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COMPUTATION OF SATELLITE ORBITS BY THE HANSEN METHOD AS MODIFIED BY MUSEN

By HOWARD T. PHELAN

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Presented in a series of lectures at Goddard Space Flight Center, Oct. 1960

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By HOWARD T. PHELAN

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PREFACE

This work attempts a complete exposition of the modified Hansen's theory developed by Dr. Peter Musen for analysis of the motion of an artificial satellite in the earth's gravitational field. However, any exposition which lays claim to being complete is subject to severe criticism, for the sheer mass of details that are involved can never be completely covered in a work of practical proportions. Nonetheless, it is the attempt of the author to provide a systematic presentation which will begin at a relatively fundamental stage of celestial mechanics. It is hoped that in this manner, the exposition may be of value to those new to the field of orbit computation and to those whose concern is primarily machine programming, as well as to those more interested in this particular theory of general perturbations.

The body of this work was presented by the author in a series of lectures at the Goddard Space Flight Center of the National Aeronautics and Space Administration in October, 1960. The questions and discussions which arose in the course of this lecture series were of value in determining what were the particularly troublesome concepts and techniques in the theory, and an attempt is made to deal with them thoroughly in this work.

The raison d'etre of this exposition is the recently generated high degree of interest in artificial satellite orbit computation, and in the Hansen approach in particular. The theory has been in use in the computation of satellite orbits since Vanguard I (1958 β) went into orbit in March, 1958, and is the basis of orbit predictions of the Goddard Space Flight Center. Despite the important role the theory has played to date, its working is not widely understood, and it is hoped that this exposition will lead to a greater understanding and use of the theory.

The author, at the request of the Data Systems Division of the Goddard Space Flight Center, undertook a study of Musen's development in order that an exposition of this type, beginning at a fairly basic level, might be made available to the growing number of those involved in satellite orbit computation.

It is with the deepest gratitude that the author acknowledges the invaluable assistance and guidance offered him by Dr. Peter Musen, who gave freely and amiably of his time, in order to make clear the more subtle points of the theory. As well, the author wishes to extend his deepest thanks to Mr. David Fisher of the Goddard Space Flight Center, at whose suggestion this work was undertaken, and whose direction, assistance, and moral support were instrumental in achieving a finished product. In addition, grateful acknowledgment is given to Dr. Alan Galbraith and Mr. Lloyd Carpenter for carefully reviewing the manuscript and rendering valuable suggestions.

CONTENTS

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SUMMARY 14437

A comprehensive description of the Hansen theory of satellite orbit calculation, as modified by Musen, is presented. The equations of the theory are developed in sufficient detail to allow the reader to relate them to fundamental laws of celestial mechanics. The physical and mathematical concepts underlying Hansen's coordinate system and auxiliary ellipse are treated. The disturbing potential function and its derivatives are developed in the derivation of equations for the perturbations in the orbit plane, as well as the perturbations of the orbit plane. The method is described for determination of the final position and velocity vectors of the real satellite and determination of the osculating elements. Finally, a brief evaluation of the theory is presented. Author

INTRODUCTION

This exposition presents a complete development of all the mathematical relationships used in Musen's theory of the motion of an artificial satellite in the gravitational field of the earth, a theory which is basically an application of Hansen's lunar theory to an artificial satellite. Since Musen's development is based upon Hansen's classical work, an understanding of the latter is most important in comprehending Musen's work. However, Hansen's theory does not at all lend itself to an easy, clear, and simple exposition; quite the contrary is true.

For the past 130 years, Hansen's theory has led to confusion and controversy in the world of celestial mechanics, and for the most part it has been avoided. The difficulty in understanding Hansen arises, as Ernest W. Brown (Reference 1) expressed it, "partly on account of the somewhat uncouth form in which it is given in the *Fundamenta* and partly on account of the very unusual way in which the perturbations are expressed." In other words, Hansen's techniques in solving lunar perturbations were extremely unorthodox, enough so that many of his contemporaries and successors violently disagreed with him. Nonetheless, the undeniable fact about Hansen's lunar theory was that it worked, and with a high degree of accuracy. Here, it is necessary to inspect what are basically Hansen's methods if we are to understand Musen's final result. Though it is impossible in a work of this length to cover all the details of Hansen's theory, it is hoped that by dealing with only the techniques incorporated in Musen's development, we will have a sufficiently clear and complete perspective on the theory as adapted to the motion of artificial satellites.

In general, Hansen's lunar theory had six distinguishing features:

1. A fictitious, or auxiliary, ellipse is introduced and placed in the plane of the instantaneous orbit, i.e., the plane containing the instantaneous radius and velocity vectors. This fictitious ellipse is of constant shape, and its perigee moves in a specific manner.
2. The angular perturbations in the plane of the orbit are added to the mean anomaly of the fictitious ellipse.
3. The radial perturbations are expressed as a ratio between the radial distance of the real satellite and that of the fictitious satellite.
4. The longitudes are measured from a "Departure Point" in the plane of the orbit.
5. One function, W, is found which expresses *all* the perturbations in the orbit plane.
6. The theory is a general one which handles lunar perturbations of all kinds.

Each of these six features is to be found in the Musen development. Only four changes are made by Musen in his development, but each is ingenious and very significant. They are as follows:

1. Musen has used the eccentric anomaly of the fictitious satellite as the independent variable instead of time. He has developed all his Fourier series expressions in terms of it, whereas Hansen's Fourier series were developed in terms of mean anomaly. The idea of using the eccentric anomaly was borrowed from Hansen's planetary theory.
2. The method of iteration is used in developing the final series forms, replacing Hansen's method of development into Maclaurin series. This change was made desirable by the existence of fast computing machines which handle iterations rapidly.
3. Parameters designated by the symbol λ are introduced. These parameters, which determine the perturbations of the orbit plane, allow the introduction of a rotation matrix, an ingenious development leading directly and simply to the final position vector.
4. The rotation matrix is introduced in place of Hansen's development in polar coordinates, obviating much of the cumbersome calculation required by Hansen.

So in the final form of Musen's development, the basic idea consists of introduction of a fictitious auxiliary satellite which describes an auxiliary rotating ellipse of constant shape and moves in this ellipse in accordance with Kepler's laws. The position of the real satellite is determined by its deviations in time, as well as in space, from the position of this auxiliary satellite. The perturbations *in* the orbit plane are relatively large compared to those *of* the orbit plane, and are separated from the latter. They are then determined by the single function W for which a differential equation of the first order is formed. The perturbations of the orbit plane are determined by four interdependent parameters, the λ parameters.

It should be noted here that the sixth distinguishing feature of Hansen's method, given above, has significance in the artificial satellite theory. Hansen's method allows inclusion of all perturbing forces on the moon. Clearly, the forces which disturb the motion of artificial satellites can be more numerous and more complex than those which disturb the motion of the moon. Nonetheless, Musen's modified Hansen theory allows easy inclusion of such forces as those due to the gravitational attraction of the sun and the moon, solar radiation pressure, the ellipticity of the earth's equator, and the motion of the earth's water masses. It is not inconceivable that even the force of atmospheric drag can be included in the theory. However, this effect, which is the most troublesome and difficult force to deal with in artificial satellite theories, is not yet sufficiently understood to allow its easy inclusion in the disturbing function.

The theory as developed by Musen is of unique and primary significance because it is exact for all orders of perturbations. This renders it most valuable for the accurate determination of long period effects, and allows long term predictions. However, it should be noted that for low altitude satellites (on which the drag effects are considerably larger than other perturbing effects) the theory gives, for practical purposes, orbit predictions good for approximately two weeks' time.

Musen has recently completed a modification of his development (Reference 2) which circumvents some of its more troublesome aspects. However, the basic approach is exactly that which is described here. The major difference is that the perturbations are developed by using the true longitude rather than the eccentric anomaly as the independent variable. The new technique avoids the necessity of "starring" and "barring" the potential function in the development of the basic perturbation function W. It also allows polynomial representation in most places where the present development uses infinite series. Much of the problem of machine truncation error is thus eliminated. The new modification is also limited to only those eccentricities for which Kepler's equation can be readily solved. Until some replacement of Kepler's equation is found for large eccentricities, this limitation will exist. The techniques used in the new modification for developing the perturbations and finding the constants of integration are exactly those described here, and the machine program is roughly equivalent to the existing one.

The notation in the following exposition is, where possible, the same as that used in Brown's discussion of Hansen's Lunar Theory (Reference 1), which was based upon Hansen's notation in the *Darlegung* (Reference 3).

SECTION I

BASIC NOTATION AND FUNDAMENTAL EQUATIONS OF CLASSICAL CELESTIAL MECHANICS

DISCUSSION

Before an attempt is made to dissect the Musen theory, it is necessary to become familiar with the basic expressions which occur in it, as well as the notation used. This section lists the notation used and then gives the derivations of five equations common in classical celestial mechanics, which are used in the Musen development. Two of these equations apply only in an "ideal" coordinate system (defined as a coordinate system which rotates, but so that the form of the equations of motion of a satellite is invariant). This is a very important consequence, and one upon which Hansen relies. Hansen's particular "ideal" rotating coordinate system is an orthogonal one in which the X and Y axes are allowed no rotation about the Z axis (see Figure 1 which is explained in greater detail in Section II).

The basic ellipse equations found in this section are derived in Appendix A, as are the equations of motion in polar form. We start our development with the classical two-body problem, and since much of Musen's development is a vectorial one, let us begin here by listing all the vectors and angles which will occur. We assume an orthogonal, xyz, inertial coordinate system whose origin is at the center of the earth, which in the first approximation we will assume to be spherical and homogeneous. The xy plane is the earth's equatorial plane, and the z axis has the positive direction of the axis of rotation. In the absence of disturbing forces, a satellite of negligible mass has as its equation of motion

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3}, \tag{1}$$

where

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

is the position vector of the satellite, r is its magnitude, and **i, j, k** are unit vectors along the x, y, z axes respectively. The velocity vector is

$$\dot{\mathbf{r}} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

and the acceleration vector is

$$\ddot{\mathbf{r}} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

In this system of equations, the equations of motion have been nondimensionalized by choosing as a unit of length the earth's mean equatorial radius, and as the unit of time the square root of the ratio of the radius to the Newtonian acceleration at a distance of one radius from a point mass (having the earth's mass). (See Section X.)

The satellite's orbit is assumed to be an ellipse. The following notation will be used, and is identical to the notation used throughout Musen's development:

$E_1 = a =$ semimajor axis of ellipse,
$E_2 = e =$ eccentricity,
$E_3 = \omega =$ argument of perigee as measured from the ascending node,
$E_4 = \theta =$ longitude of the ascending node in the equatorial system,
$E_5 = i =$ inclination of the orbit plane to the equatorial plane,
$E_6 = g_0 =$ mean anomaly at the epoch (i.e., at time $t = t_0$),
$n = \sqrt{\mu}a^{-3/2} =$ mean motion (in Vanguard units, where standard $\mu = 1$),
$g = g_0 + n(t - t_0) =$ mean anomaly,
$E =$ eccentric anomaly,
$f =$ true anomaly,
$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k} =$ unit vector directed from origin to perigee,
$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} =$ unit vector normal to orbit plane,
$\mathbf{Q} = \mathbf{R} \times \mathbf{P}$
$\mathbf{r}^0 =$ unit vector in direction of $\mathbf{r}$,
$\mathbf{n}^0 =$ unit vector normal to $\mathbf{r}$, lying in the orbit plane.

In the two-body problem, the elements $E_i (i = 1, 2, \ldots, 6)$ are the constants of integration, and the complete solution is given by the following set of classical equations (Reference 4, p. 164):

$$E - e \sin E = g, \tag{2}$$

$$r \cos f = a(\cos E - e), \tag{3}$$

$$r \sin f = a\sqrt{1 - e^2} \sin E, \tag{4}$$

$$r = a(1 - e \cos E) = \frac{a(1 - e^2)}{1 + e \cos f}. \tag{5}$$

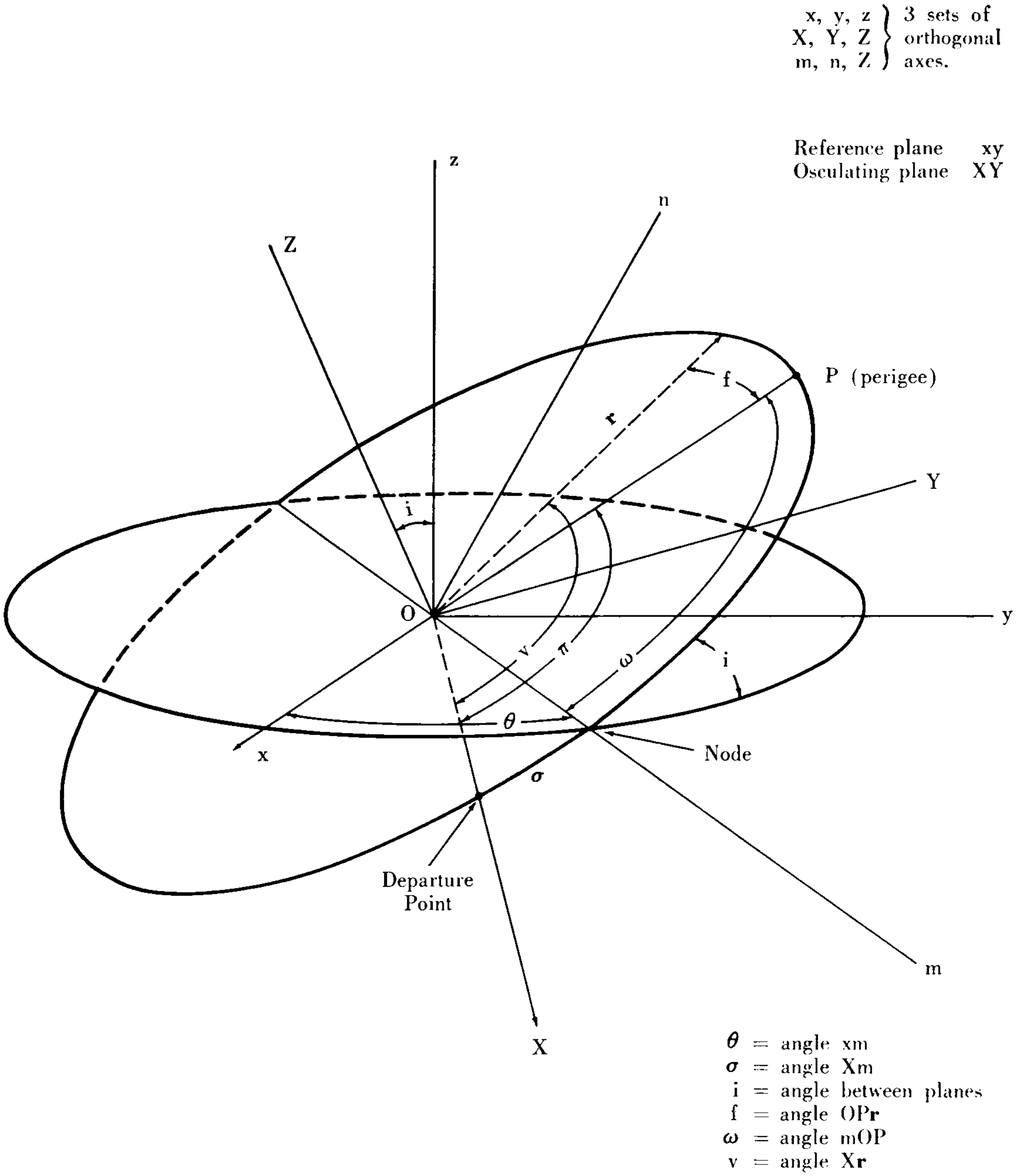


FIGURE 1.—The coordinate systems of Hansen's theory.

Equation 2 is Kepler's classical equation relating the eccentric anomaly to the mean anomaly. Equations 3, 4, and 5 are the standard ellipse equations and are derived in Appendix A. The final equations necessary to the complete solution are those of the rotation matrix and the position and velocity vectors:

$$\begin{bmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (6)$$

$$\mathbf{r} = \mathbf{P}a(\cos E - e) + \mathbf{Q}a\sqrt{1-e^2}\sin E, \quad (7)$$

$$\dot{\mathbf{r}} = \frac{1}{r\sqrt{a}}(\mathbf{Q}a\sqrt{1-e^2}\cos E - \mathbf{P}a\sin E)\cdot \quad (8)$$

The first and third matrices on the right-hand side of Equation 6 each describes a rotation about the z axis. The second matrix describes a rotation about the x axis.

In the two-body problem, the components of the position and velocity vectors at the initial time $t=t_0$ can be taken as the constants of integration, and the elements can be determined from them by means of the above set of equations. The two-body problem, however, yields only a first approximation to the motion of a planet or a satellite. The presence of some disturbing force **F** causes deviations from the simple motion of the two-body problem, and gives rise to variations in the elements of the orbit. Thus, we must find a way to deal with these variations. The classical concept of osculating elements was introduced as a device to facilitate the handling of this variation. The osculating, or instantaneous, elements of the orbit are the elements which would be found at any given instant if at that instant the planet or satellite were assumed to be traveling in a perfect ellipse and in a stationary plane.

In mathematical terms, the osculating elements are defined in such a way that if the position and velocity vectors of the orbiting body are given as functions of the six elements and time,

$$\left.\begin{aligned} \mathbf{r} &= f(E_1, E_2, \ldots, E_6, t) \\ \dot{\mathbf{r}} &= g(E_1, E_2, \ldots, E_6, t) \end{aligned}\right\}, \quad (9)$$

673-619—64——2

then the following relations must hold:

$$\sum_{i=1}^{6} \frac{dE_i}{dt}\frac{\partial \mathbf{r}}{\partial E_i} = 0, \quad (10)$$

and

$$\sum_{i=1}^{6} \frac{dE_i}{dt}\frac{\partial \dot{\mathbf{r}}}{\partial E_i} = \mathbf{F}, \quad (11)$$

where

$$\sum_{i=1}^{6} \frac{dE_i}{dt}\frac{\partial}{\partial E_i} \equiv \frac{\delta}{dt}$$

is called "Brown's operator." (See Reference 4, pp. 374–375.) This is a result of the Method of Variation of Parameters, a mathematical technique commonly used in celestial mechanics; it is not a result of deductive reasoning, but rather an educated guess as to the form of the general solution to the problem. (See Reference 5, pp. 466–473.) Two consequences of defining the osculating elements in this fashion are

$$\frac{\partial \mathbf{r}}{\partial t} = \dot{\mathbf{r}} \text{ and } \frac{\partial \dot{\mathbf{r}}}{\partial t} = \frac{-\mathbf{r}}{r^3}, \quad (12)$$

relations which appear throughout the development of certain derivatives of the elements.

Now that we have introduced a disturbing force **F** which produces variations in the elements, we must try to form expressions for these variations. To do this it is most convenient to work with the disturbing potential Ω rather than its gradient, the force. We assume that **F** has the form

$$\mathbf{F} = \text{grad }\Omega = \frac{\partial\Omega}{\partial x}\mathbf{i} + \frac{\partial\Omega}{\partial y}\mathbf{j} + \frac{\partial\Omega}{\partial z}\mathbf{k}, \quad (13)$$

which is a convenient form for the development that follows.

In Musen's development, we require expressions for the time rate of change of the angle of inclination i, the time rate of change of the right ascension of the node θ, and the periodic part of the time rate of change of the argument of perigee g. We want these expressions in terms of components of the force function. In this development, we will introduce the XYZ coordinate system where X, Y, and Z are orthogonal axes, the X axis and Y axis lying in the orbit plane, and the Z axis being normal to it. Our attempt is to obtain the expressions in terms of the gradients of the potential function along the X, Y, or Z axis. By using a vectorial development, we can readily find the desired form of these expressions.

DERIVATION OF $\frac{di}{dt}$

In the two-body problem, in which we let X and Y be orthogonal axes in the plane of the orbit, and Z the axis normal to the orbit, we have

$$\ddot{\mathbf{r}}=\frac{-\mathbf{r}}{r^3} \tag{14}$$

since the force is an inverse square force in the direction of the unit vector $\mathbf{r}^0$, where $\mathbf{r}^0=\mathbf{r}/r$.

If we operate on Equation 14 by $\mathbf{r}\times$, we get

$$\mathbf{r}\times\ddot{\mathbf{r}}=\frac{\mathbf{r}}{r^3}\times\mathbf{r}\cdot$$

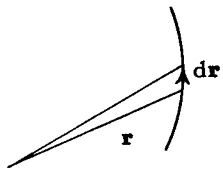
But

$$\frac{\mathbf{r}}{r^3}\times\mathbf{r}=0; \tag{15}$$

so, integrating $\mathbf{r}\times\ddot{\mathbf{r}}=0$, we have

$$\mathbf{r}\times\dot{\mathbf{r}}=\mathbf{c},$$

where $\mathbf{c}$ is the vector constant of integration. But from the sketch



we see that $|\mathbf{r}\times\mathbf{r}|$ is twice the area swept out per unit time. Also, the direction of $(\mathbf{r}\times\dot{\mathbf{r}})$ is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{dr}$; so if we call the unit normal to this plane $\mathbf{R}$, we see that

$$\mathbf{c}=\frac{\mathbf{R}}{h},$$

where $1/h$ is twice the area swept out per unit time; and from Kepler's law,

$$h=\frac{1}{\sqrt{a(1-e^2)}};$$

so we have

$$\mathbf{r}\times\dot{\mathbf{r}}=\frac{\mathbf{R}}{h}=\mathbf{c}. \tag{16}$$

Now, again taking Equation 14 and operating with $\mathbf{R}\times$, we have

$$\mathbf{R}\times\ddot{\mathbf{r}}=-\mathbf{R}\times\frac{\mathbf{r}}{r^3}. \tag{17}$$

But we see from Equation 16 that $\mathbf{R}=h(\mathbf{r}\times\dot{\mathbf{r}})$; so

$$\mathbf{R}\times\frac{\mathbf{r}}{r^3}=h(\mathbf{r}\times\dot{\mathbf{r}})\times\frac{\mathbf{r}}{r^3}$$

$$=\frac{h}{r^2}(\mathbf{r}\times\cdot)\times\frac{\mathbf{r}}{r}$$

$$=\frac{h}{r^2}\left[\left(\mathbf{r}\cdot\frac{\mathbf{r}}{r}\right)\dot{\mathbf{r}}-(\mathbf{r}\cdot\dot{\mathbf{r}})\frac{\mathbf{r}}{r}\right].$$

However, we can write

$$\dot{\mathbf{r}}=\frac{d\mathbf{r}}{dt}=\frac{dr}{dt}\mathbf{r}^0+r\frac{dv}{dt}\frac{d\mathbf{r}^0}{dt},$$

where dr/dt and rdv/dt are the components of $\dot{\mathbf{r}}$ along the radius vector and perpendicular to it, respectively. It is important here to keep in mind that the magnitude of $\dot{\mathbf{r}}$ is *not* equal to that of dr/dt, but rather

$$|\dot{\mathbf{r}}|=\left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{dv}{dt}\right)^2}.$$

So, writing $\mathbf{r}$ in terms of its components, as above, we have

$$\mathbf{r}\cdot\dot{\mathbf{r}}=\mathbf{r}\cdot\frac{dr}{dt}\mathbf{r}^0+\mathbf{r}\cdot r\frac{dv}{dt}\frac{d\mathbf{r}^0}{dt},$$

which gives

$$\mathbf{r}\cdot\dot{\mathbf{r}}=r\frac{dr}{dt}$$

because

$$\mathbf{r}\cdot\frac{d\mathbf{r}^0}{dt}=0\left[\mathbf{r}\text{ is perpendicular to }\frac{d\mathbf{r}^0}{dt}\right]$$

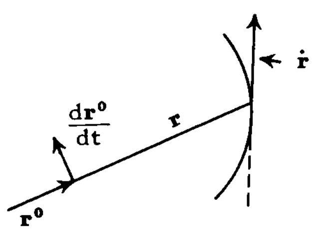
and

$$\mathbf{r}\cdot\mathbf{r}^0=r.$$

Therefore,

$$\mathbf{R}\times\frac{\mathbf{r}}{r^3}=\frac{h}{r^2}\left(r\dot{\mathbf{r}}-\mathbf{r}\frac{dr}{dt}\right). \tag{18}$$

We should again avoid confusion by remembering that in our notation,



the radius vector is $\mathbf{r}$, the velocity vector is $\dot{\mathbf{r}}$, and the unit vector along $\mathbf{r}$ is $\mathbf{r}^0$. Clearly, too, $\dot{\mathbf{r}}$ is tangent to the path. That is, $d\mathbf{r}^0/dt$ is perpendicular to the radius vector, whereas $\dot{\mathbf{r}}$ is not. So, we have from Equation 18

$$\mathbf{R}\times\frac{\mathbf{r}}{r^3}=\frac{h}{r^2}\,r\dot{\mathbf{r}}-\frac{h}{r^2}\,\mathbf{r}\,\frac{dr}{dt}$$

$$=\frac{h}{r}\,\dot{\mathbf{r}}-\frac{h}{r^2}\,\mathbf{r}\,\frac{dr}{dt}.$$

But we notice that

$$h\,\frac{d}{dt}\,(\mathbf{r}^0)=h\,\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

$$=\frac{h}{r}\,\frac{d\mathbf{r}}{dt}-\frac{h}{r^2}\,\mathbf{r}\,\frac{dr}{dt}.$$

Therefore,

$$\mathbf{R}\times\frac{\mathbf{r}}{r^3}=h\,\frac{d\mathbf{r}^0}{dt}$$

and we can write Equation 17 in the form

$$\mathbf{R}\times\ddot{\ }=-h\,\frac{d\mathbf{r}^0}{dt}. \tag{19}$$

Integrating Equation 19 and knowing that

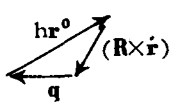
$$\int(\mathbf{R}\times\ddot{\mathbf{r}})=\mathbf{R}\times\dot{\mathbf{r}}-\int\dot{\mathbf{R}}\times\dot{\mathbf{r}},$$

where $\dot{\mathbf{R}}=0$ we have

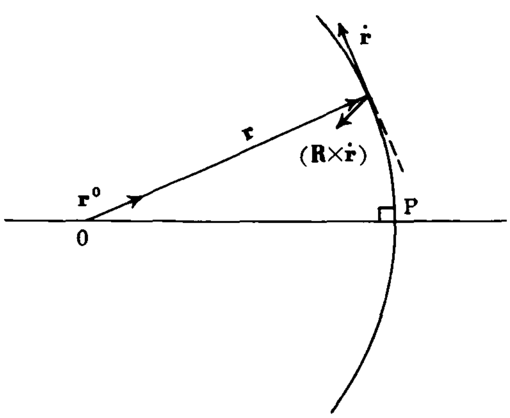
$$\mathbf{R}\times\dot{\mathbf{r}}+h\mathbf{r}^0+\mathbf{q}=0, \tag{20}$$

where $\mathbf{q}$ is a constant of integration.

Now, with Equation 20, known as the Laplacian integral, we have a closed vector triangle



with $h\mathbf{r}^0$, $(\mathbf{R}\times\dot{\mathbf{r}})$, and the constant vector $\mathbf{q}$. To find the direction of $\mathbf{q}$ it is convenient to use the following sketch:



Taking the case where $\mathbf{r}$ is in the direction of P, the perigee (by definition, the point closest to the origin), we see the path is perpendicular to the line OP at the point P. Since the velocity vector $\dot{\mathbf{r}}$ is always tangent to the path, in this case $\dot{\mathbf{r}}$ must be perpendicular to OP, and, therefore, perpendicular to the vector $\mathbf{r}$ lying along OP. Then, if $\dot{\mathbf{r}}$ is perpendicular to $\mathbf{r}$, the vector $(\mathbf{R}\times\dot{\mathbf{r}})$ must lie along $\mathbf{r}$, directed inward toward the origin. So, in the case where $\mathbf{r}$ is along OP, we have both the $(\mathbf{R}\times\dot{\mathbf{r}})$ and $h\mathbf{r}^0$ vectors lying along the same line. Therefore, in order for the Laplacian integral to hold true, $\mathbf{q}$ must also lie along the line OP and we can write it

$$\mathbf{q}=he\mathbf{P},$$

in all cases, where $\mathbf{P}$ is the unit vector along OP. From this it can be shown that e is the eccentricity of the ellipse. Now, we know that $c=1/h$ (and is therefore a function of the elements a and e); and

$$\mathbf{c}=c\mathbf{R}=\frac{1}{h}\,\mathbf{R}\text{ and }\mathbf{q}=he\mathbf{P}.$$

In applying Brown's operator,

$$\frac{\delta}{dt}=\sum_{i=1}^{6}\frac{dE_i}{dt}\,\frac{\partial}{dE_i},$$

we know that δ/dt applied to any element is equivalent to the total derivative. So from Equation 16

$$\frac{\delta}{dt}\,\mathbf{c}=\frac{\delta}{dt}\,(\mathbf{r}\times\dot{\mathbf{r}}),$$

which gives

$$\frac{d\mathbf{c}}{dt}=\left(\frac{\delta}{dt}\,\mathbf{r}\right)\times\dot{\mathbf{r}}+\mathbf{r}\times\frac{\delta\dot{\mathbf{r}}}{dt}; \tag{21}$$

but we know from Equations 10 and 11,

$$\frac{\delta}{dt}\,\mathbf{r}=0$$

and

$$\frac{\delta}{dt}\,\dot{\mathbf{r}}=\mathbf{F},$$

so we have

$$\frac{d\mathbf{c}}{dt}=\mathbf{r}\times\mathbf{F}. \tag{22}$$

Now,

$$\frac{d\mathbf{c}}{dt}=\frac{d(c\mathbf{R})}{dt}=\mathbf{R}\,\frac{dc}{dt}+c\,\frac{d\mathbf{R}}{dt}=\mathbf{r}\times\mathbf{F}.$$

Multiplying through by $\mathbf{R}\times$ and h, we have

$$h\left(\mathbf{R}\times\mathbf{R}\frac{dc}{dt}\right)+h\left(\mathbf{R}\times c\,\frac{d\mathbf{R}}{dt}\right)=h\mathbf{R}\times(\mathbf{r}\times\mathbf{F}); \tag{23}$$

but $c=1/h$, and $\mathbf{R}\times\mathbf{R}=0$, so

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=h\mathbf{R}\times(\mathbf{r}\times\mathbf{F})$$

$$=h(\mathbf{R}\cdot\mathbf{F})\mathbf{r}-h(\mathbf{R}\cdot\mathbf{r})\mathbf{F},$$

and since $\mathbf{R}\cdot\mathbf{r}=0$,

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=h(\mathbf{R}\cdot\mathbf{F})\mathbf{r}.$$

However, $\mathbf{R}\cdot\mathbf{F}$ is simply the projection of the force on the Z axis, i.e., the component normal to the plane. Therefore,

$$\mathbf{R}\cdot\mathbf{F}=\frac{\partial\Omega}{\partial Z}$$

and

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=h\frac{\partial\Omega}{\partial Z}\mathbf{r}. \tag{24}$$

Since $\mathbf{R}$ is a unit vector, $d\mathbf{R}/dt$ allows only the rotation of the osculating plane about the $\mathbf{r}$ vector, since $\mathbf{r}$ is perpendicular to $\mathbf{R}$ by definition. The instantaneous angular velocity of rotation of the osculating plane, about $\mathbf{r}$, we shall designate as ψ, a vector clearly in the direction of $\mathbf{r}$. Clearly then,

$$\frac{d\mathbf{R}}{dt}=\psi\times\mathbf{R},$$

and we can write

$$\mathbf{R}\times(\psi\times\mathbf{R})=h\frac{\partial\Omega}{\partial Z}\mathbf{r}.$$

Therefore,

$$(\mathbf{R}\cdot\mathbf{R})\psi-(\mathbf{R}\cdot\psi)\mathbf{R}=h\frac{\partial\Omega}{\partial Z}\mathbf{r}.$$

But, $\mathbf{R}\cdot\mathbf{R}=1$ and $\mathbf{R}\cdot\psi=0$ since ψ is in the direction of $\mathbf{r}$, so

$$\psi=h\frac{\partial\Omega}{\partial Z}\mathbf{r}. \tag{25}$$

This gives ψ, the rotation of the orbit plane with the plane considered to be a rigid body. However, if we have an ellipse in the osculating plane, it is allowed another motion, a rotation *in* the plane, if we disregard changes in the shape of the ellipse. If we consider the angle π between the x axis and the line from origin to perigee, then the rotation of the ellipse about $\mathbf{R}$ normal to the plane of the ellipse is clearly $(d\pi/dt)\mathbf{R}$. Therefore, the total motion of the ellipse is given by the rotation of the plane in space plus the rotation of the ellipse in the plane, and is

$$h\frac{\partial\Omega}{\partial Z}\mathbf{r}+\frac{d\pi}{dt}\mathbf{R}.$$

Considering Figure 2, we can attach three unit vectors to the osculating plane: $\mathbf{m}$ is the direction of the node; $\mathbf{K}$ is perpendicular to $\mathbf{m}$ and to the reference plane; $\mathbf{R}$ is perpendicular to the osculating plane and to $\mathbf{m}$. Using these three unit vectors, we can resolve the total rotation of the ellipse into three motions:

1. the rotation about $\mathbf{m}$, which is di/dt,
2. the rotation about $\mathbf{K}$, which is $d\theta/dt$,
3. the rotation about $\mathbf{R}$, which is $d\omega/dt$.

The sum of the three rotation vectors must equal the total rotational motion of the ellipse:

$$h\frac{\partial\Omega}{\partial Z}\mathbf{r}+\frac{d\pi}{dt}\mathbf{R}=\frac{di}{dt}\mathbf{m}+\frac{d\theta}{dt}\mathbf{K}+\frac{d\omega}{dt}\mathbf{R}. \tag{26}$$

Multiplying Equation 26 through by $\mathbf{m}\cdot$, we get

$$h\frac{\partial\Omega}{\partial Z}(\mathbf{m}\cdot\mathbf{r})+\frac{d\pi}{dt}(\mathbf{m}\cdot\mathbf{R})$$
$$=\frac{di}{dt}(\mathbf{m}\cdot\mathbf{m})+\frac{d\theta}{dt}(\mathbf{m}\cdot\mathbf{K})+\frac{d\omega}{dt}(\mathbf{m}\cdot\mathbf{R}). \tag{27}$$

But $\mathbf{m}\cdot\mathbf{m}=1$, $\mathbf{m}\cdot\mathbf{K}=0$, and $\mathbf{m}\cdot\mathbf{R}=0$; also, $\mathbf{m}\cdot\mathbf{r}=|\mathbf{m}||\mathbf{r}|\cos(f+\omega)=r\cos(v-\sigma)$. (See Figure 2.) Therefore, we have a final explicit expression for the first derivative of the angle of inclination:

$$hr\frac{\partial\Omega}{\partial Z}\cos(v-\sigma)=\frac{di}{dt}. \tag{28}$$

DERIVATION OF $\frac{d\theta}{dt}$

Multiplying Equation 26 through by $\cdot(\mathbf{m}\times\mathbf{R})$ we get

$$h\frac{\partial\Omega}{\partial Z}\mathbf{r}\cdot(\mathbf{m}\times\mathbf{R})+\frac{d\pi}{dt}\mathbf{R}\cdot(\mathbf{m}\times\mathbf{R})$$
$$=\frac{di}{dt}\mathbf{m}\cdot(\mathbf{m}\times\mathbf{R})+\frac{d\theta}{dt}\mathbf{K}\cdot(\mathbf{m}\times\mathbf{R})+\frac{d\omega}{dt}\mathbf{R}\cdot(\mathbf{m}\times\mathbf{R}). \tag{29}$$

But $(\mathbf{m}\times\mathbf{R})$ is clearly perpendicular both to $\mathbf{R}$ and to $\mathbf{m}$, so both $\mathbf{R}\cdot(\mathbf{m}\times\mathbf{R})$ and $\mathbf{m}\cdot(\mathbf{m}\times\mathbf{R})$ will simply equal 0, and we have

$$h\frac{\partial\Omega}{\partial Z}\mathbf{r}\cdot(\mathbf{m}\times\mathbf{R})=\frac{d\theta}{dt}\mathbf{K}\cdot(\mathbf{m}\times\mathbf{R}). \tag{30}$$

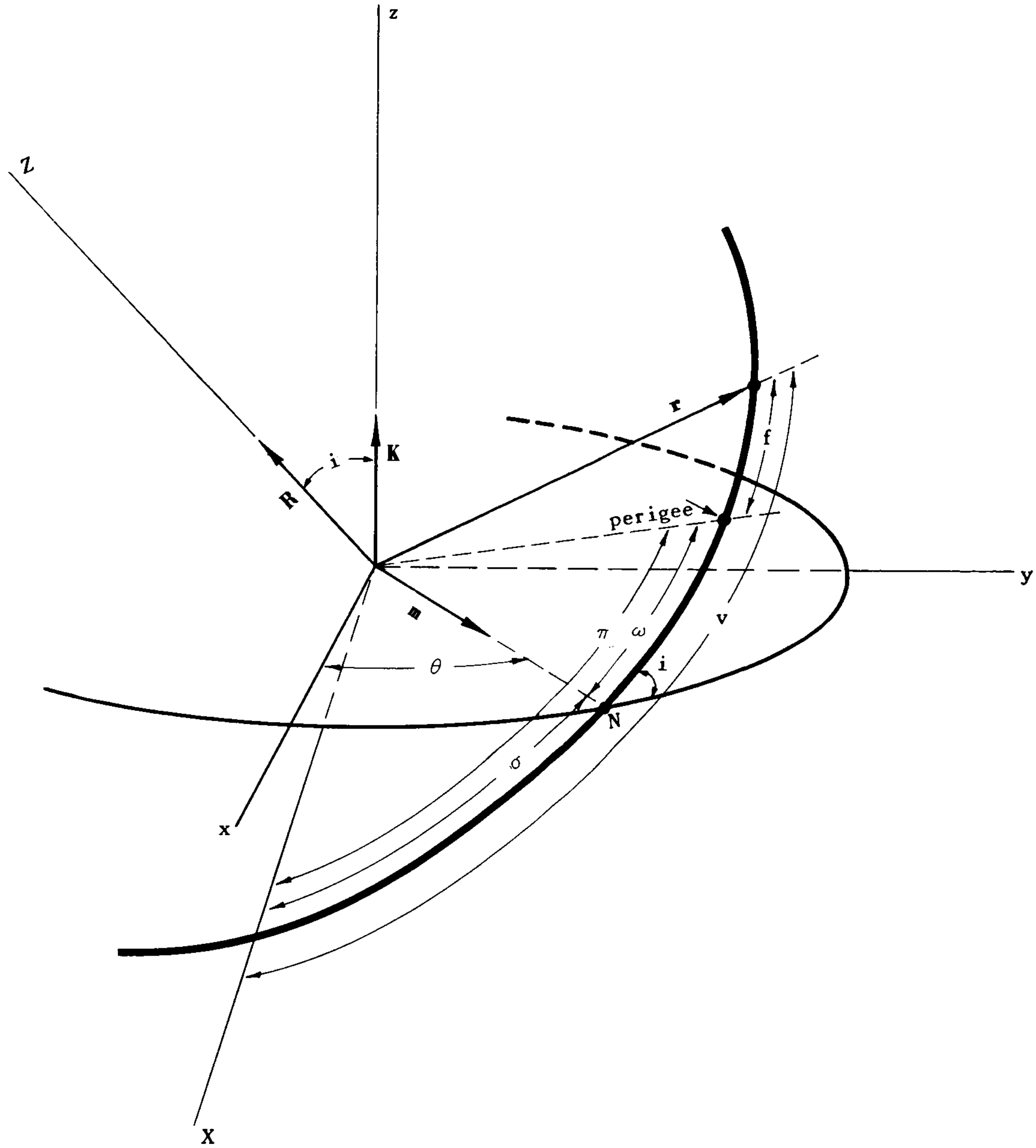


FIGURE 2.—Rotation of the ellipse.

Now, $\mathbf{K}\cdot(\mathbf{m}\times\mathbf{R})=\mathbf{m}\cdot(\mathbf{R}\times\mathbf{K})$. And since **R** and **K** are unit vectors, $\mathbf{R}\times\mathbf{K}=-(\mathbf{K}\times\mathbf{R})=-(\sin i)\mathbf{m}$ (see Figure 2), so

$$h\frac{\partial\Omega}{\partial Z}\mathbf{r}\cdot(\mathbf{m}\times\mathbf{R})=-\frac{d\theta}{dt}\sin i. \tag{31}$$

To write the triple product $\mathbf{r}\cdot(\mathbf{m}\times\mathbf{R})$ as a scalar function of r, v, and σ, we can temporarily insert a set of orthogonal axes x', y', z' along which lie the unit vectors $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{l}'$ respectively. Placing these axes so that the unit vector **m** falls along x', and **R** falls along z', we can write:

$$\mathbf{r}=r\cos(v-\sigma)\mathbf{i}'+r\sin(v-\sigma)\mathbf{j}',$$
$$\mathbf{m}=\mathbf{i}',$$
$$\mathbf{R}=\mathbf{l}'.$$

Using the standard form of solution of a triple product, we have

$$\mathbf{r}\cdot(\mathbf{m}\times\mathbf{R})=\begin{vmatrix} r\cos(v-\sigma) & r\sin(v-\sigma) & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=-r\sin(v-\sigma),$$

and finally,

$$h\frac{\partial\Omega}{\partial Z}[-r\sin(v-\sigma)]=-\frac{d\theta}{dt}\sin i$$

or

$$\frac{d\theta}{dt}\sin i=hr\frac{\partial\Omega}{\partial Z}\sin(v-\sigma). \tag{32}$$

DERIVATION OF $\frac{d\mathbf{q}}{dt}$

We have from Equation 22

$$\frac{d\mathbf{c}}{dt}=\mathbf{r}\times\mathbf{F}.$$

But $\mathbf{c}=c\mathbf{R}$, so

$$\frac{dc}{dt}\mathbf{R}+c\frac{d\mathbf{R}}{dt}=\mathbf{r}\times\mathbf{F}. \tag{33}$$

Multiplying Equation 33 through by $\cdot\,\mathbf{R}$, we get

$$\frac{dc}{dt}\mathbf{R}\cdot\mathbf{R}+c\frac{d\mathbf{R}}{dt}\cdot\mathbf{R}=(\mathbf{r}\times\mathbf{F})\cdot\mathbf{R}=\mathbf{R}\cdot(\mathbf{r}\times\mathbf{F}).$$

But $\mathbf{R}\cdot\mathbf{R}=1$ and $d\mathbf{R}/dt\cdot\mathbf{R}=0$ since $\mathbf{R}$ is a unit vector and $d\mathbf{R}/dt$ must be perpendicular to it. Thus,

$$\frac{dc}{dt}=\mathbf{R}\cdot(\mathbf{r}\times\mathbf{F}). \tag{34}$$

Then, since

$$c=\frac{1}{h}\text{ and }\frac{dc}{dt}=-\frac{1}{h^2}\frac{dh}{dt},$$

we have

$$\frac{dh}{dt}=-h^2\mathbf{R}\cdot(\mathbf{r}\times\mathbf{F}). \tag{35}$$

Now, if we take the Laplacian Integral

$$\mathbf{R}\times\frac{d\mathbf{r}}{dt}+h\mathbf{r}^0+\mathbf{q}=0\quad\text{(Equation 20)}$$

and apply Brown's operator to it, we have

$$\frac{\delta}{dt}\mathbf{R}\times\frac{d\mathbf{r}}{dt}+\mathbf{R}\times\frac{\delta}{dt}\frac{d\mathbf{r}}{dt}+\mathbf{r}^0\frac{\delta h}{dt}+h\frac{\delta\mathbf{r}^0}{dt}+\frac{\delta\mathbf{q}}{dt}=0. \tag{36}$$

But, we know that in an ideal system

$$\frac{\delta}{dt}\mathbf{R}=0, \tag{37}$$

because $\mathbf{R}$, a unit vector always normal to the XY plane, does not depend upon the elements, and Brown's operator gives the dependence of a function upon the osculating elements. We know, also, that since the XY plane is always the plane of the orbit, containing both the position and velocity vectors, there can be no component of the disturbing force normal to the XY plane; if there were, the orbiting body would move out of the plane. Thus, if we write the force

$$\mathbf{F}=(\mathbf{F})+\frac{\partial\Omega}{\partial Z}\mathbf{k}',$$

where

$$(\mathbf{F})=\frac{\partial\Omega}{\partial X}\mathbf{i}'+\frac{\partial\Omega}{\partial Y}\mathbf{j}',$$

with $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ unit vectors along X, Y, and Z, respectively, we see that $\mathbf{F}=(\mathbf{F})$ because $\partial\Omega/\partial Z$ does not appear explicitly in an ideal system. So, Equation 11 can be written:

$$\frac{\delta}{dt}\dot{\mathbf{r}}=\frac{\delta}{dt}\frac{d\mathbf{r}}{dt}=(\mathbf{F}). \tag{38}$$

Further, because h and $\mathbf{q}$ are functions of the elements alone, we have

$$\frac{\delta\mathbf{q}}{dt}=\frac{d\mathbf{q}}{dt}\text{ and }\frac{\delta h}{dt}=\frac{dh}{dt} \tag{39}$$

and, as a consequence of Equation 10,

$$\frac{\delta\mathbf{r}^0}{dt}=0. \tag{40}$$

Substitution of Equations 37 through 40 into Equation 36 yields:

$$\mathbf{R}\times(\mathbf{F})+\mathbf{r}^0\frac{dh}{dt}+\frac{d\mathbf{q}}{dt}=0, \tag{41}$$

where we have from Equation 35

$$\frac{dh}{dt}=-h^2\mathbf{R}\cdot(\mathbf{r}\times\mathbf{F})=-h^2\mathbf{F}\cdot(\mathbf{R}\times\mathbf{r}). \tag{42}$$

Now, considering the unit vector $\mathbf{n}^0$ which lies in the orbit plane and is perpendicular to $\mathbf{r}$, we know that $\mathbf{n}^0=(\mathbf{R}\times\mathbf{r}^0)$ and that $(\mathbf{R}\times\mathbf{r})=r\mathbf{n}^0$

which we put into Equation 42. Equation 41 thus becomes:

$$\mathbf{R}\times(\mathbf{F})-\mathbf{r}^0 h^2 r\mathbf{F}\cdot\mathbf{n}^0+\frac{d\mathbf{q}}{dt}=0. \tag{43}$$

If we write $(\mathbf{F})$ in terms of its components in polar form, along the radius and normal to the radius, where v is the polar angle, we have

$$\mathbf{F}=\frac{\partial\Omega}{\partial r}\,\mathbf{r}^0+\frac{1}{r}\,\frac{\partial\Omega}{\partial v}\,\mathbf{n}^0.$$

Putting this into the first term of Equation 43 gives:

$$\frac{\partial\Omega}{\partial r}\,(\mathbf{R}\times\mathbf{r}^0)+\frac{1}{r}\,\frac{\partial\Omega}{\partial v}\,(\mathbf{R}\times\mathbf{n}^0)-h^2 r(\mathbf{F}\cdot\mathbf{n}^0)\mathbf{r}^0+\frac{d\mathbf{q}}{dt}=0, \tag{44}$$

and since $(\mathbf{R}\times\mathbf{r}^0)=\mathbf{n}^0$ and $(\mathbf{R}\times\mathbf{n}^0)=-\mathbf{r}^0$, we write

$$\frac{\partial\Omega}{\partial r}\,\mathbf{n}^0-\left(\frac{1}{r}\,\frac{\partial\Omega}{\partial v}+h^2 r(\mathbf{F}\cdot\mathbf{n}^0)\right)\mathbf{r}^0+\frac{d\mathbf{q}}{dt}=0. \tag{45}$$

But $\mathbf{F}=(\mathbf{F})$, and thus

$$(\mathbf{F}\cdot\mathbf{n}^0)=\frac{\partial\Omega}{\partial r}\,\mathbf{r}^0\cdot\mathbf{n}^0+\frac{1}{r}\,\frac{\partial\Omega}{\partial v}\,\mathbf{n}^0\cdot\mathbf{n}^0$$

or

$$(\mathbf{F}\cdot\mathbf{n}^0)=\frac{1}{r}\,\frac{\partial\Omega}{\partial v},$$

since $\mathbf{n}^0\cdot\mathbf{n}^0=1$ and $\mathbf{r}^0\cdot\mathbf{n}^0=0$. Therefore, Equation 45 becomes

$$\frac{\partial\Omega}{\partial r}\,\mathbf{n}^0-\frac{1}{r}\,\frac{\partial\Omega}{\partial v}\,(1+h^2 r)\mathbf{r}^0=-\frac{d\mathbf{q}}{dt}$$

or

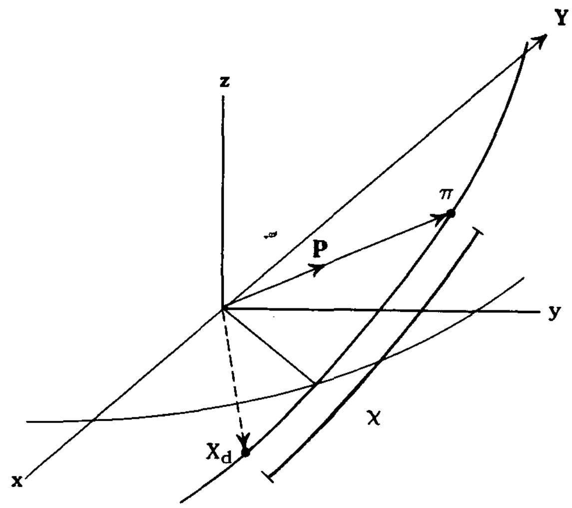
$$\frac{d\mathbf{q}}{dt}=(\mathbf{r}^0\times\mathbf{R})\,\frac{\partial\Omega}{\partial r}+\mathbf{r}^0\left(\frac{1}{r}+h^2\right)\frac{\partial\Omega}{\partial v}. \tag{46}$$

In the inertial coordinate system, the rotational term $\boldsymbol{\omega}\times\mathbf{q}$ would have to be added to this expression.

At this point, we should recall that $\mathbf{q}$ is a vector directed from the origin to the perigee, and is given by

$$\mathbf{q}=he\mathbf{P},$$

where $\mathbf{P}$ is the unit vector in the direction of the perigee. From the sketch



where χ is the argument of perigee as measured from the departure point X_d, and $\mathbf{i}'$ and $\mathbf{j}'$ are unit vectors along X and Y, respectively, we see that

$$\mathbf{P}=\mathbf{i}'\cos\chi+\mathbf{j}'\sin\chi$$

and

$$\mathbf{q}=he(\mathbf{i}'\cos\chi+\mathbf{j}'\sin\chi). \tag{47}$$

Using this last expression for $\mathbf{q}$ and the final form of $d\mathbf{q}/dt$, we could readily derive the classical equations for $d/dt(he\cos\chi)$ and $d/dt(he\sin\chi)$. However, these are not used in the Musen development.

We now have formed three of the fundamental equations of celestial mechanics which are used in Musen's development. They are:

$$\left.\begin{aligned}\frac{di}{dt}&=hr\,\frac{\partial\Omega}{\partial Z}\cos(v-\sigma),\\ \sin i\,\frac{d\theta}{dt}&=hr\,\frac{\partial\Omega}{\partial Z}\sin(v-\sigma),\\ \frac{d\mathbf{q}}{dt}&=(\mathbf{r}^0\times\mathbf{R})\,\frac{\partial\Omega}{\partial r}+\mathbf{r}^0\left(\frac{1}{r}+h^2\right)\frac{\partial\Omega}{\partial v}.\end{aligned}\right\} \tag{48}$$

In addition to these, we will require two relationships which are classical results; however, they are valid only in the ideal coordinate system such as the one Hansen devised.

PROOF THAT $\frac{d\sigma}{dt}=\frac{d\theta}{dt}\cos i$ IN THE IDEAL SYSTEM

It is apparent from Figure 1 that we can divide the rotational motion of the orthogonal XYZ system into the rotation components along three axes: the z axis normal to the reference plane, the

Z axis normal to the osculating XY plane, and the m axis along the line of intersection of the two planes.

If we examine the motion of the X axis, we see it has a rotation component about z which is $d\theta/dt$. Similarly it has a rotation component about Z which is $-d\sigma/dt$, and one about m which is di/dt. Therefore, if we let $\mathbf{z}^0$, $\mathbf{Z}^0$, and $\mathbf{m}^0$ be unit vectors along z, Z, and m respectively, we can write the total rotation velocity of the X axis, $\boldsymbol{\omega}$, as

$$\boldsymbol{\omega}=\frac{d\theta}{dt}\,\mathbf{z}^0-\frac{d\sigma}{dt}\,\mathbf{Z}^0+\frac{di}{dt}\,\mathbf{m}^0; \tag{49}$$

but

$$\mathbf{z}^0\,\frac{d\theta}{dt}=\frac{d\theta}{dt}\,(\mathbf{Z}^0\cos i+\mathbf{n}^0\sin i),$$

where $\boldsymbol{n}^0=\mathbf{Z}^0\times\boldsymbol{m}^0$, that is, $\boldsymbol{n}^0$ is a unit vector in the direction of n, which is normal to m and in the XY plane. Note that Z is perpendicular to n and that Z, z, and n are all in the same plane; thus we can write

$$\boldsymbol{\omega}=\left(\frac{d\theta}{dt}\cos i-\frac{d\sigma}{dt}\right)\mathbf{Z}^0+\left(\frac{d\theta}{dt}\sin i\right)\mathbf{n}^0+\frac{di}{dt}\,\mathbf{m}^0.$$

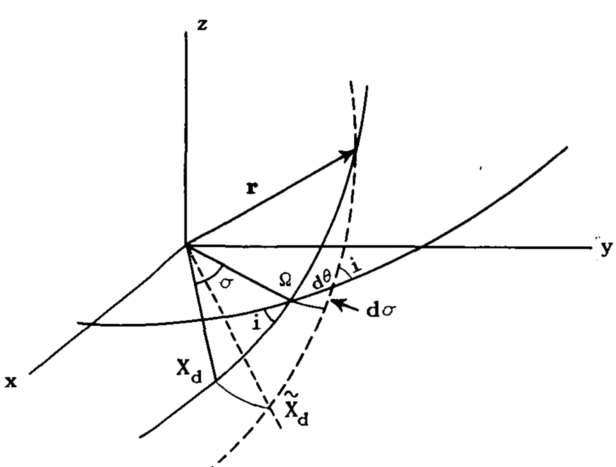
This gives the rotational velocity of the X axis in terms of three *orthogonal* components. Now, the definition of an ideal system is that the X and Y axes have no rotation about the Z axis. Therefore, we must have

$$\frac{d\theta}{dt}\cos i-\frac{d\sigma}{dt}=0$$

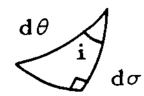
or

$$\boxed{\frac{d\sigma}{dt}=\frac{d\theta}{dt}\cos i.} \tag{50}$$

This conclusion can also be reached geometrically:



In our ideal system, we allow rotation of the XY plane only about the radius vector $\boldsymbol{r}$, in which case the plane, in some time dt, will be displaced to the dotted line. The departure point will be displaced to $\tilde{X}_d$, and the line between X_d and $\tilde{X}_d$ will be perpendicular to both the XY plane and the displaced XY plane. Thus, σ will be increased by $d\sigma$, and θ by $d\theta$. And we will have the right triangle



in which $d\sigma=(d\theta)\cos i$. Therefore, since $i+di\rightarrow i$ as $di\rightarrow 0$, we see that

$$\boxed{\frac{d\sigma}{dt}=\frac{d\theta}{dt}\cos i.}$$

It is clearly evident that in a non-ideal system, this condition cannot hold, for in such a system, θ could be held constant (see Figure 1) and the X and Y axes could be rotated about the Z axis. In this case, $d\theta/dt=0$, but $d\sigma/dt\neq 0$, so

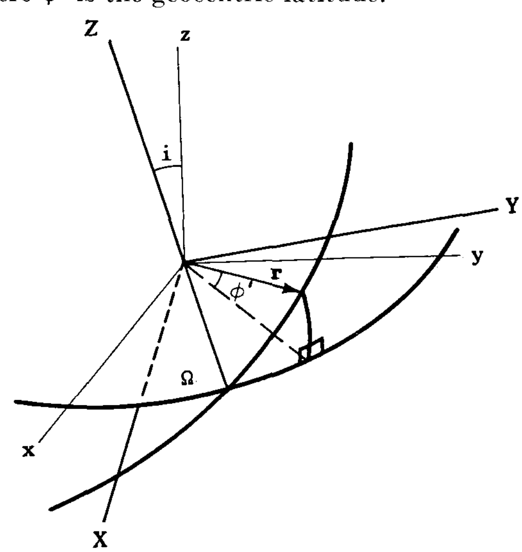
$$\frac{d\sigma}{dt}\neq\frac{d\theta}{dt}\cos i.$$

PROOF THAT IN THE IDEAL SYSTEM $r\,\frac{\partial\Omega}{\partial Z}=\frac{\partial\Omega}{\partial\Psi}\cos i$

Inasmuch as the disturbing function Ω to be used is composed of the zonal harmonics of the earth's gravitational field (see Equation 77), Ω is symmetric with respect to the earth's axis (Z axis). Thus,

$$\Omega=\Omega(r,\,\phi'),$$

where ϕ' is the geocentric latitude.



If $\mathbf{q}^0$ is the unit vector normal to $\mathbf{r}$ and in the direction of increasing ϕ', $\partial\Omega/\partial Z$ will be given by the projection of

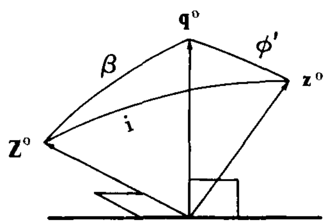
$$\frac{1}{r}\frac{\partial\Omega}{\partial\phi'}\mathbf{q}^0$$

on the Z axis. The other component of gradient Ω is in the direction of $\mathbf{r}$ which is perpendicular to the Z axis and therefore makes no contribution to $\partial\Omega/\partial Z$. Now we can write

$$\frac{\partial\Omega}{\partial Z}=\frac{1}{r}\frac{\partial\Omega}{\partial\phi'}\cos\beta,$$

where β is the angle between $\mathbf{q}^0$ and the Z axis.

From the spherical triangle determined by $\mathbf{Z}^0$, $\mathbf{z}^0$, and $\mathbf{q}^0$



where $\mathbf{Z}^0$ and $\mathbf{q}^0$ are each perpendicular to $\mathbf{r}$ and the vectors $\mathbf{q}^0$, $\mathbf{z}^0$, and $\mathbf{r}$ are coplanar, the angle β between $\mathbf{q}^0$ and $\mathbf{Z}^0$ is given by

$$\cos\beta=\frac{\cos i}{\cos\phi'}.$$

Hence,

$$\frac{\partial\Omega}{\partial Z}=\frac{\cos i}{r\cos\phi'}\frac{\partial\Omega}{\partial\phi'};$$

or, if we let $\psi=\sin\phi'$, then

$$d\psi=\cos\phi' d\phi'$$

and

$$\boxed{r\frac{\partial\Omega}{\partial Z}=\frac{\partial\Omega}{\partial\psi}\cos i.} \tag{51}$$

SECTION II

HANSEN'S COORDINATE SYSTEM AND THE AUXILIARY ELLIPSE

GENERAL OUTLINE OF THE PROCEDURE

In this section, the rotating coordinate system and the auxiliary ellipse will be introduced and discussed. At this point it seems advisable to discuss the entire problem and method of solution in order that the entire procedure be put into focus. The purpose of a theory such as Musen's is to allow analysis of the effect of forces on an artificial satellite and, from them, predict the motion and the behavior of the satellite in orbit. In this development, only the zonal harmonics of the earth's gravitational potential are taken into account, though other forces could be considered as well. We want to be able to predict the position of a satellite moving in this gravitational field, once we have established by observation its position and motion at some initial time t_0. From the initial observations of the satellite, we are able to deduce an approximation to its orbit, which will be the auxiliary ellipse. We very carefully determine this first approximation so it will have a specific motion. The process of this careful determination is essentially one of separating the secular motions from the periodic motions, both of which are caused by the *disturbing* potential of the earth. Out of this separation process come two products: (1) a set of equations which defines exactly in time the position of the fictitious satellite, and (2) a set of equations which gives exactly the relationship between the real and fictitious satellites in time. Using these two results, we are able to determine the position of the fictitious satellite at some desired future time, and then the position of the real satellite at that time.

THE COORDINATE SYSTEMS

Hansen's first step was to introduce a rotating coordinate system. He then defined the motion of the rotating system in such a way that the equations of motion were invariant in it, i.e., he made his rotating system ideal. The coordinate systems used by Musen are the same as Hansen's, so only one discussion is needed.

In the Musen development, a right-handed inertial orthogonal system xyz is introduced, with its origin at the center of the earth, its x and y axes in the earth's equatorial plane, and its z axis towards the north pole. Then a second orthogonal system, the XYZ system, is obtained by rotation through three Eulerian angles from the xyz system (see Figure 3). If the XYZ system is originally identical to xyz, its first rotation is that of the X and Y axes through an angle θ_0 about the z axis. Next, the Y and Z axes are rotated through an angle i_0 about the X axis. The third rotation is that of the X and Y axes through an angle $-\sigma_0$ about the Z axis. After these three rotations, the point where the X axis intersects the celestial sphere, named by Cayley the departure point, is the point from which all angles in the XY plane are measured (Reference 1, p. 60).

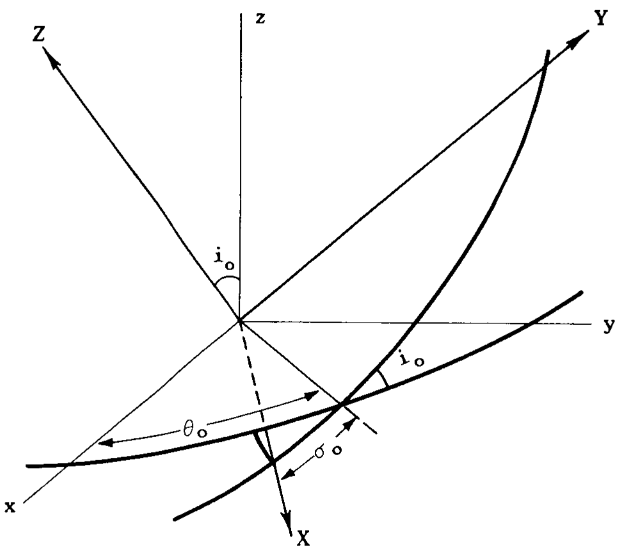


FIGURE 3.—Geometry of the departure point.

Now, with the original position of the XYZ system defined by the angles θ_0, i_0, and σ_0, we impose such conditions as to render it an ideal system. The first condition is that the XY plane be always the instantaneous plane of the satellite's orbit; that is, the XY plane *always* contains the instantaneous position vector and velocity vector of the satellite. The second condition imposed is that after the original position of the XYZ system is defined by the angles θ_0, i_0, and σ_0, the angular velocity of the system, considered as a rigid body, have a component of zero along the Z axis. These two conditions define the rotating coordinate system as ideal, and give rise to the two important relations developed at the end of Section I. It is helpful to prove this result.

PROOF THAT THE ROTATING SYSTEM IS IDEAL

The form of the equation of motion which includes the disturbing force is

$$\ddot{\mathbf{r}}=\frac{-\mathbf{r}}{r^3}+\mathbf{F}.$$

Operating with $\mathbf{r}\times$, we have

$$\mathbf{r}\times\ddot{\mathbf{r}}=-\frac{1}{r^3}(\mathbf{r}\times\mathbf{r})+\mathbf{r}\times\mathbf{F},$$

or

$$\mathbf{r}\times\ddot{\mathbf{r}}=\mathbf{r}\times\mathbf{F}, \tag{52}$$

since $(\mathbf{r}\times\mathbf{r})=0$. But

$$\mathbf{r}\times\ddot{\mathbf{r}}=\frac{d}{dt}\int\mathbf{r}\times\ddot{\mathbf{r}}=\frac{d}{dt}(\mathbf{r}\times\dot{\mathbf{r}}),$$

where in all cases

$$\mathbf{r}\times\dot{\mathbf{r}}=\frac{\mathbf{R}}{h}$$

(Equation 16) in which $1/h$ is twice the area swept out per unit time. So

$$\mathbf{r}\times\ddot{\mathbf{r}}=\frac{d}{dt}\left(\frac{\mathbf{R}}{h}\right)=\mathbf{R}\frac{d}{dt}\left(\frac{1}{h}\right)+\frac{1}{h}\frac{d\mathbf{R}}{dt},$$

and Equation 52 becomes

$$\mathbf{R}\frac{d}{dt}\left(\frac{1}{h}\right)+\frac{1}{h}\frac{d\mathbf{R}}{dt}=\mathbf{r}\times\mathbf{F}. \tag{53}$$

Operating with $\mathbf{R}\cdot$ gives

$$\mathbf{R}\cdot\mathbf{R}\frac{d}{dt}\left(\frac{1}{h}\right)+\mathbf{R}\cdot\frac{d\mathbf{R}}{dt}\left(\frac{1}{h}\right)=\mathbf{R}\cdot(\mathbf{r}\times\mathbf{F}). \tag{54}$$

But $\mathbf{R}\cdot\mathbf{R}=1$ and $\mathbf{R}\cdot d\mathbf{R}/dt=0$ since $\mathbf{R}$ is a unit vector, so Equation 54 becomes

$$\frac{d}{dt}\left(\frac{1}{h}\right)=\mathbf{R}\cdot(\mathbf{r}\times\mathbf{F}). \tag{55}$$

Now, operating on Equation 53 with $\mathbf{R}\times$ gives

$$\mathbf{R}\times\mathbf{R}\frac{d}{dt}\left(\frac{1}{h}\right)+\mathbf{R}\times\frac{d\mathbf{R}}{dt}\left(\frac{1}{h}\right)=\mathbf{R}\times(\mathbf{r}\times\mathbf{F}),$$

yielding

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=h\mathbf{R}\times(\mathbf{r}\times\mathbf{F}).$$

Expanding the triple cross product, we have

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=h[\mathbf{r}(\mathbf{R}\cdot\mathbf{F})-\mathbf{F}(\mathbf{R}\cdot\mathbf{r})].$$

But $\mathbf{R}\cdot\mathbf{r}=0$, so

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=h\mathbf{r}(\mathbf{R}\cdot\mathbf{F}). \tag{56}$$

Operating on Equation 56 with $\mathbf{R}\times$ and considering that

$$\mathbf{R}\times\left(\mathbf{R}\times\frac{d\mathbf{R}}{dt}\right)=-\frac{d\mathbf{R}}{dt},$$

we have

$$\frac{d\mathbf{R}}{dt}=h(\mathbf{R}\cdot\mathbf{F})(\mathbf{r}\times\mathbf{R}). \tag{57}$$

We know $d\mathbf{R}/dt$ is perpendicular to $\mathbf{R}$, so $(\mathbf{R}\times d\mathbf{R}/dt)$ is the vector about which we would rotate $\mathbf{R}$ to get $d\mathbf{R}/dt$. From Equation 56, this vector is $h\mathbf{r}(\mathbf{R}\cdot\mathbf{F})$, and so we see that all rotation of $\mathbf{R}$ is about $\mathbf{r}$, a very informative result. We will let this rotation vector be $\boldsymbol{\omega}$, so

$$\boldsymbol{\omega}=h(\mathbf{R}\cdot\mathbf{F})\mathbf{r}.$$

Since the limitations imposed on the rotating coordinate system allow it to have only this rotation around the instantaneous radius vector, $\boldsymbol{\omega}$ represents the *total* rotation of the rotating coordinate system. The differential operator which takes into account the rotation $\boldsymbol{\omega}$ of one system with respect to another is

$$\frac{d}{dt}+\boldsymbol{\omega}\times.$$

Therefore, the equations of motion in the rotating system are

$$\dot{\mathbf{r}}=\frac{d\mathbf{r}}{dt}+\boldsymbol{\omega}\times\mathbf{r}, \tag{58}$$

and

$$\ddot{\mathbf{r}}=\frac{d}{dt}\dot{\mathbf{r}}+\boldsymbol{\omega}\times\dot{\mathbf{r}}$$

or

$$\ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{dt^2}+\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}+2\left(\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}\right)+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=-\frac{\mathbf{r}}{r^3}+\mathbf{F}. \tag{59}$$

When it is taken into account that in our case $\boldsymbol{\omega}=h(\mathbf{R}\cdot\mathbf{F})\mathbf{r}$, the equations of motion in the rotating system become

$$\dot{\mathbf{r}}=\frac{d\mathbf{r}}{dt}; \tag{60}$$

and since

$$\ddot{\mathbf{r}}=\frac{d\dot{\mathbf{r}}}{dt}+\boldsymbol{\omega}\times\dot{\mathbf{r}},$$

we have, using Equation 60,

$$\ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{dt^2}+h(\mathbf{R}\cdot\mathbf{F})(\mathbf{r}\times\dot{\mathbf{r}}). \tag{61}$$

But again using Equation 16, we know that

$$\mathbf{r}\times\dot{\mathbf{r}}=\frac{\mathbf{R}}{h},$$

so we get

$$\ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{dt^2}+(\mathbf{R}\cdot\mathbf{F})\mathbf{R}=-\frac{\mathbf{r}}{r^3}+\mathbf{F}. \tag{62}$$

If we write $\mathbf{F}$ in terms of its components along the X, Y, and Z axes,

$$\mathbf{F}=\frac{\partial\Omega}{\partial X}\mathbf{i}'+\frac{\partial\Omega}{\partial Y}\mathbf{j}'+\frac{\partial\Omega}{\partial Z}\mathbf{k}'$$

and let

$$(\mathbf{F})=\frac{\partial\Omega}{\partial X}\mathbf{i}'+\frac{\partial\Omega}{\partial Y}\mathbf{j}',$$

we can write

$$\mathbf{F}=(\mathbf{F})+\frac{\partial\Omega}{\partial Z}\mathbf{k}'. \tag{63}$$

But $\mathbf{R}(\mathbf{R}\cdot\mathbf{F})$ is just the component of $\mathbf{F}$ along the Z axis, because $\mathbf{R}$ is identical to $\mathbf{k}'$, so

$$\mathbf{R}(\mathbf{R}\cdot\mathbf{F})=\frac{\partial\Omega}{\partial Z}\mathbf{k}'. \tag{64}$$

Upon substitution of Equations 63 and 64 into Equation 62, we have

$$\ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{dt^2}+\frac{\partial\Omega}{\partial Z}\mathbf{k}'$$

$$=-\frac{\mathbf{r}}{r^3}+(\mathbf{F})+\frac{\partial\Omega}{\partial Z}\mathbf{k}'. \tag{65}$$

However, the motion of this rotating coordinate system is fixed in such a way that the satellite always moves in the XY plane, which means that there is, in effect, *no* force component normal to the osculating plane in the XY coordinate system. This allows us to consider

$$\frac{\partial\Omega}{\partial Z}\mathbf{k}'=0,$$

and Equation 14 becomes

$$\ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{dt^2}=-\frac{\mathbf{r}}{r^3}+(\mathbf{F}). \tag{66}$$

Thus, we see that the differential equation of motion relative to the system rigidly connected to the orbit or osculating plane is of exactly the same form as in the inertial system. Although this result may at first glance seem trivial or obvious, it is not to be expected in a rotating coordinate system and must be shown. The importance of the result cannot be minimized for it leads to a great simplification of the development.

THE AUXILIARY ELLIPSE

We now have two coordinate systems defined, the inertial system and the XYZ rotating system in which the orbit plane is always the XY plane. The standard approach in celestial mechanics is to introduce, in the plane of the orbit, some first approximation to the real orbit. This intermediary orbit is determined, and then the deviations of the real orbiting body from it are determined. Hansen's method was to introduce an ellipse of constant shape into the osculating plane of the real orbit, with a_0, e_0, and $n_0=a_0^{-3/2}$ fixed. A fictitious satellite describes this ellipse as it moves in accordance with Kepler's Laws. The ellipse is allowed only one motion in the XY plane, and that is a rotation about the Z axis, directly proportional to the eccentric anomaly of the fictitious satellite. Thus, the argument of perigee π in the XY plane is given by

$$\pi=\pi_0+yE, \tag{67}$$

where y is a constant called the secular motion of the perigee, to be determined in a specific manner. The directions and lengths of the radii vectors of the real and fictitious satellites are not identical, but differ by the order of magnitude of the perturbations. (The constants y in Equation 67 and ν in Equation 69 have nothing to do with the inertial coordinates.)

The position vectors of the real and fictitious satellites are related in space and time. The introduction of the time dimension was one of the major causes of controversy among Hansen's colleagues, though it need not be such a great obstacle. The first relationship is that the unit vector along the radius of the real satellite, denoted by $\mathbf{r}^0$, has the same direction at time t that the unit radius vector of the fictitious satellite, denoted by $\bar{\mathbf{r}}^0$, has at the "pseudotime" z. Thus,

$$\mathbf{r}^0(t)=\bar{\mathbf{r}}^0(z). \tag{68}$$

The second relationship defines the ratio of the length of the real satellite's radius vector, at time t, to the fictitious satellite's radius vector at pseudotime z as $(1+\nu)$. In this ratio, ν is small and can be considered a "lengthening" or "shortening" factor; $\mathbf{r}$ is the radius vector of the real satellite, and $\bar{\mathbf{r}}$ is the radius vector of the fictitious satellite. The ratio can be written:

$$\mathbf{r}(t)=(1+\nu)\bar{\mathbf{r}}(z). \tag{69}$$

It becomes necessary, therefore, to know the relationship between the real time and the pseudotime, or "disturbed time," as well as the factor ν, to determine the position of the real satellite once the position of the auxiliary satellite is known. Let the difference in times be defined as δz,

$$\delta z=z-t, \tag{70}$$

where δz denotes the perturbation of time. We can write one further relation between the two satellites, and that is that the polar angle v of the real satellite at time t is equal to the polar angle $(\bar{f}+\pi_0+y\Delta E)$ of the fictitious satellite at time z. In this expression, $\bar{f}$ is the true anomaly of the fictitious satellite (see Figure 4), $\Delta E=E-E_0$, where E_0 is the eccentric anomaly of the fictitious satellite at the epoch, and π_0 is the argument of perigee of the fictitious satellite at the epoch. Again, y is the "secular motion" of the perigee. Both polar angles are measured from the X axis:

$$v=\bar{f}+\pi_0+y\Delta E. \tag{71}$$

With the true anomaly of the fictitious satellite given by $\bar{f}$ and its eccentric anomaly by E, the motion of the fictitious satellite is governed by the usual two-body problem equations:

$$\bar{r}\cos\bar{f}=a_0(\cos E-e_0), \tag{72}$$

$$\bar{r}\sin\bar{f}=a_0\sqrt{1-e_0^2}\sin E, \tag{73}$$

$$\bar{r}=a_0(1-e_0\cos E), \tag{74}$$

and Kepler's equation becomes

$$E-e_0\sin E=g_0+n_0(t-t_0)+n_0\delta z. \tag{75}$$

The "area integral" for the fictitious satellite retains its usual form:

$$\bar{r}^2\frac{d\bar{f}}{dz}=\frac{1}{h_0}, \tag{76}$$

where

$$h_0=\frac{1}{\sqrt{a_0(1-e_0^2)}}$$

(see Appendix A).

From the above set of equations, the motion of the fictitious ellipse in the osculating orbit plane can be uniquely determined. The essence of the problem, then, is to determine ν and δz in order that the position of the real satellite can be found.

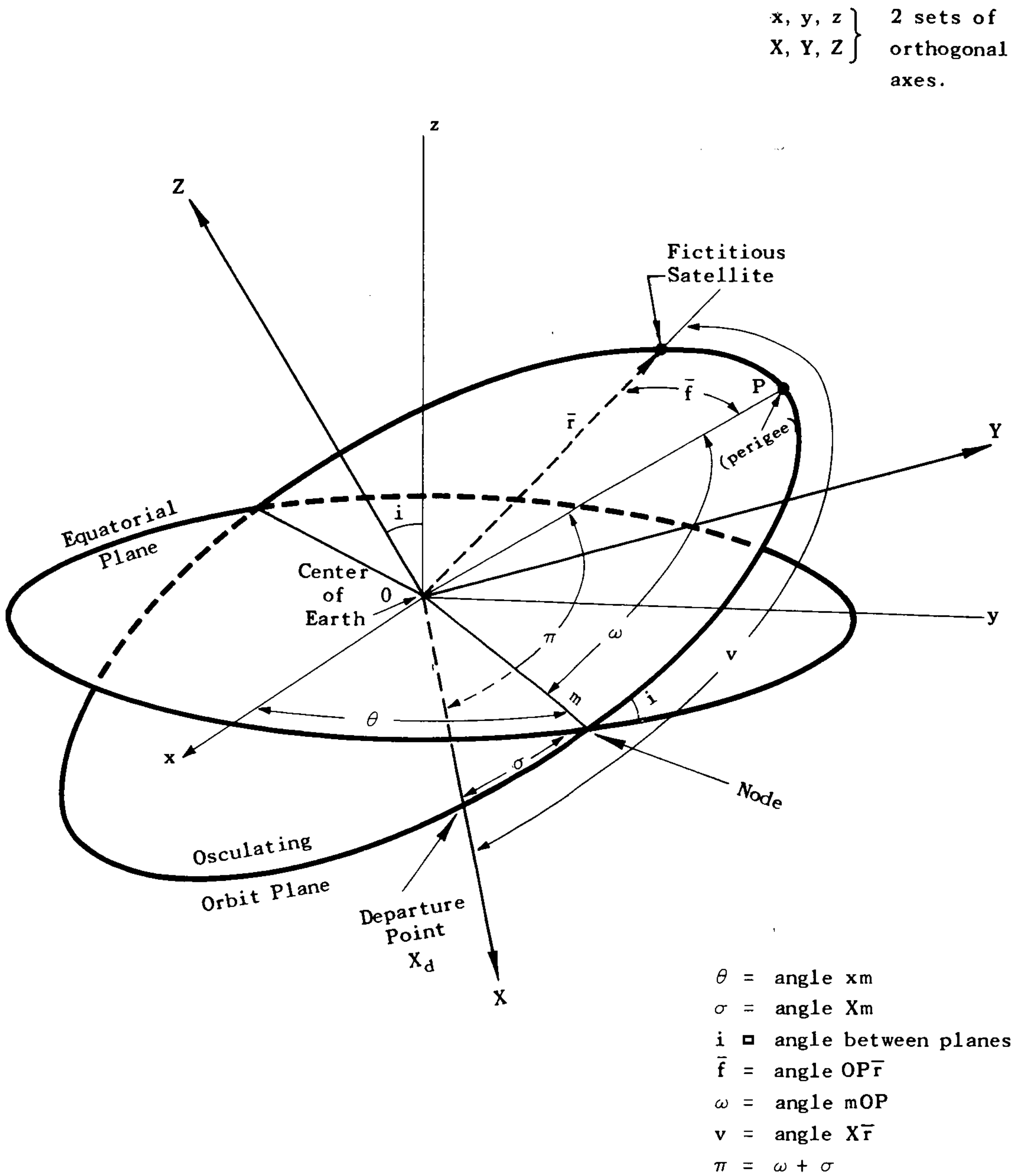


FIGURE 4.—The geometry of the auxiliary ellipse

SECTION III

THE DISTURBING POTENTIAL AND ITS PARTIAL DERIVATIVES

THE POTENTIAL FUNCTION IN TERMS OF THE GEOCENTRIC LATITUDE

The earth's disturbing potential function is defined as the negative of the difference between the earth's gravitational potential and the potential of a perfectly spherical earth of the same mass. Musen writes this disturbing potential Ω explicitly to the fourth order in zonal harmonics, where the first harmonic is obviated because the origin is taken at the center of mass. He gives

$$\Omega=\frac{k_2}{r^3}(1-3\psi^2)+\frac{k_3}{r^4}(3\psi-5\psi^3)+\frac{k_4}{r^5}(3-30\psi^2+35\psi^4), \tag{77}$$

where ψ is the sine of the geocentric latitude, and (as is shown in Appendix A) is given by

$$\psi=\sin i \sin (v-\sigma), \tag{78}$$

and k_2, k_3, and k_4 are the geodetic parameters of the earth.

As is generally true in classical celestial mechanics, it is convenient to develop the perturbation equations in terms of the partial derivatives of the disturbing function. In Musen's development, these partials should have a very particular form, that of Fourier series whose terms have arguments containing the eccentric anomaly E and the argument of perigee (measured from the node) ω. This form is required by Musen throughout the development, and makes much cumbersome algebra necessary, but the end result is that all perturbation functions can be easily handled.

In order to get ψ in this desired form, we make several transformations. The first step is to define σ_0 and θ_0, the angles which designate the original position of the XYZ system (see Figure 3), in such a way that the following two equations do not contain any constant terms:

$$2N=\sigma_0+\theta_0-\sigma-\theta-2\alpha\Delta E, \tag{79}$$

$$2K=\sigma_0-\theta_0-\sigma+\theta+2\eta\Delta E, \tag{80}$$

where $\Delta E=E-E_0$.

We will now show that N and K are periodic only. The quantities $\sigma-\sigma_0$ and $\theta-\theta_0$ by definition contain only periodic and secular terms. Secular terms are those which are proportional to time. If the secular terms contained in the sum of $(\sigma-\sigma_0)$ and $(\theta-\theta_0)$ are denoted by $-2\alpha\Delta E$, then N contains periodic terms only. This definition of $2\alpha\Delta E$ determines the constant α. Similarly, by denoting the secular terms contained in the difference of $(\sigma-\sigma_0)$ and $(\theta-\theta_0)$ by $2\eta\Delta E$, the constant η is determined and K contains only periodic terms. The determinations of α, η, and one additional determination of y, the secular motion of the perigee, lead to the development of x, y, z containing periodic terms only. These requirements are equivalent to the requirement that our expression for the "perturbation of time," $n_0\delta z$, contain periodic terms only.

Rearranging Equations 79 and 80 we can write

$$\sigma=\sigma_0-(\alpha-\eta)\Delta E-(N+K), \tag{81}$$

$$\theta=\theta_0-(\alpha+\eta)\Delta E-(N-K), \tag{82}$$

in which the constant, secular, and periodic parts of σ and θ are clearly separated in that order. Then taking Equations 71 and 81 into consideration, we have

$$v-\sigma=\bar{f}+(\pi_0-\sigma_0)+(y+\alpha-\eta)\Delta E+(N+K). \tag{83}$$

We next define the mean values of these three elements, denoted by (σ), (θ), and (ω), by the following equations:

$$(\sigma)=\sigma_0-(\alpha-\eta)\Delta E, \tag{84}$$

$$(\theta)=\theta_0-(\alpha+\eta)\Delta E, \tag{85}$$

and since $\omega=v-\sigma-\bar{f}$ (see Figure 4),

$$(\omega)=(\pi_0-\sigma_0)+(y+\alpha-\eta)\Delta E. \tag{86}$$

Considering Equations 83 and 86 we can now write for Equation 78

$$\psi=\sin i \sin [\bar{f}+(\omega)+N+K]. \tag{87}$$

We now introduce four parameters, which we can see by inspection contain the periodic parts of the three elements, σ, θ, and i, and also the constant part of i. The physics of the problem allows no secular motion of the angle of inclination, so it is of no concern to us. The four parameters introduced are

$$\left.\begin{aligned}\lambda_1&=\sin\frac{i}{2}\cos N,\\ \lambda_2&=\sin\frac{i}{2}\sin N,\\ \lambda_3&=\cos\frac{i}{2}\sin K,\\ \lambda_4&=\cos\frac{i}{2}\cos K,\end{aligned}\right\}\qquad(88)$$

and clearly we have the condition that

$$\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2=1.\qquad(89)$$

Now if we expand Equation 87, we have

$$\psi=\sin i\cos(N+K)\sin[\bar{f}+(\omega)]+\sin i\sin(N+K)\cos[\bar{f}+(\omega)]\qquad(90)$$

which, after further expansion and use of the relationship

$$\sin i=2\sin\frac{i}{2}\cos\frac{i}{2},$$

leads to

$$\psi=2(\lambda_1\lambda_4-\lambda_2\lambda_3)\sin[\bar{f}+(\omega)]+2(\lambda_2\lambda_4+\lambda_1\lambda_3)\cos[\bar{f}+(\omega)],\qquad(91)$$

leaving only $\bar{f}$ and (ω) in the argument.

As Musen points out in his original paper (Reference 6), the advantage of using the four λ parameters is one of symmetry, allowing the eventual use of the neat rotation matrix. However, the parameters chosen give rise to trouble if the satellite has an inclination in the region around $\pi/2$, that is, it is a polar satellite. For this case the three Hansen parameters p, q, and s which still contain only $\bar{f}$ and (ω) in their arguments are useful. They are defined in terms of the λ parameters (Reference 6) as follows:

$$\left.\begin{aligned}p&=\frac{\lambda_1}{\lambda_4}=\left(\tan\frac{i}{2}\right)\frac{\cos N}{\cos K},\\ q&=-\frac{\lambda_2}{\lambda_4}=-\left(\tan\frac{i}{2}\right)\frac{\sin N}{\cos K},\\ s&=\frac{\lambda_3}{\lambda_4}=\tan K.\end{aligned}\right\}\qquad(92)$$

This exposition will not deal with this special case, but the approach followed is exactly parallel to that using the four λ parameters.

We perform two additional transformations to arrive at a final form of ψ in which the arguments are written in terms of E and (ω). The first of these is to introduce two functions l and m which have the forms

$$l=\frac{\bar{r}}{a_0}\cos[\bar{f}+(\omega)],\qquad(93)$$

$$m=\frac{\bar{r}}{a_0}\sin[\bar{f}+(\omega)].\qquad(94)$$

Considering that

$$\bar{r}=a_0(1-e_0\cos E),\qquad\text{(Equation 74)}$$

$$\bar{r}\cos\bar{f}=a_0(\cos E-e_0),\qquad\text{(Equation 72)}$$

$$\bar{r}\sin\bar{f}=a_0\sqrt{1-e_0^2}\sin E,\qquad\text{(Equation 73)}$$

we can write

$$l=\frac{1}{2}\left(1+\sqrt{1-e_0^2}\right)\cos[E+(\omega)]+\frac{1}{2}\left(1-\sqrt{1-e_0^2}\right)\cos[E-(\omega)]-e_0\cos(\omega)\qquad(95)$$

and

$$m=\frac{1}{2}\left(1+\sqrt{1-e_0^2}\right)\sin[E+(\omega)]-\frac{1}{2}\left(1-\sqrt{1-e_0^2}\right)\sin[E-(\omega)]-e_0\sin(\omega).\qquad(96)$$

Substituting Equations 93 and 94 into Equation 91, we have for ψ

$$\psi=2\left(\frac{a_0}{\bar{r}}\right)(\lambda_1\lambda_4-\lambda_2\lambda_3)\,m+2\left(\frac{a_0}{\bar{r}}\right)(\lambda_2\lambda_4+\lambda_1\lambda_3)\,l,\qquad(97)$$

where m and l have been expressed in trigonometric series whose arguments are in terms of E and (ω). We will next find such a form for $a_0/\bar{r}$ and in the course of the development will show how the λ parameters evolve into Fourier series in E and (ω).

DEVELOPMENT OF $\frac{a_0}{\bar{r}}$ IN SERIES FORM

Brown and Shook (Reference 7, page 70) describe the development of $(a_0/\bar{r})^n$ into a Fourier series in E, but for our purposes only the first power of the expression is needed. First we define two functions ϕ and β:

$$\phi=\sin^{-1} e_0 \text{ or } e_0=\sin \phi \ (e_0 \text{ is eccentricity}), \quad (98)$$

and

$$\beta=\tan\frac{\varphi}{2}=\frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}. \quad (99)$$

Hence,

$$\cos\varphi=\frac{1-\beta^2}{1+\beta^2} \text{ and } e_0=\frac{2\beta}{1+\beta^2}. \quad (100)$$

Using Equation 100, we know that

$$\frac{a_0}{r}=\frac{1}{1-e_0\cos E}=\frac{1+\beta^2}{1+\beta^2-2\beta\cos E}.$$

Writing $\cos E=(e^{iE}+e^{-iE})/2$, where e is the base of natural logarithms, and rearranging, we have

$$\frac{a_0}{r}=\frac{(1+\beta^2)e^{iE}}{(e^{iE}-\beta)(1-\beta e^{iE})}.$$

This can be expressed as the sum of partial fractions in the usual manner, and for convenience, since β lies between zero and one, can be written

$$\frac{a_0}{r}=\frac{1+\beta^2}{1-\beta^2}\left[\frac{1}{1-\beta e^{-iE}}+\frac{\beta e^{iE}}{1-\beta e^{iE}}\right]$$

$$=\frac{1+\beta^2}{1-\beta^2}\left[[1+\beta e^{-iE}+\beta^2 e^{-2iE}+\ldots\right.$$
$$\left.+\beta e^{iE}\ (1+\beta e^{iE}+\beta^2 e^{2iE}+\ldots)\right]$$

$$=\frac{2(1+\beta^2)}{1-\beta^2}\left[\frac{1}{2}+\beta\cos E+\beta^2\cos 2E+\ldots\right].$$

But from Equation 100,

$$\beta=\frac{e_0}{1+\sqrt{1-e_0^2}} \quad (101)$$

so

$$\frac{a_0}{r}=\frac{2}{\sqrt{1-e_0^2}}\left(\frac{1}{2}+\beta\cos E+\beta^2\cos 2E+\ldots\right). \quad (102)$$

With the $a_0/\overline{r}$ series written in terms of the eccentric anomaly E, and with the λ parameters eventually developed in E and (ω), it is clear that ψ is in the final series form desired. If placed in the potential function Ω, and with $1/r$ written as

$$\frac{1}{r}=\frac{1}{a_0}\frac{a_0}{\overline{r}}\frac{1}{(1+\nu)},$$

where $1/(1+\nu)$ is a Fourier series in E and (ω) alone, the potential function Ω and its partial derivatives could be given as trigonometric series in E and (ω) alone.

SEPARATION OF THE TWO ECCENTRIC ANOMALIES

At this point in the development, we come to one of the most difficult operations, one which usually is a major obstacle to a clear understanding of the theory. The fact is that we have to distinguish the E entering into the development of the perturbations from the "elliptic" E entering into equations derived by using E as a geometric angle. This distinction is a very subtle and confusing one, and must be handled very carefully throughout the development and the computational process. As Musen points out in his paper (Reference 6), E has the usual geometrical meaning, when describing the motion of the fictitious satellite in its ellipse. However, the perturbation expressions are developed using the eccentric anomaly E as the independent variable replacing time. These two types of E must be distinguished from each other because the partial derivative of the potential function $\partial\Omega/\partial E$ is taken with respect to the "elliptic" E.

The reason the separation is made will not be found in the physics of the problem. Rather, this separation is a mathematical trick to facilitate the development of *one* differential equation which requires only *one* integration to find the perturbations in the orbit plane. The expression $\overline{W}$ containing the perturbations in the orbit plane is an explicit expression of the three variable elements, h, e, and χ, where χ is the argument of the perigee (sometimes denoted by π). However, the development of the elements in terms of the potential function provides only the derivatives of these elements. Therefore, rather than perform these integrations to find h, e, and χ which are then substituted into $\overline{W}$, it is preferable to form $d\overline{W}/dE$ which is written explicitly in terms of the three derivatives. This allows us to find $\overline{W}$ by the single integration of its derivative. In order to perform this process, $\overline{r}$ and $\overline{f}$, both functions of E which appear in the $\overline{W}$ function, are considered constant.

The method used to distinguish between the two types of eccentric anomaly is to temporarily let the "elliptic" E be F, and to consider F

constant. Replacing $\bar{r}$, f, and E by $\bar{\rho}$, $\bar{\phi}$, and F, respectively, in Equations 72, 73, and 74 we have

$$\bar{\rho}\cos\bar{\phi}=a_0(\cos F-e_0), \tag{103}$$

$$\bar{\rho}\sin\bar{\phi}=a_0\sqrt{1-e_0^2}\sin F, \tag{104}$$

$$\bar{\rho}=a_0(1-e_0\cos F). \tag{105}$$

Furthermore m and l, in the actual computation of ψ, will be written in terms of F and will be called m^* and l^*. The "star" operation means that E has been replaced by F in the proper places.

After the integration to get W, which contains E and F, the temporary substitution is dropped and the F's are replaced by E's. Throughout the development, this replacement of F by E is done by the "bar" operator; thus, $f(E)=\overline{f(F)}$. The procedure was introduced by Hansen for the same reason described here; however, his problem was to distinguish between the time t entering into $\bar{r}$ and $\bar{f}$ through z, and the time in the elements. (See Reference 8, p. 304 or Reference 1, p. 169.)

FORMATION OF THE PARTIAL DERIVATIVES OF THE POTENTIAL FUNCTION

So, in order to find the partial derivative of the potential function which will enter the development of the perturbation expression $d\bar{W}/dE$, we take ψ as given in Equation 97. In the expression used, called ψ^*, we will have $a_0/\bar{r}$, m, and l replaced by $a_0/\bar{\rho}$, m^*, and l^*, respectively. It is important to note that the λ parameters do *not* contain F, but are always expressed in terms of E. So we could write

$$\psi^*=2\left(\frac{a_0}{\bar{\rho}}\right)(\lambda_1\lambda_4-\lambda_2\lambda_3)m^*+2\left(\frac{a_0}{\bar{\rho}}\right)(\lambda_2\lambda_4+\lambda_1\lambda_3)l^*, \tag{106}$$

where a_0 / is a series in F, and m^* and l^* are series in F and (ω). After ψ^* is placed in the potential function, one further replacement is necessary to get the final form Ω^* which will be used to find the derivatives. Since the $1/r^n$ factors in the potential function are found from the form

$$\left(\frac{1}{1+\nu}\right)^n\left(\frac{1}{a_0}\right)^n\left(\frac{a_0}{\bar{r}}\right)^n,$$

$a_0/\bar{r}$ is replaced by $a_0/\bar{\rho}$, and the final form of the potential function is:

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$$\begin{aligned}\Omega^*=&\frac{k_2}{a_0^3}\frac{1}{(1+\nu)^3}\left(\frac{a_0}{\bar{\rho}}\right)^3(1-3\psi^{*2})\\&+\frac{k_3}{a_0^4}\frac{1}{(1+\nu)^4}\left(\frac{a_0}{\bar{\rho}}\right)^4(3\psi^*-5\psi^{*3})\\&+\frac{k_4}{a_0^5}\frac{1}{(1+\nu)^5}\left(\frac{a_0}{\bar{\rho}}\right)^5(3-30\psi^{*2}+35\psi^{*4});\end{aligned} \tag{107}$$

where ψ^* is given in Equation 106 with

$$\frac{a_0}{\bar{\rho}}=\frac{2}{\sqrt{1-e_0^2}}\left(\tfrac{1}{2}+\beta\cos F+\beta^2\cos 2F+\beta^3\cos 3F+\ldots\right), \tag{108}$$

$$\begin{aligned}m^*=&\tfrac{1}{2}(1+\sqrt{1-e_0^2})\sin[F+(\omega)]\\&-\tfrac{1}{2}(1-\sqrt{1-e_0^2})\sin[F-(\omega)]-e_0\sin(\omega),\end{aligned} \tag{109}$$

$$\begin{aligned}l^*=&\tfrac{1}{2}(1+\sqrt{1-e_0^2})\cos[F+(\omega)]\\&+\tfrac{1}{2}(1-\sqrt{1-e_0^2})\cos[F-(\omega)]-e_0\cos(\omega).\end{aligned} \tag{110}$$

The $(1+\nu)$ factor in Ω^*, as is true of the λ parameters, is always a series in E and (ω); no replacement of E by F is ever made in the series for ν. The form of this series for ν will be developed in later sections. In the development of dW/dE, the partial derivatives $r(\partial\Omega/\partial r)$ and $\partial\Omega/\partial E$ are needed. Later on, the derivatives of the λ functions will contain explicitly the partial derivatives $\partial\Omega/\partial\psi$. The partial derivative $\partial\Omega/\partial E$ is obtained by the formal differentiation of Ω^* with respect to F and application of the "bar" operator

$$\frac{\partial\Omega}{\partial E}=\overline{\frac{\partial\Omega^*}{\partial F}}. \tag{111}$$

The partial $r(\partial\Omega/\partial r)$ is obtained very simply from the differentiation of Equation 77, and use of

$$\left(\frac{1}{r}\right)^n=\left(\frac{1}{1+\nu}\right)^n\left(\frac{1}{a_0}\right)^n\left(\frac{a_0}{\bar{r}}\right)^n. \tag{112}$$

This gives

$$\begin{aligned}r\frac{\partial\Omega}{\partial r}=&-\frac{3k_2}{a_0^3}\left(\frac{1}{1+\nu}\right)^3\left(\frac{a_0}{\bar{r}}\right)^3(1-3\psi^2)\\&-\frac{4k_3}{a_0^4}\left(\frac{1}{1+\nu}\right)^4\left(\frac{a_0}{\bar{r}}\right)^4(3\psi-5\psi^3)\\&-\frac{5k_4}{a_0^5}\left(\frac{1}{1+\nu}\right)^5\left(\frac{a_0}{\bar{r}}\right)^5(3-30\psi^2+35\psi^4)\end{aligned} \tag{113}$$

which is easily seen to be

$$r\frac{\partial\Omega}{\partial r}=-3\Omega_2-4\Omega_3-5\Omega_4, \tag{114}$$

where Ω_2, Ω_3, and Ω_4 are the 1st, 2nd, and 3rd terms of Equation 77, respectively. This is identical to

$$r\frac{\partial\Omega}{\partial r}=-\overline{3\Omega_2^*}-\overline{4\Omega_3^*}-\overline{5\Omega_4^*}, \tag{115}$$

where Ω_2^*, Ω_3^*, and Ω_4^* are the first three terms of Equation 107, respectively.

The partial derivative $\partial\Omega/\partial\psi$ is obtained again by the formal differentiation of Ω^* with respect to ψ^*, and is then "barred":

$$\frac{\partial\Omega}{\partial\psi}=\overline{\frac{\partial\Omega^*}{\partial\psi^*}}, \tag{116}$$

where

$$\begin{aligned}\frac{\partial\Omega^*}{\partial\psi^*}=&-\frac{6k_2}{a_0^3}\left(\frac{1}{1+\nu}\right)^3\left(\frac{a_0}{\rho}\right)^3\psi^*\\&+\frac{k^3}{a_0^4}\left(\frac{1}{1+\nu}\right)^4\left(\frac{a_0}{\rho}\right)^4(3-15\psi^{*2})\\&+\frac{k_4}{a_0^5}\left(\frac{1}{1+\nu}\right)^5\left(\frac{a_0}{\rho}\right)^5(-60\psi^*+140\psi^{*3}).\end{aligned} \tag{117}$$

At this point, it should be noted that careful development of the terms in Ω^* can result in the following Fourier series form for Ω^*:

$$\begin{aligned}\Omega^*=&\sum_i\sum_j\sum_k C_{i,j,k}\cos\left[iE+2j(\omega)+kF\right]\\&+\sum_i\sum_j\sum_k S_{i,j,k}\sin\left[iE+(2j+1)(\omega)+kF\right].\end{aligned} \tag{118}$$

The fact that the coefficients of (ω) are in this form is purely a result of the form in which (ω) enters into m^*, l^*, $1/(1+\nu)$, and the λ's. It should be noted that even though ψ and Ω are starred, they contain E. This is again a result of the fact that not *all* the E's are replaced by F's; the λ's and the $1/(1+\nu)$ term retain their form in E. The star really indicates only that F's are present.

After Ω^* is differentiated with respect to F, and then operated on by the bar operator to get $\partial\Omega/\partial E$, the form is

$$\begin{aligned}\frac{\partial\Omega}{\partial E}=&\sum_i\sum_j C_{i,j}\cos\left[iE+(2j+1)(\omega)\right]\\&+\sum_i\sum_j S_{i,j}\sin\left[iE+2j(\omega)\right].\end{aligned} \tag{119}$$

The partials $\partial\Omega/\partial\psi$ and $r(\partial\Omega/\partial r)$ will have nearly the same form as $\partial\Omega/\partial E$, but the indices of (ω) will differ. (See Section VII.)

Now that we have determined the expressions for the partial derivatives of the potential function, we are able to begin our development of the W function, the basic perturbation function which itself contains all the perturbations of the satellite motion in the orbit plane.

SECTION IV

EQUATIONS FOR THE PERTURBATIONS IN THE ORBIT PLANE

Hansen, and correspondingly Musen, have divided the perturbations of the orbiting body's motion into the perturbations *in* the orbit plane, and those *of* the orbit plane. In the orbit plane, the deviations from the two-body path are contained in ν, the shortening or lengthening factor, and $n_0\delta z$, the perturbation of the mean anomaly. The factor ν can be thought of as containing the variation of the elements a and e, whereas the angle $n_0\delta z$ contains the variations of the argument of perigee (the periodic variations) and the mean anomaly of the real satellite. The determination of both ν and $n_0\delta z$ is done by finding *one* function, the W function, which includes the perturbations of all four elements.

In Musen's development, the object is to express the quantity for dW/dE as a Fourier series in E, (ω), and F. Getting an expression of this form involves a considerable amount of manipulation, transformation, and algebra; and as a result it is virtually impossible to relate the developed equations to the physics of the problem. In addition, the variable elements themselves are lost in the transformations, and the final form might appear meaningless. It should be kept in mind that after $\overline{W}$ is originally defined, all efforts are made to find its derivative in a Fourier series with arguments containing integral multiples of E, (ω), and F.

In the process for deriving dW/dE, we will obtain an expression for $dn_0\delta z/dE$, which we will integrate to get a final series form for $n_0\delta z$. Also, in the course of the development, we will produce a series form for W which isolates the terms containing sin F and cos F. This series will be used in the iteration processes to determine ν.

One more remark would be helpful here. In this part of the exposition, we are only developing expressions, series expressions, for dW/dE, W, and $dn_0\delta z/dE$. The methods by which these series are used to calculate perturbations are discussed in later sections. The purpose of this section is only to show the origin of the equations. It will be noticed that the final explicit expressions are in a form suitable for iteration; that is, the W function will have W in it. It is to be understood that $W_{n+1}=f(W_n)$, where n is the number of the iteration.

Our first step is to define the W function. We do this by setting up an expression for the "perturbation of time" $n_0\delta z$. We take Equation 71, which gives the polar angle of the real satellite at time t equal to that of the fictitious satellite at pseudotime z, that is,

$$v=\bar{f}+\pi_0+y\Delta E,$$

where $\Delta E=E-E_0$ and y is the secular motion of the perigee of the auxiliary ellipse in the XY plane.

Differentiating, we have

$$\frac{dv}{dt}=\frac{d\bar{f}}{dz}\frac{dz}{dt}+y\frac{dE}{dt}. \tag{120}$$

But we know from our "area integrals" (see Appendix A) that

$$\frac{dv}{dt}=\frac{1}{hr^2}$$

and

$$\frac{d\bar{f}}{dz}=\frac{1}{h_0\bar{r}^2}.$$

Substituting into Equation 120 we have

$$\frac{1}{hr^2}=\frac{dz}{dt}\frac{1}{h_0\bar{r}^2}+y\frac{dE}{dt}$$

or

$$\frac{dz}{dt}=\frac{h_0}{h}\left(\frac{\bar{r}}{r}\right)^2-yh_0\bar{r}^2\frac{dE}{dt}. \tag{121}$$

But we know that

$$\frac{\bar{r}}{r}=\frac{1}{1+\nu}$$

by definition, and

$$h_0=\frac{1}{\sqrt{a_0(1-e_0^2)}}$$

(see Appendix A); and, from Kepler's law,

$$n_0=a_0^{-3/2},$$

which gives us

$$\frac{1}{h_0}=a_0^2n_0\sqrt{1-e_0^2}.$$

Substituting these into Equation 121, we have

$$\frac{dz}{dt}=\left(\frac{h_0}{h}\right)\frac{1}{(1+\nu)^2}-\frac{y}{n_0\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2\frac{dE}{dt} \tag{122}$$

or, since $\delta z=z-t$ and

$$\frac{d\delta z}{dt}=\frac{dz}{dt}-1,$$

we have

$$\frac{d\delta z}{dt}=-1+\left(\frac{h_0}{h}\right)\frac{1}{(1+\nu)^2}-\frac{y}{n_0\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2\frac{dE}{dt}. \tag{123}$$

We see that Equation 123 gives the time derivative of the "perturbation of time" and is expressed in terms of two other perturbations h_0/h and $1/(1+\nu)$, as well as the secular motion y of the perturbed perigee. However, we wish to separate the higher and lower order perturbations, as a first step toward developing the expression in a form suitable for iteration. We can write

$$\frac{1}{(1+\nu)^2}=-\frac{(1+2\nu+\nu^2)}{(1+\nu)^2}+\frac{2+2\nu}{(1+\nu)^2}+\frac{\nu^2}{(1+\nu)^2}$$

$$=-1+\frac{2}{1+\nu}+\frac{\nu^2}{(1+\nu)^2},$$

which is put into Equation 123 and gives

$$\frac{d\delta z}{dt}=-1-\frac{h_0}{h}+\left(\frac{h_0}{h}\right)\frac{2}{(1+\nu)}+\left(\frac{h_0}{h}\right)\frac{\nu^2}{(1+\nu)^2}$$
$$-\frac{y}{n_0\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2\frac{dE}{dt}. \tag{124}$$

Now, if we collect the first three terms in a single function which we call $\overline{W}$, we have

$$\overline{W}=-1-\frac{h_0}{h}+\left(\frac{h_0}{h}\right)\frac{2}{1+\nu}, \tag{125}$$

where we can see that $\overline{W}$ is of the order of the perturbations, and

$$\frac{d\delta z}{dt}=\overline{W}+\left(\frac{h_0}{h}\right)\frac{\nu^2}{(1+\nu)^2}-\frac{y}{n_0\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2\frac{dE}{dt}. \tag{126}$$

But now, since we know that the equation of the orbit is

$$r=\frac{a(1-e^2)}{1+e\cos f}\quad \text{(See Appendix A)},$$

where in Musen's method $r=(1+\nu)\bar{r}$, we can proceed as follows:

We have shown previously (page 7) that $\mathbf{q}=he\mathbf{P}$, where $\mathbf{P}$ is a unit vector directed from the origin to the perigee, and we know $\mathbf{r}$ is the vector directed from the origin to the real satellite. Therefore, the angle between $\mathbf{r}$ and $\mathbf{P}$ is the true anomaly f. And, by the definition of the dot product, we see

$$\frac{\mathbf{r}}{r}\cdot\frac{\mathbf{q}}{he}=\cos f \tag{127}$$

or

$$\mathbf{r}\cdot\mathbf{q}=her\cos f.$$

Now from the equation,

$$\frac{1}{h}=\sqrt{a(1-e^2)},$$

we see that our equation of the orbit can be written

$$hr+her\cos f=\frac{1}{h};$$

and if we divide this through by $(1+\nu)$,

$$h\left(\frac{r}{1+\nu}\right)+\frac{1}{1+\nu}\,(her\cos f)=\frac{1}{h(1+\nu)}.$$

But we have shown in Equation 127 that $her\cos f=\mathbf{r}\cdot\mathbf{q}$, so

$$h\left(\frac{r}{1+\nu}\right)+\frac{1}{1+\nu}\,(\mathbf{r}\cdot\mathbf{q})=\frac{1}{h(1+\nu)}.$$

But

$$\frac{r}{1+\nu}=\bar{\mathbf{r}},$$

and

$$\frac{1}{1+\nu}\,(\mathbf{r}\cdot\mathbf{q})=\bar{\mathbf{r}}\cdot\mathbf{q},$$

and thus

$$h\bar{r}+\bar{\mathbf{r}}\cdot\mathbf{q}=\frac{1}{h(1+\nu)}. \tag{128}$$

From Equation 128, we can write

$$\overline{W}=-1-\frac{h_0}{h}+2h_0h\bar{r}+2h_0\,\bar{\mathbf{r}}\cdot\mathbf{q}$$

$$=-1-\frac{h_0}{h}+\frac{2h_0^2}{h_0}\,h\bar{r}+\frac{2h_0^2}{h_0}\,\bar{\mathbf{r}}\cdot\mathbf{q};$$

but since

$$h_0^2=\frac{1}{a_0(1-e_0^2)},$$

we have

$$\overline{W}=-1-\frac{h_0}{h}+\frac{2h\bar{r}}{h_0a_0(1-e_0^2)}+\frac{2\bar{\mathbf{r}}\cdot\mathbf{q}}{h_0a_0(1-e_0^2)}. \tag{129}$$

Now, as has been previously discussed, for the development of the differential equation for $\overline{W}$, it is preferable to separate the perturbations from the elliptic motion. Thus, we will replace the E by F, $\bar{\mathbf{r}}$ by $\bar{\boldsymbol{\rho}}$, and $\bar{f}$ by $\bar{\phi}$, so that we have the basic ellipse equations written as

$$\bar{\rho}\cos\bar{\phi}=a_0(\cos F-e_0),$$

$$\bar{\rho}\sin\bar{\phi}=a_0\sqrt{1-e_0^2}\sin F,$$

$$\bar{\rho}=a_0(1-e_0\cos F),$$

and

$$\bar{\boldsymbol{\rho}}=\mathbf{i}'\bar{\rho}\cos(\pi_0+\bar{\phi}+y\Delta E)+\mathbf{j}'\bar{\rho}\sin(\pi_0+\bar{\phi}+y\Delta E).$$

Making the proper replacements in $\overline{W}$, we now have our first expression for W. To be consistent, we should call this W^*, but neither Hansen nor Musen has used the W^* notation. Thus, we shall use W here, remembering that W has the appropriate E's replaced by F's, and that $\overline{W}$ has all the F's replaced by E's. We have

$$W=-1-\frac{h_0}{h}+\frac{2h\bar{\rho}}{h_0a_0(1-e_0^2)}+\frac{2\bar{\boldsymbol{\rho}}\cdot\mathbf{q}}{h_0a_0(1-e_0^2)}. \tag{130}$$

But since

$$\mathbf{q}=he\mathbf{P}=he(\mathbf{i}'\cos\chi+\mathbf{j}'\sin\chi),$$

where χ is the osculating argument of the perigee as measured from the departure point (see sketch on page 11), and since

$$\bar{\boldsymbol{\rho}}\cdot\mathbf{q}=he\bar{\rho}[\cos(\bar{\phi}+\pi_0+y\Delta E)\cos\chi$$
$$+\sin(\bar{\phi}+\pi+y\Delta E)\sin\chi]$$
$$=he\bar{\rho}[\cos(\bar{\phi}+\pi_0+y\Delta E-\chi)], \tag{131}$$

we have the classic equation

$$W=-1-\frac{h_0}{h}+\frac{2h}{h_0}\,\frac{\bar{\rho}}{a_0}\left[\frac{1+e\cos(\bar{\phi}+\pi_0+y\Delta E-\chi)}{(1-e_0^2)}\right]. \tag{132}$$

Using Equations 103, 104, and 105, we have for W,

$$W=-1-\frac{h_0}{h}+\left(\frac{2h}{h_0}\right)\frac{(1-e_0\cos F)}{(1-e_0^2)}$$
$$+\left(\frac{2h}{h_0}\right)\frac{e}{(1-e_0^2)}[(\cos F-e_0)\cos(\pi_0+y\Delta E-\chi)$$
$$-\sqrt{1-e_0^2}\sin F\sin(\pi_0+y\Delta E-\chi)]. \quad (133)$$

Expanding Equation 133 gives

$$W=\left[-1-\frac{h_0}{h}+\left(\frac{2h}{h_0}\right)\frac{1}{(1-e_0^2)}\right.$$
$$\left.-\left(\frac{2h}{h_0}\right)\frac{e}{(1-e_0^2)}e_0\cos(\pi_0+y\Delta E-\chi)\right]$$
$$+\left[\left(\frac{2h}{h_0}\right)\frac{1}{(1-e_0^2)}(e\cos[\pi_0+y\Delta E-\chi]-e_0)\right]\cos F$$
$$+\left[-\left(\frac{2he}{h_0}\right)\frac{1}{\sqrt{1-e_0^2}}\sin(\pi_0+y\Delta E-\chi)\right]\sin F, \quad (134)$$

which we can write as

$$W=\Xi+\Upsilon\cos F+\Psi\sin F \quad (135)$$

if we let Ξ, Υ, and Ψ equal the first, second, and third bracketed terms, respectively. We see here that Ξ, Υ, and Ψ are functions of E and contain no F's. Now, if we multiply Υ through by e_0, and add it to Ξ, we will have

$$\Xi+e_0\Upsilon=-1-\frac{h_0}{h}+\frac{2h}{h_0}. \quad (136)$$

This is a very important relation, to be used in the determination of h and h_0, and it should again be noted that it gives h_0/h and h/h_0 in terms of E and (ω) alone.

But for now let us turn our attention to the development of the differential equation for W. We operate on Equation 130 with Brown's operator δ/dt:

$$\frac{\delta W}{dt}=-h_0\frac{\delta}{dt}\left(\frac{1}{h}\right)+\frac{2}{h_0a_0(1-e_0^2)}\left[\bar{\rho}\frac{\delta h}{dt}+\frac{\delta\bar{\rho}}{dt}h\right]$$
$$+\frac{2}{h_0a_0(1-e_0^2)}\left[\frac{\delta\bar{\boldsymbol{\rho}}}{dt}\cdot\mathbf{q}+\bar{\boldsymbol{\rho}}\cdot\frac{\delta\mathbf{q}}{dt}\right]. \quad (137)$$

Since h is a function of the elements alone, and h_0 is a function only of the constant elements a_0 and e_0, we know that

$$\left.\begin{aligned}&\frac{\delta}{dt}h_0=0,\\&\frac{\delta h}{dt}=\frac{dh}{dt},\\&\frac{\delta}{dt}\left(\frac{1}{h}\right)=\frac{d}{dt}\left(\frac{1}{h}\right).\end{aligned}\right\} \quad (138)$$

and

We also know from the equations of motion in polar form (see Appendix A), that the component of the acceleration in the orbital plane perpendicular to the radius vector is

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{dv}{dt}\right).$$

Written in terms of the disturbing potential (this component depends on the disturbing potential alone), this gives

$$\frac{\partial\Omega}{r\partial v}=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{dv}{dt}\right);$$

but we have defined the "area" as

$$r^2\frac{dv}{dt}\equiv\frac{1}{h},$$

so we have

$$\frac{d}{dt}\left(\frac{1}{h}\right)=\frac{\partial\Omega}{\partial v}. \quad (139)$$

From Equation 139, another result appears:

$$\frac{d}{dt}\left(\frac{1}{h}\right)=-\frac{1}{h^2}\frac{dh}{dt}=\frac{\partial\Omega}{\partial v}$$

or

$$\frac{dh}{dt}=-h^2\frac{\partial\Omega}{\partial v}. \quad (140)$$

Then, we recall that $\mathbf{q}=he\mathbf{P}$ where $\mathbf{P}$ is a unit vector, and that

$$\frac{\delta\mathbf{q}}{dt}=\frac{d\mathbf{q}}{dt}=(\mathbf{r}^0\times\mathbf{R})\frac{\partial\Omega}{\partial r}+\mathbf{r}^0\left(\frac{1}{r}+h^2\right)\frac{\partial\Omega}{\partial v} \quad (141)$$

from Equation 46. Furthermore, we know the perturbations in $\bar{\boldsymbol{\rho}}$ are present only in the vector's rotation due to the rotation of the auxiliary ellipse in the XY plane. This rotation of $\bar{\boldsymbol{\rho}}$ around the Z axis will be given by

$$\frac{\delta\bar{\boldsymbol{\rho}}}{dt}=(\mathbf{R}\times\bar{\boldsymbol{\rho}})y\frac{dE}{t}, \quad (142)$$

since $y(dE/dt)$ is the angular velocity of the ellipse in the XY plane. Clearly, the scalar $\bar{\rho}$ is not affected by the perturbations; that is,

$$\frac{\delta\bar{\rho}}{dt}=0. \tag{143}$$

Taking Equations 138 through 142 into consideration, we are able to rewrite Equation 137 as follows:

$$\begin{aligned}\frac{\delta W}{dt}=&-h_0\frac{\partial\Omega}{\partial v}+\frac{2}{h_0a_0(1-e_0^2)}\bar{\rho}\left(-h^2\frac{\partial\Omega}{\partial v}\right)\\&+\frac{2}{h_0a_0(1-e_0^2)}(\mathbf{R}\times\bar{\boldsymbol{\rho}})\cdot\mathbf{q}y\frac{dE}{dt}\\&+\frac{2}{h_0a_0(1-e_0^2)}\frac{\partial\Omega}{\partial r}\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})\\&+\frac{2}{h_0a_0(1-e_0^2)}\frac{1}{r}\frac{\partial\Omega}{\partial v}\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0\\&+\frac{2h^2}{h_0a_0(1-e_0^2)}\frac{\partial\Omega}{\partial v}\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0;\end{aligned} \tag{144}$$

and since $h_0^2a_0(1-e_0^2)=1$ we can write (noting that $\delta W/dt=dW/dt$)

$$\begin{aligned}\frac{dW}{dt}=&h_0\frac{\partial\Omega}{\partial v}\left[-1-\frac{2h^2}{h_0^2a_0(1-e_0^2)}\bar{\rho}\right.\\&\left.+\frac{2}{r}\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{h_0^2a_0(1-e_0^2)}+\frac{2h^2\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{h_0^2a_0(1-e_0^2)}\right]\\&+\frac{2h_0}{h_0^2a_0(1-e_0^2)}\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})\frac{\partial\Omega}{\partial r}\\&+\frac{2}{h_0a_0(1-e_0^2)}(\mathbf{R}\times\bar{\boldsymbol{\rho}})\cdot\mathbf{q}y\frac{dE}{dt}\end{aligned} \tag{145}$$

or, finally,

$$\begin{aligned}\frac{dW}{dt}=&h_0\frac{\partial\Omega}{\partial v}\left[\frac{2\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{r}-1+\left(\frac{2h^2}{h_0^2}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\rho}{a_0(1-e_0^2)}\right]\\&+2h_0\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})\frac{\partial\Omega}{\partial r}+\frac{2\mathbf{q}\cdot(\mathbf{R}\times\bar{\boldsymbol{\rho}})}{h_0a_0(1-e_0^2)}y\frac{dE}{dt}.\end{aligned} \tag{146}$$

We wish to simplify this expression further. To find $d\bar{\boldsymbol{\rho}}/dt$, note that $\bar{\boldsymbol{\rho}}=\bar{\rho}\boldsymbol{\rho}^0$. From Equations 103, 104, and 105, we obtain by differentiation and some algebra

$$\frac{d\bar{\rho}}{dF}=a_0e_0\sin F, \tag{147}$$

$$\frac{d\bar{\phi}}{dF}=\frac{a_0\sqrt{1-e_0^2}}{\bar{\rho}}. \tag{148}$$

From these, and the fact that $d\boldsymbol{\rho}^0/d\bar{\phi}=\mathbf{R}\times\boldsymbol{\rho}^0$ (by the definition of the derivative of a unit vector in a plane), we get

$$\begin{aligned}\frac{d\bar{\boldsymbol{\rho}}}{dF}&=\bar{\rho}\frac{d\boldsymbol{\rho}^0}{d\phi}\frac{d\bar{\phi}}{dF}+\boldsymbol{\rho}^0\frac{d\bar{\rho}}{dF}\\&=\left[(\mathbf{R}\times\bar{\boldsymbol{\rho}})\frac{a_0\sqrt{1-e_0^2}}{\bar{\rho}}\right]+\boldsymbol{\rho}^0a_0e_0\sin F.\end{aligned} \tag{149}$$

Now, differentiating Equation 130

$$W=-1-\frac{h_0}{h}+\left(\frac{2h}{h_0}\right)\frac{\rho}{a_0(1-e_0^2)}+\frac{2\bar{\boldsymbol{\rho}}\cdot\mathbf{q}}{h_0a_0(1-e_0^2)}$$

with respect to F, we have

$$\begin{aligned}\frac{\partial W}{\partial F}=&\frac{2h}{h_0a_0(1-e_0^2)}\frac{d\bar{\rho}}{dF}+\frac{2}{h_0a_0(1-e_0^2)}\frac{d\bar{\boldsymbol{\rho}}}{dF}\cdot\mathbf{q}\\=&\frac{2ha_0e_0\sin F}{h_0a_0(1-e_0^2)}+\frac{2}{h_0a_0(1-e_0^2)}\mathbf{q}\\&\cdot\left[(\mathbf{R}\times\bar{\boldsymbol{\rho}})\frac{a_0}{\bar{\rho}}\sqrt{1-e_0^2}+\boldsymbol{\rho}^0a_0e_0\sin F\right].\end{aligned} \tag{150}$$

This last equation for $\partial W/\partial F$ can be rearranged and written

$$\frac{2\mathbf{q}\cdot(\mathbf{R}\times\bar{\boldsymbol{\rho}})}{\bar{\rho}h_0(1-e_0^2)}=\left[\frac{\partial W}{\partial F}-\frac{2ha_0e_0\sin F}{h_0a_0(1-e_0^2)}-\frac{2\mathbf{q}\cdot\boldsymbol{\rho}^0a_0e_0\sin F}{h_0a_0(1-e_0^2)}\right]\frac{1}{\sqrt{1-e_0^2}}.$$

Multiplied through by $\bar{\rho}$ and divided by a_0, this becomes

$$\frac{2\mathbf{q}\cdot(\mathbf{R}\times\boldsymbol{\rho})}{h_0a_0(1-e_0^2)}=\left[\frac{\bar{\rho}\partial W}{a_0\partial F}-e_0\sin F\left(\frac{2h\bar{\rho}+2\mathbf{q}\cdot\bar{\boldsymbol{\rho}}}{h_0a_0(1-e_0^2)}\right)\right]\frac{1}{\sqrt{1-e_0^2}}.$$

This, with the aid of Equation 130, can be written

$$\frac{2\mathbf{q}\cdot(\mathbf{R}\times\bar{\boldsymbol{\rho}})}{h_0a_0(1-e_0^2)}=\left[\frac{\bar{\rho}}{a_0}\frac{\partial W}{\partial F}-e_0\sin F\left(W+\frac{h_0}{h}+1\right)\right]\frac{1}{\sqrt{1-e_0^2}}.$$

Substituting this into our last equation for dW/dt (Equation 146), we have

$$\begin{aligned}\frac{dW}{dt}=&h_0\frac{\partial\Omega}{\partial v}\left(\frac{2\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{r}-1+\left(\frac{2h^2}{h_0^2}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0(1-e_0^2)}\right)\\&+2h_0\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})\frac{\partial\Omega}{\partial r}+\frac{y}{\sqrt{1-e_0^2}}\left[\frac{\bar{\rho}}{a_0}\frac{\partial W}{\partial F}\right.\\&\left.-\left(W+\frac{h_0}{h}+1\right)e_0\sin F\right]\frac{dE}{dt}.\end{aligned} \tag{151}$$

Now taking our original definition of $\overline{W}$ (Equation 125) we see that

$$\overline{W}+1=\frac{-h_0(1+\nu)+2h_0}{h(1+\nu)}=\frac{h_0(1-\nu)}{h(1+\nu)}$$

and

$$\frac{h_0}{h}=\left(\frac{1+\nu}{1-\nu}\right)(\overline{W}+1).$$

Putting this into our last equation for $d\delta z/dt$ (Equation 126), we have

$$\frac{d\delta z}{dt}=\overline{W}+\left(\frac{1+\nu}{1-\nu}\right)(\overline{W}+1)\frac{\nu^2}{(1+\nu)^2}$$
$$-\frac{y}{n_0\sqrt{1-e_0^2}}\left(\frac{r}{a_0}\right)^2\frac{dE}{dt}.$$

If we multiply through by n_0 and rearrange, this becomes

$$\frac{dn_0\delta z}{dt}=\frac{n_0\overline{W}(1-\nu^2)+n_0\nu^2(\overline{W}+1)}{1-\nu^2}-\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\overline{r}}{a_0}\right)^2\frac{dE}{dt},$$

or the generalized Hill formula

$$\frac{dn_0\delta z}{dt}=n_0\frac{\overline{W}+\nu^2}{1-\nu^2}-\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\overline{r}}{a_0}\right)^2\frac{dE}{dt}. \qquad (152)$$

Now, differentiating Kepler's equation

$$M=E-e_0\sin E=g_0+n_0(t-t_0)+n_0\delta z$$

(Equation 75)

with respect to E we have

$$1-e_0\cos E=n_0\frac{dt}{dE}+\frac{dn_0\delta z}{dE}.$$

But since $\overline{r}=a_0(1-e_0\cos E)$, we have

$$n_0\frac{dt}{dE}=\frac{\overline{r}}{a_0}-\frac{dn_0\delta z}{dE}. \qquad (153)$$

Multiplying Equation 152 by dt/dE and using Equation 153 in the result gives

$$\frac{dn_0\delta z}{dE}=\left(\frac{\overline{W}+\nu^2}{1-\nu^2}\right)\left(\frac{\overline{r}}{a_0}-\frac{dn_0\delta z}{dE}\right)-\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\overline{r}}{a_0}\right)^2,$$

which simplifies to

$$\frac{dn_0\delta z}{dE}\left(1+\frac{\overline{W}+\nu^2}{1-\nu^2}\right)-\frac{\overline{W}+\nu^2}{1-\nu^2}\left(\frac{\overline{r}}{a_0}\right)$$
$$-\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\overline{r}}{a_0}\right)^2=\frac{dn_0\delta z}{dE}\left(\frac{1+\overline{W}}{1-\nu^2}\right).$$

Therefore,

$$\frac{dn_0\delta z}{dE}=\frac{\overline{W}+\nu^2}{1+\overline{W}}\left(\frac{\overline{r}}{a_0}\right)-\left(\frac{1-\nu^2}{1+\overline{W}}\right)\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\overline{r}}{a_0}\right)^2. \qquad (154)$$

This is close to the final form of the derivative which we must integrate to find the perturbations of time. We will discuss its properties later, but for now we must use it in the further development of dW/dt. We next substitute it into Equation 153 to get

$$n_0\frac{dt}{dE}=\frac{\overline{r}}{a_0}-\frac{\overline{W}+\nu^2}{1+\overline{W}}\left(\frac{\overline{r}}{a_0}\right)+\left(\frac{1-\nu^2}{1+\overline{W}}\right)\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\overline{r}}{a_0}\right)^2$$
$$=\frac{\overline{r}}{a_0}\left(1-\frac{\overline{W}+\nu^2}{1+\overline{W}}\right)+\left(\frac{1-\nu^2}{1+\overline{W}}\right)\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\overline{r}}{a_0}\right)^2,$$

or, finally,

$$n_0\frac{dt}{dE}=\frac{\overline{r}(1-\nu^2)}{a_0(1+\overline{W})}\left(1+\frac{y\overline{r}}{a_0\sqrt{1-e_0^2}}\right). \qquad (155)$$

This is an important equation which will also be used in the development of the derivatives of the λ parameters.

Now, as a final step, we must express our derivative in terms of E. We know that

$$\frac{\partial\Omega}{\partial E}=\frac{\partial\Omega}{\partial\overline{r}}\frac{\partial\overline{r}}{\partial E}+\frac{\partial\Omega}{\partial\overline{f}}\frac{\partial\overline{f}}{\partial E}, \qquad (156)$$

where, by differentiation of Equations 72, 73, and 74,

$$\frac{\partial\overline{f}}{\partial E}=\frac{a_0\sqrt{1-e_0^2}}{\overline{r}}\text{ and }\frac{\partial\overline{r}}{\partial E}=a_0e_0\sin E.$$

And given $v=\pi_0+f$, we have

$$\frac{\partial\Omega}{\partial v}=\frac{\partial\Omega}{\partial\overline{f}}.$$

Also, since $r=(1+\nu)\overline{r}$,

$$r\frac{\partial\Omega}{\partial r}=\overline{r}\frac{\partial\Omega}{\partial\overline{r}}.$$

So we can write for Equation 156

$$\frac{\partial\Omega}{\partial E}=a_0e_0\sin E\frac{r}{\overline{r}}\frac{\partial\Omega}{\partial r}+\frac{\partial\Omega}{\partial v}\frac{a_0\sqrt{1-e_0^2}}{\overline{r}},$$

or

$$\frac{\partial\Omega}{\partial v}=\frac{r}{a_0\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial E}-\frac{e_0-\sin E}{\sqrt{1-e_0^2}}r\frac{\partial\Omega}{\partial r}. \qquad (157)$$

Taking Equation 157 and our last equation for $n_0(dt/dE)$, Equation 155, we want to substitute into our equation for dW/dt, Equation 151. First multiplying through by dt/dE, and using Equation 155, we have

$$\frac{dW}{dE}=\left\{h_0\left[\frac{\bar{r}}{a_0\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial E}-\frac{e_0\sin E}{\sqrt{1-e_0^2}}r\frac{\partial\Omega}{\partial r}\right]\left[\frac{2\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{r}-1+\left(\frac{2h^2}{h_0^2}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0(1-e_0^2)}\right]\left[\left(\frac{\bar{r}}{n_0a_0}\right)\frac{1-\nu^2}{1+\overline{W}}\left(1+\frac{y}{\sqrt{1-e_0^2}}\frac{\bar{r}}{a_0}\right)\right]\right\}$$
$$+\left\{2h_0\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})\frac{\partial\Omega}{\partial r}\left[\left(\frac{\bar{r}}{n_0a_0}\right)\frac{1-\nu^2}{1+\overline{W}}\left(1+\frac{y}{\sqrt{1-e_0^2}}\frac{\bar{r}}{a_0}\right)\right]\right\}+\frac{y}{\sqrt{1-e_0^2}}\left[\frac{\bar{\rho}}{a_0}\frac{\partial W}{\partial F}-\left(W+\frac{h_0}{h}+1\right)e_0\sin F\right]. \tag{158}$$

SECTION V

FINAL EXPANSION OF $\frac{dW}{dE}$ IN TERMS OF E AND F

We now turn to expanding Equation 158 in terms of the two eccentric anomalies E and F.

If we let

$$\left.\begin{aligned}\Lambda&=\frac{1-\nu^2}{1+\overline{W}}\left(1+\frac{\bar{r}}{a_0}\frac{y}{\sqrt{1-e_0^2}}\right),\\ S&=\left[\frac{\bar{\rho}}{a_0}\frac{\partial W}{\partial F}-\left(W+\frac{h_0}{h}+1\right)e_0\sin F\right],\end{aligned}\right\} \tag{159}$$

we have

$$\frac{dW}{dE}=h_0\frac{\bar{r}}{n_0a_0}\Lambda\frac{\partial\Omega}{\partial E}\left[\frac{2\bar{r}}{r}\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{a_0\sqrt{1-e_0^2}}-\frac{\bar{r}}{a_0\sqrt{1-e_0^2}}+\frac{\bar{r}}{a_0\sqrt{1-e_0^2}}\left(\frac{2h^2}{h_0^2}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0(1-e_0^2)}\right]$$
$$+\frac{h_0\bar{r}}{n_0a_0}\Lambda\frac{r\partial\Omega}{\partial r}\left[\frac{e_0\sin E}{\sqrt{1-e_0^2}}-\frac{2e_0\sin E\,\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{\sqrt{1-e_0^2}\,r}-\frac{e_0\sin E}{\sqrt{1-e_0^2}}\left(\frac{2h^2}{h_0^2}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0(1-e_0^2)}+\frac{2\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})}{r}\right]$$
$$+\frac{Sy}{\sqrt{1-e_0^2}}. \tag{160}$$

The process of expanding Equation 160 is a long and troublesome one. It is desirable for us to expand it term by term, so we write

$$\frac{dW}{dE}=A'+B'+C'+D'+E'+F'+\frac{Sy}{\sqrt{1-e_0^2}}, \tag{161}$$

where

$$A'=\frac{h_0\bar{r}}{n_0a_0}\Lambda\frac{\partial\Omega}{\partial E}\left[\left(\frac{2\bar{r}}{r}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{a_0\sqrt{1-e_0^2}}\right],$$

$$B'=h_0\frac{\bar{r}}{n_0a_0}\Lambda\frac{\partial\Omega}{\partial E}\left[-\frac{\bar{r}}{a_0\sqrt{1-e_0^2}}\right],$$

$$C'=h_0\frac{\bar{r}}{n_0a_0}\Lambda\frac{\partial\Omega}{\partial E}\left[\frac{\bar{r}}{a_0\sqrt{1-e_0^2}}\left(\frac{2h^2}{h_0^2}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0(1-e_0^2)}\right],$$

$$D'=\frac{h_0\bar{r}}{n_0a_0}\Lambda\frac{r\partial\Omega}{\partial r}\left[\frac{e_0\sin E}{\sqrt{1-e_0^2}}\right],$$

$$E'=\frac{h_0\bar{r}}{n_0a_0}\Lambda\frac{r\partial\Omega}{\partial r}\left[\frac{2\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})}{r}-\frac{2e_0\sin E\,\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{\sqrt{1-e_0^2}\,r}\right],$$

$$F'=\frac{h_0\bar{r}}{n_0a_0}\Lambda\frac{r\partial\Omega}{\partial r}\left[-\frac{e_0\sin E}{\sqrt{1-e_0^2}}\left(\frac{2h^2}{h_0^2}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0(1-e_0^2)}\right].$$

Now, we will deal with A' alone. Using the known relationships

$$\left.\begin{aligned}&\frac{\bar{r}}{r}=\frac{1}{(1+\nu)},\\ &n_0=a_0^{-3/2},\\ &h_0=\frac{1}{\sqrt{a_0(1-e_0^2)}},\\ &\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0=\bar{\rho}\cos(\bar{\phi}-\bar{f})=\bar{\rho}(\cos\bar{\phi}\cos\bar{f}\\ &\qquad+\sin\bar{\phi}\sin\bar{f}),\\ &\bar{r}=a_0(1-e_0\cos E),\\ &\frac{h_0}{n_0}=\frac{a_0}{\sqrt{1-e_0^2}},\\ &\bar{\rho}=a_0(1-e_0\cos F),\\ &\bar{r}\cos\bar{f}=a_0(\cos E-e_0),\\ &\bar{r}\sin\bar{f}=a_0\sqrt{1-e_0^2}\sin E,\\ &\bar{\rho}\cos\bar{\phi}=a_0(\cos F-e_0),\\ &\bar{\rho}\sin\bar{\phi}=a_0\sqrt{1-e_0^2}\sin F.\end{aligned}\right\} \tag{162}$$

We can write A' in terms of E and F. First we have

$$A'=\Lambda\frac{\partial a_0\Omega}{\partial E}\frac{1}{(1+\nu)}\frac{1}{(1-e_0^2)}\left[\left(\frac{2\bar{r}}{a_0}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{a_0}\right]. \tag{163}$$

The term in brackets in Equation 163 can be written

$$\left[\left(\frac{2\bar{r}}{a_0}\right)\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0}{a_0}\right]=\left(\frac{2\bar{r}}{a_0}\right)\frac{\bar{\rho}}{a_0}\cos(\bar{\phi}-\bar{f})$$

$$=\frac{2\bar{r}\cos\bar{f}}{a_0}\frac{\bar{\rho}\cos\bar{\phi}}{a_0}+\frac{2\bar{r}\sin\bar{f}}{a_0}\frac{\bar{\rho}\sin\bar{\phi}}{a_0}$$

$$=2(\cos E-e_0)(\cos F-e_0)$$
$$+2\sqrt{1-e_0^2}\sin E\sqrt{1-e_0^2}\sin F$$
$$=\cos(E-F)+\cos(E+F)$$
$$-2e_0\cos F-2e_0\cos E+2e_0^2$$
$$+(1-e_0^2)[\cos(E-F)-\cos(E+F)]$$
$$=(2-e_0^2)\cos(F-E)+e_0^2\cos(E+F)$$
$$-2e_0\cos F-2e_0\cos E+2e_0^2;$$

and we have A' in terms of E and F:

$$A'=\Lambda\frac{\partial a_0\Omega}{\partial E}\left(\frac{1}{1+\nu}\right)\frac{1}{(1-e_0^2)}[(2-e_0^2)\cos(F-E)$$
$$+e_0^2\cos(E+F)-2e_0\cos F-2e_0\cos E+2e_0^2]. \tag{164}$$

Next, we take term B' which, using Equation 162, we can write

$$B'=\Lambda\frac{\partial a_0\Omega}{\partial E}\frac{1}{(1-e_0^2)}\left[-1+2e_0\cos E-\frac{1}{2}e_0^2\cos 2E-\frac{e_0^2}{2}\right]. \tag{165}$$

Now, C' can be written

$$C'=\Lambda\frac{\partial a_0\Omega}{\partial E}\frac{h^2}{h_0^2}\frac{1}{(1-e_0^2)}\left[\frac{\bar{r}}{a_0\sqrt{1-e_0^2}}\frac{2\bar{r}}{a_0\sqrt{1-e_0^2}}\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0}\right]. \tag{166}$$

Since $\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0=\bar{\rho}\cos(\bar{\phi}-\bar{f})$, we can write the bracketed term of Equation 166 as

$$[\ \ ]=\left[\frac{2}{\sqrt{1-e_0^2}}\frac{\bar{r}}{a_0}\frac{\bar{r}}{a_0}\frac{1}{\sqrt{1-e_0^2}}\frac{\bar{\rho}}{a_0}[\cos(\bar{\phi}-\bar{f})-1]\right]$$
$$=\frac{2\bar{r}}{a_0(1-e_0^2)}\frac{\bar{r}\cos\bar{f}}{a_0}\frac{\bar{\rho}}{a_0}\cos\bar{\phi}$$
$$+\frac{2\bar{r}}{a_0}\frac{\bar{\rho}\sin\bar{\phi}}{a_0\sqrt{1-e_0^2}}\frac{\bar{r}\sin\bar{f}}{a_0\sqrt{1-e_0^2}}$$
$$-\frac{2}{\sqrt{1-e_0^2}}\frac{\bar{r}}{a_0}\frac{\bar{r}}{a_0}\frac{\bar{\rho}}{a_0\sqrt{1-e_0^2}}.$$

Applying Equation 162, and substituting the result in Equation 166, we get for C':

$$C'=\Lambda\frac{\partial a_0\Omega}{\partial E}\frac{h^2}{h_0^2}\frac{1}{(1-e_0^2)}[2\cos(E-F)-2$$
$$-e_0\cos(2E-F)-e_0\cos F+2e_0\cos E]. \tag{167}$$

Next, term D' of Equation 161 can be written

$$D'=\Lambda r\frac{\partial a_0\Omega}{\partial r}\frac{1}{(1-e_0^2)}\left[\frac{\bar{r}}{a_0}e_0\sin E\right]$$

$$D'=\Lambda r\frac{\partial a_0\Omega}{\partial r}\frac{1}{(1-e_0^2)}[(1-e_0\cos E)e_0\sin E],$$

and so

$$D'=\Lambda r\frac{\partial a_0\Omega}{\partial r}\frac{1}{(1-e_0^2)}\left[e_0\sin E-\frac{e_0^2}{2}\sin 2E\right]. \tag{168}$$

The next term E' of Equation 161 becomes, after consideration of Equation 162 and using the fact that $\bar{\boldsymbol{\rho}}\cdot(\mathbf{r}^0\times\mathbf{R})=\bar{\rho}\sin(\bar{f}-\bar{\phi})$,

$$E'=\Lambda r\frac{\partial\Omega}{\partial r}\left[\frac{2\bar{\rho}\bar{r}}{r\sqrt{1-e_0^2}}\sin(\bar{f}-\bar{\phi})-\frac{2\bar{\rho}}{r}\cos(\bar{f}-\bar{\phi})\frac{\bar{r}}{1-e_0^2}e_0\sin E\right]. \tag{169}$$

For the bracketed part of this equation, we write

$$[\ \ ]=\frac{1}{(1-e_0^2)}\frac{\bar{r}}{\bar{r}(1+\nu)}[2\bar{\rho}\sin(\bar{f}-\bar{\phi})\sqrt{1-e_0^2}$$
$$-2\bar{\rho}\cos(\bar{f}-\bar{\phi})e_0\sin E]$$
$$=\frac{1}{(1-e_0^2)}\frac{a_0}{(1+\nu)}\frac{2a_0}{\bar{r}}[(1-e_0^2)\sin E(\cos F-e_0)$$
$$-(1-e_0^2)\sin F(\cos E-e_0)$$
$$-(e_0\sin E)(\cos E-e_0)(\cos F-e_0)$$
$$-e_0\sin^2 E(1-e_0^2)\sin F]$$

which after rearranging becomes

$$=\frac{1}{(1-e_0^2)}\frac{a_0}{(1+\nu)}\frac{2a_0}{\bar{r}}\{\sin E[(1-e_0^2)(\cos F-e_0)$$
$$-e_0(\cos F-e_0)(\cos E-e_0)]$$
$$-\sin F[(1-e_0^2)(\cos E-e_0)+e_0(1-e_0^2)\sin^2 E]\}.$$

But $\sin^2 E=1-\cos^2 E$, and $a_0/\bar{r}=1/(1-e_0\cos E)$, so we have

$$\frac{1}{(1-e_0^2)}\frac{a_0}{(1+\nu)}2[\sin E(\cos F-e_0)-\sin F(1-e_0^2)\cos E]$$

$$=\frac{1}{(1-e_0^2)}\frac{a_0}{1+\nu}2\left[\frac{1}{2}\sin(E+F)+\frac{1}{2}\sin(E-F)-e_0\sin E-\frac{(1-e_0^2)}{2}\sin(F+E)-\frac{(1-e_0^2)}{2}\sin(F-E)\right].$$

Thus, Equation 169 for E' becomes

$$E'=\Lambda r\frac{\partial a_0\Omega}{\partial r}\left[\frac{1}{(1-e_0^2)}\frac{1}{(1+\nu)}(e_0^2\sin(F+E)-(2-e_0^2)\sin(F-E)-2e_0\sin E)\right]. \quad (170)$$

We next determine term F' in terms of E and F: first we rewrite F' as

$$F'=\Lambda r\frac{\partial a_0\Omega}{\partial r}\frac{h^2}{h_0^2}\frac{1}{(1-e_0^2)}\left[-\frac{2\bar{r}}{a_0}\frac{e_0\sin E}{(1-e_0^2)}\left(\frac{\bar{\boldsymbol{\rho}}\cdot\mathbf{r}^0-\bar{\rho}}{a_0}\right)\right]. \quad (171)$$

Considering the bracketed term, we see that

$$[\quad]=-\frac{2\bar{r}}{a_0}\frac{\bar{\rho}}{a_0}\frac{e_0\sin E}{(1-e_0^2)}[\cos(\bar{\phi}-\bar{f})-1]$$

$$=-\frac{2\bar{r}}{a_0}\frac{\bar{\rho}}{a_0}\frac{e_0\sin E}{(1-e_0^2)}(\cos\bar{\phi}\cos\bar{f}+\sin\bar{\phi}\sin\bar{f}-1)$$

which, after some algebra, becomes

$$e_0\sin(F-2E)-e_0\sin F+2e_0\sin E.$$

Putting this into Equation 171, we have

$$F'=\Lambda r\frac{\partial a_0\Omega}{\partial r}\frac{h^2}{h_0^2}\frac{1}{(1-e_0^2)}[e_0\sin(F-2E)-e_0\sin F+2e_0\sin E]. \quad (172)$$

We now return to Equation 161 which was

$$\frac{dW}{dE}=A'+B'+C'+D'+E'+F'+\frac{Sy}{\sqrt{1-e_0^2}}.$$

If we replace A', B', C', D', E', and F' by the values given by Equations 164, 165, 167, 168, 170, and 172, respectively, we can write our final equation for dW/dE as follows:

$$\frac{dW}{dE}=N\Lambda r\frac{\partial a_0\Omega}{\partial r}+M\Lambda\frac{\partial a_0\Omega}{\partial E}+\frac{Sy}{\sqrt{1-e_0^2}}, \quad (173)$$

where Λ and S are given in Equation 159 as

$$S=\left[\frac{\bar{\rho}}{a_0}\frac{\partial W}{\partial F}-\left(W+\frac{h_0}{h}+1\right)e_0\sin F\right],$$

$$\Lambda=\frac{1-\nu^2}{1+\bar{W}}\left(1+\frac{\bar{r}}{a_0}\frac{y}{\sqrt{1-e_0^2}}\right),$$

and M and N are given by

$$(1-e_0^2)M=\frac{h^2}{h_0^2}[2\cos(E-F)-2-e_0\cos(2E-F)-e_0\cos F+2e_0\cos E]+\frac{1}{1+\nu}[(2-e_0^2)\cos(F-E)+e_0^2\cos(E+F)-2e_0\cos F-2e_0\cos E+2e_0^2]-1+2e_0\cos E-\frac{1}{2}e_0^2\cos 2E-\frac{e_0^2}{2}$$

and

$$(1-e_0^2)N=\frac{h^2}{h_0^2}[e_0\sin(F-2E)-e_0\sin F+2e_0\sin E]+\frac{1}{1+\nu}[(2-e_0^2)\sin(E-F)-2e_0\sin E+e_0^2\sin(E+F)]+e_0\sin E-\frac{e_0^2}{2}\sin 2E.$$

SECTION VI

DERIVATIVES OF λ PARAMETERS, EXPRESSIONS FOR $\frac{h_0}{h}$, $\frac{h}{h_0}$, AND ν

THE λ DERIVATIVES

In Section V, we arrived at the final form of the differential equation for dW/dE. We developed it in a form such that all its components are trigonometric series with arguments containing only E, F, and (ω). Now we turn our attention to the much simpler development of similar forms for the derivatives of the λ parameters. Several relationships we have found before are required in this development.

We have previously defined $2N$ and $2K$ as

$$2N=\sigma_0+\theta_0-\sigma-\theta-2\alpha\Delta E \quad \text{(Equation 79)}$$

and

$$2K=\sigma_0-\theta_0-\sigma+\theta+2\eta\Delta E, \quad \text{(Equation 80)}$$

and we have developed the expressions

$$\sin i\frac{d\theta}{dt}=h\frac{\partial\Omega}{\partial Z}r\sin(v-\sigma), \quad \text{(Equation 32)}$$

$$\frac{d\sigma}{dt}=\frac{d\theta}{dt}\cos i, \qquad \text{(Equation 50)}$$

$$\frac{di}{dt}=h\frac{\partial\Omega}{\partial Z}r\cos(v-\sigma), \qquad \text{(Equation 28)}$$

$$r\frac{\partial\Omega}{\partial Z}=\frac{\partial\Omega}{\partial\psi}\cos i. \qquad \text{(Equation 51)}$$

Also, we have

$$n_0\frac{dt}{dE}=\frac{\bar{r}}{a_0}\frac{1-\nu^2}{1+\overline{W}}\left(1+\frac{y}{\sqrt{1-e_0^2}}\frac{\bar{r}}{a_0}\right). \qquad \text{(Equation 155)}$$

But in Equation 159 we defined

$$\Lambda=\frac{1-\nu^2}{1+\overline{\overline{W}}}\left(1+\frac{y}{\sqrt{1-e_0^2}}\frac{\bar{r}}{a_0}\right),$$

so we can write

$$\frac{dt}{dE}=\frac{\bar{r}}{n_0a_0}\Lambda \tag{174}$$

$$\frac{dN}{dt}=-\frac{1}{2}\frac{d\sigma}{\partial t}-\frac{1}{2}\frac{d\theta}{dt}-\alpha\frac{dE}{dt}. \tag{175}$$

Substituting Equations 32 and 50 into this yields

$$\frac{dN}{dt}=-\frac{1}{2}\frac{d\theta}{dt}\cos i-\frac{1}{2}\frac{1}{\sin i}h\frac{\partial\Omega}{\partial Z}r\sin(v-\sigma)-\alpha\frac{dE}{dt}$$

or

$$\frac{dN}{dt}=-\alpha\frac{dE}{dt}-\frac{1}{2}h\frac{\partial\Omega}{\partial Z}r\sin(v-\sigma)(\cot i+\csc i)$$

$$=-\alpha\frac{dE}{dt}-\frac{1}{2}h\frac{\partial\Omega}{\partial Z}r\sin(v-\sigma)\cot\frac{i}{2}. \tag{176}$$

We now differentiate our original definition from Equation 88,

$$\lambda_1=\sin\frac{i}{2}\cos N,$$

and get

$$\frac{d\lambda_1}{dt}=\frac{1}{2}\cos\frac{i}{2}\frac{di}{dt}\cos N-\sin\frac{i}{2}\sin N\frac{dN}{dt}$$

from which, using Equations 28 and 176, we get

$$\frac{d\lambda_1}{dt}=\frac{1}{2}\cos\frac{i}{2}\cos N\left(h\frac{\partial\Omega}{\partial Z}r\cos(v-\sigma)\right)$$
$$+\left(\sin\frac{i}{2}\right)\alpha\frac{dE}{dt}\sin N$$
$$+\left(\sin\frac{i}{2}\right)\frac{1}{2}h\frac{\partial\Omega}{\partial Z}r\cot\frac{i}{2}\sin(v-\sigma)\sin N$$

or

$$\frac{d\lambda_1}{dt}=\alpha\lambda_2\frac{dE}{dt}+\frac{1}{2}h\frac{\partial\Omega}{\partial Z}r\left[\cos\frac{i}{2}\cos N\cos(v-\sigma)+\cos\frac{i}{2}\sin(v-\sigma)\sin N\right], \tag{177}$$

since λ_2 was originally defined as

$$\lambda_2=\sin\frac{i}{2}\sin N.$$

Recasting the second term of Equation 177 we have

$$\frac{d\lambda_1}{dt}=\alpha\lambda_2\frac{dE}{dt}+\frac{1}{2}a_0h(1+\nu)\frac{\partial\Omega}{\partial Z}\left[\frac{\bar{r}}{a_0}\cos\frac{i}{2}\cos N\cos(v-\sigma)+\frac{\bar{r}}{a_0}\cos\frac{i}{2}\sin(v-\sigma)\sin N\right]. \tag{178}$$

The bracketed term of Equation 178 can be written

$$[\quad]=\frac{\bar{r}}{a_0}\cos\frac{i}{2}\cos[N-(v-\sigma)]. \tag{179}$$

Since $v-\sigma=\bar{f}+(\omega)+N+K$, from Equations 83 and 86, the right-hand side of Equation 179 is

$$\frac{\bar{r}}{a_0}\cos\frac{i}{2}\{\cos K\cos[\bar{f}+(\omega)]-\sin K\sin[\bar{f}+(\omega)]\}$$
$$=\frac{\bar{r}}{a_0}\cos[\bar{f}+(\omega)]\cos\frac{i}{2}\cos K$$
$$-\frac{\bar{r}}{a_0}\sin[\bar{f}+(\omega)]\cos\frac{i}{2}\sin K.$$

But we know from original definitions in Section III that

$$\frac{\bar{r}}{a_0}\cos[\bar{f}+(\omega)]=l, \qquad \text{(Equation 93)}$$

$$\frac{\bar{r}}{a_0}\sin[\bar{f}+(\omega)]=m, \qquad \text{(Equation 94)}$$

$$\left.\begin{aligned}\cos\frac{i}{2}\cos K&=\lambda_4,\\ \cos\frac{i}{2}\sin K&=\lambda_3,\end{aligned}\right\} \qquad \text{(Equation 88)}$$

so the right-hand side becomes

$$\{\lambda_4 l-\lambda_3 m\};$$

and we have from Equation 178 that

$$\frac{d\lambda_1}{dt}=\alpha\lambda_2\frac{dE}{dt}+\frac{1}{2}a_0h(1+\nu)\frac{\partial\Omega}{\partial Z}(\lambda_4 l-\lambda_3 m). \quad (180)$$

But, since

$$r\frac{\partial\Omega}{\partial Z}=\frac{\partial\Omega}{\partial\psi}\cos i \qquad \text{(Equation 51)}$$

and

$$\frac{dt}{dE}=\frac{\bar{r}}{n_0a_0}\Lambda, \qquad \text{(Equation 174)}$$

we can write Equation 180, after multiplying through by dt/dE, as

$$\frac{d\lambda_1}{dE}=\alpha\lambda_2+\frac{1}{2}\frac{h}{n_0}(1+\nu)\frac{\bar{r}}{r}\frac{\partial\Omega}{\partial\psi}\cos i(l\lambda_4-m\lambda_3)\Lambda.$$

But we know

$$n_0=a_0^{-3/2},$$

$$h_0=\frac{1}{\sqrt{a_0(1-e_0^2)}},$$

and

$$r=\bar{r}(1+\nu),$$

so finally we have

$$\frac{d\lambda_1}{dE}=\alpha\lambda_2+\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i(l\lambda_4-m\lambda_3)\Lambda. \quad (181)$$

The derivatives of the other three λ parameters are developed in precisely the same manner as was $d\lambda_1/dE$. The final forms they assume are:

$$\frac{d\lambda_2}{dE}=-\alpha\lambda_1+\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i(-\lambda_3 l-\lambda_4 m)\Lambda, \quad (182)$$

$$\frac{d\lambda_3}{dE}=\eta\lambda_4+\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i(\lambda_2 l+\lambda_1 m)\Lambda, \quad (183)$$

$$\frac{d\lambda_4}{dE}=-\eta\lambda_3+\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i(-\lambda_1 l+\lambda_2 m)\Lambda. \quad (184)$$

In each of these four derivatives, $\cos i$ is expressed as

$$\cos i=\lambda_4^2+\lambda_3^2-\lambda_2^2-\lambda_1^2. \quad (185)$$

By inspection, we notice that these four expressions are set up in such a way that they allow iteration. Each derivative is expressed in terms of the λ's themselves, and uses the series determined in the previous iteration for these λ's. Also, it is readily seen that every component of the derivatives can be expressed as a trigonometric series in E and (ω). No F's will occur anywhere in these derivatives. These are the final forms which are formally integrated to obtain the new values of the λ's in any iteration.

EXPRESSIONS FOR $\frac{h_0}{h}$, $\frac{h}{h_0}$, AND ν

We have shown previously (Equations 135 and 137) that the W function can be separated as follows:

$$W=\Xi+\Upsilon\cos F+\Psi\sin F,$$

where Ξ, Υ, and Ψ are series containing E and (ω) alone, and are thus independent of F. And we have also shown that

$$\Xi+e_0\Upsilon=-1-\frac{h_0}{h}+\frac{2h}{h_0}.$$

If we set

$$\frac{h_0}{h}=1+\Delta, \quad (186)$$

we can write

$$\frac{h}{h_0}=1-\Delta+\Delta^2-\Delta^3+\ldots, \quad (187)$$

which gives

$$\Xi+e_0\Upsilon=-1-(1+\Delta)+2(1-\Delta+\Delta^2-\Delta^3+\ldots),$$

or

$$\Xi+e_0\Upsilon=-3\Delta+2(\Delta^2-\Delta^3+\Delta^4-\ldots).$$

Writing this in a form suitable for iteration we have

$$\Delta=-\frac{1}{3}(\Xi+e_0\Upsilon)+\frac{2}{3}(\Delta^2-\Delta^3+\Delta^4-\ldots). \quad (188)$$

From Equations 136 and 186, it immediately follows that

$$\frac{h}{h_0}=1+\frac{1}{2}(\Xi+e_0\Upsilon+\Delta). \quad (189)$$

The series for W determines Ξ and Υ. Then, from Equations 186, 188, and 189 the remaining quantities Δ, h_0/h, and h/h_0 are found. This method of iteration is desirable in that it avoids the use of further integration to determine these series. Once the series h_0/h and h/h_0 are found, the perturbations of the radius vector ν can be readily found from our original definition of $\overline{W}$,

$$\overline{W}=-1-\frac{h_0}{h}+\frac{2h_0}{h}\frac{1}{1+\nu} \quad \text{(Equation 125)}$$

which, after substitution of $h_0/h=(1+\Delta)$ can be written suitably for iteration as follows:

$$\nu=\frac{1}{2}(\Delta-\overline{W})-\frac{1}{2}(\overline{W}+\Delta)\nu. \qquad (190)$$

Again we are avoiding additional integration by the use of this iteration process. The $\overline{W}$ series used in this iteration process is obtained simply by replacing the F's by E's in the series generated for W.

We now have developed all the necessary expressions which are used in the theory. In Equations 173, 154, 181, 182, 183, 184, 186, 189 and 190, we have developed series forms for

$$\frac{dW}{dE}, \frac{dn_0\delta z}{dE}, \frac{d\lambda_1}{dE}, \frac{d\lambda_2}{dE}, \frac{d\lambda_3}{dE}, \frac{d\lambda_4}{dE}, \frac{h_0}{h}, \frac{h}{h_0}, \text{ and } \nu,$$

respectively. Our next problem is to add the constants of integration which are determined in very specific ways; the method of determination of these constants has led to much of the controversy and confusion circulating about Hansen's theory.

SECTION VII

DETERMINATION OF THE SECULAR MOTIONS y, α, AND η AND THE CONSTANTS OF INTEGRATION

DISCUSSION

In Musen's original paper (Reference 6) he points out that the "real constants of integration are the six elements $a_0, e_0, g_0, \theta_0, i_0$, and $\omega_0=\pi_0-\sigma_0$." It is understood that any theory which attempts to give the motion of an orbiting body in a disturbing force field requires a complete solution of the differential equations for the time rate of change of the six elements which define the orbit. The constants of integration of the set of six differential equations would clearly be some starting values of the six elements. In Hansen's theory, the differential equations of the elements are disguised and combined in the differential equations for W, $n_0\delta z$, and the λ's. Intuitively, therefore, we would expect the constants of integration of these new differential equations to be functions of the elements alone. This is another way of stating that Hansen's theory introduces no new constants of integration, other than the elements.

Musen's next statement is that these elements "do not have any simple kinematical or geometrical meaning; in particular, no moment of time exists for which these elements are osculating." The point here is that the "real constants of integration" are mean values of the osculating elements, which Hansen assumes to be known exactly. However, in practice, the mean values are never known as an initial condition.

The standard technique for determining them is to assume a set of elements at some time t_0, use them to predict an orbit, and then to make an orbit correction assuming that residuals are a result of inaccurate starting values of the elements. Then by some correction device, new starting values of the elements are found and the process is repeated. In general, the first approximation is the set of osculating elements determined by observation at some time t_0. After several corrections, the values of the elements should converge to the "mean elements" which are Musen's "real constants of integration." However, since these elements do not appear explicitly in the constants of integration of our development, let us turn to the matter at hand.

At this stage of the exposition, it behooves us to recall the basic ideas that have governed the development thus far. After having introduced an auxiliary ellipse into the osculating orbit plane, we established that the ellipse moves uniformly in this XY plane with respect to the eccentric anomaly. That is, the perigee moves with an angular velocity $y(dE/dt)$ in the XY plane. The perturbations of the real satellite's motion in the orbit plane are combined in the two perturbations, ν and $n_0\delta z$. The perturbations of the orbit plane, that is, of the elements θ, σ, and i, are contained in the four interdependent λ parameters. By the way in which we introduced and defined the secular motions y, α, and η, we have established the following important condition: The four parameters λ_1, λ_2, λ_3, λ_4 and the two perturbations ν and $n_0\delta z$ can contain *no* secular terms. The secular motions α and η are defined as containing together all the secular motions of σ and θ, and we know that i physically has no secular motion. Also, ν clearly has no secular motion, and the secular motion of $n_0\delta z$ is all contained in y.

Since the development uses the iteration process, the above condition dictates the evaluation of the various constants of integration. In most cases, if one of the above six series contained a constant term, integration of the series would produce a secular term, which is not allowed. This is the logic behind the determination of the secular motions, and the arbitrary constants of integration.

DETERMINATION OF y

Now let us examine each integral separately. The first integration performed is that of dW/dE. As was explained in Sections IV and V, we finally were able to develop the differential equation dW/dE as a trigonometric series whose arguments are of the form $[iE+j(\omega)+kF]$. From the integration of this derivative, we will be able to perform the important determination of y, the secular motion of the perigee. The condition that allows this result has been given: $n_0\delta z$ cannot contain secular terms.

The determination of the constant of integration of dW/dE is a result of the condition stated above, as well as the additional condition that $n_0\delta z$ can contain no term of the form sin E. This second condition derives simply from the original *definition* that $n_0\delta z$ *is the deviation of the real satellite from Kepler's equation,* and Kepler's equation already contains a sin E term. It should be emphasized that this is strictly a result of the way in which $n_0\delta z$ is defined. We will presently show that these two conditions dictate that the constant of integration in W be $c_0+c_1 \cos F$.

Thus, the final form of W is as follows:

$$W=\sum_i\sum_j\sum_k C_{i,j,k} \cos [iE+2j(\omega)+kF] +\sum_i\sum_j\sum_k S_{i,j,k} \sin [iE+(2j+1)(\omega)+kF] +c_0+c_1 \cos F, \quad (191)$$

where $k=-1, 0, +1$.

The limits on k and the form of the coefficients of (ω) are not arbitrary but are purely results of the way in which various terms combine. The expression above is the general form of W in every iteration. We need to know this form before we can explain the determination of y.

Now, as can be seen above, and in accordance with the condition that $n_0\delta z$ have no secular terms, W is not allowed to contain any secular terms. Therefore, since the integration is performed with respect to E, all terms which have F alone in the argument must be removed before integration. If they were not removed, W would contain terms of the form (sin F)E, a secular term. There are no constant terms in dW/dE, as will be seen by inspection of Equation 173, and so no secular terms arise from this source. We shall presently show that the only term of the series for dW/dE which contains F alone in the argument is of the form (A_1+yA_2) sin F where A_1 and A_2 are constants. Thus, y is determined in such a way that this term disappears.

To show that dW/dE contains a term of this form as the only term in F alone, we must examine Equation 173 which is

$$\frac{dW}{dE}=N\Lambda r \frac{\partial a_0\Omega}{\partial r}+M\Lambda \frac{\partial a_0\Omega}{\partial E}+\frac{Sy}{\sqrt{1-e_0^2}}.$$

We know the following:

1. The quantities $r(\partial a_0\Omega/\partial r)$ and $\partial a_0\Omega/\partial E$ are series containing arguments of the form $iE+j(\omega)$ only, because both are the results of "barring" $r(\partial\Omega^*/\partial r)$ and $\partial\Omega^*/\partial E$, that is, replacing F by E.
2. Also, $\overline{W}$ and ν are series in E and (ω) alone, since $\overline{W}$ is W with the F's replaced by E's and ν is a function of $\overline{W}$.
3. The quantity $\overline{r}/a_0=1-e_0 \cos E$.
4. The quantity $\overline{\rho}/a_0=1-e_0 \cos F$.
5. The quantities h_0/h, h^2/h_0^2, and $1/(1+\nu)$ are series in E and (ω) alone since they are all functions of $\overline{W}$.

We will now show that the products $N\Lambda r(\partial a_0\Omega/\partial r)$, and $M\Lambda(\partial a_0\Omega/\partial E)$ can have only terms in $\pm F$ alone, that is, no terms of $\pm kF$ can appear with $|k|>1$. After this we show that $N\Lambda r(\partial a_0\Omega/\partial r)$ and $M\Lambda(\partial a_0\Omega/\partial E)$ can contain only sin F terms and can have no cos F terms.

Examining $N\Lambda r(\partial a_0\Omega/\partial r)$, we can see from items 2 and 3 above that Λ contains only terms of E and (ω) since

$$\Lambda=\frac{1-\nu^2}{1+\overline{W}}\left(1+\frac{y}{\sqrt{1-e_0^2}}\frac{\overline{r}}{a_0}\right),$$

and we know from item 1 that $r(\partial a_0\Omega/\partial r)$ contains only E and (ω), *not* F. Furthermore, we know that M and N contain terms of the form $\cos (mE+nF)$ where $-\infty<m<\infty$ and $n=-1, 0, 1$ (see Equation 173). This means no multiples of F other than $-1, 0, 1$ exist in terms of M and N. Therefore, it is evident that the product $N\Lambda r(\partial a_0\Omega/\partial r)$ contains no terms with multiples of F other than $-1, 0, 1$. Exactly the same arguments can be used in the case of $M\Lambda(\partial a_0\Omega/\partial E)$.

Having seen that no terms of arguments kF with $|k|>1$ can appear in the first two terms of dW/dE, we want to show that only $\sin F$ terms can appear, and no $\cos F$ terms are possible. The proof of this rests squarely on two facts that are ascertained by close inspection of the series forms that go into Ω and Λ. This inspection shows the forms to be as follows. (In these series expressions, C and S represent only general coefficients.)

$$\Omega=\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)] \quad \text{(See Equation 118).}$$

The two partials of Ω then are easily found to be of the form:

$$\frac{\partial\Omega}{\partial E}=\sum_i\sum_j C_{i,j}\cos[iE+(2j+1)(\omega)]-\sum_i\sum S_{i,j}\sin[iE+2j(\omega)]$$

and

$$r\frac{\partial\Omega}{\partial r}=\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)],$$

and the series Λ can be expressed as

$$\Lambda=\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)].$$

We know further from Equation 173 that M and N take the general form

$$M=\sum_i\sum_k C_{i,k}\cos(iE+kF)$$

and

$$N=\sum_i\sum_k S_{i,k}\sin(iE+kF).$$

Using these five general forms, we can see that the following general forms of the product series are produced:

$$N\Lambda r\frac{\partial a_0\Omega}{\partial r}=\sum_i\sum_j\sum_k C_{i,j,k}\cos[iE+(2j+1)(\omega)+kF]+\sum_i\sum_j\sum_k S_{i,j,k}\sin[iE+2j(\omega)+kF]$$

and

$$M\Lambda\frac{\partial a_0\Omega}{\partial E}=\sum_i\sum_j\sum_k C_{i,j,k}\cos[iE+(2j+1)(\omega)+kF]+\sum_i\sum_j\sum_k S_{i,j,k}\sin[iE+2j(\omega)+kF].$$

It is evident from these series forms that neither of the two terms can contain $\cos F$ terms since all cosine terms must also have an (ω). The sine terms can clearly be in F alone, but only $\pm F$, as we have shown previously. Thus, we know that only one term in the series for

$$N\Lambda r\frac{\partial a_0\Omega}{\partial r}+M\Lambda\frac{\partial a_0\Omega}{\partial E}$$

can contain F alone, and that term will be of the form $A_1\sin F$.

Let us now examine the third term of dW/dE, that is,

$$\frac{Sy}{\sqrt{1-e_0^2}},\ \text{where } S=\frac{\bar\rho}{a_0}\frac{\partial W}{\partial F}-\left(W+1+\frac{h_0}{h}\right)e_0\sin F. \quad \text{(See Equation 173).}$$

We can examine S alone, since y and e_0 are constants. First let us look at W. We can show, by examination of Equation 191 that

$$W=c_0+c_1\cos F+\text{Periodic Terms},$$

but contains no terms other than $c_1\cos F$ which have F alone in the argument. We know this because all such terms are removed from dW/dE before it is integrated. The $c_1\cos F$ term is part of the added constant of integration. From this fact we know that

$$\frac{\partial W}{\partial F}=-c_1\sin F+\text{Periodic Terms},$$

again containing no terms in F alone other than $-c_1\sin F$. So we see that, since

$$\frac{\bar\rho}{a_0}=1-e_0\cos F,$$

the first term of S,

$$\frac{\bar\rho}{a_0}\frac{\partial W}{\partial F},$$

can be written

$$\frac{\bar{\rho}}{a_0}\frac{\partial W}{\partial F}=(1-e_0\cos F)(-c_1\sin F+\text{Periodic Terms in } E \text{ and } F)$$

$$=-c_1\sin F+\frac{1}{2}c_1e_0\sin 2F+\text{Periodic Terms in } E \text{ and } F.$$

The second term of S is

$$\left(W+1+\frac{h_0}{h}\right)e_0\sin F=We_0\sin F+e_0\sin F+\frac{h_0}{h}e_0\sin F$$

$$=c_0e_0\sin F+c_1e_0\cos F\sin F+e_0\sin F+\text{Periodic Terms in } E \text{ and } F,$$

since h_0/h is a series in E. Therefore, S becomes

$$S=[-c_1-e_0(c_0+1)]\sin F+\text{Periodic Terms in } E \text{ and } F.$$

So,

$$\frac{dW}{dE}=(A_1+yA_2)\sin F+\text{Periodic Terms in } E \text{ and } F, \quad (192)$$

where

$$-A_2=[c_1+e_0(c_0+1)]\sqrt{1-e_0^2}.$$

Thus, from Equation 192 we are able to see that proper adjustment of y leaves dW/dE with the desired periodic terms in E, (ω), and F. We now turn to the constant of integration problem.

CONSTANT OF INTEGRATION OF W

As we have stated before, the condition that dictates the form of the constant of integration in W is that $n_0\delta z$ is defined to be the deviation from Kepler's equation:

$$E-e_0\sin E=g_0+n_0(t-t_0)+n_0\delta z. \quad \text{(Equation 75)}$$

Therefore, since a constant term, a secular term, and a term of the form $\sin E$ all appear in Kepler's equation, none of these is allowed in $n_0\delta z$.

Returning to the differential equation (Equation 154) we have

$$\frac{dn_0\delta z}{dE}=\frac{\overline{W}+\nu^2}{1+\overline{W}}\frac{\bar{r}}{a_0}-\left(\frac{1-\nu^2}{1+\overline{W}}\right)\frac{y}{\sqrt{1-e_0^2}}\left(\frac{\bar{r}}{a_0}\right)^2$$

which after minor rearranging can be written

$$\frac{dn_0\delta z}{dE}=\overline{W}\frac{\bar{r}}{a_0}+\left(\frac{\nu^2-\overline{W}^2}{1+\overline{W}}\frac{\bar{r}}{a_0}-\frac{y}{\sqrt{1-e_0^2}}\frac{1-\nu^2}{1+\overline{W}}\frac{\bar{r}^2}{a_0^2}\right). \quad (193)$$

From this equation, we can derive a form convenient for the use of the iteration process. In any given iteration loop, the factors and the terms in parentheses are all of the previous iteration, with the exception of y which is the latest value found. For $\overline{W}$ in the first term on the right-hand side of Equation 193 we "bar" operate on Equation 191 and get:

$$\overline{W}=c_0+c_1\cos E+\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)]. \quad (194)$$

It should now be clear that we set the form of the constant of integration in W to be $c_0+c_1\cos F$ in order that we could use it to remove the constant terms and the $\cos E$ term of the series for $dn_0\delta z/dE$. If not removed, these terms would give secular terms and a $\sin E$ term after integration. Thus, c_0 and c_1 are set in such a way as to make the terms disappear before the integration.

The subsequent expressions for c_0 and c_1 are obtained as follows. Since we are using previously generated series for the terms in parentheses in Equation 193, and the new series for $\overline{W}$ in the term $\overline{W}(\bar{r}/a_0)$, we have:

$$\frac{dn_0\delta z}{dE}=(c_0+c_1\cos E)(1-e_0\cos E)+\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)], \quad (195)$$

since $\bar{r}/a_0=1-e_0\cos E$. By means of the standard trigonometric identities, Equation 195 becomes

$$\frac{dn_0\delta z}{dE}=c_0-\frac{1}{2}c_1e_0+(c_1-c_0e_0)\cos E-\frac{1}{2}c_1e_0\cos 2E+\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)]. \quad (196)$$

The two conditions that the constant terms and terms of the form $\cos E$ must be eliminated dictate that

$$c_0-\frac{1}{2}c_1e_0+C_{0,0}=0 \quad (197)$$

and

$$c_1-c_0e_0+C_{1,0}=0. \quad (198)$$

And a further consequence is that the coefficient of $\cos 2E$ in Equation 196 must be added to the coefficient $C_{2,0}$ of the series. From Equations 197 and 198, we are able to write

$$c_0=-\left(\frac{e_0C_{1,0}+2C_{0,0}}{2-e_0^2}\right) \tag{199}$$

and

$$c_1=-\left(\frac{2e_0C_{0,0}+2C_{1,0}}{2-e_0^2}\right). \tag{200}$$

[Note: These values can also be found by iteration. See Appendix B, steps 61–67.]

Putting these values into the series (Equation 194) gives a final series form of $\overline{W}$. We now are able to write that $dn_0\delta z/dE$ consists of periodic terms only. When $dn_0\delta z/dE$ is integrated, the constant of integration in $n_0\delta z$ is set equal to zero, as a result of the condition that $n_0\delta z$ contain no constants. And so we are able to develop series for $\overline{W}$ and $n_0\delta z$ for each iteration. We now must turn to the secular motions α and η and the constants involved in the integration of the λ differential equations.

DETERMINATION OF α AND η

The differential equations for the λ's, Equations 181 through 184, provide an easy determination of α and η. Recalling that the λ's were defined in such a way that they contain no secular terms, it is at once clear that the derivatives can have no constant terms. The expressions for $d\lambda_2/dE$ and $d\lambda_3/dE$ are especially convenient here. They are given by

$$\frac{d\lambda_2}{dE}=-\alpha\lambda_1+\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i(-\lambda_3 l-\lambda_4 m)\Lambda \tag{201}$$

and

$$\frac{d\lambda_3}{dE}=+\eta\lambda_4+\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i(\lambda_2 l+\lambda_1 m)\Lambda. \tag{202}$$

We know that the series for λ_1 and λ_4 contain constant terms, since the first approximations to λ_1 and λ_4 are $\sin(i_0/2)$ and $\cos(i_0/2)$, respectively. We also know, from the way they are defined, that λ_2 and λ_3 can contain no constant terms. The first approximations are $\lambda_2=\lambda_3=0$. (We point out again that in Equations 201 and 202, the λ's that appear on the right-hand sides are λ's of the previous iteration.)

If we let

$$T=\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}(\cos i)\Lambda, \tag{203}$$

$$b_1=\text{constant part of } \lambda_1, \tag{204}$$

and

$$b_2=\text{constant part of } T(-\lambda_3 l-\lambda_4 m), \tag{205}$$

then we remove the constant part of $d\lambda_2/dE$ by setting

$$\alpha=\frac{b_2}{b_1}. \tag{206}$$

In a similar manner, taking the expression for $d\lambda_3/dE$, Equation 202, and letting

$$b_3=\text{constant part of } T(\lambda_2 l+\lambda_1 m) \tag{207}$$

and

$$b_4=\text{constant part of } \lambda_4, \tag{208}$$

we have

$$\eta=-\frac{b_3}{b_4}. \tag{209}$$

We now are able to integrate formally all the λ derivatives. Upon integration, we are able to set the constants of integration in λ_2 and λ_3 equal to zero, since our conditions dictate that λ_2 and λ_3 contain periodic terms only. It is a little more involved to find the arbitrary constants in λ_1 and λ_4. Two further conditions must be satisfied, and these govern the arbitrary constants in λ_1 and λ_4. They are:

1. That $\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2=1$. (See Equation 89.)
2. That the principal term in the latitude must have the form $\sin i_0 \sin[\bar{f}+(\omega)]$. (See Equation 87).

Since we had the form of the latitude

$$\psi=2(\lambda_1\lambda_4-\lambda_2\lambda_3)\sin[\bar{f}+(\omega)]$$
$$+(2\lambda_2\lambda_4+\lambda_1\lambda_3)\cos[\bar{f}+(\omega)], \quad \text{(see Equation 91)}$$

condition 2 dictates that the constant part of $2(\lambda_1\lambda_4-\lambda_2\lambda_3)$ be $\sin i_0$.

Now, we have stated previously that parts of the constant terms of λ_1 and λ_4 are the first approximationss, in $(i_0/2)$ and $\cos(i_0/2)$, respectively, and we know that neither λ_1 nor λ_4 can contain secular terms. Let us arbitrarily pick a form for the constants of integration which will enable us to satisfy the two conditions above. We write the final forms of λ_1 and λ_4:

$$\lambda_1=\sin\frac{i_0}{2}+\frac{1}{2}(A+B)+\delta\lambda_1, \tag{210}$$

$$\lambda_4=\cos\frac{i_0}{2}+\frac{1}{2}(A-B)+\delta\lambda_4, \tag{211}$$

where A and B must be determined; and

$$\delta\lambda_1=\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)], \tag{212}$$

$$\delta\lambda_4=\sum_i\sum_j C_{i,j}\cos[iE+2j(\omega)]+\sum_i\sum_j S_{i,j}\sin[iE+(2j+1)(\omega)]. \tag{213}$$

These forms of λ_1 and λ_4 are actual series forms arrived at after several iterations. Likewise, we find that

$$\delta\lambda_2=\sum_i\sum_j S_{i,j}\sin[iE+2j(\omega)]+\sum_i\sum_j C_{i,j}\cos[iE+(2j+1)(\omega)], \tag{214}$$

$$\delta\lambda_3=\sum_i\sum_j S_{i,j}\sin[iE+2j(\omega)]+\sum_i\sum_j C_{i,j}\cos[iE+(2j+1)(\omega)], \tag{215}$$

where in final form,

$$\lambda_2=\delta\lambda_2, \tag{216}$$

$$\lambda_3=\delta\lambda_3. \tag{217}$$

Taking the final forms of the λ's (Equations 210, 211, 216, and 217), we apply them into the two conditions

$$\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2=1 \tag{218}$$

and

$$\text{constant part of } 2(\lambda_1\lambda_4-\lambda_2\lambda_3)=\sin i_0. \tag{219}$$

In this procedure we must remember that even though λ_2 and λ_3 do not contain constant terms, their product terms will. The following results are obtained by setting the constant parts of each side of Equations 218 and 219 equal, and by setting the periodic parts of each side equal. Clearly, in this manner, the periodic parts vanish from the equations and we have the forms from which $A/2$ and $B/2$ are determined.

After making the substitutions, we find that

$$\frac{1}{2}(A^2+B^2)+A\left(\cos\frac{i_0}{2}+\sin\frac{i_0}{2}\right)-B\left(\cos\frac{i_0}{2}-\sin\frac{i_0}{2}\right)+[\text{constant part of }(\delta\lambda_1^2+\delta\lambda_2^2+\delta\lambda_3^2+\delta\lambda_4^2)]=0 \tag{220}$$

and

$$\frac{1}{2}(A^2-B^2)+A\left(\cos\frac{i_0}{2}+\sin\frac{i_0}{2}\right)+B\left(\cos\frac{i_0}{2}-\sin\frac{i_0}{2}\right)+[\text{constant part of }2(\delta\lambda_1\delta\lambda_4-\delta\lambda_2\delta\lambda_3)]=0. \tag{221}$$

Solving Equations 220 and 221 for A and B, we get:

$$A^2+2A\left(\cos\frac{i_0}{2}+\sin\frac{i_0}{2}\right)+\text{constant part of }[(\delta\lambda_1+\delta\lambda_4)^2+(\delta\lambda_2-\delta\lambda_3)^2]=0 \tag{222}$$

and

$$B^2-2B\left(\cos\frac{i_0}{2}-\sin\frac{i_0}{2}\right)+\text{constant part of }[(\delta\lambda_1-\delta\lambda_4)^2+(\delta\lambda_2+\delta\lambda_3)^2]=0. \tag{223}$$

It is preferable to solve for A and B by iteration rather than by the quadratic method, so we rewrite Equations 222 and 223 as

$$\frac{A}{2}=\frac{-\frac{H}{4}-\left(\frac{A}{2}\right)^2}{\cos\frac{i_0}{2}+\sin\frac{i_0}{2}}, \tag{224}$$

$$\frac{B}{2}=-\frac{\frac{G}{4}+\left(\frac{B}{2}\right)^2}{\cos\frac{i_0}{2}-\sin\frac{i_0}{2}}, \tag{225}$$

where H and G are the bracketed terms of Equations 222 and 223, respectively, and the $A/2$ and $B/2$ used in the right-hand sides are the values obtained in the previous iterations.

We have now described completely the technique and reasoning used in the determination of the secular motions y, α, and η, and the constants of integration in W, $n_0\,\delta z$, and the four λ's. These six formal integrations are the only ones which are performed. All other determinations use the iteration process, as we have shown. The next section attempts to outline in detail the entire procedure used in the generation of the final series forms which are used in the eventual predictions.

SECTION VIII

PROCEDURE USED IN THE GENERATION OF FINAL SERIES FORMS

DISCUSSION

In the preceding sections we have developed explicit expressions for all the equations which must be solved in the problem. It is now possible to discuss the entire procedure for the generation of final series forms, using the various expressions found.

It is important to emphasize one point. Throughout the process of iteration, our goal is to ascertain final, "good" series expressions for the following functions: $n_0\delta z$, ν, λ_1, λ_2, λ_3, and λ_4. We start out with approximations to these series as follows:

$$
\begin{aligned}
n_0\delta z &= 0, \\
\nu &= 0, \\
\lambda_1 &= \sin \frac{\dot{i}_0}{2}, \\
\lambda_2 &= 0, \\
\lambda_3 &= 0, \\
\lambda_4 &= \cos \frac{\dot{i}_0}{2}.
\end{aligned}
$$

Out of the iteration process, we eventually converge to series which express accurately these six functions as functions of the eccentric anomaly of the fictitious satellite E and the mean argument of perigee (ω), which is in turn a function of E. Out of the process come the three secular motions y, α, and η. Conceptually speaking, the entire procedure is one of separating all secular motions from the periodic motions. Thus, series are generated which give the motion of the fictitious satellite as defined, and give the relationship between the two coordinate systems. And so, the final value of y and the final series form of $n_0\delta z$ provide that all the periodic angular perturbations of the satellite in the orbit plane are contained in the mean anomaly, and the secular motion in the plane due to disturbing forces is contained in the secular motion of the perigee.

We must remember, therefore, a fact that is often overlooked and consequently is the cause of consternation in studying this theory. None, absolutely none, of the series is *evaluated* until the final series forms are determined after several iterations. Only the *coefficients* of the trigonometric terms are calculated; the terms themselves are not, but are carried along throughout the iteration process. The iteration process is used to arrive at final series forms expressed with numerical coefficients multiplying trigonometric terms of indexed arguments. Once the final series are developed, we go into the final stage where we can predict the position and velocity vectors of the real satellite at some future time t. Thus, there are two major phases of the development: the generation of final series forms, and the calculation of position and velocity at various times.

The first of these phases has been written as a machine program by G. E. Collins of IBM, in collaboration with Paul Herget of the Cincinnati Observatory. Its IBM code name is the General Oblateness Perturbations (GOP) program. The second phase, considerably more straightforward than the first, has been written as a program by the IBM Space Computing Center and is known as the IKINT program. Now, let us trace through the procedure used in the first of these two phases.

For any given satellite, we must begin with nominal values of the three elements a, e, and i. The theory calls for "mean" values of these osculating elements, which we will assume we have. In actuality, these are determined by starting with nominal values, predicting an orbit, running an orbit correction, and then correcting the values of a, e, and i used. But let us say we have good numerical "mean" values of a, e, and i which we designate as a_0, e_0, and i_0. Only these three elements are needed in the development of the series. We also need values of the three geodetic parameters k_2, k_3, and k_4. Next, we need the following first series approximations, each of which is used in an iteration process:

$$
\left.
\begin{array}{ll}
\lambda_1 = \sin \dfrac{\dot{i}_0}{2}; & n_0\delta z = 0; \\[2ex]
\lambda_2 = \lambda_3 = 0; & \dfrac{h_0}{h} = 1; \\[2ex]
\lambda_4 = \cos \dfrac{\dot{i}_0}{2}; & \dfrac{h}{h_0} = 1; \\[2ex]
W = 0; & \\
 & \Delta = 0; \\
\nu = 0; &
\end{array}
\right\} \qquad (226)
$$

and we need first approximations to the constants,

$$y = 0 \text{ and } \frac{A}{2} = \frac{B}{2} = 0.$$

The first step is to develop the series for dW/dE. This involves the partials $\overline{\partial\Omega^*/\partial F}$ and $\overline{r(\partial\Omega^*/\partial r)}$ which are readily found, once Ω^* is developed in terms of ψ^*, where the λ's are those listed in Equation 226 for the first approximation, after which the last series obtained for the λ's are used. Taking Equation 173,

$$\frac{dW}{dE}=N\Lambda r\frac{\partial a_0\Omega}{\partial r}+M\Lambda\frac{\partial a_0\Omega}{\partial E}+\frac{Sy}{\sqrt{1-e_0^2}},$$

we see that N and M contain h/h_0 and $1/(1+\nu)$, so we must use the series forms obtained in the previous iteration. The same is true of the ν, W, and h_0/h that appear in Λ and S. And finally, we see that y appears twice in dW/dE; as a coefficient of S in Equation 173, and in the expression for Λ. In the expression for Λ, the value of y previously determined is used, whereas the y multiplying S is the y to be determined by the removal of sin F terms from dW/dE (see Section VII).

Once dW/dE is determined and the sin F terms are removed, it is formally integrated and the constant of integration $c_0+c_1\cos F$ is added. Keeping c_0 and c_1 undetermined as yet, we "bar" the new series for W to get $\overline{W}$ which has now the terms c_0 and $c_1\cos E$. Keeping this series for $\overline{W}$ available, with c_0 and c_1 still undetermined, we turn to our expression for $dn_0\delta z/dE$

$$\frac{dn_0\delta z}{dE}=\overline{W}\frac{\overline{r}}{a_0}+\left(\frac{\nu^2-\overline{W}^2}{1+\overline{W}}\frac{\overline{r}}{a_0}-\frac{y}{\sqrt{1-e_0^2}}\frac{1-\nu^2}{1+\overline{W}}\frac{\overline{r}^2}{a_0^2}\right).$$

(Equation 193)

Into this expression we put the ν series determined in the previous iteration (or, in the case of the first iteration, $\nu=0$). In the term in parentheses, we also put for $\overline{W}$ the series determined previously. The y is the number just found prior to the integration of dW/dE. Then, in the first term, $\overline{W}(\overline{r}/a_0)$, we put the $\overline{W}$ series just determined and find c_0 and c_1 so that no constant terms or terms of the form cos E appear in $dn_0\delta z/dE$. The expressions for c_0 and c_1 are given in Equations 199 and 200. With c_0 and c_1 evaluated, we formally integrate to get the series for $n_0\delta z$, which contains no additive constant of integration for reasons given in Section VII. We now also have the new series for W and $\overline{W}$ since c_0 and c_1 have been determined.

Our next step is to take the new series for W and write it in the form,

$$W=\Xi+\Upsilon\cos F+\Psi\sin F.$$

This is accomplished by scanning the series W and first picking out all the terms which have no F in the argument, a series we call Ξ. Next we are able to find Υ by putting $F=0$ in all the terms in which F occurs. It should be remembered that k only takes on values of -1, 0, 1 in the series for W (Equation 191). This splitting process simply uses the trigonometric identity for the sine or cosine of the sum of two angles. With the series for Ξ and Υ determined, we turn to our equation for Δ

$$\Delta=-\frac{1}{3}(\Xi+e_0\Upsilon)+\frac{2}{3}(\Delta^2-\Delta^3+\ldots).$$

(Equation 188)

Using the value of Δ of the previous iteration, or for the first iteration $\Delta=0$, in the right-hand side, we solve to get a new series for Δ. Properly, for any iteration, the expression should be written

$$\Delta_{n+1}=-\frac{1}{3}(\Xi+e_0\Upsilon)+\frac{2}{3}(\Delta_n^2-\Delta_n^3+\ldots).\quad(227)$$

In any given iteration of the whole program, that is, when new series for W, $\overline{W}$, $n_0\delta z$, etc., are found, we iterate around Equation 227 n times, until $|\Delta_{n+1}-\Delta_n|<\epsilon$ where ϵ is some arbitrarily chosen small number. This Δ_{n+1} is the Δ to be used in the determination of h/h_0, h_0/h, and ν.

With the Δ series determined, and with the Ξ and Υ series used above, h_0/h and h/h_0 are found simply by substitution in

$$\frac{h_0}{h}=1+\Delta\qquad\text{(Equation 186)}$$

and

$$\frac{h}{h_0}=1+\frac{1}{2}(\Delta+\Xi+e_0\Upsilon).\qquad\text{(Equation 189)}$$

These series for h_0/h, h/h_0, and Δ are then put aside with W and $n_0\delta z$ for use in the next overall iteration. The last series to be found is that for ν,

$$\nu=\frac{1}{2}(\Delta-\overline{W})-\frac{1}{2}(\overline{W}+\Delta)\nu.\qquad\text{(Equation 190)}$$

We use the same type of iteration of this equation that was used to find Δ. Here, the Δ used is the one just found and stored, and $\overline{W}$ is simply the last $\overline{W}$ determined. Again, the expression should properly be written

$$\nu_{n+1}=\frac{1}{2}(\Delta-\overline{W})-\frac{1}{2}(\overline{W}+\Delta)\nu_n, \qquad (228)$$

where n is the number of the iteration of the above equation. The series ν_n originally put on the right-hand side is that determined in the last iteration of the whole program.

THE λ DERIVATIVES

We can now turn to the computation of the λ series and their constants of integration. Taking the expression for $d\lambda_1/dE$ as an example, we can illustrate the procedure used in all the four derivatives (Equations 181 through 184). Thus,

$$\frac{d\lambda_1}{dE}=\alpha\lambda_2+\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i(\lambda_4 l-\lambda_3 m)\Lambda. \qquad (229)$$

It is readily seen that the factor

$$\frac{1}{2}\frac{h}{h_0}\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}(\cos i)\Lambda \qquad (230)$$

occurs in each of the four derivatives. This factor is developed using the h/h_0 series determined in the previous iteration, not the one found in the steps immediately before this. We use the previous series to assure consistency through the overall iteration process. The $\partial\Omega/\partial\psi$ was found by operating with the "bar" operator on $\partial\Omega^*/\partial\psi^*$ and $\cos i$ we have noted to be

$$\cos i=\lambda_4^2+\lambda_3^2-\lambda_2^2-\lambda_1^2,$$

where the λ's are those resulting from the previous iteration. The Λ factor is the same as that used in the formation of dW/dE. As was true of the λ's that occurred in the expression for $\cos i$, all the other λ's on the right-hand side of Equation 229 are those of the previous iteration. This leaves only α undetermined.

The determination of α and η was described in Section VII, on the basis that $d\lambda_2/dE$ and $d\lambda_3/dE$ must contain no constant terms. In any iteration, α and η are determined as described before any of the λ derivatives are integrated, since the expressions contain the value of α or η just determined. Thus, $d\lambda_2/dE$ and $d\lambda_3/dE$ are formed before $d\lambda_1/dE$ or $d\lambda_4/dE$. Once α and η are found, the four derivatives are formally integrated. No constants of integration are added to λ_2 or λ_3, for reasons previously discussed, so the integrated forms are stored as the new λ_2 and λ_3.

The constants of integration added to

$$\int\frac{d\lambda_1}{dE} \text{ and } \int\frac{d\lambda_4}{dE}$$

have also been discussed previously. The forms are

$$\lambda_1=\sin\frac{i_0}{2}+\frac{1}{2}(A+B)+\int\frac{d\lambda_1}{dE}, \qquad (231)$$

$$\lambda_4=\cos\frac{i_0}{2}+\frac{1}{2}(A-B)+\int\frac{d\lambda_4}{dE}, \qquad (232)$$

where A and B are determined by iterations of Equations 224 and 225. As was described in Section VII, H and G in the latter are functions of

$$\int\frac{d\lambda_1}{dE}, \int\frac{d\lambda_2}{dE}, \int\frac{d\lambda_3}{dE}, \text{ and } \int\frac{d\lambda_4}{dE}$$

before the constants of integration are added. The iteration process is to be handled in exactly the same manner as in the cases of Δ and ν, so we could write Equations 224 and 225 as

$$\frac{A_{n+1}}{2}=\frac{-\frac{H}{4}-\left(\frac{A_n}{2}\right)^2}{\cos\frac{i_0}{2}+\sin\frac{i_0}{2}},$$

$$\frac{B_{n+1}}{2}=\frac{\frac{G}{4}+\left(\frac{B_n}{2}\right)^2}{\cos\frac{i_0}{2}-\sin\frac{i_0}{2}}.$$

After the iteration process gives convergence to some $A/2$ and $B/2$, the new λ_1 and λ_4 series are given by Equations 231 and 232.

We have now traced through one entire iteration of the first phase of the solution. We have generated in this iteration new series for W, $\overline{W}$, $n_0\delta z$, h_0/h, h/h_0, ν, Δ, λ_1, λ_2, λ_3, and λ_4. We have also evaluated new values of the constants y, α, η, $A/2$, and $B/2$. With the iteration complete, the test for convergence is made with the secular motions. If

$$|y_{n+1}-y_n|<\epsilon,$$

$$|\alpha_{n+1}-\alpha_n|<\epsilon,$$

and

$$|\eta_{n+1}-\eta_n|<\epsilon,$$

(where ϵ is some arbitrary small number) the series are considered to have converged. If not, the entire process is repeated as outlined. This completes the description of the first phase, that incorporated in the IBM program known as GOP. Appendix B has been included as an aid to those interested in the programming of this development. It lists one possible method of attack but does not represent the only valid computational procedure. As can be seen from Appendix B, this program develops the λ's before it finds the series for h_0/h, h/h_0, and ν. This order is preferable in that the h/h_0 series used in the λ's is that of the previous iteration, as discussed earlier.

We next turn to the second phase of the solution, that of producing a position vector of the real satellite at certain values of t.

SECTION IX

DETERMINATION OF FINAL POSITION AND VELOCITY VECTORS OF THE REAL SATELLITE AND ITS OSCULATING ELEMENTS

INTRODUCTION

The entire development thus far has led us to series which express exactly the perturbations of the real satellite and the motions of the orbit plane as functions of the eccentric anomaly. We now turn to the second phase, the determination of the position vector as a function of time. We know the motion of the fictitious satellite to be given by

$$\bar{r}\sin\bar{f}=a_0\sqrt{1-e_0^2}\sin E$$

and

$$\bar{r}\cos\bar{f}=a_0(\cos E-e_0).$$

Also, we know the relationship between r and $\bar{r}$ to be

$$r=(1+\nu)\bar{r},$$

and the relationship between the eccentric anomaly E of the fictitious satellite and the real time t of the real satellite to be in Kepler's equation

$$E-e_0\sin E=g_0+n_0(t-t_0)+n_0\delta z.$$

Thus, we are in a position to find the radius vector of the real satellite in the rotating XYZ coordinate system. However, we have to perform a rotation operation in order to express the position vector in terms of xyz coordinates, that is, inertial coordinates.

THE ROTATION MATRIX

The XYZ system has been rotated through three angles with respect to the xyz system. If we rotate the position vector through these same three angles with respect to the xyz system, the new components in the XYZ system are the same as those of the nonrotated position vector in the xyz system. Therefore, if we can find a rotation matrix operator which expresses the rotation of XYZ with respect to xyz, we can operate on $\bar{\mathbf{r}}$ with it to get $\mathbf{r}$ in the inertial system.

One important thing to note here is that, for purposes of developing this matrix, we want to rotate our X axis from the x axis to the line in the plane of the orbit, from the origin to the perigee. This allows us to express $\bar{\mathbf{r}}$ in the XYZ system in two components: $\bar{r}\sin\bar{f}$ and $\bar{r}\cos\bar{f}$, where $\bar{f}$ is the true anomaly of the ellipse, that is, the angle between the position vector $\mathbf{r}$ and the line from origin to perigee.

Therefore, to rotate from the xyz inertial, or equatorial system, into the XYZ system where the XY plane is the plane of the orbit, and the X axis goes through the perigee, we rotate through three angles. Starting with the X, Y, and Z axes lying along the x, y, and z axes, we first rotate the XYZ system about Z, through the angle θ. This leaves the X axis lying along the line of the nodes. Next we rotate the XY plane about X, through an angle i, putting the XY plane coincident with the orbit plane. Finally, we rotate the x and y axes about Z again, until X passes through the perigee. Clearly, this final rotation is through an angle $(\pi_0+y\Delta E-\sigma)$.

A rotation about the Z axis through an angle α is given by

$$A_3[\alpha]=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{233}$$

and a rotation about the X axis through an angle β is

$$A_1[\beta]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\beta & -\sin\beta\\ 0 & \sin\beta & \cos\beta\end{bmatrix}. \tag{234}$$

Therefore, our total, or triple rotation, designated by Γ, is clearly:

$$\Gamma = A_3[\theta]A_1[i]A_3[\pi_0 + y\Delta E - \sigma].$$

But from Equations (82), (85), (71), (83) and (86) we know that

$$\theta = (\theta) - N + K \text{ and } (\pi_0 + y\Delta E - \sigma) = (\omega) + N + K,$$

so we write

$$\Gamma = A_3[(\theta)]A_3[K-N]A_1[i]A_3[N+K]A_3[(\omega)],$$

knowing that $A_3[\alpha_1+\alpha_2] = A_3(\alpha_1)A_3(\alpha_2)$. We wish to express $A_3[K-N]A_1[i]A_3[K+N]$ as one matrix in terms of λ_1, λ_2, λ_3, and λ_4. We shall call this matrix ψ.

Making the transformations

$$\begin{Bmatrix} K-N=2\alpha \\ K+N=2\beta \end{Bmatrix} \text{ or } \begin{Bmatrix} K=\alpha+\beta \\ N=\beta-\alpha \end{Bmatrix},$$

we have

$$\psi = A_3[2\alpha]A_1[i]A_3[2\beta].$$

This is given explicitly by

$$\psi = \begin{bmatrix} \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta \cos i & -\cos 2\alpha \sin 2\beta - \sin 2\alpha \cos 2\beta \cos i & \sin 2\alpha \sin i \\ \sin 2\alpha \cos 2\beta + \cos 2\alpha \sin 2\beta \cos i & -\sin 2\alpha \sin 2\beta + \cos 2\alpha \cos 2\beta \cos i & -\cos 2\alpha \sin i \\ \sin 2\beta \sin i & \cos 2\beta \sin i & \cos i \end{bmatrix}$$

which we write as

$$\psi \equiv \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}.$$

Throughout the following development of the $\lambda_{m,n}$ terms, we use the definitions

$$\lambda_1 = \sin \frac{i}{2} \cos N, \qquad \lambda_3 = \cos \frac{i}{2} \sin K,$$

$$\lambda_2 = \sin \frac{i}{2} \sin N, \qquad \lambda_4 = \cos \frac{i}{2} \cos K,$$

and

$$K \equiv \alpha + \beta,$$

$$N \equiv \beta - \alpha,$$

and we have

$$\lambda_1 = \sin \frac{i}{2} \cos (\beta - \alpha),$$

$$\lambda_2 = \sin \frac{i}{2} \sin (\beta - \alpha),$$

$$\lambda_3 = \cos \frac{i}{2} \sin (\beta + \alpha),$$

$$\lambda_4 = \cos \frac{i}{2} \cos (\beta + \alpha).$$

Now, we put

$$\lambda_{11} \equiv \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta \cos i$$

$$= \left[\sin^2\left(\frac{i}{2}\right) + \cos^2\left(\frac{i}{2}\right)\right] \cos 2\alpha \cos 2\beta - \left[\cos^2\left(\frac{i}{2}\right) - \sin^2\left(\frac{i}{2}\right)\right] \sin 2\alpha \sin 2\beta$$

or

$$\lambda_{11} = \cos^2\left(\frac{i}{2}\right)(\cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta) + \sin^2\left(\frac{i}{2}\right)(\cos 2\alpha \cos 2\beta + \sin 2\alpha \sin 2\beta)$$

$$= \cos^2\left(\frac{i}{2}\right) \cos 2(\alpha+\beta) + \sin^2\left(\frac{i}{2}\right) \cos 2(\beta-\alpha)$$

$$= \cos^2\left(\frac{i}{2}\right)[\cos^2(\alpha+\beta) - \sin^2(\alpha+\beta)] + \sin^2\left(\frac{i}{2}\right)[\cos^2(\beta-\alpha) - \sin^2(\beta-\alpha)].$$

Then, employing $\lambda_1, \ldots, \lambda_4$, we have

$$\lambda_{11} = \lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2.$$

The same type of substitution and manipulation gives

$$\lambda_{12} = -2(\lambda_3\lambda_4 + \lambda_1\lambda_2),$$

$$\lambda_{13} = 2(\lambda_1\lambda_3 - \lambda_2\lambda_4),$$

$$\lambda_{21} = 2(\lambda_3\lambda_4 - \lambda_1\lambda_2),$$

$$\lambda_{22} = \lambda_4^2 - \lambda_3^2 - \lambda_1^2 + \lambda_2^2,$$

$$\lambda_{23} = -2(\lambda_1\lambda_4 + \lambda_3\lambda_2),$$

$$\lambda_{31}=2(\lambda_1\lambda_3+\lambda_2\lambda_4),$$

$$\lambda_{32}=2(\lambda_4\lambda_1-\lambda_2\lambda_3),$$

$$\lambda_{33}=\lambda_4^2+\lambda_3^2-\lambda_1^2-\lambda_2^2.$$

We now have our Γ matrix

$$\Gamma=A_3[(\theta)]\begin{bmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\end{bmatrix}A_3[(\omega)], \tag{235}$$

which represents the transformation of coordinates from the osculating system to the equatorial system. Consequently, we will get the position vector in the equatorial system by the operation

$$\mathbf{r}=(1+\nu)\Gamma\begin{bmatrix}a_0(\cos E-e_0)\\ a_0\sin E\sqrt{1-e_0^2}\\ 0\end{bmatrix}, \tag{236}$$

since $a_0(\cos E-e_0)$ and $a_0 \sin E\sqrt{1-e_0^2}$ are the components of $\bar{\mathbf{r}}$ in the X and Y directions, respectively, considering X to be along the line from the origin (focus) to the perigee. Once the x, y, and z components of $\mathbf{r}$ are known, any other coordinates of the satellite can be readily found by elementary trigonometry.

DETERMINATION OF r FOR A GIVEN t

It is immediately evident from Equations 235 and 236 that we can find $\mathbf{r}$ for any E if we know ω_0, θ_0, and E_0 since we have series for ν and the λ's in terms of E and (ω), where

$$(\omega)=\omega_0+(y+\alpha-\eta)E-(y+\alpha-\eta)E_0.$$

We need θ_0 in the Γ matrix, where

$$A_3[(\theta)]=A_3[\theta_0-(\alpha+\eta)E+(\alpha+\eta)E_0].$$

We have already found the final values for y, α, and η. However, to get the components of the position vector at some time t, we must determine the value E which corresponds to that time t. As well, to evaluate any of the series, we need to know the value E_0 which corresponds to t_0.

We proceed as follows: Taking Kepler's equation, we find the E_0 at t_0 by iteration. In this case, Kepler's equation gives

$$E_0-e_0\sin E_0=g_0+n_0(t_0-t_0)+n_0\delta z, \tag{237}$$

where g_0, the mean anomaly at the epoch, must be given. In the first iteration, we solve Equation 237 by letting $n_0\delta z=0$. This gives us a value E_0, which we immediately put into the series generated for $n_0\delta z$:

$$n_0\delta z=\text{Fourier series in}$$

$$\left[iE+j\Big(\omega_0+(y+\alpha-\eta)E\Big)-j(y+\alpha-\eta)E_0\right], \tag{238}$$

which becomes

$$n_0\delta z=\text{Fourier series in }[iE_0+j\omega_0].$$

Taking the $n_0\delta z$, a numerical value, and putting it back into Equation 237, we solve to get the second approximation to E_0, which we put back into Equation 238 for a second approximation to $n_0\delta z$. We repeat this iteration process until

$$|(E_0)_{n+1}-(E_0)_n|<\epsilon,$$

at which point we take E_0 to be known for $t=t_0$. Next, we turn to some time t for which we wish to know the position of the real satellite. Turning first to Kepler's equation, in the first approximation we let $n_0\delta z=0$ and solve for E. Then, we put E and E_0 into the series for $n_0\delta z$ and solve for a numerical value, which we then put into Kepler's equation to solve for a second value of E. We continue in this iteration process until

$$|E_{n+1}-E_n|<\epsilon.$$

Once we have determined the value of E corresponding to t, we are able to compute numerical values for λ_1, λ_2, λ_3, λ_4, $(1+\nu)$, and can find (ω) and (θ) as well as $a_0(\cos E-e_0)$ and $a_0\sqrt{1-e_0}^2 \sin E$. Thus, we can easily put all of these values into Equation 236, and the components of the radius vector of the real satellite in the inertial coordinate system will emerge.

DETERMINATION OF THE VELOCITY VECTOR AND OSCULATING ELEMENTS AT TIME t

Bailie and Bryant have published (Reference 9) the method by which the velocity vector of the real satellite and the osculating elements at some time t are determined. The relationships follow directly and simply from the development discussed here.

Recalling that the W function was split up to give

$$W=\Xi+\Upsilon\cos F+\Psi\sin F, \quad \text{(Equation 135)}$$

Bailie and Bryant define the two quantities

$$\beta\equiv e_0+\frac{h_0}{h}\left(\frac{1-e_0^2}{2}\right)\Upsilon, \tag{239}$$

and

$$\gamma=\frac{h_0}{h}\frac{\sqrt{1-e_0^2}}{2}\Psi. \tag{240}$$

Further, they use the components of the Γ rotation matrix written as

$$\Gamma=\begin{bmatrix}(P_x) & (Q_x) & (R_x)\\ (P_y) & (Q_y) & (R_y)\\ (P_z) & (Q_z) & (R_z)\end{bmatrix}, \tag{241}$$

where (P_x), (Q_x), etc., are the components in the inertial system of unit vectors **P, Q,** and **R.** The unit vector **P** is in the osculating plane and is directed toward the perigee, **R** is normal to the plane, and $\mathbf{Q}=\mathbf{R}\times\mathbf{P}$.

Bailie and Bryant give the components as:

$$\begin{aligned}(P_x)=&+(\lambda_4^2-\lambda_3^2)\cos[(\omega)+(\theta)]-2\lambda_3\lambda_4\sin[(\omega)+(\theta)]\\&+(\lambda_1^2-\lambda_2^2)\cos[(\omega)-(\theta)]\\&-2\lambda_1\lambda_2\sin[(\omega)-(\theta)],\end{aligned}$$

$$\begin{aligned}(P_y)=&+(\lambda_4^2-\lambda_3^2)\sin[(\omega)+(\theta)]+2\lambda_3\lambda_4\cos[(\omega)+(\theta)]\\&-(\lambda_1^2-\lambda_2^2)\sin[(\omega)-(\theta)]\\&-2\lambda_1\lambda_2\cos[(\omega)-(\theta)],\end{aligned}$$

$$(P_z)=+2(\lambda_1\lambda_4-\lambda_2\lambda_3)\sin(\omega)+2(\lambda_2\lambda_4+\lambda_1\lambda_3)\cos(\omega),$$

$$\begin{aligned}(Q_x)=&-(\lambda_4^2-\lambda_3^2)\sin[(\omega)+(\theta)]-2\lambda_3\lambda_4\cos[(\omega)+(\theta)]\\&-(\lambda_1^2-\lambda_2^2)\sin[(\omega)-(\theta)]\\&-2\lambda_1\lambda_2\cos[(\omega)-(\theta)],\end{aligned}$$

$$\begin{aligned}(Q_y)=&+(\lambda_4^2-\lambda_3^2)\cos[(\omega)+(\theta)]-2\lambda_3\lambda_4\sin[(\omega)+(\theta)]\\&-(\lambda_1^2-\lambda_2^2)\cos[(\omega)-(\theta)]\\&+2\lambda_1\lambda_2\sin[(\omega)-(\theta)],\end{aligned}$$

$$(Q_z)=+2(\lambda_1\lambda_4-\lambda_2\lambda_3)\cos(\omega)-2(\lambda_2\lambda_4+\lambda_1\lambda_3)\sin(\omega),$$

$$(R_x)=+2(\lambda_1\lambda_4+\lambda_2\lambda_3)\sin(\theta)-2(\lambda_2\lambda_4-\lambda_1\lambda_3)\cos(\theta),$$

$$(R_y)=-2(\lambda_1\lambda_4+\lambda_2\lambda_3)\cos(\theta)-2(\lambda_2\lambda_4-\lambda_1\lambda_3)\sin(\theta),$$

$$(R_z)=\lambda_4^2+\lambda_3^2-\lambda_2^2-\lambda_1^2.$$

After some straightforward analysis the following results are obtained giving the velocity vector **v** of the real satellite and the osculating elements in terms of β, γ, the components of **P, Q,** and **R,** h_0/h, h/h_0, and the λ's:

$$\mathbf{v}=\frac{1}{\sqrt{a_0}}\Gamma\begin{bmatrix}-\dfrac{\sin E}{1-e_0\cos E}\dfrac{h}{h_0}-\dfrac{1}{2}Z\\[2ex] \dfrac{\sqrt{1-e_0^2}\cos E}{1-e_0\cos E}\dfrac{h}{h_0}+\dfrac{\sqrt{1-e_0^2}}{2}\bar{Y}\\[2ex] 0\end{bmatrix}, \tag{242}$$

$$a=\frac{a_0(1-e_0^2)}{1-\beta^2-\gamma^2}\left(\frac{h_0}{h}\right)^2, \tag{243}$$

$$e=\sqrt{\beta^2+\gamma^2}, \tag{244}$$

$$\tan i=\frac{\sqrt{(P_z)^2+(Q_z)^2}}{\lambda_4^2+\lambda_3^2-\lambda_2^2-\lambda_1^2}, \tag{245}$$

$$\tan\omega=\frac{\beta(P_z)+\gamma(Q_z)}{\beta(Q_z)-\gamma(P_z)}, \tag{246}$$

$$\tan\theta=\frac{(P_y)(Q_z)-(Q_y)(P_z)}{(P_x)(Q_z)-(Q_x)(P_z)}. \tag{247}$$

And from the osculating value of the eccentric anomaly

$$\sqrt{\frac{1+e}{1-e}}\tan\frac{E_{osc}}{2}=\frac{\sqrt{\dfrac{1+e_0}{1-e_0}}\tan\dfrac{E}{2}-\dfrac{e-\beta}{\gamma}}{1+\dfrac{e-\beta}{\gamma}\sqrt{\dfrac{1+e_0}{1-e_0}}\tan\dfrac{E}{2}}, \tag{248}$$

the osculating mean anomaly M is found in Kepler's equation:

$$M=E_{osc}-e\sin E_{osc}. \tag{249}$$

The components of the three unit vectors are easily arrived at, but are given explicitly by Bailie and Bryant, in terms of the four λ parameters and the mean values (ω) and (θ).

SECTION X

EVALUATION AND CONCLUSION

This theory contains certain areas in which trouble may arise for particular values of the elements. The three areas of difficulty in this theory are (1) small eccentricity, (2) large eccentricity, and (3) an angle of inclination in the neighborhood of the critical angle.

SMALL ECCENTRICITY

The first of these difficulties is not a weakness of the theory itself, but rather of the way in which it has been adapted for machine use. We recall that in the determination of the secular motion y, the technique involved was that of finding y such that no sin F terms appeared in the series for dW/dE. We solved an equation for y (see Equation 192):

$$A_1+yA_2=0, \tag{250}$$

where A_1 and A_2 are numerical values of coefficients. Upon close inspection of the explicit expression for dW/dE, it becomes evident that this equation can also be written as

$$eA_1'+y(eA_2')=0,$$

or

$$y=-\frac{eA_1'}{eA_2'}, \tag{251}$$

since the eccentricity appears explicitly in the coefficients which are summed to give A_1:

$$A_1=(e_0a_1+e_0^3a_2+e_0^5a_3+e_0^7a_4+\ \ldots)\sin F. \tag{252}$$

From the last term of dW/dE (see Equation 173), we see that

$$A_2=e_0A_2'\sin F. \tag{253}$$

The difficulty arises, not specifically from the appearance of e_0 in the denominator of Equation 251 but from the fact that in Equation 252 the machine is adding a series of very small numbers during which process a large error can accumulate. When this sum is divided by a small number, e_0A_2', the accumulated error is greatly magnified, and can easily exceed error limitations. An educated guess as to the lower limit of the eccentricity would be on the order of 0.001.

LARGE ECCENTRICITY

The problem with large eccentricities is that the series for a_0/ρ and a_0/r converge very slowly, owing to the presence of the factor $2/\sqrt{1-e_0^2}$:

$$\frac{a_0}{\rho}=\frac{2}{\sqrt{1-e_0^2}}\left(\frac{1}{2}+\beta\cos F+\beta^2\cos 2F+\ \ldots\right).$$

(Equation 108)

It would be desirable to have different series expansions for a_0/r and a_0/ρ for large eccentricities. It is impossible to find one series expansion which will give reasonably fast convergence for the entire range of eccentricity, $0<e_0<1$. Also, the factor $1/(1-e_0^2)$ appears in the development, for example, in the M and N factors of the derivative of W. The factor places a limitation on e_0. The numerical procedure described in this development is not satisfactory for eccentricities larger than approximately 0.90.

THE CRITICAL ANGLE OF INCLINATION

It is a physical fact involving the particular oblateness of the earth, that at some angle of inclination in the region close to 63.4 degrees the forces disappear which gives rise to the secular motion of the perigee of a satellite. This is a result of oblateness symmetry with respect to a plane passed through the earth at this angle to the equator. The angle will vary slightly from 63.4 degrees depending upon the geodetic parameters used in the potential function. But, nonetheless, there is some angle at which the secular motion of the perigee vanishes, or, in our notation,

$$y+\alpha-\eta=0.$$

This leads to a problem primarily in the integration of the series $dn_0\delta z/dE$; the same problem is present in all the integrations, but in those of dW/dE, and $d\lambda/dE$, the coefficients of the terms involved are very small and the difficulty is not readily apparent. We recall the series $dn_0\delta z/dE$ is of the general form

$$\frac{dn_0\delta z}{dE}=\sum_i\sum_j\{C_{i,j}\cos[iE+2j(\omega)]+S_{i,j}\sin[iE+(2j+1)(\omega)]\}.$$

(Equation 195)

If we take the term of the series where $i=0$ and $j=0$, we have the sin (ω) term, where we know

$$(\omega)=\omega_0+(y+\alpha-\eta)\Delta E,$$

with $\Delta E = E - E_0$. So we must integrate with respect to E, that is,

$$\int \sin(\omega) dE = \int \sin [\omega_0 + (y+\alpha-\eta)E - (y+\alpha-\eta)E_0] dE,$$

which gives

$$\int \sin(\omega) dE = -\frac{\cos(\omega)}{y+\alpha-\eta}.$$

It is evident that this term becomes meaningless when $(y+\alpha-\eta)\to 0$, as it does at the critical angle. This renders the theory unsatisfactory in a very small region about the critical angle.

We see that the term $\sin(\omega)$ is actually a very long period term in the region, and a constant at the critical angle. It might at first seem possible to separate the constant part of the term from its periodic part, but the periodic part would have to be considered secular and disallowed first order secular motions would result in the λ parameters. This weakness in the theory seems to be inherent and unavoidable.

In the other integrations, these long period terms are of a very small magnitude, and the limitations due to series truncation have thus far made it difficult to evaluate their significance.

ACCURACY

The theory, as stated in the introduction, is an exact one, which includes all orders of perturbations. The accuracy is that of the geodetic parameters k_2, k_3, and k_4 in the potential function. However, in practice the accuracy is greatly affected by the truncation of series and the number of significant figures carried in the machine. The great number of series multiplications taxes the storage capacity of even the largest machines, so the question of series truncation is a serious one.

UNITS

It has been found that the Vanguard system of units is quite satisfactory in this development, although by no means is it the only valid system. The theory can use any system so long as it is used consistently throughout. The Vanguard units allow that the product of the mass of the earth and the universal gravitational constant is unity. The unit of time, also a Vanguard unit, is the amount of time it takes an orbiting satellite at a distance of one mean equatorial radius from the earth's center to travel one radian. This is

1 Unit of Time=806.814 mean solar seconds.

The unit of length, considered one mean equatorial radius, is

1 Unit Length=6,378.165 kilometers.

CONCLUSION

The theory described has been adapted to make optimum use of the capacity and speed of modern computing machines. It is a purely numerical theory, the accuracy of which is determined only by the accuracy of the geodetic parameters, and in practicality, by the limitations of series truncation in the machine. The theory described deals only with general oblateness perturbations, but is easily adaptable to the inclusion of other perturbing forces, including solar radiation pressure, solar and lunar perturbations, and the effects of the tesseral harmonics in the earth's gravitational potential. Though this exposition has dealt only with the first three harmonics of the earth's potential function, higher harmonics can be easily included.

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APPENDIX A

COROLLARY DERIVATIONS

BASIC EQUATIONS OF AN ELLIPSE

A basic property of an ellipse is

$$a^2=b^2+c^2, \tag{A1}$$

where a is the semimajor axis, b is the semiminor axis, and c is the distance from the center of the ellipse to one focus. The eccentricity of the ellipse is defined as the ratio

$$e\equiv\frac{c}{a}. \tag{A2}$$

If we let the center of the ellipse be the center of a Cartesian coordinate system, and let the x axis lie along the semimajor axis, we can write the equation of the ellipse as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{A3}$$

and we can then solve for y, which is the distance $r \sin f$ (see Figure A1), by the equation

$$y=\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}=r\sin f. \tag{A4}$$

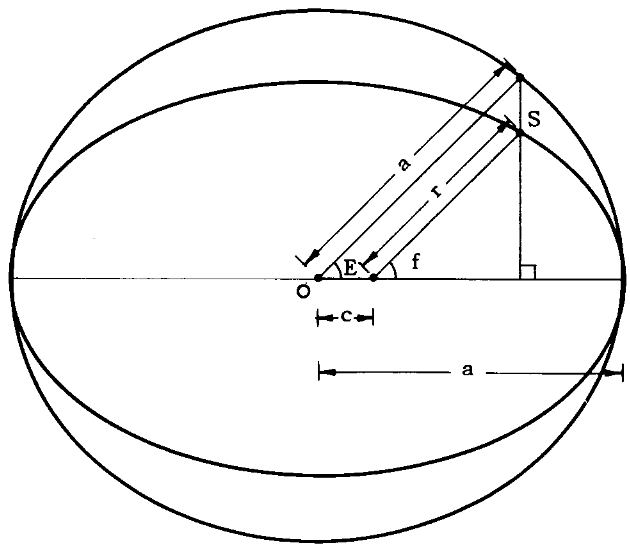


FIGURE A1.—The geometry of the ellipse.

We know as well that

$$x=a\cos E$$

or

$$x^2=a^2\cos^2 E;$$

therefore, we can write

$$r \sin f=\sqrt{b^2\left(1-\frac{a^2 \cos^2 E}{a^2}\right)}$$
$$=b\sqrt{1-\cos^2 E}$$
$$=b \sin E.$$

But from Equations A1 and A2, we have

$$b=\sqrt{a^2(1-e^2)}, \tag{A5}$$

which gives

$$\boxed{r \sin f=a\sqrt{1-e^2} \sin E.} \tag{A6}$$

Similarly, from Figure A1, we see that

$$\boxed{r \cos f=a \cos E-c;} \tag{A7}$$

but since $e=c/a$, we can write this

$$\boxed{r \cos f=a(\cos E-e).} \tag{A8}$$

Now, squaring Equations A6 and A8 and adding them together, we have

$$r^2=a^2[(1-e^2) \sin^2 E+\cos^2 E-2e \cos E+e^2]$$
$$=a^2[1+e^2(1-\sin^2 E)-2e \cos E]$$
$$=a^2[1-2e \cos E+e^2 \cos^2 E],$$

so then

$$r^2=a^2(1-e \cos E)^2$$

and

$$\boxed{r=a(1-e \cos E).} \tag{A9}$$

Now, we can rewrite Equation A9 as

$$1=\frac{a}{r}(1-e \cos E).$$

When this is multiplied through by e and rearranged, we have

$$e-\frac{ea}{r}(1-e \cos E)=0, \tag{A10}$$

which can be written

$$e-\cos E+\cos E-\frac{ea}{r}(1-e \cos E)=0. \quad \text{(A10)}$$

But we know from Equation A9 that

$$\frac{a}{r}=\frac{1}{1-e \cos E},$$

so we can write Equation A11 as

$$e-\cos E+\frac{a}{r} \cos E-\frac{ae}{r} \cos^2 E-\frac{ea}{r}+\frac{e^2a}{r} \cos E=0.$$

From Equation A8 we know that

$$\cos f=\frac{a}{r}(\cos E-e),$$

and so we have

$$e-\cos E+\left(\frac{a}{r} \cos E-\frac{ae}{r}\right)$$
$$-e \cos E\left(\frac{a}{r} \cos E-\frac{ae}{r}\right)=0,$$

or

$$e-\cos E+\cos f-e \cos E \cos f=0. \tag{A12}$$

If we now subtract e from both sides of Equation A12, multiply through by e, and add 1 to both sides, we get

$$1-e \cos E+e \cos f-e^2 \cos E \cos f=1-e^2,$$

or

$$(1-e \cos E)(1+e \cos f)=1-e^2.$$

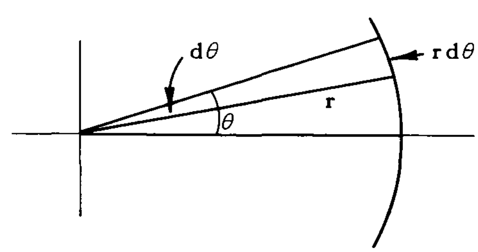
Multiplying through by $a/(1+e \cos f)$, we have

$$a(1-e\cos E)=\frac{a(1-e^2)}{1+e \cos f}; \tag{A13}$$

but since the left-hand side of Equation A13 is r by Equation A9, we have the ellipse equation

$$\boxed{r=\frac{a(1-e^2)}{1+e \cos f}.} \tag{A14}$$

The next equation of interest states that the total area swept out per unit time by the radius vector of an ellipse is $1/h$ where $h=1/\sqrt{a(1-e^2)}$.



In polar coordinates, the area swept out in time dt is

$$\frac{1}{2} r(rd\theta)$$

since the area is that of a triangle. So $r^2(d\theta/dt)$ is twice the area swept out per unit time. Now, the area swept out per *unit* time is the total area divided by the period of one revolution of the radius vector. Thus,

$$r^2\frac{d\theta}{dt}=2\frac{A}{P}. \tag{A15}$$

But $A=\pi ab$ where a and b are the semimajor and semiminor axes, respectively. We know from Equation A5 that $b=\sqrt{a^2(1-e^2)}$, so the area becomes

$$A=\pi a^2\sqrt{(1-e^2)}.$$

We know also that the period P is $2\pi/n$, where n is the mean motion (average angular velocity) of the radius vector, and from Kepler's law, $n=a^{-3/2}$. Therefore, $P=2\pi/a^{-3/2}$ and Equation A15 becomes

$$r^2\frac{d\theta}{dt}=\frac{2\pi a^2\sqrt{(1-e^2)}}{2\pi}a^{-3/2}=\sqrt{a(1-e^2)} \tag{A16}$$

which we have defined as $1/h$. Thus,

$$h=\frac{1}{\sqrt{a(1-e^2)}}.$$

Proof That $\psi=\sin i \sin (v-\sigma)$

In spherical trigonometry, the law of sines is: "In any spherical triangle, the sines of the sides are proportional to the sines of the corresponding opposite angles." The sine of the geocentric latitude of a satellite, denoted by ψ, is defined as the sine of the angle between the earth's equatorial plane and the line drawn from the center of the earth to the satellite.

So, in the spherical triangle NQS (Figure A2), we have two sides $(v-\sigma)$ and SQ and two angles i and SQN. If we designate the side SQ as α, the angle which subtends it, we can write the law of sines:

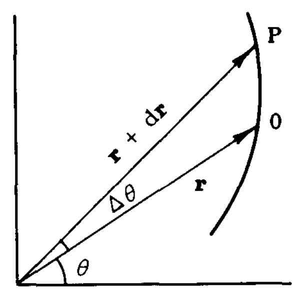
$$\frac{\sin SQN}{\sin (v-\sigma)}=\frac{\sin i}{\sin \alpha}.$$

But we know that angle SQN is a right angle, and we have defined $\sin \alpha$ as ψ, so we have

$$\boxed{\psi=\sin i \sin (v-\sigma).}$$

DERIVATION OF EQUATIONS OF MOTION IN POLAR FORM

Consider the motion of a particle in motion along a smooth path C, as shown.



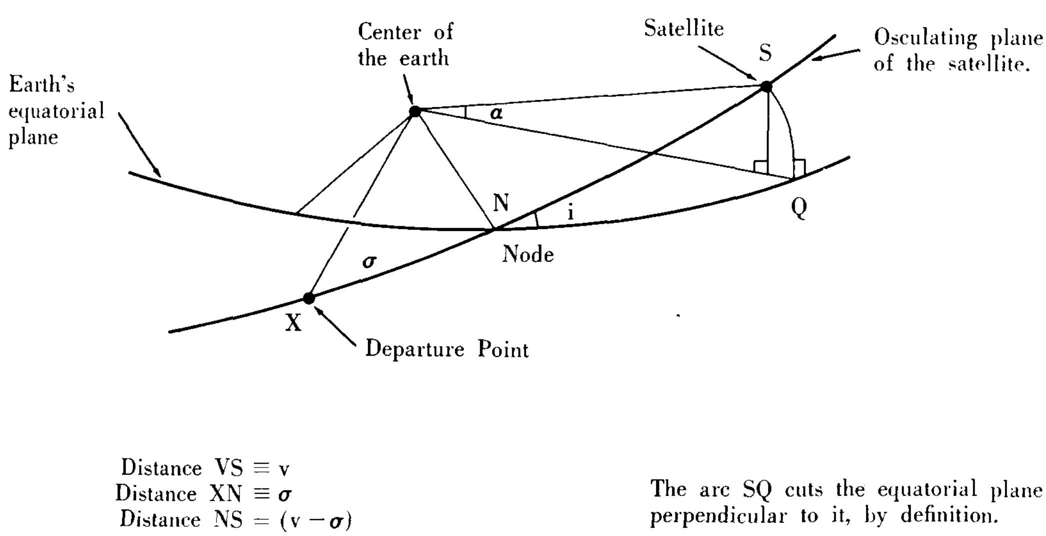


FIGURE A2.—Hansen's coordinate system.

APPENDIX B

THE COMPUTATIONAL PROCEDURE USED IN THE IBM GOP PROGRAM FOR THE GENERATION OF FINAL SERIES FORMS

The following program was used in the actual computation of orbits. The format is the same as used in the program.

1. Store a_0, e_0, and i_0.
2. Store truncating values ϵ and ϵ'.
3. Store limit m to number of iterations.
4. Store y.
5. Leave space for α and η.
6. Set up storage for 10 Fourier series: λ_1, λ_2, λ_3, λ_4, W, ν, h/h_0, h_0/h, $n_0\delta z$, and Δ.
7. Store k_2, k_3, k_4, $A/2$, and $B/2$.
8. Compute $i_0/2$, sin $(i_0/2)$, and cos $(i_0/2)$.
9. Store as follows: sin $(i_0/2)\rightarrow\lambda_1$

$$\begin{aligned} 0&\rightarrow\lambda_2\\ 0&\rightarrow\lambda_3\\ \cos(i_0/2)&\rightarrow\lambda_4\\ 0&\rightarrow W\\ 0&\rightarrow\nu\\ 1&\rightarrow h_0/h\\ 1&\rightarrow h/h_0\\ 0&\rightarrow n_0\delta z\\ 0&\rightarrow\Delta. \end{aligned}$$

10. Compute and store $\beta=\dfrac{e_0}{1+\sqrt{1-e_0^2}}$.
11. Compute and store the Fourier series

$$l^*=\frac{1}{2}(1+\sqrt{1-e_0^2})\cos[F+(\omega)]+\frac{1}{2}(1-\sqrt{1-e_0^2})\cos[F-(\omega)]-e_0\cos(\omega).$$

12. Compute and store the Fourier series

$$m^*=\frac{1}{2}(1+\sqrt{1-e_0^2})\sin[F+(\omega)]-\frac{1}{2}(1-\sqrt{1-e_0^2})\sin[F-(\omega)]-e_0\sin(\omega).$$

13. Compute and store

$$\frac{1}{1+\nu}=1-\nu+\nu^2-\nu^3+\ldots+(-1)^n\nu^n. \qquad n=1,2,3,\ldots$$

14. Compute and store the Fourier series

$$\frac{a_0}{\bar\rho}=\frac{2}{\sqrt{1-e_0^2}}\left(\frac{1}{2}+\beta\cos F+\beta^2\cos 2F+\ldots+\beta^n\cos nF\right). \qquad n=1,2,3,\ldots$$

15. Compute the Fourier series product

$$\psi^*=2\frac{a_0}{\rho}[(\lambda_1\lambda_4-\lambda_2\lambda_3)m^*+(\lambda_2\lambda_4+\lambda_1\lambda_3)l^*].$$

16. Compute ψ^{*2}.
17. Compute

$$\frac{a_0}{\rho}=\frac{a_0}{\bar\rho}\left(\frac{1}{1+\nu}\right),\quad\left(\frac{a_0}{\rho}\right)^2,\quad\left(\frac{a_0}{\rho}\right)^3.$$

18. Compute the Fourier series $(1-3\psi^{*2})$.
19. Compute the Fourier series $(3-5\psi^{*2})\psi^*$.
20. Compute the Fourier series $(35\psi^{*2}-30)\psi^{*2}+3$.
21. Compute

$$\frac{k_2}{a_0^3}\left(\frac{a_0}{\rho}\right)^3.$$

22. Compute

$$A=\frac{k_2}{a_0^3}\left(\frac{a_0}{\rho}\right)^3(1-3\psi^{*2}).$$

23. Compute

$$\frac{k_3}{a_0^4}\left(\frac{a_0}{\rho}\right)^4.$$

24. Compute

$$B=\frac{k_3}{a_0^4}\left(\frac{a_0}{\rho}\right)^4(3-5\psi^{*2})\psi^*.$$

25. Compute

$$\frac{k_4}{a_0^5}\left(\frac{a_0}{\rho}\right)^5.$$

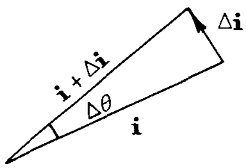
26. Compute

$$C=\frac{k_4}{a_0^5}\left(\frac{a_0}{\rho}\right)^5[(35\psi^{*2}-30)\psi^{*2}+3].$$

27. Compute $\Omega^*=A+B+C$.
28. Compute $\dfrac{\partial\Omega^*}{\partial F}$ by differentiating the Fourier series Ω^*.
29. Compute

$$\rho\frac{\partial\Omega^*}{\partial\rho}=-3A-4B-5C.$$

As the particle moves from 0 to P, **r** goes to **r**+**dr**. If we let **i** be a unit vector along **r**, and **i**+Δ**i** be a unit vector along **r**+Δ**r** we can draw the following isosceles triangle, since both sides are unit vectors:



We know from the definition of plane trigonometry,

$$\Delta i = 2 \sin \frac{\Delta\theta}{2}.$$

So we can write

$$\frac{\Delta i}{\Delta\theta} = \frac{2 \sin \frac{\Delta\theta}{2}}{\Delta\theta},$$

and since

$$\frac{di}{d\theta} = \lim_{\Delta\theta\to 0} \frac{\Delta i}{\Delta\theta} = 1,$$

we see the magnitude of the vector $di/d\theta = 1$. Also clearly, from the diagram, as $\Delta\theta \to 0$, $\Delta\mathbf{i}$ becomes perpendicular to **i**; so in the limit,

$$\frac{d\mathbf{i}}{d\theta} = \lim_{\Delta\theta\to 0} \frac{\Delta\mathbf{i}}{\Delta\theta} = \text{a unit vector perpendicular to } \mathbf{i},$$

which we will designate as **j**, and write

$$\frac{d\mathbf{i}}{d\theta} = \mathbf{j}.$$

Now, the radius vector $\mathbf{r} = r\mathbf{i}$, and the velocity vector is simply

$$\frac{d}{dt}(\mathbf{r}) = \frac{d}{dt}(r\mathbf{i}) = \frac{dr}{dt}\mathbf{i} + r\frac{d\mathbf{i}}{dt}.$$

But since

$$\frac{d\mathbf{i}}{d\theta} = \mathbf{j}, \quad d\mathbf{i} = \mathbf{j}\,d\theta, \quad \text{and } \frac{d\mathbf{i}}{dt} = \frac{d\theta}{dt}\mathbf{j},$$

we can write the velocity vector

$$\mathbf{v} = \frac{dr}{dt}\mathbf{i} + r\frac{d\mathbf{i}}{dt} = \frac{dr}{dt}\mathbf{i} + r\frac{d\theta}{dt}\mathbf{j}.$$

Then the acceleration vector **a** is $d\mathbf{v}/dt$, or

$$\mathbf{a} = \frac{d^2r}{dt^2}\mathbf{i} + \frac{dr}{dt}\frac{d\mathbf{i}}{dt} + \left(\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{j} + r\frac{d\theta}{dt}\frac{d\mathbf{j}}{dt}.$$

However, using exactly the same reasoning as above, we can show that

$$\frac{d\mathbf{j}}{d\theta} = -\mathbf{i}$$

or

$$\frac{d\mathbf{j}}{dt} = -\frac{d\theta}{dt}\mathbf{i}.$$

So, substituting this expression for $d\mathbf{j}/dt$ in the **a** equation and replacing $d\mathbf{i}/dt$ with $(d\theta/dt)\mathbf{j}$, then rearranging, we have:

$$\mathbf{a} = \frac{d^2r}{dt^2}\mathbf{i} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{j} + \left(\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{j} - r\frac{d\theta}{dt}\frac{d\theta}{dt}\mathbf{i}$$

or

$$\mathbf{a} = [\ddot{r} - r(\dot{\theta})^2]\mathbf{i} + [2\dot{r}\dot{\theta} + r\ddot{\theta}]\mathbf{j}.$$

However,

$$[2\dot{r}\dot{\theta} + r\ddot{\theta}] = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$= \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right],$$

so we can write

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{i} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\mathbf{j}}$$

where **i** and **j** are unit vectors along the radius and perpendicular to the radius vector, respectively. Therefore the component of the acceleration along **r** is

$$\boxed{\ddot{r} - r\dot{\theta}^2,}$$

and the component of the acceleration perpendicular to the radius vector is

$$\boxed{\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).}$$

30. Compute

$$\frac{\partial\Omega^*}{\partial\psi^*}=-6\psi^*\left(\frac{k_2}{a_0^3}\right)\left(\frac{a_0}{\rho}\right)^3+\frac{k_3}{a_0^4}\left(\frac{a_0}{\rho}\right)^4(3-15\psi^{*2})$$
$$+\frac{k_4}{a_0^5}\left(\frac{a_0}{\rho}\right)^5(-60\psi^*+140\psi^{*3}).$$

31. Find

$$r\frac{\partial\Omega}{\partial r}=\rho\overline{\frac{\partial\Omega^*}{\partial\rho}}$$

where the bar operation means that all F's in the argument of the Fourier series become E's.

32. Find

$$\frac{\partial\Omega}{\partial\psi}=\overline{\frac{\partial\Omega^*}{\partial\psi^*}}.$$

33. Find

$$\frac{\partial\Omega}{\partial E}=\overline{\frac{\partial\Omega^*}{\partial F}}.$$

34. Find $l=\overline{l^*}$ and $m=\overline{m^*}$.

35. Compute M',

$$M'=-1-2\left(\frac{h}{h_0}\right)^2+(3-\nu)\left(\frac{1}{1+\nu}\right)\left(\frac{e_0^2}{2}\right)$$
$$+\left[\left(\frac{h}{h_0}\right)^2+\nu\left(\frac{1}{1+\nu}\right)\right]2e_0\cos E$$
$$-\left(\frac{e_0^2}{2}\right)\cos 2E+\frac{1}{1+\nu}[e_0^2\cos(F+E)]$$
$$+\left[(2-e_0^2)\frac{1}{1+\nu}+2\left(\frac{h}{h_0}\right)^2\right]\cos(F-E)$$
$$-\left(\frac{h}{h_0}\right)^2[e_0\cos(F-2E)]$$
$$-\left[\left(\frac{h}{h_0}\right)^2+\frac{2}{1+\nu}\right]e_0\cos F.$$

36. Compute N',

$$N'=\left[\left(\frac{h}{h_0}\right)^2-\frac{1}{1+\nu}+\frac{1}{2}\right]2e_0\sin E-\left(\frac{h}{h_0}\right)^2 e_0\sin F$$
$$-\frac{e_0^2}{2}\sin 2E+\frac{1}{1+\nu}[e_0^2\sin(F+E)]$$
$$-\frac{1}{1+\nu}(2-e_0^2)\sin(F-E)$$
$$+\left(\frac{h}{h_0}\right)^2[e_0\sin(F-2E)].$$

37. Take the W Fourier series stored, and compute $\frac{\partial W}{\partial F}$.

38. Compute

$$S=(1-e\cos F)\left[\frac{\partial W}{\partial F}-\left(1+W+\frac{h_0}{h}\right)e_0\sin F\right].$$

39. Compute

$$\Lambda=\left(\frac{h}{h_0}\right)(1+\nu)^2\left[1+\frac{y}{\sqrt{1-e_0^2}}-\frac{e_0 y}{\sqrt{1-e_0^2}}\cos E\right].$$

40. Compute

$$S_1=\frac{a_0}{(1-e_0^2)}\Lambda\left[N'\left(r\frac{\partial\Omega}{\partial r}\right)+M'\frac{\partial\Omega}{\partial E}\right].$$

41. Compute

$$S_2=\frac{1}{\sqrt{1-e_0^2}}S.$$

42. Find the coefficients a_1 and a_2 of the $\underline{\sin F}$ terms in S_1 and S_2, respectively.

43. Solve for y in the equation

$$a_1+a_2y=0;\qquad \left(y=-\frac{a_1}{a_2}\right).$$

44. Construct

$$\frac{dW}{dE}=S_1+yS_2.$$

45. Compute

$$\cos i=-\lambda_1^2-\lambda_2^2+\lambda_3^2+\lambda_4^2.$$

46. Compute

$$T=\left(\frac{a_0}{2\sqrt{1-e_0^2}}\right)\cos i\left(\frac{h}{h_0}\right)\frac{\partial\Omega}{\partial\psi}\Lambda.$$

47. Compute

$$T_2=T(-\lambda_3 l-\lambda_4 m).$$

48. Compute

$$T_3=T(\lambda_2 l+\lambda_1 m).$$

49. Find the constant terms in the Fourier series for λ_1, T_2, T_3, and λ_4 and call them b_1, b_2, b_3, and b_4, respectively.

50. Compute

$$\alpha=\frac{b_2}{b_1}\text{ and }\eta=-\frac{b_3}{b_4}.$$

51. Compute

$$\frac{d\lambda_1}{dE}=\alpha\lambda_2+T(\lambda_4 l-\lambda_3 m),$$

$$\frac{d\lambda_2}{dE}=-\alpha\lambda_1+T_2,$$

$$\frac{d\lambda_3}{dE}=\eta\lambda_4+T_3,$$

$$\frac{d\lambda_4}{dE}=-\eta\lambda_3+T(-\lambda_1 l+\lambda_2 m).$$

52. Compute

$$\tilde{\lambda}_1=\int\frac{d\lambda_1}{dE},$$

$$\lambda_2=\int\frac{d\lambda_2}{dE},$$

$$\lambda_3=\int\frac{d\lambda_3}{dE},$$

$$\tilde{\lambda}_4=\int\frac{d\lambda_4}{dE},$$

where the tilde indicates that the constant of integration has not yet been added. We have the equivalence:

$$\tilde{\lambda}_1=\delta\lambda_1,$$

$$\lambda_2=\delta\lambda_2,$$

$$\lambda_3=\delta\lambda_3,$$

$$\tilde{\lambda}_4=\delta\lambda_4.$$

53. Compute $U=(\tilde{\lambda}_1+\tilde{\lambda}_4)^2+(\lambda_2-\lambda_3)^2$.

54. Compute $V=(\tilde{\lambda}_1-\tilde{\lambda}_4)^2+(\lambda_2+\lambda_3)^2$.

55. Find H and G, which are the constant parts of U and V, respectively.

56. Compute $\frac{A}{2}$ and $\frac{B}{2}$ by iteration, where

$$\frac{A}{2}=\frac{-\frac{H}{4}-\left(\frac{A}{2}\right)^2}{\cos\frac{i_0}{2}+\sin\frac{i_0}{2}},$$

and

$$\frac{B}{2}=\frac{\frac{G}{4}+\left(\frac{B}{2}\right)^2}{\cos\frac{i_0}{2}-\sin\frac{i_0}{2}}.$$

57. Compute

$$\lambda_1=\left(\sin\frac{i_0}{2}+\frac{A}{2}+\frac{B}{2}\right)+\tilde{\lambda}_1.$$

58. Compute

$$\lambda_4=\left(\cos\frac{i_0}{2}+\frac{A}{2}-\frac{B}{2}\right)+\tilde{\lambda}_4.$$

59. Store λ_1 and λ_4 found in steps 57 and 58, along with λ_2 and λ_3 found in step 52, as the new λ's to be used in the next complete iteration.

60. Compute $\tilde{W}=\int\frac{dW}{dE}$ (constant of integration not yet included).

61. Operate to get $\tilde{\bar{W}}$.

62. Compute

$$\frac{1}{1+\bar{\tilde{\bar{W}}}}=1-\bar{\tilde{\bar{W}}}+\left(\bar{\tilde{\bar{W}}}\right)^2-\left(\bar{\tilde{\bar{W}}}\right)^3+\ldots.$$

63. Compute

$$\frac{dn_0\delta z}{dE}=\left[\frac{1}{1+\bar{\tilde{\bar{W}}}}\right]\left[\left(\bar{\tilde{\bar{W}}}+\nu^2\right)(1-e_0\cos E)-\frac{y}{\sqrt{1-e_0^2}}(1-\nu^2)(1-e_0\cos E)^2\right].$$

64. Find the coefficients α and β of the cos (0) and cos (E) terms, respectively, in $\frac{dn_0\delta z}{dE}$.

65. Compute

$$\Delta C_0=\frac{-2\alpha-e_0\beta}{2-e_0^2},$$

$$\Delta C_1=\frac{-2\beta-2e_0\alpha}{2-e_0^2}.$$

66. Compute $W=\tilde{W}+(\Delta C_0+\Delta C_1\cos F)$—store as new W.

67. Compare α and β with ϵ, and if $|\alpha|>\epsilon$ or $|\beta|>\epsilon$, return to step 61 and use the W computed in step 66 for the $\tilde{W}$ seen in step 61.

68. Compute $n_0\delta z=\int\frac{dn_0\delta z}{dE}$ where no constant of integration is added.

69. Compute Ξ, by splitting W series to get X, that part of W which contains terms independent of F, that is, select and collect all

Fourier series terms whose coefficients of F in the argument are zero.

70. Compute Y, by writing $W=\Xi+\Upsilon \cos F+\psi \sin F$, then letting $F=0$, and subtracting X contained in step 69.

71. Form $\Xi+e_0\Upsilon$.

72. Compute Δ by iteration:

$$\Delta=-\frac{1}{3}(\Xi+e_0\Upsilon)+\frac{2}{3}(\Delta^2-\Delta^3+\Delta^4-\Delta^5+\ldots).$$

73. Compute $\frac{h}{h_0}=1+\frac{1}{2}(\Delta+\Xi+e_0\Upsilon)$—store as new $\frac{h_0}{h}$.

74. Compute $\frac{h_0}{h}=1+\Delta$—store as new $\frac{h_0}{h}$.

75. Compute $\overline{W}$ by replacing F's in step 66 by E's.

76. Compute ν by iteration, where

$$\nu=\Delta-\frac{1}{2}(\Delta+\overline{W})(1+\nu)\text{—store as new } \nu.$$

77. Compare

$$|y_{n+1}-y_n| \text{ with } \epsilon,$$
$$|\alpha_{n+1}-\alpha_n| \text{ with } \epsilon,$$
$$|\eta_{n+1}-\eta| \text{ with } \epsilon,$$

and if any is greater, return to step 13 for next iteration.

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