2}\left[\left(\frac{1}{r}\right)^{1-\epsilon} - (1+\epsilon)\left(\frac{1}{r}\right)^{1-\epsilon} \ln\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right)^{1+\epsilon} - (1-\epsilon)\left(\frac{1}{r}\right)^{1+\epsilon} \ln\left(\frac{1}{r}\right)\right]\right\}$$

$$= 1 - \frac{1}{r} + \frac{1}{r}\ln\frac{1}{r} = 1 - \frac{1}{r}\left(1 - \ln\frac{1}{r}\right) = 1 - \frac{1+\ln r}{r}$$

Unfortunately it is not possible to express acceleration, velocity and distance also by the time t, as in the special case of constant proper acceleration, since eq. (161) cannot be inverted to give t_o as a function of t. Developing this equation in a power series

$$t = \frac{M_o}{2\mu_o}\left[\frac{1-(1-\mu_o t_o/M_o)^{1-(v_{eo}/c)}}{1-(v_{eo}/c)} + \frac{1-(1-\mu_o t_o/M_o)^{1+(v_{eo}/c)}}{1+(v_{eo}/c)}\right],$$

$$= t_o\left(1 + \left(\frac{v_{eo}}{c}\right)^2\left\{\frac{1}{3!}\left(\frac{\mu_o}{M_o}t_o\right)^2 + \frac{3}{4!}\left(\frac{\mu_o}{M_o}t_o\right)^3 + \frac{1}{5!}\left[11 + \left(\frac{v_{eo}}{c}\right)^2\right]\left(\frac{\mu_o}{M_o}t_o\right)^4 + \cdots\right\}\right) \qquad (165)$$

then the reversed power series reads

$$t_o = t\left(1 - \left(\frac{v_{eo}}{c}\right)^2\left\{\frac{1}{3!}\left(\frac{\mu_o}{M_o}t\right)^2 + \frac{3}{4!}\left(\frac{\mu_o}{M_o}t\right)^3 + \frac{1}{5!}\left[11 - 9\left(\frac{v_{eo}}{c}\right)^2\right]\left(\frac{\mu_o}{M_o}t\right)^4 + \cdots\right\}\right) \qquad (166)$$

All preceding relations can be written in dimensionless form when choosing M_o/μ_o as the unit of time, c as the unit of velocity and $c\, M_o/\mu_o$ as the unit of length. The following three tables show numerically the characteristic difference between the case of constant thrust acceleration a_o and the case of constant mass flow rate μ_o (the exhaust velocity v_{eo} was taken constant in all cases). For the second case (μ_o = const.) the calculations were made for two special exhaust velocities, namely for $v_{eo} = c/10$ and $v_{eo} = c$ (photon rocket). The assumption $v_{eo}/c = 0.1$ is not typical for ion rockets; it is probably an upper limit. For fusion processes, the mass conversion is <0.009 corresponding to v_{eo}/c <0.134.

Table 2

Data of Rocket in Free Space Without External Forces

Data of rocket in free space without external forces	In system S of stationary earth observer	In rest system S_o of astronaut centered in the moving rocket itself
Velocity of rocket	$\mathbf{u}$	0
Actual mass of rocket	$m = \frac{m_o}{\sqrt{1-u^2/c^2}} = \frac{m_o}{\alpha}$	m_o
Time element	$dt = \frac{dt_o}{\sqrt{1-u^2/c^2}} = \frac{dt_o}{\alpha}$	dt_o
Acceleration of rocket	$\mathbf{a} = \alpha^2 \left[\mathbf{a}_o - (1-\alpha)\left(\frac{\mathbf{a}_o \cdot \mathbf{u}}{u^2}\right)\mathbf{u}\right]$	$\mathbf{a}_o$
Exhaust velocity of expelled gases	$\mathbf{v}_e = \frac{\alpha \mathbf{v}_{eo} + \mathbf{u}[(1-\alpha)(\mathbf{v}_{eo} \cdot \mathbf{u})/u^2 - 1]}{1 - (\mathbf{v}_{eo} \cdot \mathbf{u})/c^2}$	v_{eo}
Element of mass flow (after ejection)	$dm = \frac{dm_o^*}{\sqrt{1-v_e^2/c^2}}$	$dm_o = \frac{dm_o^*}{\sqrt{1-v_{eo}^2/c^2}}$
Mass flow rate	$\mu = -\frac{dm}{dt} = \frac{\alpha\mu_o^*}{\sqrt{1-v_e^2/c^2}}$	$\mu_o = -\frac{dm_o}{dt_o} = \frac{\mu_o^*}{\sqrt{1-v_{eo}^2/c^2}}$
Thrust force of rocket	$\mathbf{F} = \mu \mathbf{v}_e = \frac{\alpha\mu_o^*}{\sqrt{1-v_e^2/c^2}}\mathbf{v}_e$ $= \alpha \mathbf{F}_o + \mathbf{u}\,[(1-\alpha)(\mathbf{F}_o \cdot \mathbf{u})/u^2 - \mu_o]$	$\mathbf{F}_o = \mu_o \mathbf{v}_{eo} = \frac{\mu_o^*}{\sqrt{1-v_{eo}^2/c^2}}\mathbf{v}_{eo}$

Table 3. Constant Thrust Acceleration a_o

$\beta=\frac{u}{c}$	$\alpha=\frac{m_o}{m}=\frac{dt_o}{dt}$	$\frac{a}{a_o}$	$\frac{x}{c^2/a_o}$	$\frac{t_o}{c/a_o}$	$\frac{t_o}{c/a_o}$	$\frac{t_o}{t}$	$r^{v_o/c}$
0.0	1	1	0	0	0	1	1
0.1	0.9950	0.9850	0.0050	0.1005	0.1003	0.9980	1.1054
0.2	0.9798	0.9406	0.0206	0.2041	0.2030	0.9944	1.2247
0.3	0.9539	0.8680	0.0483	0.3145	0.3097	0.9847	1.3627
0.4	0.9165	0.7699	0.0911	0.4364	0.4240	0.9715	1.5274
0.5	0.8660	0.6495	0.1547	0.5774	0.5493	0.9513	1.7321
0.6	0.8000	0.5120	0.2500	0.7500	0.6932	0.9242	2.0000
0.7	0.7141	0.3642	0.4004	0.9803	0.8675	0.8850	2.3806
0.8	0.6000	0.2160	0.6667	1.3333	1.0986	0.8242	3.0000
0.9	0.4359	0.0828	1.2941	2.0647	1.4722	0.7130	4.3589
0.995	0.1	0.001	9.000	9.950	2.9932	0.3008	19.975
0.99995	0.01	0.000001	99.00	100.00	5.298	0.053	200.00
1.0	0	0	∞	∞	∞	0	∞

Table 4. Constant Thrust $v_o/c = 0.1$ (Ion Rocket)

$\beta = \mu/c$	$\frac{x}{cM_o/\mu_o}$	$\frac{t}{M_o/\mu_o}$	$\frac{t}{M_o/\mu_o}$	$\frac{t_o}{t}$	r
0.0	0	0	0	1	1
0.1	0.0269	0.6334	0.6329	0.9992	2.724×10^0
0.2	0.0603	0.8719	0.8684	0.9960	7.599×10^0
0.3	0.0816	0.9604	0.9547	0.9941	2.208×10^1
0.4	0.0930	0.9935	0.9855	0.9919	6.912×10^1
0.5	0.0981	1.0051	0.9959	0.9908	2.430×10^2
0.6	0.1002	1.0088	0.9990	0.9903	1.024×10^3
0.7	0.1008	1.0098	0.9999	0.9902	5.844×10^3
0.8	0.1010	1.0101	1.0000	0.9900	5.905×10^4
0.9	0.1010	1.0101	1.0000	0.9900	2.476×10^6
0.995	0.1010	1.0101	1.0000	0.9900	1.024×10^{13}
1.0	0.1010	1.0101	1.0000	0.9900	∞

Table 5. Constant Thrust $\nu_o/c = 1$ (Photon Rocket)

$\beta = \mu/c$	$\frac{x}{cM_o/\mu_o}$	$\frac{t}{M_o/\mu_o}$	$\frac{t_o}{M_o/\mu_o}$	$\frac{t_o}{t}$	r
0.0	0	0	0	1	1
0.1	0.0047	0.0955	0.0953	0.9979	1.1054
0.2	0.0180	0.1847	0.1835	0.9935	1.2247
0.3	0.0394	0.2701	0.2662	0.9856	1.3627
0.4	0.0690	0.3547	0.3453	0.9735	1.5274
0.5	0.1080	0.4414	0.4227	0.9576	1.7321
0.6	0.1591	0.5341	0.5000	0.9362	2.0000
0.7	0.2278	0.6396	0.5799	0.9067	2.3806
0.8	0.3271	0.7715	0.6667	0.8642	3.0000
0.9	0.4993	0.9730	0.7706	0.7920	4.3589
0.995	1.2476	1.7464	0.9499	0.5439	19.9750
1.0	∞	∞	1	0	∞

PART II

RELATIVISTIC PERTURBATION THEORY OF AN ARTIFICIAL SATELLITE IN AN ARBITRARY ORBIT ABOUT THE ROTATING OBLATED EARTH SPHEROID AND THE TIME DILATATION EFFECT FOR THIS SATELLITE

SUMMARY

In this part Einstein's general theory of relativity (gravitational theory) is applied to the motion of an artificial satellite revolving in an arbitrary orbit around a central body and the time dilatation effect for this satellite is given. This relativistic perturbation theory is based on Einstein's general field theory, differential geometry of non-Euclidean spaces, potential theory, and celestial mechanics. The short periodic perturbations are excluded by using time average values over a revolution. The secular and long-periodic (non-relativistic as well as relativistic) perturbations of the osculating orbital elements, which represent deviations from the elliptic orbit, are presented here for the case of a rotating, non-homogeneous, oblated spheroidal central body. This is an extension of the work of Einstein (1915) who considered motion around a mass point as well as the work of deSitter (1916) and, independently, of Lense and Thirring (1918), who treated the relativistic motion around a rotating, homogeneous, spherical central body, omitting the terms due to the square of the angular velocity.

A formula for the relative difference of the time rates of a satellite clock, compared against a standard earth clock (time dilatation effect) is derived for orbits of any eccentricity and equatorial inclinations, thus extending the paper of Winterberg (1955), Singer (1956) and Hoffmann (1957).

3. APPLICATION OF THE GENERAL THEORY OF RELATIVITY TO ARTIFICIAL SATELLITES

A. Relativistic Perturbation Theory

In this section Einstein's general theory of relativity will be applied to determine the motion of an artificial satellite revolving around the rotating earth as well as the difference in time rates of a satellite clock and a standard earth clock.

In Einstein's general theory of relativity gravitation is determined by the 10 different components g_{kl} of a symmetric covariant tensor of the second rank called the fundamental or metric tensor. These components g_{kl} are functions of the coordinates x_1; x_2; x_3, x_4 and they appear in the formula for the four dimensional line-element of the non-Enclidean time-space world, namely

$$ds^2 = \sum_k \sum_l g_{kl}\, dx_k\, dx_l \qquad (g_{kl} = g_{lk}) \tag{167}$$

In the following, a spherical polar coordinate system $(x_1 = r;\ x_2 = \theta;\ x_3 = \phi;\ x_4 = ct)$ will be used, where r is the radius vector, θ the longitude, ϕ the latitude, t the time and c the light velocity. The line element has the following form

$$\begin{aligned} ds^2 &= g_{11}\, dx_1^2 + g_{22}\, dx_2^2 + g_{33}\, dx_3^2 + g_{44}\, dx_4^2 + 2\, g_{24}\, dx_2\, dx_4 \\ &= -(1+\alpha)\, dr^2 - (1+\beta)\,(r^2 \cos^2\phi\, d\theta^2 + r^2 d\phi^2) + (1+\gamma)\, c^2 dt^2 + 2 g_{24}\, d\theta\, cdt \\ &= -(1+\beta)\, d\sigma^2 + (\beta - \alpha)\, dr^2 + (1+\gamma)\, c^2 dt^2 + 2\, g_{24}\, d\theta\, cdt \end{aligned} \tag{168}$$

thus

$$g_{11} = -(1+\alpha)\ ;\quad g_{22} = -(1+\beta)\, r^2 \cos^2\phi\ ;\quad g_{33} = -(1+\beta) r^2\ ;\quad g_{44} = 1+\gamma$$

$$g_{ik} = 0\ (i \neq k) \ \text{ except }\ g_{24} = g_{42} = \frac{b \cos^2\phi}{r} \quad (b = \text{const}) \tag{169}$$

where α and β are functions of r alone while g_{44} will be assumed to be a function of r and ϕ because it is needed to the second order.

The fundamental metric tensor has now the following covariant components:

$$g_{kl} = g_{lk} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & g_{24} \\ 0 & 0 & g_{33} & 0 \\ 0 & g_{42} & 0 & g_{44} \end{pmatrix}$$

with the determinant

$$g = |g_{kl}| = g_{11} \begin{vmatrix} g_{22} & 0 & g_{24} \\ 0 & g_{33} & 0 \\ g_{42} & 0 & g_{44} \end{vmatrix} = g_{11}\, g_{22} \begin{vmatrix} g_{33} & 0 \\ 0 & g_{44} \end{vmatrix} + g_{11}\, g_{42} \begin{vmatrix} 0 & g_{24} \\ g_{33} & 0 \end{vmatrix}$$

$$= g_{11}\, g_{22}\, g_{33}\, g_{44} - g_{11}\, g_{33}\, g_{24}^2 \approx g_{11}\, g_{22}\, g_{33}\, g_{44}$$

The contravariant components are now the minors of the correspondent covariant components divided by the determinant, namely

$$g^{kl} = \frac{\text{minor of } g_{kl}}{g} = \frac{\text{cofactor of } g_{kl}}{g}$$

Therefore

$$g^{11} = \frac{1}{g} \begin{vmatrix} g_{22} & 0 & g_{24} \\ 0 & g_{33} & 0 \\ g_{42} & 0 & g_{44} \end{vmatrix} = \frac{1}{g_{11}}\,; \quad g^{33} = \frac{1}{g} \begin{vmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & g_{24} \\ 0 & g_{42} & g_{44} \end{vmatrix} = \frac{g_{11}(g_{22} g_{44} - g_{24}^2)}{g_{11} g_{33} (g_{22} g_{44} - g_{24}^2)} = \frac{1}{g_{33}}$$

$$g^{22} = \frac{1}{g} \begin{vmatrix} g_{11} & 0 & 0 \\ 0 & g_{33} & 0 \\ 0 & 0 & g_{44} \end{vmatrix} = \frac{g_{11}\, g_{33}\, g_{44}}{g_{11}\, g_{22}\, g_{33}\, g_{44} - g_{11}\, g_{33}\, g_{24}^2} = \frac{1}{g_{22} - g_{24}^2 / g_{44}} \approx \frac{1}{g_{22}}$$

$$g^{44} = \frac{1}{g} \begin{vmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{vmatrix} = \frac{g_{11}\, g_{22}\, g_{33}}{g_{11}\, g_{22}\, g_{33}\, g_{44} - g_{11}\, g_{33}\, g_{24}^2} = \frac{1}{g_{44} - g_{24}^2 / g_{22}} \approx \frac{1}{g_{44}}$$

$$g^{24} = g^{42} = \frac{1}{g} \begin{vmatrix} g_{11} & 0 & 0 \\ 0 & 0 & g_{24} \\ 0 & g_{33} & 0 \end{vmatrix} = \frac{-g_{11}\, g_{33}\, g_{24}}{g_{11}\, g_{33}\, (g_{22}\, g_{44} - g_{24}^2)} = \frac{-g_{24}}{g_{22}\, g_{44} - g_{24}^2} \approx -\frac{g_{24}}{g_{22}\, g_{44}}$$

The Christoffel three-index symbols of first and second kind can now be defined as:

$$\Gamma_{i\,kl} = \begin{bmatrix} k\,l \\ i \end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_l} + \frac{\partial g_{il}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_i}\right) = \begin{bmatrix} l\,k \\ i \end{bmatrix} \tag{170}$$

and

$$\Gamma^i_{kl} = \begin{Bmatrix} k\,l \\ i \end{Bmatrix} = \sum_{r=1}^{4} g^{ir} \begin{bmatrix} k\,l \\ r \end{bmatrix} = \frac{1}{2} \sum_{r=1}^{4} g^{ir} \left(\frac{\partial g_{kr}}{\partial x_l} + \frac{\partial g_{lr}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_r}\right) = \begin{Bmatrix} l\,k \\ i \end{Bmatrix}$$

$$= \frac{1}{2} g^{i1} \left(\frac{\partial g_{k1}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_k} - \frac{\partial g_{k1}}{\partial x_1} \right) + \frac{1}{2} g^{i2} \left(\frac{\partial g_{k2}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_k} \right)$$

$$+ \frac{1}{2} g^{i3} \left(\frac{\partial g_{k3}}{\partial x_1} + \frac{\partial g_{13}}{\partial x_k} - \frac{\partial g_{k1}}{\partial x_3} \right) + \frac{1}{2} g^{i4} \left(\frac{\partial g_{k4}}{\partial x_1} + \frac{\partial g_{14}}{\partial x_k} \right)$$

$$= \frac{1}{2} g^{ii} \left(\frac{\partial g_{ki}}{\partial x_1} + \frac{\partial g_{1i}}{\partial x_k} - \frac{\partial g_{k1}}{\partial x_i} \right) + \begin{cases} \frac{1}{2} g^{24} \frac{\partial g_{k4}}{\partial x_1} + \frac{\partial g_{14}}{\partial x_k} & \text{for } i = 2 \\ \frac{1}{2} g^{42} \frac{\partial g_{k2}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_k} & \text{for } i = 4 \\ 0 & \text{for } i = 1 \text{ or } 3 \end{cases} \quad (171)$$

where the g_{k1} and g^{k1} were taken as functions of x_1 and x_3 only. Special cases for $l = i$, $l = k$ and $k = l = i$ are

$$\Gamma^i_{ki} = \frac{1}{2} \sum_{r=1}^{4} g^{ir} \left(\frac{\partial g_{kr}}{\partial x_i} + \frac{\partial g_{ir}}{\partial x_k} - \frac{\partial g_{ki}}{\partial x_r} \right) = \frac{1}{2} \sum_{r=1}^{4} g^{ir} \frac{\partial g_{ir}}{\partial x_k}$$

$$= \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial x_k} + \begin{cases} \frac{1}{2} g^{24} \frac{\partial g_{24}}{\partial x_k} & (i = 2 \text{ or } 4) \\ 0 & (i = 1 \text{ or } 3) \end{cases}$$

thus $\Gamma^i_{2i} = 0$ and $\Gamma^i_{4i} = 0.$

$$\Gamma^i_{kk} = \frac{1}{2} \sum_{r=1}^{4} g^{ir} \left(2 \frac{\partial g_{kr}}{\partial x_k} - \frac{\partial g_{kk}}{\partial x_r} \right) = \frac{1}{2} g^{ii} \left(2 \frac{\partial g_{ki}}{\partial x_k} - \frac{\partial g_{kk}}{\partial x_i} \right) = \begin{cases} \neq 0 & (i = 1 \text{ or } 3) \\ 0 & (i = 2 \text{ or } 4) \end{cases}$$

$$\Gamma^i_{ii} = \frac{1}{2} \sum_{r=1}^{4} g^{ir} \left(2 \frac{\partial g_{ir}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_r} \right) = \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial x_i} = \begin{cases} \neq 0 & (i = 1 \text{ or } 3) \\ 0 & (i = 2 \text{ or } 4) \end{cases}$$

When all the functions g_{11}, g_{22}, g_{33}, g_{44} and $g_{24} = g_{42}$ are functions of $x_1 = r$ and $x = \phi$, then the $\Gamma^i_{k1} \neq 0$ are

$$\Gamma^1_{11} = \frac{1}{2 g_{11}} \frac{\partial g_{11}}{\partial x_1}; \quad \Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2 g_{11}} \frac{\partial g_{11}}{\partial x_3} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2 g_{22}} \frac{\partial g_{22}}{\partial x_1}; \quad \Gamma^2_{14} = \Gamma^2_{41} = \frac{1}{2 g_{22}} \frac{\partial g_{24}}{\partial x_1}$$

$$\Gamma^1_{22} = -\frac{1}{2 g_{11}} \frac{\partial g_{22}}{\partial x_1}; \quad \Gamma^1_{24} = \Gamma^1_{42} = -\frac{1}{2 g_{11}} \frac{\partial g_{24}}{\partial x_1} \qquad \Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{2 g_{22}} \frac{\partial g_{22}}{\partial x_3}$$

$$\Gamma^1_{33} = -\frac{1}{2 g_{11}} \frac{\partial g_{33}}{\partial x_1}; \quad \Gamma^1_{44} = -\frac{1}{2 g_{11}} \frac{\partial g_{44}}{\partial x_1} \qquad \Gamma^2_{34} = \Gamma^2_{43} = \frac{1}{2 g_{22}} \frac{\partial g_{24}}{\partial x_3}$$

$$\Gamma^3_{11} = -\frac{1}{2g_{33}}\frac{\partial g_{11}}{\partial x_3}\,;\; \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial x_1}$$

$$\Gamma^3_{22} = -\frac{1}{2g_{33}}\frac{\partial g_{22}}{\partial x_3}\,;\; \Gamma^3_{24} = \Gamma^3_{42} = -\frac{1}{2g_{33}}\frac{\partial g_{24}}{\partial x_3}$$

$$\Gamma^3_{33} = \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial x_3}\,;\; \Gamma^3_{44} = -\frac{1}{2g_{33}}\frac{\partial g_{44}}{\partial x_3}$$

$$\Gamma^4_{12} = \Gamma^4_{21} = \frac{1}{2g_{44}}\left(\frac{\partial g_{24}}{\partial x_1} - \frac{g_{24}}{g_{22}}\frac{\partial g_{22}}{\partial x_1}\right);\; \Gamma^4_{14} = \Gamma^4_{41} = \frac{1}{2g_{44}}\frac{\partial g_{44}}{\partial x_1}$$

$$\Gamma^4_{23} = \Gamma^4_{32} = \frac{1}{2g_{44}}\left(\frac{\partial g_{24}}{\partial x_3} - \frac{g_{24}}{g_{22}}\frac{\partial g_{22}}{\partial x_3}\right)$$

$$\Gamma^4_{34} = \Gamma^4_{43} = \frac{1}{2g_{44}}\frac{\partial g_{44}}{\partial x_3}$$

and the following Christoffel symbols vanish

$$\Gamma^i_{12} = \Gamma^i_{14} = \Gamma^i_{21} = \Gamma^i_{23} = \Gamma^i_{32} = \Gamma^i_{34} = \Gamma^i_{41} = \Gamma^i_{43} = 0 \quad (i = 1 \text{ or } 3)$$

$$\Gamma^i_{11} = \Gamma^i_{13} = \Gamma^i_{22} = \Gamma^i_{24} = \Gamma^i_{31} = \Gamma^i_{33} = \Gamma^i_{42} = \Gamma^i_{44} = 0' \quad (i = 2 \text{ or } 4)$$

By contraction of the Riemann-Christoffel curvature tensor of rank 4 for the non-Euclidean space-time world a symmetrical curvature tensor of rank 2 is obtained by the explicit expression

$$\sum_r R^r_{\;krl} = R_{kl} = R_{lk} = \sum_r \frac{\partial \Gamma^r_{kr}}{\partial x_l} - \sum_r\sum_s \Gamma^r_{kl}\,\Gamma^s_{rs} - \sum_r \frac{\partial \Gamma^r_{kl}}{\partial x_r} + \sum_r\sum_s \Gamma^s_{kr}\,\Gamma^r_{ls} \tag{172}$$

Using the relation

$$\sum_r \Gamma^r_{kr} = \frac{1}{2}\sum_r \frac{1}{g_{rr}}\frac{\partial g_{rr}}{\partial x_k} = \frac{1}{2}\sum_r \frac{\partial \ln|g_{ll}|}{\partial x_k} = \frac{1}{2}\frac{\partial \ln|g|}{\partial x_k} = \frac{\partial \ln\sqrt{|g|}}{\partial x_k} \tag{173}$$

the contracted curvature tensor can now be expressed by

$$R_{kl} = R_{lk} = \frac{\partial^2 \ln\sqrt{|g|}}{\partial x_k \partial x_l} - \sum_r \Gamma^r_{kl}\frac{\partial \ln\sqrt{|g|}}{\partial x_r} - \sum_r \frac{\partial \Gamma^r_{kl}}{\partial x_r} + \sum_r\sum_s \Gamma^s_{kr}\,\Gamma^r_{ls}$$

$$= \frac{\partial^2 \ln\sqrt{|g|}}{\partial x_k \partial x_l} - \Gamma^1_{kl}\frac{\partial \ln\sqrt{|g|}}{\partial x_1} - \Gamma^3_{kl}\frac{\partial \ln\sqrt{|g|}}{\partial x_3} - \frac{\partial \Gamma^1_{kl}}{\partial x_1} - \frac{\partial \Gamma^3_{kl}}{\partial x_3}$$

$$+ \left[\Gamma^1_{k1}\Gamma^1_{11} + \Gamma^2_{k1}\Gamma^1_{12} + \Gamma^3_{k1}\Gamma^1_{13} + \Gamma^4_{k1}\Gamma^1_{14}\right] + \left[\Gamma^1_{k2}\Gamma^2_{11} + \Gamma^2_{k2}\Gamma^2_{12} + \Gamma^3_{k2}\Gamma^2_{13} + \Gamma^4_{k2}\Gamma^2_{14}\right]$$

$$+ \left[\Gamma^1_{k3}\Gamma^3_{11} + \Gamma^2_{k3}\Gamma^3_{12} + \Gamma^3_{k3}\Gamma^3_{13} + \Gamma^4_{k3}\Gamma^3_{14}\right] + \left[\Gamma^1_{k4}\Gamma^4_{11} + \Gamma^2_{k4}\Gamma^4_{12} + \Gamma^3_{k4}\Gamma^4_{13} + \Gamma^4_{k4}\Gamma^4_{14}\right]$$

and thus for $l = k$

$$R_{kk} = \frac{\partial^2 \ln\sqrt{|g|}}{\partial x_k^2} - \Gamma^1_{kk}\frac{\partial \ln\sqrt{|g|}}{\partial x_1} - \Gamma^3_{kk}\frac{\partial \ln\sqrt{|g|}}{\partial x_3} - \frac{\partial \Gamma^1_{kk}}{\partial x_1} - \frac{\partial \Gamma^3_{kk}}{\partial x_3}$$

$$+ \left(\Gamma^1_{k1}\right)^2 + 2\,\Gamma^2_{k1}\,\Gamma^1_{k2} + 2\,\Gamma^3_{k1}\,\Gamma^1_{k3} + 2\,\Gamma^4_{k1}\,\Gamma^1_{k4} + \left(\Gamma^2_{k2}\right)^2 + 2\,\Gamma^3_{k2}\,\Gamma^2_{k3}$$

$$+ 2\,\Gamma^4_{k2}\,\Gamma^2_{k4} + \left(\Gamma^3_{k3}\right)^2 + 2\,\Gamma^4_{k3}\,\Gamma^3_{k4} + \left(\Gamma^4_{k4}\right)^2$$

Neglecting higher order terms the ten components of the contracted curvature tensor are ($\Gamma^1_{13} = \Gamma^3_{11} = \Gamma^3_{33} = \Gamma^4_{23} = 0$):

$$R_{11} = \frac{\partial^2 \ln\sqrt{|g|}}{\partial x} - \Gamma^1_{11}\frac{\partial \ln\sqrt{|g|}}{\partial x} - \frac{\partial \Gamma^1_{11}}{\partial x} + (\Gamma^1_{11})^2 + (\Gamma^2_{12})^2 + (\Gamma^3_{13})^2 + (\Gamma^4_{14})^2$$

$$R_{22} = -\Gamma^1_{22}\frac{\partial \ln\sqrt{|g|}}{\partial x_1} - \Gamma^3_{22}\frac{\partial \ln\sqrt{|g|}}{\partial x_3} - \frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^3_{22}}{\partial x_3} + 2\,\Gamma^2_{21}\,\Gamma^1_{22} + 2\,\Gamma^3_{22}\,\Gamma^2_{23}$$

$$R_{33} = \frac{\partial^2 \ln\sqrt{|g|}}{\partial x_3} - \Gamma^1_{33}\frac{\partial \ln\sqrt{|g|}}{\partial x_1} - \frac{\partial \Gamma_{33}}{\partial x_1} + 2\,\Gamma^3_{31}\,\Gamma^1_{33} + (\Gamma^2_{32})^2 + (\Gamma^4_{34})^2$$

$$R_{44} = -\Gamma^1_{44}\frac{\partial \ln\sqrt{|g|}}{\partial x_1} - \Gamma^3_{44}\frac{\partial \ln\sqrt{|g|}}{\partial x_3} - \frac{\partial \Gamma^1_{44}}{\partial x_1} - \frac{\partial \Gamma^3_{44}}{\partial x_3} + 2\,\Gamma^4_{41}\,\Gamma^1_{44} + 2\,\Gamma^4_{43}\,\Gamma^3_{44}$$

$$R_{13} = R_{31} = \frac{\partial^2 \ln\sqrt{|g|}}{\partial x_1 \partial x_3} - \Gamma^3_{13}\frac{\partial \ln\sqrt{|g|}}{\partial x_3} + \Gamma^2_{12}\,\Gamma^2_{32} + \Gamma^2_{14}\,\Gamma^4_{32} + \Gamma^4_{14}\,\Gamma^4_{34}$$

$$R_{24} = R_{42} = -\Gamma^1_{24}\frac{\partial \ln\sqrt{|g|}}{\partial x_1} - \Gamma^3_{24}\frac{\partial \ln\sqrt{|g|}}{\partial x_3} - \frac{\partial \Gamma^1_{24}}{\partial x_1} - \frac{\partial \Gamma^3_{24}}{\partial x_3} + \Gamma^2_{21}\,\Gamma^1_{42} + \Gamma^4_{21}\,\Gamma^1_{44}$$

$$+ \Gamma^1_{22}\,\Gamma^2_{41} + \Gamma^3_{22}\,\Gamma^2_{43} + \Gamma^2_{23}\,\Gamma^3_{42} + \Gamma^1_{24}\,\Gamma^4_{41} + \Gamma^3_{24}\,\Gamma^4_{43}$$

$$R_{12} = R_{21} = 0 \;;\quad R_{14} = R_{41} = 0 \;;\quad R_{23} = R_{32} = 0 \;;\quad R_{34} = R_{43} = 0$$

The most important problem in Einstein's theory was to set up the general equations determining the gravitational field variables or the g_{kl} when the distribution of mass is given. Einstein finally solved this problem in 1915 after several attempts and succeeded in finding the general field equations in covariant form corresponding to Poisson's equation in Newton's mechanics. Based on the theorems of conservation of energy and momentum he (Ref.19) found

$$R_{kl} - \frac{1}{2} R g_{kl} = -\kappa\, T_{kl} \tag{174}$$

where

$$R = \sum_k \sum_l g^{kl} R_{kl} \tag{175}$$

is the curvature scalar and

$$T_{kl} = \rho \sum_r \sum_s g_{rk}\, g_{sl} \frac{dx_r}{ds} \frac{dx_s}{ds} \tag{176}$$

is Einstein's energy-momentum tensor neglecting the small contributions of pressure and elastic stresses (ρ = density). The constant of proportionality

$$\kappa = \frac{8\pi G}{c^2} \tag{177}$$

follows from a comparison of the general field equations with Poisson's equation (G is Newton's gravitational constant).

Multiplying Einstein's field equations (174) by g^{kl} and summing over k and l (contraction) then follows

$$R = \kappa T \tag{178}$$

because

$$T = \sum_k \sum_l g^{kl}\, T_{kl} \quad \text{and} \quad \sum_k \sum_l g^{kl}\, g_{kl} = 4 \tag{179}$$

Therefore, the field equations can be also written in the form

$$R_{kl} = -\kappa \cdot \left(T_{kl} - \frac{1}{2}\, g_{kl}\, T\right) \tag{180}$$

Neglecting the pressure gives

$$T_{11} = T_{22} = T_{33} = 0 \;, \quad T_{44} = \rho\, (1 + \gamma) \;, \quad T = \rho \tag{181}$$

Einstein's field equations are a system of nonlinear partial differential equations of the second order which must be solved simultaneously to obtain the components g_{kl} of the metric tensor. An approximate solution for a field with spherical symmetry was first given by A. Einstein (Ref. 20) in 1915, making the assumption that

$$\sqrt{|g|} = 1 \quad \text{or} \quad \beta + \frac{1}{2}\,(\alpha + \gamma) \doteq 0;$$

$$\alpha = -\gamma \;, \quad \beta = 0 \;, \quad \gamma = -\frac{2m}{r}$$

where

$$m = \frac{GM}{c^2} = \begin{cases} 1.475 \text{ km (Sun)} \\ 0.4435 \text{ cm (Earth)} \end{cases} \tag{182}$$

is the gravitational radius of the central mass M. In 1916, K. Schwarzschild (Ref. 21) gave the correct solution (with $\sqrt{|g|} = 1$).

$$1 + \alpha = \frac{1}{1+\gamma}; \quad \beta = 0, \quad \gamma = -\frac{2m}{r}$$

The line element has now the form

$$ds^2 = -\frac{dr^2}{1 - 2m/r} - r^2(\cos^2\phi \; d\theta^2 + d\phi^2) + \left(1 - \frac{2m}{r}\right)c^2 dt^2 \tag{183}$$

Introducing harmonic or isotropic coordinates $\bar{r}$, θ, ϕ, ct defined by the transformation

$$r = \bar{r}\left(1 + \frac{m}{2\bar{r}}\right)^2 \approx \bar{r} + m \tag{184}$$

the line element assumes the form

$$ds^2 = -\left(1 + \frac{m}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 \cdot \cos^2\phi \; d\theta^2 + \bar{r}^2 d\phi^2) + \left(\frac{1 - m/2\bar{r}}{1 + m/2\bar{r}}\right)^2 c^2 dt^2 \tag{185}$$

Including only first order terms in $\alpha = \beta$ and second-order terms in γ and neglecting the small difference between r and $\bar{r}$ the line element is now given by

$$ds^2 \approx -\left(1 + \frac{2m}{r}\right)(dr^2 + r^2\cos^2\phi \; d\theta^2 + r^2 d\phi^2) + \left(1 - \frac{2m}{r} + \frac{2m^2}{r^2}\right)c^2 dt^2 \tag{186}$$

thus

$$\alpha = \beta = -\gamma \quad , \quad \gamma = -\frac{2m}{r} + \frac{2m^2}{r^2}$$

This particular form was derived and used by deSitter (Ref. 22) in 1916. It determines the metric of the non-Euclidean space-time world in the neighborhood of the mass M and the gravitational field and thus governs also the motion of satellites around this mass.

In 1916 W. deSitter (Ref. 22) has also shown how the motion of satellites is influenced by the rotation of the central body according to Einstein's gravitational theory by also introducing the component

$$T_{24} = -\rho\,\Omega\; r^2\cos^2\phi$$

of the energy-momentum tensor in his field equations (Ω = angular velocity of the central body). The solution then gives the component

$$g_{24} = \frac{b\cos^2\phi}{r} \quad (b = \text{const.})$$

In 1918, J. Lense and H. Thirring (Ref. 23) independently solved the same problem using Einstein's linear approximation solution for weak fields (Ref. 24) and they arrived at the same result as de Sitter. All these authors neglect terms proportional to Ω^2.

In this paper the g_{kl} will be determined including terms due to Ω and Ω^2 as well as terms due to the oblateness of the central body. This then leads to the relativistic perturbation theory for the motion of a satellite around a rotating oblated central body. Again, Einstein's linear approximation method for weak fields will be used.

A Cartesian coordinate system $(x_1 = x;\ x_2 = y;\ x_3 = z)$ with $x_4 = ict$ for the imaginary time coordinate will be used. In this system all the g_{kk} have the same value -1 in zero approximation. If

$$g_{kl} = \delta_{kl} + \gamma_{kl}\ ; \quad \delta_{kl} = \begin{cases} -1 & (k = l) \\ 0 & (k \neq l) \end{cases} \tag{187}$$

and

$$\gamma_{kl} = \gamma'_{kl} + \frac{1}{2}\ \delta_{kl} \sum_{\alpha=1}^{4} \gamma'_{\alpha\alpha} \tag{188}$$

then Einstein's solution for weak fields is given by

$$\gamma'_{kl} = -\frac{\kappa}{2\pi} \int \frac{T_{kl}\ dV_o}{\Delta} = -\frac{4G}{c^2} \int \frac{T_{kl}\ dV_o}{\Delta} \tag{189}$$

along with the energy-momentum tensor

$$T_{kl} = \rho_o \frac{dx_k}{ds} \frac{dx_l}{ds} = \rho \left(\frac{dx_4}{ds}\right)^2 \frac{dx_k}{dx_4} \frac{dx_l}{dx_4}, \tag{190}$$

the volume element

$$d\,V_o = i\ \frac{dx_4}{ds}\ dV = i\ \frac{dx_4}{ds}\ dx'\,dy'\,dz' = i\ \frac{dx_4}{ds}\ r'^2\,dr' \cos\phi'\,d\phi'\,d\theta' \tag{191}$$

and the distance of the mass element dm' from the attracted point $(x = r\cos\phi\cos\theta;\ y = r\cos\phi \cdot \sin\theta\ ;\ z = r\sin\phi)$ under consideration

$$\Delta = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} = r\left[\,1 - 2\,\frac{r'}{r}\cos\sigma + \left(\frac{r'}{r}\right)^2\right]^{1/2} \tag{192}$$

with

$$\cos\sigma = \sin\phi'\sin\phi + \cos\phi'\cos\phi\cos(\theta' - \theta) \tag{193}$$

From potential theory for $r > r'$

$$\frac{1}{\Delta} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n\ (\cos\sigma) \tag{194}$$

where, according to Legendre, the surface harmonic $P_n(\cos\sigma)$ is given by the addition theorem

$$P_n\ (\cos\sigma) = P_n\ (\sin\phi)\ P_n\ (\sin\phi') + 2\sum_{s=1}^{n}\frac{(n-s)!}{(n+s)!}\ P_n^s\ (\sin\phi)\ P_n^s\ (\sin\phi')\cos s\ (\theta' - \theta) \tag{195}$$

and

$$P_n^s\ (\sin\phi) = \cos^s\phi\ \frac{d^s P_n\ (\sin\phi)}{d(\sin\phi)^s} \tag{196}$$

are the associated Legendre functions of Ferrer.

Under the assumption that the central body rotates around the z-axis the velocity components of the mass element are:

$$\frac{dx_1}{dx_4} = -i\ \frac{dx'}{c\,dt} = i\ \frac{\Omega y'}{c} = i\ \frac{\Omega r'}{c} \cos\phi' \sin\theta'$$

$$\frac{dx_2}{dx_4} = -i\ \frac{dy'}{c\,dt} = -i\ \frac{\Omega x'}{c} = -i\ \frac{\Omega r'}{c} \cos\phi' \cos\theta'$$

$$\frac{dx_3}{dx_4} = 0 \quad ; \quad \frac{dx_4}{dx_4} = 1$$

The components of the energy-momentum tensor are therefore

$$T_{kl} = \rho_o \left(\frac{dx_4}{ds}\right)^2 \begin{pmatrix} -\left(\frac{\Omega r'}{c}\right)^2 \cos^2\phi' \sin^2\theta' & \left(\frac{\Omega r'}{c}\right)^2 \cos^2\phi \sin\theta' \cos\theta' & 0 & i\left(\frac{\Omega r'}{c}\right)\cos\phi' \sin\theta' \\ \left(\frac{\Omega r'}{c}\right)^2 \cos^2\phi' \sin\theta' \cos\theta' & -\left(\frac{\Omega r'}{c}\right)^2 \cos^2\phi' \cos^2\theta' & 0 & -i\left(\frac{\Omega r'}{c}\right)\cos\phi' \cos\theta' \\ 0 & 0 & 0 & 0 \\ i\left(\frac{\Omega r'}{c}\right)\cos\phi' \sin\theta' & -i\left(\frac{\Omega r'}{c}\right) \cos\phi' \cos\theta' & 0 & 1 \end{pmatrix} \tag{197}$$

Introducing eqs. (190), (191) and (194) into eq. (189) yields

$$\gamma'_{kl} = -\frac{4G}{c^2 r} \sum_{n=0}^{\infty} \int_0^M i \left(\frac{dx_4}{ds}\right)^3 \frac{dx_k}{dx_4}\ \frac{dx_l}{dx_4} \left(\frac{r'}{r}\right)^n P_n(\cos\sigma)\ dm'$$

In first approximation there is

$$\frac{ds}{dx_4} = i \ , \quad \frac{dx_4}{ds} = \frac{1}{i} = -i \ ; \quad \frac{dx_4}{ds}^3 = i$$

thus

$$\gamma'_{kl} = \frac{4G}{c^2 r} \sum_{n=0}^{\infty} \int_0^M \frac{dx_k}{dx_4}\ \frac{dx_l}{dx_4} \left(\frac{r'}{r}\right)^n P_n(\cos\sigma)\ dm' \tag{198}$$

where the components of the tensor $(dx_k/dx_4)(dx_l/dx_4)$ are given by the matrix in eq. (197).

It follows at once that

$$\gamma'_{13} = \gamma'_{31} = 0 \ ; \ \gamma'_{23} = \gamma'_{32} = 0 \ ; \ \gamma'_{33} = 0 \ ; \ \gamma'_{34} = \gamma'_{43} = 0$$

while the other components are given by

$$\gamma'_{\substack{11\\22}} = -\frac{4G}{c^2 r} \left(\frac{\Omega}{c}\right)^2 \sum_{n=0}^{\infty} \frac{1}{r^n} \int_0^M r'^{\,n+2} \cos^2\phi' \ \begin{matrix}\sin^2\theta' \\ \cos^2\theta'\end{matrix}\ P_n(\cos\sigma)\ dm' \tag{199}$$

$$\gamma'_{44} = \frac{4G}{c^2 r} \sum_{n=0}^{\infty} \frac{1}{r^n} \int_0^M r'^{\,n} P_n (\cos \sigma)\, dm' \tag{200}$$

$$\gamma'_{12} = \frac{4G}{c^2 r} \left(\frac{\Omega}{c}\right)^2 \sum_{n=0}^{\infty} \frac{1}{r^n} \int_0^M r'^{\,n+2} \cos^2 \phi' \sin \theta' \cos \theta' P_n (\cos \sigma)\, dm' \tag{201}$$

$$\gamma''_{\substack{14\\24}} = \pm i \frac{4G}{c^2 r} \left(\frac{\Omega}{r}\right) \sum_{n=0}^{\infty} \frac{1}{r^n} \int_0^M r'^{\,n+1} \cos \phi' \begin{matrix}\sin \theta' \\ \cos\theta'\end{matrix} P_n (\cos \sigma)\, dm' \tag{202}$$

The expression for P_n $(\cos \sigma)$ in eq. (195) and the mass element, $dm' = \rho\, r'^2 \cos \phi'\, dr'\, d\phi'\, d\theta'$, will now be introduced into eqs. (197) to (202). The integration with respect to the local radius r, will be made from 0 to R; with respect to the latitude, ϕ', from $-\pi/2$ to $\pi/2$; and with respect to the longitude, θ', from 0 to 2π. For a spherically symmetric model the density is independent of the longitude, θ', and will be assumed to be given by $\rho = \rho\ (r', \phi')$. Therefore the integration with respect to θ' can be performed immediately. In eq. (199) there occurs the integrals:

$$\int_0^{2\pi} \begin{matrix}\sin \theta' \\ \cos\theta'\end{matrix}\, d\theta' = 0$$

$$\int_0^{2\pi} \begin{matrix}\sin^2 \theta' \\ \cos^2\theta'\end{matrix} \cos s\,(\theta' - \theta)\, d\theta' = \frac{1}{2} \int_0^{2\pi} [1 \mp \cos 2\theta'] \cos s\,(\theta' - \theta)\, d\theta'$$

$$= \mp \frac{1}{4} \int_0^{2\pi} \{\cos [(s+2)\,\theta' - s\theta] + \cos [(s-2)\theta' - s\theta]\}\, d\theta' =$$

$$= \begin{cases} \mp \frac{\pi}{2} \cos 2\theta & (s = 2) \\ 0 & (s \neq 0) \end{cases}$$

Therefore

$$\gamma'_{\substack{11\\22}} = -\frac{4G}{c^2 r} \left(\frac{\Omega}{c}\right)^2 \left\{ \sum_{n=0}^{\infty} \left[\frac{P_n (\sin \phi)}{2r^n} \int_0^M r'^{\,n+2} \cos^2 \phi'\ P_n (\sin \phi') dm' \right] \right.$$

$$\left. \mp \sum_{n=2}^{\infty} \left[\frac{(n-2)!}{(n+2)!}\ \frac{P_n^2 (\sin \phi)}{2r^n} \cos 2\theta \int_0^M r'^{\,n+2} \cos \phi' P_n^2 (\sin \phi')\, dm' \right] \right\} \tag{203}$$

The following integral appears in eq. (200):

$$\int_0^{2\pi} \cos\ s\ (\theta' - \theta)\ d\theta' = 0$$

thus

$$\gamma'_{44} = \frac{4G}{c^2 r} \sum_{n=0}^{\infty} \left[\frac{P_n (\sin \phi)}{r^n} \int_0^M r'^{\,n} P_n (\sin \phi')\, dm' \right] \tag{204}$$

In eq. (201) the following integrals can be found:

$$\int_0^{2\pi} \sin \theta' \cos \theta'\, d\theta' = 0$$

$$\int_0^{2\pi} \sin \theta' \cos \theta' \cos s\,(\theta' - \theta)\, d\theta' = \tfrac{1}{2} \int_0^{2\pi} \sin 2\theta' \cos s\,(\theta' - \theta)\, d\theta'$$

$$= \frac{1}{4} \int_0^{2\pi} \{ \sin [(s+2)\theta' - s\,\theta] - \sin [(s-2)\,\theta' - s\theta] \}\, d\theta' = \begin{cases} \frac{\pi}{2} \sin 2\theta & (s = 2) \\ 0 & (s \neq 2) \end{cases}$$

therefore

$$\gamma'_{12} = \frac{4G}{c^2 r}\left(\frac{\Omega}{c}\right)^2 \sum_{n=2}^{\infty} \left[\frac{(n-2)!}{(n+2)!} \quad \frac{P_n^2\,(\sin\phi)}{2r^n} \quad \sin 2\theta \int_0^M r'^{\,1+2} \cos^2\phi'\, P_n^2(\sin\phi')dm' \right. \tag{205}$$

In eq. (202) occurs the integrals

$$\int_0^{2\pi} \begin{matrix}\sin\theta' \\ \cos\theta'\end{matrix}\; d\theta' = 0$$

$$\int_0^{2\pi} \begin{matrix}\sin\theta' \\ \cos\theta'\end{matrix}\; \cos s\,(\theta'-\theta)\; d\theta' = \tfrac{1}{2} \int_0^{2\pi} \left\{ \begin{matrix}\sin \\ \cos\end{matrix} \left[(s+1)\,\theta' - s\theta\right] \mp \begin{matrix}\sin \\ \cos\end{matrix}\left[(s-1)\theta' - s\,\theta\right] \right\} d\theta'$$

$$= \begin{cases} \pi \cdot \begin{matrix}\sin\theta \\ \cos\theta\end{matrix} & (s=1) \\ 0 & (s \neq 1) \end{cases}$$

thus

$$\gamma'_{\substack{14\\24}} = \pm\, i\; \frac{4G}{c^2 r}\left(\frac{\Omega}{c}\right) \sum_{n=1}^{\infty} \left[\frac{(n-1)!}{(n+2)!} \quad \frac{P_n^1\,(\sin\phi)}{r} \quad \begin{matrix}\sin\theta \\ \cos\theta\end{matrix} \int_0^M r'^{\,n+1} \cos\phi'\, P_n^1(\sin\phi')\,dm' \right. \tag{206}$$

Introducing

$$I_n = \frac{1}{M R^{n+2}} \int_0^M r'^{\,n+2} \cos^2\phi'\; P_n\,(\sin\phi')\,dm' = \begin{cases} \dfrac{C}{M R^2} = \Gamma \approx \dfrac{2}{5} & (n=0) \\ \quad\approx 0 & (n=1) \\ \Delta \approx -\dfrac{2}{35} & (n=2) \end{cases}$$

$$J_n = -\frac{1}{M R^n} \int_0^M r'^{\,n} P_n\,(\sin\phi')\;dm' = \begin{cases} -1 & (n=0) \\ 0 & (n=1) \\ \dfrac{C-A}{M R^2} = \dfrac{2}{3}\,J & (n=2) \end{cases}$$

$$K_n = \frac{1}{M R^{n+1}} \int_0^M r'^{\,n+1} \cos\phi'\; P_n^1\,(\sin\phi')\;dm' = \begin{cases} \dfrac{C}{M R^2} = \Gamma \approx \dfrac{2}{5} & (n=1) \\ \quad\approx 0 & (n=2) \end{cases}$$

$$L_n = \frac{1}{M R^{n+2}} \int_0^M r'^{\,n+2} \cos^2\phi'\; P_n^2(\sin\phi')\;dm' = \Big\{ \approx \frac{24}{35} \quad (n=2)$$

(approximate values refer to a homogeneous body) and using the gravitational radius $m = GM/c^2$ eqs. (203) to (206) become

$$\gamma'_{\substack{11\\22}} = -\frac{4m}{r}\left(\frac{\Omega R}{c}\right)^2 \sum_{n=0}^{\infty} \left[\frac{I_n}{2}\left(\frac{R}{r}\right)^n P_n\,(\sin\phi) \mp \frac{(n-2)!}{(n+2)!}\; \frac{L_n}{2}\left(\frac{R}{r}\right)^n P_n^2\,(\sin\phi)\,\cos 2\theta \right] \tag{207}$$

$$\gamma'_{44} = \frac{4m}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n\,(\sin\phi) \right] \tag{208}$$

$$g_{12} = \gamma'_{12} = \frac{4m}{r}\left(\frac{\Omega R}{c}\right)^2 \sum_{n=1}^{\infty} \left[\frac{(n-2)!}{(n+2)!}\; \frac{L_n}{2}\left(\frac{R}{r}\right)^n P_n^2\,(\sin\phi)\;\sin 2\theta \right] \tag{209}$$

$$g_{\substack{14\\24}} = \gamma'_{\substack{14\\24}} = \pm\, i\ \frac{4m}{r}\left(\frac{\Omega R}{c}\right)\sum_{n=1}^{\infty}\left[\frac{(n-1)!}{(n+1)!}\, K_n\left(\frac{R}{r}\right)^n P_n^1\ (\sin\phi)\ \begin{matrix}\sin\theta\\ \cos\theta\end{matrix}\right] \tag{210}$$

and therefore

$$\sum_{a=1}^{4}\gamma'_{aa} = \frac{4m}{r}\left[1-\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n\,(\sin\phi) - \left(\frac{\Omega R}{c}\right)^2\sum_{n=0}^{\infty} I_n\left(\frac{R}{r}\right)^n P_n\,(\sin\phi)\right]$$

Because

$$g_{kk} = -1+\gamma'_{kk} - \frac{1}{2}\sum_{a=1}^{4}\gamma'_{aa}$$

there is

$$g_{\substack{11\\22}} = -1-\frac{2m}{r}\left[1-\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n\,(\sin\phi) \mp \left(\frac{\Omega R}{c}\right)^2\frac{(n-2)!}{(n+2)!}L_n\left(\frac{R}{r}\right)^n P_n^2\,(\sin\phi)\cos 2\theta\right] \tag{211}$$

$$g_{33} = -1-\frac{2m}{r}\left[1-\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n\ (\sin\phi) - \left(\frac{\Omega R}{c}\right)^2\sum_{n=0}^{\infty} I_n\left(\frac{R}{r}\right)^n P_n\ (\sin\phi)\right] \tag{212}$$

$$g_{44} = -1+\frac{2m}{r}\left[1-\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n\,(\sin\phi) + \left(\frac{\Omega R}{c}\right)^2\sum_{n=0}^{\infty} I_n\left(\frac{R}{r}\right)^n P_n\,(\sin\phi)\right] \tag{213}$$

Taking only the terms up to $n = 2$ then the components of the fundamental tensor are

$$g_{\substack{11\\22}} = -1-\frac{2m}{r}\left[1-J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi) \mp \left(\frac{\Omega R}{c}\right)^2\frac{L_2}{8}\left(\frac{R}{r}\right)^2 \cos^2\phi\cos 2\theta\right] \tag{214}$$

$$g_{33} = -1-\frac{2m}{r}\left\{1-J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi) - \left(\frac{\Omega R}{c}\right)^2\left[\Gamma+\Delta\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\} \tag{215}$$

$$g_{44} = -1+\frac{2m}{r}\left\{1-J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi) + \left(\frac{\Omega R}{c}\right)^2\left[\Gamma+\Delta\left(\frac{R}{r}\right)^2 P_2\,(\sin\phi)\right]\right\} \tag{216}$$

$$g_{12} = \frac{4m}{r}\left(\frac{\Omega R}{c}\right)^2\frac{L_2}{16}\left(\frac{R}{r}\right)^2\cos^2\phi\sin 2\theta \tag{217}$$

$$g_{\substack{14\\24}} = \pm\, i\ \frac{2m}{r}\left(\frac{\Omega R}{c}\right)\Gamma\left(\frac{R}{r}\right)\cos\phi\ \begin{matrix}\sin\theta\\ \cos\theta\end{matrix} \tag{218}$$

In order to have a spherically symmetric field the very small terms proportional to m/r and $(\Omega R/c)^2$ will be neglected. In g_{44} only, the term will be retained and the second order term

$$2m^2/r^2\cdot[1-2J_2\,(R/r)^2\,P_2\,(\sin\phi)]$$

will be added according to de Sitter (eq. 186). The reason why g_{44} is required with higher accuracy, is that it appears in the equations of motion multiplied with the large factor c^2. The line element can now be written

$$
\begin{aligned}
ds^2 &= g_{11}(dx^2+dy^2+dz^2) - g_{44}c^2dt^2 + g_{14}dx\,icdt + g_{24}dy\,icdt \\
&= -\left\{1+\frac{2m}{r}\left[1-J_2\left(\frac{R}{r}\right)^2 P_2\left(\frac{z}{r}\right)\right]\right\}(dx^2+dy^2+dz^2) \\
&\quad + \left\langle 1-\frac{2m}{r}\left\{1-J_2\left(\frac{R}{r}\right)^2 P_2\left(\frac{z}{r}\right)+\left(\frac{\Omega R}{c}\right)^2\left[\Gamma+\Delta\left(\frac{R}{r}\right)^2 P_2\left(\frac{z}{r}\right)\right]\right\}\right. \\
&\quad \left. + 2\left(\frac{m}{r}\right)^2\left[1-2J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\rangle c^2dt^2 \\
&\quad - \frac{2m}{r}\left(\frac{\Omega R}{c}\right)\Gamma\left(\frac{R}{r}\right)\frac{y}{r}\ dx\,c\,dt + \frac{2m}{r}\left(\frac{\Omega R}{c}\right)\Gamma\left(\frac{R}{r}\right)\frac{x}{r}\ dy\,c\,dt
\end{aligned}
\tag{219}
$$

Because

$$
d\sigma^2 = dx^2+dy^2+dz^2 = dr^2 + r^2(\cos^2\phi\, d\theta^2 + d\phi^2) \tag{220}
$$

$$
x\,dy - y\,dx = r^2\cos^2\phi\, d\theta\ ,\quad \frac{z}{r} = \sin\phi \tag{221}
$$

the line element can be written

$$
\begin{aligned}
ds^2 &= -\left\{1+\frac{2m}{r}\left[1-J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\}(dr^2+r^2\cos^2\phi\, d\theta^2+r^2 d\phi^2) \\
&\quad + \left\langle 1-\frac{2m}{r}\left\{1-J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)+\left(\frac{\Omega R}{c}\right)^2\left[\Gamma+\Delta\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\}\right. \\
&\quad \left. + 2\left(\frac{m}{r}\right)^2\left[1-2J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\rangle c^2dt^2 \\
&\quad + \frac{2m}{r}\Gamma\left(\frac{\Omega R}{c}\right) R\ \cos^2\phi\ d\theta\, c\,dt
\end{aligned}
\tag{222}
$$

This is now the square of the four-dimensional line-element for a non-Euclidean space outside a rotating oblated unhomogenous spheroid of mass M. Writing

$$
ds^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 + g_{24}dx_2dx_4
$$

where now

$$
x_1 = r\ ;\ x_2 = \theta\ ;\ x_3 = \phi\ ;\ x_4 = ct
$$

then the components of the fundamental tensor are

$$
g_{11} = -\left\{1+\frac{2m}{r}\left[1-J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\}\ ;\ g_{22} = g_{11}r^2\cos^2\phi
$$

$$
g_{33} = g_{11}r^2\ ;\ g_{24} = g_{42} = \frac{2m}{r}\Gamma\left(\frac{\Omega R}{c}\right) R\cos^2\phi
$$

$$
\begin{aligned}
g_{44} &= 1-\frac{2m}{r}\left\{1-J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)+\left(\frac{\Omega R}{c}\right)\left[\Gamma+\Delta\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\} \\
&\quad + 2\left(\frac{m}{r}\right)^2\left[1-2J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]
\end{aligned}
$$

Their derivatives are given by

$$\frac{\partial g_{11}}{\partial r} = \frac{2m}{r^2}\left[1 - 3J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]$$

$$\frac{\partial g_{22}}{\partial r} = 2r\cos^2\phi\left(g_{11} + \frac{r}{2}\frac{\partial g_{11}}{\partial r}\right);\quad \frac{1}{g_{22}}\frac{\partial g_{22}}{\partial r} = \frac{2}{r} + \frac{1}{g_{11}}\frac{\partial g_{11}}{\partial r}$$

$$\frac{\partial g_{24}}{\partial r} = -\frac{2m}{r^2}\Gamma\left(\frac{\Omega R}{c}\right)R\cos^2\phi = -\frac{g_{24}}{r}$$

$$\frac{\partial g_{33}}{\partial r} = 2r\left(g_{11} + \frac{r}{2}\frac{\partial g_{11}}{\partial r}\right);\quad \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial r} = \frac{2}{r} + \frac{1}{g_{11}}\frac{\partial g_{11}}{\partial r}$$

$$\frac{\partial g_{44}}{\partial r} = \frac{2m}{r^2}\left\{1 - 3J_2\left(\frac{R}{r}\right)^2 P_2(\sin\phi) + \left(\frac{\Omega R}{c}\right)^2\left[\Gamma + 3\Delta\left(\frac{R}{r}\right)^2 P_2(\sin\phi)\right] - \right.$$

$$\left. - \frac{2m}{r}\left[1 - 4J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]\right\}$$

and

$$\frac{\partial g_{11}}{\partial\phi} = \frac{2m}{r}J_2\left(\frac{R}{r}\right)^2 P_2^1\ (\sin\phi) = \frac{6m}{r}J_2\left(\frac{R}{r}\right)^2\sin\phi\cos\phi$$

$$\frac{\partial g_{22}}{\partial\phi} = 2r^2\sin\phi\cos\phi\left(-g_{11} + \frac{1}{2}\cot\phi\frac{\partial g_{11}}{\partial\phi}\right);\quad \frac{1}{g_{22}}\frac{\partial g_{22}}{\partial\phi} =$$

$$= \frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\phi} - 2\tan\phi$$

$$\frac{\partial g_{24}}{\partial\phi} = -\frac{4m}{r}\Gamma\left(\frac{\Omega R}{c}\right)R\sin\phi\cos\phi = -2g_{24}\tan\phi = 2r\tan\phi\frac{\partial g_{24}}{\partial r}$$

$$\frac{\partial g_{33}}{\partial\phi} = r^2\frac{\partial g_{11}}{\partial\phi};\quad \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial\phi} = \frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\phi}$$

$$\frac{\partial g_{44}}{\partial\phi} = \frac{6m}{r}\left[J_2\left(1 - \frac{2m}{r}\right) + \left(\frac{\Omega R}{c}\right)^2\Delta\right]\left(\frac{R}{r}\right)^2\sin\phi\cos\phi$$

It is now easy to write the values for the Christoffel symbols of second kind:

$$\Gamma^1_{11} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial r} \approx -\frac{1}{2}\frac{\partial g_{11}}{\partial r} = -\frac{m}{r^2}\left[1 - 3J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right]$$

$$\Gamma^1_{13} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial\phi} \approx -\frac{1}{2}\frac{\partial g_{11}}{\partial\phi} = -\frac{3m}{r}J_2\left(\frac{R}{r}\right)^2 \sin\phi\cos\phi$$

$$\Gamma^1_{22} = \frac{1}{2g_{11}}\frac{\partial g_{22}}{\partial r} \approx -r\cos^2\phi\left(1+\frac{r}{2}\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial r}\right) = -r\cos^2\phi - r^2\cos^2\phi\,\Gamma^1_{11}$$

$$\Gamma^1_{24} = -\frac{1}{2g_{11}}\frac{\partial g_{24}}{\partial r} \approx \frac{1}{2}\frac{\partial g_{24}}{\partial r} = -\frac{g_{24}}{2r} = -\frac{m}{r^2}\Gamma\left(\frac{\Omega R}{c}\right)R\cos^2\phi$$

$$\Gamma^1_{33} = -\frac{1}{2g_{11}}\frac{\partial g_{33}}{\partial r} = -r\left(1+\frac{r}{2}\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial r}\right) = -r - r^2\Gamma^1_{11} = \frac{1}{\cos\phi}\Gamma^1_{22}$$

$$\Gamma^1_{44} = -\frac{1}{2g_{11}}\frac{\partial g_{44}}{\partial r} =$$

$$= \frac{m}{r^2}\,\frac{1-3J_2\left(\frac{R}{r}\right)^2 P_2(\sin\phi)+\left(\frac{\Omega R}{c}\right)\left[\Gamma+3\Delta\left(\frac{R}{r}\right)^2 P_2(\sin\phi)\right]-\frac{2m}{r}\left[1-4J_2\left(\frac{R}{r}\right)^2 P_2(\sin\phi)\right]}{1+\frac{2m}{r}\left[1-J_2\left(\frac{R}{r}\right)^2 P_2(\sin\phi)\right]}$$

$$= \frac{m}{r^2}\left\{1-3J_2\left(\frac{R}{r}\right)^2 P_2(\sin\phi)+\left(\frac{\Omega R}{c}\right)\left[\Gamma+3\Delta\left(\frac{R}{r}\right)^2 P_2(\sin\phi)\right]-\frac{4m}{r}\left[1-4J_2\left(\frac{R}{r}\right)^2 P_2(\sin\phi)\right]\right\}$$

and

$$\Gamma^2_{12} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial r} = \frac{1}{r}+\Gamma^1_{11}; \quad \Gamma^2_{14} = \frac{1}{2g_{22}}\frac{\partial g_{24}}{\partial r} = -\frac{\Gamma^1_{24}}{r^2\cos^2\phi}$$

$$\Gamma^2_{23} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial\phi} = -\tan\phi+\Gamma^1_{13}; \quad \Gamma^2_{34} = \frac{1}{2g_{22}}\frac{\partial g_{24}}{\partial\phi} = +2r\tan\phi\,\Gamma^2_{14} = -\frac{2\Gamma^1_{24}\tan\phi}{r\cos^2\phi}$$

and

$$\Gamma^3_{11} = -\frac{1}{2g_{33}}\frac{\partial g_{11}}{\partial\phi} = -\frac{1}{r^2}\Gamma^1_{13}; \quad \Gamma^3_{13} = \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial r} = -\frac{1}{r^2}\Gamma^1_{33} = \frac{1}{r}+\Gamma^1_{11}$$

$$\Gamma^3_{22} = -\frac{1}{2g_{33}}\frac{\partial g_{22}}{\partial\phi} = -\cos^2\phi\,\Gamma^2_{23} = \sin\phi\cos\phi - \cos^2\phi\,\Gamma^1_{13}$$

$$\Gamma^3_{24} = -\frac{1}{2g_{33}}\frac{\partial g_{24}}{\partial\phi} = -\cos^2\phi\,\Gamma^2_{34} = \frac{2}{r}\tan\phi\,\Gamma^1_{24}$$

$$\Gamma^3_{33} = \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial\phi} = \Gamma^1_{13} = -r^2\Gamma^3_{11}; \quad \Gamma^3_{44} = -\frac{1}{2g_{33}}\frac{\partial g_{44}}{\partial\phi} = \frac{1}{r^2}\Gamma^4_{34}$$

and finally

$$\Gamma^4_{12} = \frac{g_{24}}{2g_{44}}\left[\frac{1}{g_{24}}\frac{\partial g_{24}}{\partial r} - \frac{1}{g_{22}}\frac{\partial g_{22}}{\partial r}\right] = -r\Gamma^1_{24}\left[-\frac{1}{r}-\frac{2}{r}-2\Gamma^1_{11}\right] \approx 3\Gamma^1_{24}$$

$$\Gamma^4_{14} = \frac{1}{2g_{44}}\frac{\partial g_{44}}{\partial r} \approx \frac{1}{2}\frac{\partial g_{44}}{\partial r} \approx \frac{m}{r^2}\left[1 - 3J_2\left(\frac{R}{r}\right)^2 P_2\ (\sin\phi)\right] = -\Gamma^1_{11}$$

$$\Gamma^4_{23} = \frac{g_{24}}{2g_{44}}\left[\frac{1}{g_{24}}\frac{\partial g_{24}}{\partial \phi} - \frac{1}{g_{22}}\frac{\partial g_{22}}{\partial \phi}\right] = -r\,\Gamma^1_{24}\left[-2\tan\phi - 2\,\Gamma^1_{13} + 2\tan\phi\right] = 0\,(m^2) \approx 0$$

$$\Gamma^4_{34} = \frac{1}{2g_{44}}\frac{\partial g_{44}}{\partial \phi} \approx \frac{1}{2}\frac{\partial g_{44}}{\partial \phi} = \frac{3m}{r}\left[J_2 - \Delta\left(\frac{\Omega R}{c}\right)^2\right]\left(\frac{R}{r}\right)^2 \sin\phi\cos\phi$$

These Christoffel symbols will now be substituted into the eqs. of motion (geodesic line)

$$\frac{d^2x_i}{ds^2} + \sum_{k=1}^{4}\sum_{l=1}^{4}\Gamma^i_{kl}\frac{dx_k}{ds}\frac{dx_l}{ds} = 0 \quad [\,i = 1,\ 2,\ 3,\ 4\,] \tag{223}$$

but first of all the line element ds can be eliminated by the relation

$$\frac{d^2x_4}{ds^2} = -\sum_{k=1}^{4}\sum_{l=1}^{4}\Gamma^4_{kl}\frac{dx_k}{dx_4}\frac{dx_l}{dx_4}\cdot\left(\frac{dx_4}{ds}\right)^2$$

which follows immediately from eq. (223). Therefore

$$\frac{d^2x_i}{ds^2} = \frac{d}{ds}\left(\frac{dx_i}{ds}\right) = \frac{dx_4}{ds}\cdot\frac{d}{dx_4}\left(\frac{dx_i}{dx_4}\cdot\frac{dx_4}{ds}\right) = \left(\frac{dx_4}{ds}\right)^2\frac{d^2x_i}{dx_4^2} + \frac{dx_i}{dx_4}\frac{d^2x_4}{ds^2}$$

$$= \frac{d^2x_i}{dx_4^2}\left(\frac{dx_4}{ds}\right)^2 - \frac{dx_i}{dx_4}\sum_{k=1}^{4}\sum_{l=1}^{4}\Gamma^4_{kl}\frac{dx_k}{dx_4}\frac{dx_l}{dx_4}\cdot\left(\frac{dx_4}{ds}\right)^2 =$$

$$-\sum_{k=1}^{4}\sum_{l=1}^{4}\Gamma^i_{kl}\frac{dx_k}{dx_4}\frac{dx_l}{dx_4}\left(\frac{dx_4}{ds}\right)^2$$

or after dividing by $(dx_4/ds)^2$

$$\frac{d^2x_i}{dx_4^2} + \sum_{k=1}^{4}\sum_{l=1}^{4}\left(\Gamma^i_{kl} - \Gamma^4_{kl}\frac{dx_i}{dx_4}\right)\frac{dx_k}{dx_4}\frac{dx_l}{dx_4} = 0 \quad [i = 1,\ 2,\ 3] \tag{224}$$

Because

$$x_1 = r\ ;\ x_2 = \theta\ ;\ x_3 = \phi\ ;\ x_4 = ct\ \ (\dot{x}_4 = c)$$

the final form of the equations of motion is

$$\ddot{x}_i + \sum_{k=1}^{4}\sum_{l=1}^{4}\left(\Gamma^i_{kl} - \Gamma^4_{kl}\frac{\dot{x}_i}{c}\right)\dot{x}_k\,\dot{x}_l = 0 \quad [\,i = 1,\ 2,\ 3\,] \tag{225}$$

The equations of motion in spherical polar coordinates are thus, up to terms of order m,

$$\ddot{r} + (\Gamma^1_{11} - 2\,\Gamma^4_{14})\,\dot{r}^2 + \Gamma^1_{22}\,\dot{\theta}^2 + \Gamma^1_{33}\,\dot{\phi}^2 + \Gamma^1_{44}\,c^2 + 2\,(\Gamma^1_{13} - \Gamma^4_{34})\,\dot{r}\,\dot{\phi} + 2\,\Gamma^1_{24}\,c\,\dot{\theta} = 0$$

and

$$\ddot{\theta} + 2\,\Gamma^2_{12}\,\dot{r}\,\dot{\theta} + 2\,\Gamma^2_{23}\,\dot{\theta}\,\dot{\phi} + 2\left(\Gamma^2_{14} - \Gamma^4_{14}\frac{\dot{\theta}}{c}\right)c\,\dot{r} + 2\left(\Gamma^2_{34} - \Gamma^4_{34}\frac{\dot{\theta}}{c}\right)c\,\dot{\phi} = 0$$

$$\ddot{\phi} + \Gamma^3_{11}\dot{r}^2 + \Gamma^3_{22}\dot{\theta}^2 + (\Gamma^3_{33} - 2\Gamma^4_{34})\dot{\phi}^2 + 2(\Gamma^3_{13} - \Gamma^4_{14})\dot{r}\dot{\phi} + \Gamma^3_{44}c^2 + 2\Gamma^3_{24}c\dot{\theta} = 0$$

Using the relations between the different Christoffel symbols and bringing the terms of the unperturbed motion to the left side of each equation, the right side will then be the perturbing acceleration (radial component R, lateral component P and meridional component Q, respectively) due to oblateness and relativistic effects. Rearrangement of the equations of motion yields

$$\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 + \frac{GM}{r^2} = R \equiv \left(\frac{m}{r^2} - \Gamma^1_{44}\right)c^2 + \Gamma^1_{11}v^2$$

$$+ 2(\Gamma^4_{14} - \Gamma^1_{11})\dot{r}^2 + 2(\Gamma^4_{34} - \Gamma^1_{13})\dot{r}\dot{\phi} - 2\Gamma^1_{24}c\dot{\theta} \qquad (226)$$

and

$$\frac{1}{r\cos\phi}\frac{d}{dt}(r^2\dot{\theta}\cos^2\phi) \equiv r\ddot{\theta}\cos\phi + 2\dot{r}\dot{\theta}\cos\phi - 2r\dot{\theta}\dot{\phi}\sin\phi = P$$

$$\equiv 2(\Gamma^4_{14} - \Gamma^1_{11})r\dot{r}\dot{\theta}\cos\phi + 2(\Gamma^4_{34} - \Gamma^1_{13})r\dot{\theta}\dot{\phi}\cos\phi + \frac{2c\Gamma^1_{24}}{r\cos\phi}(\dot{r} + 2r\dot{\phi}\tan\phi) \qquad (227)$$

and

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}) + r\dot{\theta}^2\sin\phi\cos\phi \equiv r\ddot{\phi} + 2\dot{r}\dot{\phi} + r\dot{\theta}^2\sin\phi\cos\phi = Q$$

$$\equiv -\Gamma^3_{44}rc^2 + \Gamma^1_{13}\frac{v^2}{r} + 2(\Gamma^4_{14} - \Gamma^1_{11})r\dot{r}\dot{\phi} + 2(\Gamma^4_{34} - \Gamma^1_{13})r\dot{\phi}^2 - \frac{4c\Gamma^1_{24}}{r\cos\phi}(r\dot{\theta}\sin\phi) \qquad (228)$$

In order to investigate the perturbations of the osculating orbit elements the perturbing acceleration must be given by its radial components R, its transversal component S, and its orthogonal component W. Definitions and transformation equations may be found in the author's report (ref. 25). Introducing γ, the inclination of the orbit with respect to the equator of the primary, χ, the velocity or flight azimuth angle or the angle between the local meridian and the orbit plane, and $u = \omega + w$, the argument of latitude or the sum of the argument of the pericenter and the true anomaly, there exist the following relations for the angles:

$$\begin{aligned} \cos\phi\ \sin\chi &= \cos\gamma \\ \cos\phi\ \cos\chi &= \sin\gamma\ \cos u \\ \sin\phi &= \sin\gamma\ \sin u \end{aligned} \qquad (229)$$

and the following equations for the angle rates

$$\begin{aligned} \dot{\theta}\cos^2\phi &= \dot{u}\cos\gamma \quad &;\quad \dot{\theta}\cos\phi &= \dot{u}\sin\chi \\ \dot{\phi}\cos\phi &= \dot{u}\sin\gamma\cos u \quad &;\quad \dot{\phi} &= \dot{u}\cos\chi \end{aligned} \qquad (230)$$

The total velocity is therefore

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2\cos^2\phi + r^2\dot{\phi}^2 = \dot{r}^2 + r^2\dot{u}^2 \qquad (231)$$

Its components

$$v_r = \dot{r} \; ; \; v_\theta = r\,\dot{\theta}\cos\phi \;\; , \;\; v_\phi = r\,\dot{\phi} \tag{232}$$

are now transformed to the radial, transversal, and orthogonal velocity components, respectively:

$$v_r = \dot{r}$$

$$v_u = v_\theta \sin X + v_\phi \cos X = r\,\dot{\theta}\cos\phi \sin X + r\,\dot{\phi}\cos X = r\,\dot{u}$$

$$v_\gamma = -\,v_\theta \cos X + v_\phi \sin X = -\,r\,\dot{\theta}\cos\phi\cos X + r\,\dot{\phi}\sin X = 0 \tag{233}$$

The transformation equations for the components of the perturbing acceleration are

$$R = R \; ; \; S = P\sin X + Q\cos X \; ; \; W = -\,P\cos X + Q\sin X \tag{234}$$

From the above mentioned relations, there follow the equations

$$\sin 2\phi \cos X = \sin^2\gamma \sin 2u \; ; \qquad \dot{\phi}\sin 2\phi = \dot{u}\sin^2\gamma\sin 2u$$

$$\sin 2\phi \sin X = \sin 2\gamma \sin u \; ; \tag{235}$$

which will be used in applying the transformation equations. Substituting the values of P (right side of eq. 227) and Q (right side of eq. 228) into eqs. (234) and taking the expression for R (right side of eq. 226) then (after using eqs. (229), (230), (235) the components of the perturbing acceleration are

$$R = \left(\frac{m}{r^2} - \Gamma^1_{44}\right)c^2 + \Gamma^1_{11}\,v^2 + 2\left(\Gamma^4_{14} - \Gamma^1_{11}\right)\dot{r}^2 + 2\left(\frac{\Gamma^4_{34} - \Gamma^1_{13}}{\sin 2\phi}\right)\dot{r}\,\dot{u}\sin^2\gamma\sin 2u$$

$$-\,2\left(\frac{c\,\Gamma^1_{24}}{\cos^2\phi}\right)\dot{u}\cos\gamma \tag{236}$$

$$S = -\left(\frac{\Gamma^4_{34}}{\sin 2\phi}\right)\frac{c^2}{r}\sin^2\gamma\sin 2u + \left(\frac{\Gamma^1_{13}}{\sin 2\phi}\right)\frac{v^2}{r}\sin^2\gamma\sin 2u + 2\left(\Gamma^4_{14} - \Gamma^1_{11}\right)r\,\dot{r}\,\dot{u}$$

$$+\,2\left(\frac{\Gamma^4_{34} - \Gamma^1_{13}}{\sin 2\phi}\right)r\,\dot{u}^2\sin^2\gamma\sin 2u + 2\left(\frac{c\,\Gamma^1_{24}}{\cos^2\phi}\right)\frac{\dot{r}\cos\gamma}{r} \tag{237}$$

$$W = -\left(\frac{\Gamma^4_{34}}{\sin 2\phi}\right)\frac{c^2}{r}\sin 2\gamma\sin u + \left(\frac{\Gamma^1_{13}}{\sin 2\phi}\right)\frac{v^2}{r}\sin 2\gamma\sin u$$

$$-\,2\left(\frac{c\,\Gamma^1_{24}}{\cos^2\phi}\right)\left(\dot{r}\cos u + 2\,r\,\dot{u}\sin u\right)\frac{\sin\gamma}{r} \tag{238}$$

or, after substituting the values of the Christoffel symbols and writing

$$P_2 = P_2(\sin\phi) = P_2(\sin\gamma\sin u) \; ; \; c^2 m = GM = \mu$$

there is

$$R = \frac{\mu}{r^2}\left\{3J_2\left(\frac{R}{r}\right)^2 P_2 - \left(\frac{\Omega R}{c}\right)^2\left[\Gamma + 3\Delta\left(\frac{R}{r}\right)^2 P_2\right] + \frac{4m}{r}\left[1 - 4J_2\left(\frac{R}{r}\right)^2 P_2\right]\right\}$$

$$- \frac{m}{r}\frac{v^2}{r}\left[1 - 3J_2\left(\frac{R}{r}\right)^2 P_2\right] + \frac{4m}{r}\frac{\dot{r}^2}{r}\left[1 - 3J_2\left(\frac{R}{r}\right)^2 P_2\right]$$

$$+ \frac{6m}{r} J_2\left(\frac{R}{r}\right)^2 \dot{r}\dot{u}\sin^2\gamma \sin 2u + \frac{2m}{r}\Gamma\left(\frac{\Omega R}{c}\right)\left(\frac{R}{r}\right) c\dot{u}\cos\gamma \tag{239}$$

and

$$S = -\frac{3}{2}\frac{\mu}{r^2}\left[J_2 - \Delta\left(\frac{\Omega R}{c}\right)^2\right]\left(\frac{R}{r}\right)^2 \sin^2\gamma \sin 2u - \frac{3}{2}\frac{m}{r}J_2\left(\frac{R}{r}\right)^2\frac{v^2}{r}\sin^2\gamma\sin 2u$$

$$+ \frac{4m}{r}\left[1 - 3J_2\left(\frac{R}{r}\right)^2 P_2\right]\dot{r}\dot{u} + \frac{6m}{r}J_2\left(\frac{R}{r}\right)^2 r\dot{u}^2\sin^2\gamma\sin 2u$$

$$- \frac{2m}{r}\Gamma\left(\frac{\Omega R}{c}\right)\left(\frac{R}{r}\right)\frac{c\dot{r}\cos\gamma}{r} \tag{240}$$

and

$$W = -\frac{3}{2}\frac{\mu}{r^2}\left[J_2 - \Delta\left(\frac{\Omega R}{c}\right)^2\right]\left(\frac{R}{r}\right)^2 \sin 2\gamma\sin u - \frac{3}{2}\frac{m}{r}J_2\left(\frac{R}{r}\right)^2\frac{v^2}{r}\sin 2\gamma\sin u$$

$$+ \frac{2m}{r}\Gamma\left(\frac{\Omega R}{c}\right)\left(\frac{R}{r}\right)\frac{c\sin\gamma}{r}(\dot{r}\cos u + 2r\dot{u}\sin u) \tag{241}$$

These expressions for the perturbing accelerations can be simplified by using relations for unperturbed Keplerian motion in order to obtain first order perturbational effects, namely

$$\dot{r} = \frac{h}{p} e \sin w \quad ; \quad \dot{u} = \frac{h}{r^2} \quad ; \quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\dot{r}^2 = v^2 - r^2\dot{u}^2 = v^2 - \frac{h^2}{r^2} = \mu\left(\frac{2}{r} - \frac{1}{a} - \frac{p}{r^2}\right)$$

or

$$\frac{v^2}{r} = \frac{\mu}{r^2}\left(2 - \frac{r}{a}\right) \quad ; \quad \frac{\dot{r}^2}{r} = \frac{\mu}{r^2}\left(2 - \frac{r}{a} - \frac{p}{r}\right)$$

$$\dot{r}\dot{u} = \frac{\dot{r}(r\dot{u})}{r} = \frac{\mu}{r^2}(e\sin w) \quad ; \quad r\dot{u}^2 = \frac{(r\dot{u})^2}{r} = \frac{h^2}{r^3} = \frac{\mu}{r^2}\left(\frac{p}{r}\right) \tag{242}$$

Eqs. (239), (240) and (241) can now be written in the form:

$$R = \frac{\mu}{r^2}\left\langle 3J_2\left(\frac{R}{r}\right)^2 P_2 + \frac{4m}{r}\left\{\left(\frac{5}{2} - \frac{3}{4}\frac{r}{a} - \frac{p}{r}\right) - \right.\right.$$
$$- \frac{3}{2}J_2\left(\frac{R}{r}\right)^2\left[\left(\frac{17}{3} - \frac{3}{2}\frac{r}{a} - 2\frac{p}{r}\right)P_2 - e\sin^2\gamma\sin w\sin 2u\right]$$
$$\left.\left. + \frac{1}{2}\Gamma\left(\frac{\Omega R^2}{h}\right)\frac{p}{r}\cos\gamma - \frac{1}{4}\left(\frac{\Omega R^2}{h}\right)^2\left(\frac{p}{R}\right)^2\frac{r}{p}\left[\Gamma + 3\Delta\left(\frac{R}{r}\right)^2 P_2\right]\right\}\right\rangle \tag{243}$$

and

$$S = \frac{\mu}{r^2}\left\langle -\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2 \sin^2\gamma\sin 2u + \frac{m}{r}\left\{4\,e\sin w - \right.\right.$$

$$-\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2\left[\left(2-\frac{r}{a}-4\,\frac{p}{r}\right)\sin^2\gamma\sin 2u + 8\,e\sin w\;P_2\right]$$

$$\left.\left. -\,2\,\Gamma\left(\frac{\Omega R^2}{h}\right)e\cos\gamma\sin w + \frac{3}{2}\,\Delta\,\left(\frac{\Omega R^2}{h}\right)^2\frac{p}{r}\sin^2\gamma\sin 2u\right\}\right\rangle \tag{244}$$

and

$$W = \frac{\mu}{r^2}\left\{-\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2\sin 2\gamma\sin u + \frac{m}{r}\left[-\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2\left(2-\frac{r}{a}\right)\sin 2\gamma\,\sin u + \right.\right.$$

$$\left.\left. +\,2\,\Gamma\left(\frac{\Omega R^2}{h}\right)\sin\gamma\left(e\sin w\cos u + 2\,\frac{p}{r}\sin u\right) + \frac{3}{2}\,\Delta\left(\frac{\Omega R^2}{h}\right)^2\frac{p}{r}\sin 2\gamma\sin u\right]\right\} \tag{245}$$

The Lagrangian method of the variation of parameters gives the following equations for the time rate (variation) of the osculating orbital elements:

$$\frac{da}{dt} = \frac{2\,a^2}{h}\left[(e\sin w)\,R + (p/r)\,S\right] \quad ; \quad \frac{dp}{dt} = \frac{2\,p}{h}\,r\,S \tag{246}$$

$$\frac{de^2}{dt} = 2\,e\,\frac{de}{dt} = \frac{1-e^2}{a}\,\frac{da}{dt} - \frac{1}{a}\,\frac{dp}{dt} = \frac{2p}{h}\left[(e\sin w)\;R + \left(\frac{p}{r}-\frac{r}{a}\right)S\right] \tag{247}$$

$$\frac{d\pi}{dt} = \frac{d\omega}{dt} + \cos\gamma\,\frac{d\Omega}{dt} = -\,\frac{p}{h\,e^2}\left[\left(\frac{p}{r}-1\right)\;R - e\sin w\left(1+\frac{r}{p}\right)S\right] \tag{248}$$

$$\frac{d\sigma}{dt} = -\,n\,\frac{dt_p}{dt} = \frac{dM}{dt} - n = -\sqrt{1-e^2}\left[\frac{2\,r}{h}\;R + \frac{d\pi}{dt}\right] \tag{249}$$

$$\frac{d\gamma}{dt} = \frac{r\cos u}{h}\;W \quad ; \quad \frac{d\Omega}{dt} = \frac{r\sin u}{h\sin\gamma}\;W \tag{250}$$

Substituting the expressions for the perturbing accelerations R, S, W into the eqs. (246) to (250) that yields after reduction:

$$\frac{da}{dt} = 2\,\frac{h}{p}\left(\frac{a}{r}\right)^2\left\langle\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2\left[2\,e\sin w\cdot P_2 - \frac{p}{r}\sin^2\gamma\sin 2u\right] + \right.$$

$$+\,\frac{m}{r}\left\{4\,e\sin w\left(\frac{5}{2}-\frac{3}{4}\,\frac{r}{a}\right) - \frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2\left[\left(\frac{68}{3}-6\,\frac{r}{a}\right)e\sin w\;P_2\right.\right.$$

$$\left. -\,3\,\frac{p}{r}\left(2-\frac{r}{a}\right)\sin^2\gamma\sin 2u\right] - \left(\frac{\Omega R^2}{h}\right)^2\left[\left(\frac{p}{R}\right)^2\frac{r}{p}\,e\sin w\left(\Gamma + 3\,\Delta\,\left(\frac{R}{r}\right)^2 P_2\right)\right.$$

$$\left.\left.\left. -\,\frac{3}{2}\,\Delta\,\left(\frac{p}{r}\right)^2\sin^2\gamma\sin 2u\right]\right\}\right\rangle \tag{251}$$

$$\frac{dp}{dt} = \frac{2h}{r}\left\langle -\frac{3}{2} J_2 \left(\frac{R}{r}\right)^2 \sin^2\gamma \sin 2u + \frac{m}{r}\left\{4\, e \sin w \right.\right.$$

$$- \frac{3}{2} J_2\left(\frac{R}{r}\right)^2 \left[\left(2 - \frac{r}{a} - 4\,\frac{p}{r}\right) \sin^2\gamma \sin 2u + 8\, e \sin w \cdot P_2\right]$$

$$\left.\left. - 2\,\Gamma\left(\frac{\Omega R^2}{h}\right) e \cos\gamma \sin w + \frac{3}{2}\,\Delta\left(\frac{\Omega R^2}{h}\right)^2 \frac{p}{r} \sin^2\gamma \sin 2u\right\}\right\rangle \tag{252}$$

$$\frac{d\pi}{dt} = -\frac{h}{e^2 r^2}\left\langle \frac{3}{2} J_2\left(\frac{R}{r}\right)^2\left[2\left(\frac{p}{r} - 1\right) P_2 + \left(1 + \frac{r}{p}\right) e \sin^2\gamma \sin w \sin 2u\right]\right.$$

$$+ \frac{m}{r}\left\{\left[10\left(\frac{p}{r} - 1\right) - (7 + e^2) + 7\,(1 - e^2)\,\frac{r}{p}\right] - \frac{3}{2} J_2\left(\frac{R}{r}\right)^2\left[\frac{68}{3}\left(\frac{p}{r} - 1\right) - 2\,(7 + e^2)\right.\right.$$

$$\left. + 14\,(1 + e^2)\,\frac{r}{p}\right] \cdot P_2 - \frac{3}{2} J_2\left(\frac{R}{r}\right)^2\left[6 - (1 + e^2)\,\frac{r}{p} + (1 - e^2)\,\frac{r^2}{p^2}\right] e \sin^2\gamma \sin w \sin 2u$$

$$+ 2\,\Gamma\left(\frac{\Omega R^2}{h}\right)\left[(1 + e^2) - (1 - e^2)\,\frac{r}{p}\right]\cos\gamma - \left(\frac{\Omega R^2}{h}\right)^2\left(\frac{p}{R}\right)^2\left(1 - \frac{r}{p}\right)\left[\Gamma + 3\,\Delta\left(\frac{R}{r}\right)^2 P_2\right]$$

$$\left.\left. - \frac{3}{2}\,\Delta\left(\frac{\Omega R^2}{h}\right)^2\left(\frac{p}{r} + 1\right) e \sin^2\gamma \sin w \sin 2u\right\}\right\rangle \tag{253}$$

$$\frac{d\sigma}{dt} + \sqrt{1 - e^2}\,\frac{d\pi}{dt} = -2\,\frac{h}{p}\,\frac{\sqrt{1 - e^2}}{r}\left\langle 3 J_2\left(\frac{R}{r}\right)^2 P_2 + \frac{4m}{r}\left\{\left(\frac{5}{2} - \frac{3}{4}\,\frac{r}{a} - \frac{p}{r}\right) - \right.\right.$$

$$- \frac{3}{2} J_2\left(\frac{R}{r}\right)^2\left[\left(\frac{17}{3} - \frac{3}{2}\,\frac{r}{a} - 2\,\frac{p}{r}\right) P_2 - e \sin^2\gamma \sin w \sin 2u\right]$$

$$\left.\left. + \frac{1}{2}\,\Gamma\left(\frac{\Omega R^2}{h}\right)\frac{p}{r} \cos\gamma - \frac{1}{4}\left(\frac{\Omega R^2}{h}\right)^2\left(\frac{p}{R}\right)^2 \frac{r}{p}\left[\Gamma + 3\,\Delta\left(\frac{R}{r}\right)^2 P_2\right]\right\}\right\rangle \tag{254}$$

$$\frac{d\gamma}{dt} = \frac{h}{pr}\left\langle -\frac{3}{4} J_2\left(\frac{R}{r}\right)^2 \sin 2\gamma \sin 2u + \frac{m}{r}\left\{-\frac{3}{4} J_2\left(\frac{R}{r}\right)^2\left(2 - \frac{r}{a}\right) \sin 2\gamma \sin 2u\right.\right.$$

$$\left.\left. + 2\,\Gamma\left(\frac{\Omega R^2}{h}\right)\sin\gamma\left(e \sin w \cos^2 u + \frac{p}{r} \sin 2u\right) + \frac{3}{4}\,\Delta\left(\frac{\Omega R^2}{h}\right)^2 \frac{p}{r} \sin 2\gamma \sin 2u\right\}\right\rangle \tag{255}$$

$$\frac{d\,a_\Omega}{dt} = \frac{h}{pr}\left\langle -3\,J_2\left(\frac{R}{r}\right)^2 \cos\gamma \sin^2 u + \frac{m}{r}\left\{-3 J_2\left(\frac{R}{r}\right)^2\left(2 - \frac{r}{a}\right) \cos\gamma \sin^2 u\right.\right.$$

$$\left.\left. + 2\,\Gamma\left(\frac{\Omega R^2}{h}\right)\left(\frac{e}{2} \sin w \sin 2u + 2\,\frac{p}{r} \sin^2 u\right) + 3\,\Delta\left(\frac{\Omega R^2}{h}\right)^2 \frac{p}{r} \cos\gamma \sin^2 u\right\}\right\rangle \tag{256}$$

The differential equations (251) to (256) have to be integrated with respect to the time or better with respect to the true anomaly w using again the relations

$$dt = \frac{r^2}{h}\, dw \quad , \quad \frac{p}{r} = 1 + e \cos w \quad , \quad u = \omega + w$$

of Keplerian motion on the right side of the above-mentioned differential equations. Instead of doing this, time mean values of the variable terms over one revolution will be used on the right side of eqs. (251) to (256) in order to cancel out short-periodic perturbations. Using

$$d\,M = n\,dt = \frac{n}{h}\, r^2 d\,w \quad , \quad \frac{p}{r} = 1 + e \cos w$$

there is

$$\overline{\left(r^{-\nu} \begin{matrix}\sin\\ \cos\end{matrix}\, \kappa\, w\right)} = \frac{1}{2\,\pi} \int_0^{2\pi} r^{-\nu} \begin{matrix}\sin\\ \cos\end{matrix}\, \kappa\, w\; dM = \frac{n}{2\,\pi\, h\, p^{n-2}} \int_0^{2\pi} (1 + e \cos w)^{\nu-2} \begin{matrix}\sin\\ \cos\end{matrix}\, \kappa\, w\, d\,w$$

The evaluation of this integral for ν, κ as integers shows that

$$\overline{\left(\frac{\cos \kappa\, w}{r^{\nu}}\right)} = 0 \;\; (\text{for } \kappa > \nu - 2 \geqslant 0) \quad ; \quad \overline{\left(\frac{\sin \kappa\, w}{r^{\nu}}\right)} = 0 \;\; (\text{for } \nu \geqslant 2)$$

and that

$$\overline{\left(\frac{\cos \kappa\, w}{r^2}\right)} = \frac{n}{h} \cdot \begin{cases} 1 & \text{for } \kappa = 0 \\ 0 & \kappa > 0 \end{cases}$$

$$\overline{\left(\frac{\cos \kappa\, w}{r^3}\right)} = \frac{n}{h\,p} \cdot \begin{cases} 1 & \text{for } \kappa = 0 \\ \frac{1}{2}\, e & \kappa = 1 \\ 0 & m > 1 \end{cases}$$

$$\overline{\left(\frac{\cos \kappa\, w}{r^4}\right)} = \frac{n}{h\,p^2} \cdot \begin{cases} 1 + \frac{e^2}{2} & \text{for } \kappa = 0 \\ e & \kappa = 1 \\ \frac{1}{4}\, e^2 & \kappa = 2 \\ 0 & \kappa > 2 \end{cases}$$

$$\overline{\left(\frac{\cos \kappa\, w}{r^5}\right)} = \frac{n}{h\,p^3} \cdot \begin{cases} 1 + \frac{3}{2}\, e^2 & \text{for } \kappa = 0 \\ \frac{3}{2}\, e \left(1 + \frac{e^2}{4}\right) & \kappa = 1 \\ \frac{3}{4}\, e^2 & \kappa = 2 \\ \frac{1}{8}\, e^3 & \kappa = 3 \\ 0 & \kappa > 3 \end{cases}$$

$$\overline{\left(\frac{\cos \kappa w}{r^6}\right)} = \frac{n}{b p^4} \cdot \begin{cases} 1 + 3 e^2 + \frac{3}{8} e^4 & \text{for } \kappa = 0 \\ 2 e \left(1 + \frac{3}{4} e^2\right) & \kappa = 1 \\ \frac{3}{2} e^2 \left(1 + \frac{1}{6} e^2\right) & \kappa = 2 \\ \frac{1}{2} e^3 & \kappa = 3 \\ \frac{1}{16} e^4 & \kappa = 4 \\ 0 & \kappa > 4 \end{cases}$$

$$\overline{\left(\frac{\cos \kappa w}{r^7}\right)} = \frac{n}{b p^5} \cdot \begin{cases} 1 + 5 e^2 + \frac{15}{8} e^4 & \text{for } \kappa = 0 \\ \frac{5}{2} e \left(1 + \frac{3}{2} e^2 + \frac{1}{8} e^4\right) & \kappa = 1 \\ \frac{5}{2} e^2 \left(1 + \frac{e^2}{2}\right) & \kappa = 2 \\ \frac{5}{4} e^3 \left(1 + \frac{1}{8} e^2\right) & \kappa = 3 \\ \frac{5}{16} e^4 & \kappa = 4 \\ \frac{1}{32} e^5 & \kappa = 5 \\ 0 & \kappa > 5 \end{cases}$$

Due to $u = \omega + w$, the addition theorems

$$\sin w \cdot \sin 2u = -\frac{1}{2} \cos (2\omega + 3w) + \frac{1}{2} \cos (2\omega + w)$$

$$\sin w \cdot \cos 2u = \frac{1}{2} \sin (2\omega + 3w) - \frac{1}{2} \sin (2\omega + w)$$

and the expression for the second harmonics

$$P_2 = \frac{1}{2} (3 \sin^2 \gamma \sin^2 u - 1) = -\frac{1}{2} \left[\left(1 - \frac{3}{2} \sin^2 \gamma\right) + \frac{3}{2} \sin^2 \gamma \cos 2u\right]$$

there is

$$\overline{\left(\frac{\sin 2u}{r^\nu}\right)} = \sin 2\omega \;\; \overline{\left(\frac{\cos 2w}{r^\nu}\right)} \neq 0 \quad \text{for} \quad \nu \geqslant 4$$

$$\overline{\left(\frac{\cos 2\omega}{r^\nu}\right)} = \cos 2\omega \;\; \overline{\left(\frac{\cos 2w}{r^\nu}\right)} \neq 0 \quad \text{for} \quad \nu \geqslant 4$$

$$\overline{\left(\frac{P_2}{r^\nu}\right)} = -\frac{1}{2}\left[\left(1-\frac{3}{2}\sin^2\gamma\right)\overline{\left(\frac{1}{r^\nu}\right)} + \frac{3}{2}\sin^2\gamma\cos 2\omega\,\overline{\left(\frac{\cos 2w}{r^\nu}\right)}\right]$$

$$\overline{\left(\frac{P_2 \sin w}{r^\nu}\right)} = -\frac{3}{4}\sin^2\gamma\;\overline{\left(\frac{\sin w\cos 2u}{r^\nu}\right)} = -\frac{3}{8}\sin^2\gamma\sin 2\omega\left[\overline{\left(\frac{\cos 3w}{r^\nu}\right)} - \overline{\left(\frac{\cos w}{r^\nu}\right)}\right]$$

$$\overline{\left(\frac{\sin w\sin 2u}{r^\nu}\right)} = -\cot 2\omega\;\overline{\left(\frac{\sin w\cos 2u}{r^\nu}\right)} = -\frac{1}{2}\cos 2\omega\left[\overline{\left(\frac{\cos 3w}{r^\nu}\right)} - \overline{\left(\frac{\cos w}{r^\nu}\right)}\right]$$

$$\neq 0 \quad \text{for} \quad \nu \geqslant 3$$

These general formulas provide now the following special time mean values applied to eqs. (249) to (254):

$$\overline{\left(\frac{\sin 2u}{r^4}\right)} = \frac{1}{4}\,\frac{n}{h p^2}\,e^2\sin 2\omega \quad ; \quad \overline{\left(\frac{\sin 2u}{r^5}\right)} = \frac{3}{4}\,\frac{n}{h p^3}\,e^2\sin 2\omega$$

$$\overline{\left(\frac{\sin 2u}{r^6}\right)} = \frac{3}{2}\,\frac{n}{h p^4}\,e^2\left(1+\frac{e^2}{6}\right)\sin 2\omega$$

$$\overline{\left(\frac{P_2\sin w}{r^4}\right)} = \frac{3}{8}\,\frac{n}{h p^2}\,e\sin^2\gamma\sin 2\omega \quad ; \quad \overline{\left(\frac{P_2\sin w}{r^5}\right)} = \frac{9}{16}\,\frac{n}{h p^3}\,e\left(1+\frac{e^2}{6}\right)\sin^2\gamma\sin 2\omega$$

$$\overline{\left(\frac{P_2}{r^3}\right)} = -\frac{1}{2}\,\frac{n}{h p}\left(1-\frac{3}{2}\sin^2\gamma\right)$$

$$\overline{\left(\frac{P_2}{r^4}\right)} = -\frac{1}{2}\,\frac{n}{h p^2}\left[\left(1+\frac{e^2}{2}\right)\left(1-\frac{3}{2}\sin^2\gamma\right) + \frac{3}{8}\sin^2\gamma\cos 2\omega\right]$$

$$\overline{\left(\frac{P_2}{r^5}\right)} = -\frac{1}{2}\,\frac{n}{h p^3}\left[\left(1+\frac{3}{2}e^2\right)\left(1-\frac{3}{2}\sin^2\gamma\right) + \frac{9}{8}e^2\sin^2\gamma\cos 2\omega\right]$$

$$\overline{\left(\frac{P_2}{r^6}\right)} = -\frac{1}{2}\,\frac{n}{h p^4}\left[\left(1+3e^2+\frac{3}{8}e^4\right)\left(1-\frac{3}{2}\sin^2\gamma\right) + \frac{9}{4}e^2\left(1+\frac{e^2}{6}\right)\sin^2\gamma\cos 2\omega\right]$$

$$\overline{\left(\frac{\sin w\sin 2u}{r^3}\right)} = \frac{1}{4}\,\frac{n}{h p}\,e\cos 2\omega \quad ; \quad \overline{\left(\frac{\sin w\sin 2u}{r^4}\right)} = \frac{1}{2}\,\frac{n}{h p^2}\,e\cos 2\omega$$

$$\overline{\left(\frac{\sin w \sin 2u}{r^5}\right)} = \frac{3}{4}\frac{n}{h p^3}\, e\left(1+\frac{e^2}{6}\right)\cos 2\omega$$

Substituting these time mean values into eqs. (251) to (256) then, after reduction, the equations can be written

$$\frac{\dot a}{a_0} = -\frac{45}{4}\, n_0\left(\frac{m}{p_0}\right) J_2\left(\frac{R}{p_0}\right)^2 \frac{1+e^2/6}{1-e_0^2}\, e_0^2 \sin^2\gamma_0 \sin 2\omega \qquad (257)$$

$$\frac{\dot p}{p_0} = -\frac{3}{2}\, n_0\left(\frac{m}{p_0}\right) J_2\left(\frac{R}{p_0}\right)^2 e_0^2 \sin^2\gamma_0 \sin 2\omega \qquad (258)$$

$$\frac{de^2}{dt} = 2e_0\dot e = (1-e_0^2)\left(\frac{\dot a}{a_0}-\frac{\dot p}{p_0}\right) = -\frac{3}{4}\, n_0\left(\frac{m}{p_0}\right) J_2\left(\frac{R}{p_0}\right)^2 e_0^2\left(13+\frac{9}{2}e_0^2\right)\sin^2\gamma_0 \sin 2\omega$$

or

$$\dot e = -\frac{3}{8}\, n_0\left(\frac{m}{p_0}\right) J_2\left(\frac{R}{p_0}\right)^2 e_0\left(13+\frac{9}{2}e_0^2\right)\sin^2\gamma_0 \sin 2\omega \qquad (259)$$

$$\dot\pi = \dot\omega + \dot a_{\Omega}\cos\gamma = n_0\left\langle \frac{3}{2} J_2\left(\frac{R}{p_0}\right)^2\left(1-\frac{3}{2}\sin^2\gamma_0\right) + \left(\frac{m}{p_0}\right)\left\{3 - \right.\right.$$

$$-\frac{3}{2} J_2\left(\frac{R}{p_0}\right)^2\left[\left(16+\frac{25}{4}e_0^2\right)\left(1-\frac{3}{2}\sin^2\gamma_0\right)+\frac{1}{4}(13+23e_0^2)\sin^2\gamma_0\cos 2\omega\right]$$

$$\left.\left.-4\Gamma\left(\frac{\Omega R^2}{h_0}\right)\cos\gamma_0 - \frac{3}{2}\Delta\left(\frac{\Omega R^2}{h_0}\right)^2\left(1-\frac{3}{2}\sin^2\gamma_0\right)\right\}\right\rangle \qquad (260)$$

thus

$$\dot\omega = n_0\left\langle \frac{3}{2} J_2\left(\frac{R}{p_0}\right)^2\left(2-\frac{5}{2}\sin^2\gamma_0\right) + \left(\frac{m}{p_0}\right)\left\{3 - \frac{3}{2} J_2\left(\frac{R}{p_0}\right)^2\left[\left(17+\frac{33}{4}e_0^2\right)\left(1-\frac{3}{2}\sin^2\gamma_0\right)\right.\right.\right.$$

$$\left. -(1+2e_0^2)\left(2-\frac{5}{2}\sin^2\gamma_0\right)+\frac{1}{4}(13+23e_0^2)\sin^2\gamma_0\sin 2\omega + \frac{1}{2}e_0^2\cos^2\gamma_0\cos 2\omega\right]$$

$$\left.\left.-6\Gamma\left(\frac{\Omega R^2}{h_0}\right)\cos\gamma_0 - \frac{3}{2}\Delta\left(\frac{\Omega R^2}{h_0}\right)^2\left(2-\frac{5}{2}\sin^2\gamma_0\right)\right\}\right\rangle \qquad (261)$$

$$\dot\sigma = -n_0\dot t_p = \dot M - n_0 = \sqrt{1-e_0^2}\, n_0\left\langle \frac{3}{2} J_2\left(\frac{R}{p_0}\right)^2\left(1-\frac{3}{2}\sin^2\gamma_0\right) - \left(\frac{m}{p_0}\right)\left\{3\left(5-2\sqrt{1-e_0^2}\right) - \right.\right.$$

$$-\frac{3}{2} J_2\left(\frac{R}{p_0}\right)^2\left[\frac{22}{3}\left(1+\frac{1}{8}e_0^2\right)\left(1-\frac{3}{2}\sin^2\gamma_0\right) - \frac{1}{4}(13+37e_0^2)\sin^2\gamma_0\cos 2\omega\right] +$$

$$+\frac{3}{2}\,\Delta\left(\frac{\Omega R^2}{h_0}\right)^2\left(1-\frac{3}{2}\sin^2\gamma_0\right)\Bigg\}\Bigg\rangle+n_0\left(\frac{m}{p_0}\right)2\Gamma\left(\frac{\Omega R p_0}{h_0}\right)^2 \tag{262}$$

$$\dot{\gamma}=-\frac{3}{8}\,n_0\left(\frac{m}{p_0}\right)J_2\left(\frac{R}{p_0}\right)^2 e_0^2\,\sin 2\,\gamma_0\,\sin 2\,\omega \tag{263}$$

$$\dot{\alpha}_\Omega=-\,n_0\Bigg\langle\frac{3}{2}\,J_2\left(\frac{R}{p}\right)^2\cos\gamma_0+\left(\frac{m}{p_0}\right)\Bigg\{\frac{3}{2}\,J_2\left(\frac{R}{p_0}\right)^2\cos\gamma_0\left[1+2\,e_0^2-\frac{1}{2}\,e_0^2\cos 2\,\omega\right]$$

$$-2\Gamma\left(\frac{\Omega R^2}{h_0}\right)-\frac{3}{2}\,\Delta\left(\frac{\Omega R^2}{h_0}\right)^2\cos\gamma\Bigg\}\Bigg\rangle \tag{264}$$

It is easy to integrate these equations where the orbital elements on the right side have their unperturbed constant values (subscript 0) except ω, which is a linear function of time t, namely

$$\omega=\omega_0+n_0\,\frac{3}{2}\,J_2\left(\frac{R}{p_0}\right)^2\left(2-\frac{5}{2}\sin^2\gamma_0\right)(t-t_0)=\omega_0+\dot{\omega}_{\text{non-rel.}}(t-t_0)$$

The integrals

$$\int\cos 2\,\omega\,dt=\frac{\sin 2\omega}{2\,\dot{\omega}_{\text{non-rel.}}}\qquad\qquad\int\sin 2\,\omega\,dt=-\frac{\cos 2\omega}{2\,\dot{\omega}_{\text{non-rel.}}}$$

show that long-periodic perturbations appear in addition to the secular perturbations.

B. The Time-Dilatation Effect

Using the line element of eq. (222) the ratio of the proper time element $d\tau$ to the local time element dt is given by

$$\left(\frac{d\tau}{dt}\right)^2 = \left(\frac{ds}{c\,dt}\right)^2 = 1 - \frac{2m}{r}\left\{1 - J_2\left(\frac{R_e}{r}\right)^2 P_2\ (\sin\phi) + \left(\frac{\Omega R_e}{c}\right)^2\left[\Gamma + \Delta\left(\frac{R_e}{r}\right)^2 P_2\ (\sin\phi)\right]\right\}$$

$$+ 0\ (m^2) - \left\{1 + \frac{2m}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2\ (\sin\phi)\right]\right\}\frac{v^2}{c^2} + \frac{2m}{r}\ \Gamma\left(\frac{\Omega R_e^2}{c^2}\right)\cos^2\phi\ \frac{d\theta}{dt}$$

or accurately enough

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{2m}{r}\left[1 + \frac{1}{2}J_2\left(\frac{R_e}{r}\right)^2(1 - 3\sin^2\phi)\right] - \frac{v^2}{c^2}$$

thus

$$\frac{d\tau}{dt} = 1 - \frac{m}{r}\left[1 + \frac{1}{2}J_2\left(\frac{R_e}{r}\right)^2(1 - 3\sin^2\phi)\right] - \frac{1}{2}\frac{v^2}{c^2} = 1 - \frac{v^2/2 + U}{c^2} \tag{265}$$

This formula, containing the invariant proper time element $d\tau = ds/c$, is now applied to an earth clock ($r = R$, $v = \Omega R \cos\phi$):

$$\frac{d\tau}{dt_E} = 1 - \frac{m}{R}\left[1 + \frac{1}{2}J_2\left(\frac{R}{R}\right)^2(1 - 3\sin^2\phi)\right] - \frac{1}{2}\frac{\Omega^2 R^2\cos^2\phi}{c^2} = 1 - \frac{W_0}{c^2} = \text{const.}$$

Assuming the Earth's surface an equipotential surface, the potential W_0 is constant and can be expressed by its value at the equator, thus

$$\frac{d\tau}{dt_E} = 1 - \frac{m}{R_e}\left[1 + \frac{1}{2}J_2\right] - \frac{1}{2}\left(\frac{\Omega R_e}{c}\right)^2 = 1 - \frac{m}{R_e}\left[1 + \frac{1}{2}J_2 + \frac{1}{2}\chi\right] \tag{266}$$

where $\chi = \Omega^2 R_e^3/\mu$ has been introduced.

Applying eq. (265) to a satellite clock moving with the satellite in an arbitrary orbit

$$(\sin\phi = \sin\delta = \sin\gamma_0 \sin u\ ;\ v^2/c^2 = m\,(2/r - 1/a_0)\ ;\ u = \omega + w)\ ,$$

that yields

$$\frac{d\tau}{dt_s} = 1 - \frac{m}{r}\left[1 + \frac{1}{2}J_2\left(\frac{R_e}{r}\right)^2(1 - 3\sin^2\gamma\ \sin^2 u)\right] - \frac{m}{2}\left(\frac{2}{r} - \frac{1}{a_0}\right)$$

$$= 1 + \frac{1}{2}\frac{m}{a_0} - \frac{m}{r}\left[2 + \frac{1}{2}J_2\left(\frac{R_e}{r}\right)^2\left(1 - 3\sin^2\gamma_0\,\frac{1 - \cos 2u}{2}\right)\right]$$

Using the time mean values

$$\overline{\left(\frac{a_0}{r}\right)} = 1\ ;\ \overline{\left(\frac{p_0}{r}\right)^3} = (1 - e_0^2)^{3/2}\ ;\ \overline{\left(\frac{p_0}{r}\right)^3\cos 2u} = 0$$

then results

$$\frac{d\tau}{dt_s} = 1 - \frac{m}{2a_0}\left[3 + J_2\left(\frac{R_e}{p_0}\right)^2 (1 - e_0^2)^{1/2}\left(1 - \frac{3}{2}\sin^2\gamma_0\right)\right] \tag{267}$$

Dividing eq. (266) by eq. (267) gives for the relative difference in the time rates of a satellite clock compared with a standard earth clock

$$\frac{dt_s}{dt_E} - 1 = -\frac{m}{R_e}\left[1 + \frac{1}{2}J_2 + \frac{1}{2}\chi\right] + \frac{m}{2a_0}\left[3 + J_2\left(\frac{R_e}{p_0}\right)^2 (1 - e_0^2)^{1/2}\left(1 - \frac{3}{2}\sin^2\gamma_0\right)\right]$$

$$= \frac{m}{R_e}\left\{\left(\frac{3}{2}\frac{R_e}{a_0} - 1\right) - \frac{1}{2}J_2\left[1 - \left(\frac{R_e}{p_0}\right)^3 (1 - e_0^2)^{3/2}\left(1 - \frac{3}{2}\sin^2\gamma_0\right)\right] - \frac{1}{2}\chi\right\} \tag{268}$$

Eq. (268) consists of three main terms. The first term (gravitational red shift and time dilatation) was first given by Winterberg (1955, Ref. 26) and Singer (1956, Ref. 27). The former author also added the last term due to the rotation of the Earth. The second term, due to the oblateness of the Earth, corrects and generalizes the term given by Hoffmann (1957, Ref. 28) for circular orbits in the equatorial plane.

This paper will be concluded by listing certain constants of the Earth's gravity field which appear in the text, namely

c	light velocity in vacuo, 299792.50 + 0.10 km/sec (K.D. Froome, 1958)
μ	gravity factor of the Earth, GM = 398613.52 km^3/sec^2 (Herrick, Baker, 1957)
R_e	equatorial radius of the Earth, 6378.150 + 0.050 km (Baker, 1961)
J_2	oblateness constant of the Earth, (1082.190 + 0.023) $\times 10^{-6}$ (Kozai, 1960)
Γ	inhomogeneity factor of the Earth, $0.3336 \approx 1/3$
Δ	inhomogeneity factor of the Earth, $-0.0429 \approx -3/70$
m	gravitational radius of the Earth, $\mu/c^2 = 0.4435$ cm
m/R_e	potential energy factor of the Earth, 6.95377×10^{-10}
Ω	angular velocity of the Earth's rotation, $7.292115083 \times 10^{-5}$ sec^{-1}
ΩR_e	rotational velocity at the Earth's equator 465.102 m/sec
ΩR_e^2	angular momentum (per mass unit) at the Earth's equator, $2.96649 \times 10^9 m^2$/sec
χ	centrifigal factor at the Earth's equator, $\Omega^2 R_e^3/\mu = 3461.30 \times 10^{-6}$
$\Omega R_e/c$	rotational velocity ratio, 1.55142×10^{-6}
$(\Omega R_e/c)^2$	square of the rotational velocity ratio, 2.40691×10^{-12}

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Redstone Arsenal, Huntsville, Alabama, May 22, 1962

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