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GODDARD MINIMUM VARIANCE ORBIT DETERMINATION PROGRAM

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SPECIAL PROJECTS BRANCH
THEORETICAL DIVISION

S. Pines, H. Wolf, L. Woolston, and R. Squires

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GODDARD SPACE FLIGHT CENTER
GREENBELT, MD.

GODDARD
MINIMUM VARIANCE
ORBIT DETERMINATION PROGRAM

Prepared by
Special Projects Branch
Theoretical Division

This report was prepared by Messrs. S. Pines and H. Wolf of Analytical Mechanics Associates (under contract to the Goddard Space Flight Center) and Messrs. D. Woolston and R. Squires. It includes analyses performed by Mrs. A. Bailie, and programming by Miss E. Fisher.

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CONTENTS

I. Introduction

The task of providing an adequate coverage for a space mission of the extent of the Apollo project immediately lays bare the striking deficiences of the existing tracking and orbit prediction programs in use today. The very nature of the Apollo mission, with its near earth circular parking orbit, its highly elliptic translunar trajectory, its lunar capture, rendezvous and earth return legs, and, finally, its critical re-entry phase, all require individual attention. Perhaps the most stringent requirement in the entire mission is the need for precision predictions of the orbit over its entire 7-14 day period. This requirement alone is sufficient to eliminate most programs presently using the conventional Cowell's integration procedure. In addition, the effect of small changes over large prediction time arcs calls for a choice of differential correction parameters which do not lose information due to their secular character. Since the information content of the observation over different portions of the trajectories will vary greatly with the vehicle position, it also becomes necessary to absorb and utilize information coming from inhomogeneous observations of many different types, such as range, range rate, azimuth and elevation, optical and other instrument readings made in mid-course. Furthermore, the need for real time decision processes requires the ability for rapidly carrying out iteration procedures which converge rapidly. In particular, the conventional least square techniques now in common use may be taxed beyond their limits of linear assumptions. Thus, the requirements are to develop a general purpose real time tracking and orbit prediction program with the following features:

1. A rapid, basic precision orbit prediction program with high numerical accuracy not subject to numerical round off error over long time periods.

2. The capability of handling circular, elliptical and hyperbolic orbits with equal flexibility under the action of perturbation forces. Special treatment will be necessary to handle the extremely rare cases of exactly rectilinear and parabolic cases.

3. The ability to absorb and utilize inhomogeneous observations from many different sources with equal facility. In particular, this requires forming matrices of partial derivatives of the observations with respect to the variational parameters rapidly.

4. A theoretically sound statistical process for obtaining information from observations rapidly without requiring large stretches of observations over long time arcs which unduly tax the linear assumption of the least square techniques.

5. A program combining these features is a necessary prerequisite for adequate ground coverage of the Apollo mission and moreover provides a ready and existing tracking program for almost any existing space mission.

II. Notation

R	Vehicle position vector
r	Magnitude of position vector, $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$
F	Perturbation acceleration vector
μ	Mass parameter
v_o	Velocity of vehicle
ξ	Perturbation displacement vector
t, t_o, t_r	Present time, initial time, time of rectification, respectively
a	Semi-major axis
e	Eccentricity of orbit
i	Inclination of orbital plane
Ω	Right ascension of ascending node
ω	Argument of perigee
P, Q	Unit vectors in the orbit plane directed, respectively, toward perigee and 90° from perigee in the direction of motion, as used in Section III.
P_x, P_y, P_z	Components of vector P
Q_x, Q_y, Q_z	Components of vector Q
E	Eccentric anomaly as used in Section III
E	Elevation angle as used in Section VIII
n	Mean motion as used in Section III
d_o	$R_o \cdot \dot{R}_o$
$f, g, \dot{f}, \dot{g}$	Coordinate functions in the Kepler problem
$\theta = E - E_o$	Incremental eccentric anomaly

f_0, f_1, f_2, f_3, f_4	Functions of θ defined by Eq. (15) through (18)
α	Orbit parameter
x_i	Orbit variable
y	Observed quantity
ϵ	Observation errors, uncorrelated with position errors
Γ^2	Covariance matrix of the ϵ's
P, P(t)	Covariance matrix of the orbit variables, Section VI
Y(t)	Covariance matrix of the observations
M, M(t)	Matrix of partial derivatives of the observables with respect to the orbit variables
K(t)	Optimum linear estimator for the orbit variables
$\Delta x(t)$, $\Delta y(t)$	Deviations in the orbit variables and in the observations, respectively.
θ'	Right ascension of the observation station meridian
C, S	Parameters related to polar oblateness of earth, Section VIII
h	Height of observation station above sea level
ρ	Range measured from observation station
$\dot{\rho}$	Range rate measured from observation station
φ	Geodetic latitude
A	Azimuth angle
x_s, y_s, z_s	Geocentric coordinates of observation station
ω_e	Angular rate of the earth's rotation
l, m, n	Minitrack direction cosines, Eq. (69)
x''', y''', z'''	Topocentric coordinates of the vehicle in the local horizon, local vertical system with x''' positive south, y''' positive east, and z''' positive upward along the local vertical.

$S(t, t)$	Point transformation matrix of the partial derivatives of the state variables with respect to the orbit parameters
$\Omega(t, t_o)$	State transition matrix of the orbit parameters
$M(t)$	Matrix of partial derivatives of the observations with respect to the orbit parameters
$Q(t)$	Covariance matrix of the orbit parameters
$\Delta\alpha(t)$	Deviations in the orbit parameters
$L(t)$	Optimum linear estimator for the orbit parameters
e_1, e_2, e_3	Unit vectors defined by Figure 3
ρ_a	Apparent range as defined in Section X
r_o, r_v	Radial distances to observer and vehicle, respectively, as used in Section X
$n, \bar{n}$	Index of refraction and average index of refraction as used in Section X
c	Vacuum speed of light
v	Signal velocity in medium as used in Section X
α	Path angle of signal, see Fig. 7
λ_o	Ground station sight angle, Fig. 7
f	Frequency of emitted signal as used in Section X
Δf	Difference in frequency between emitted and received signals
h_o, h_v	Altitude of observer and of vehicle, respectively

Subscripts

v	vehicle
i	i^{th} attracting body
c	reference body as used in Section III
k	Kepler, i.e., quantity obtained from solution of two-body problem
o	value at initial time
r	value at time of rectification

Superscript

*	transposed matrix

III. Precision Orbit Prediction

A thorough analysis comparing the various special perturbation methods in use today for orbit prediction has been carried out in Reference 1. The general conclusions drawn in this study may be stated simply as follows:

1. Cowell's method, while simple to program, consumes larger machine computing times and is subject to an unavoidable accumulation of the round off error.

2. The elimination of truncation and round off error can be accomplished through either the variation of parameters or the Encke method.

3. The preference for the Encke method over the variation of parameters comes from the simplicity of the equations and a great reduction in computing time. Thus, it is possible to generate a precision program using the Encke method which will produce a solution of the equations of motion as precisely as required in shorter computing time than any other available method.

As is well known, the Encke method solves the best local two body problem and integrates the deviation from this nominal trajectory. Since the round off error occurs only in the integrated position, it is possible to eliminate this difficulty from the answer by periodic orbit rectification.

Additional difficulties of the conventional Encke method, such as numerical inaccuracies for circular orbits, etc., have been eliminated by using a solution of the Kepler problem in terms of the initial position and velocit vectors.

A. Equations of Motion

In a Newtonian system, the equations of motion of a particle in the gravitational field of n attracting bodies and subject to other perturbing accelerations such as thrust, drag, oblateness, radiation pressure, etc. are given by

$$\ddot{R}_v = -\sum_{\substack{i=1 \\ i \neq m}}^{n} \mu_i \frac{R_{vi}}{r^3_{vi}} + \sum_j F_j \quad (1)$$

These equations are put into observable form by referring them to a reference body c(= m). The equations of motion of the reference body are

$$\ddot{R}_c = -\sum_{\substack{i=1 \\ i \neq m}}^{n} \mu_i \frac{R_{ci}}{r^3_{ci}} \quad (2)$$

Subtraction of Eq. (2) from Eq. (1) results in the equation of motion in the vehicle with respect to the reference body c

$$\ddot{R}_{vc} = -(\mu_v + \mu_c) \frac{R_{vc}}{r^3_{vc}} - \sum_{\substack{i=1 \\ i \neq m}}^{n} \mu_i \left[\frac{R_{vi}}{r^3_{vi}} - \frac{R_{ci}}{r^3_{ci}} \right] + \sum_j F_j \quad (3)$$

B. Method of Integration

If Eq. (3) is integrated directly by some numerical scheme, there results, after a number of step-by-step integrations, an accumulation of error which leads to inaccurate results. To avoid this loss in precision, it is convenient to write Eq. (3) in the form

$$\ddot{R}_{vc} = \ddot{R}_k + \ddot{\xi} \qquad (4a)$$

The velocity and displacement vectors can be written as

$$\dot{R}_{vc} = \dot{R}_k + \dot{\xi} \qquad (4b)$$

$$R_{vc} = R_k + \xi \qquad (4c)$$

The reference body is chosen so as to minimize the perturbations, i.e., the one in whose sphere of influence the vehicle travels.

In this method $\ddot{R}$ is taken as

$$\ddot{R}_k = -(\mu_v + \mu_c)\frac{R_k}{r_k^3} \qquad (5)$$

and

$$\ddot{\xi} = -(\mu_v + \mu_c)\left[\frac{R_{vc}}{r_{vc}^3} - \frac{R_k}{r_k^3}\right] - \sum_{\substack{i=1\\ i\neq m}}^{n} \mu_i \left[\frac{R_{vi}}{r_{vi}^3} - \frac{R_{ci}}{r_{ci}^3}\right] + \sum_j F_j \qquad (6)$$

Eq. (5) is solved as described below; Eq. (6) is integrated numerically.

C. Solution of the Kepler Problem

1. Classical Solution

The two-body orbit which results from the solution of Eq. (5) with the initial conditions:

$$R_k(t_o) = R_{vc}(t_o) = R(t_o) \tag{7}$$

$$\dot{R}_k(t_o) = \dot{R}_{vc}(t_o) = \dot{R}(t_o)$$

may be represented by many different sets of "elements", such as the semi-major axis, a, the eccentricity, e, the inclination, i, the right ascension of ascending node, Ω, and the argument of perigee, ω and a time variable such as the time of perigee passage or the time of nodal crossing, to locate the vehicle in its orbit. A difficulty arises with this particular set if orbits of zero or small inclination are to be used, in which case the right ascension of the ascending node is not defined or poorly defined. Further, for orbits of zero or small eccentricity the argument of perigee is ill defined. The first of these difficulties is removed by replacing i, ω and Ω by two unit vectors P and Q in the plane of the motion, P directed toward perigee and Q at right angles to it.

The expressions connecting i, ω, and Ω with P and Q are determined by a transformation of the coordinate system of the orbital plane to a geocentric coordinate system by means of three successive rotations (Eulerian angles). This results in:

$$
\begin{aligned}
P_x &= \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i \\
P_y &= \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i \\
P_z &= \sin \omega \sin i \\
Q_x &= - \cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i \\
Q_y &= - \sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i \\
Q_z &= \cos \omega \sin i
\end{aligned}
\qquad (8)
$$

In terms of the P and Q the solution of the two-body problem is given as a function of the eccentric anomaly by

$$
\begin{aligned}
R_k &= x_k P + y_k Q \\
\dot{R}_k &= \dot{x}_k P + \dot{y}_k Q
\end{aligned}
\qquad (9)
$$

where x_k, y_k, $\dot{x}_k$, $\dot{y}$, are given by:

$$
\left.
\begin{aligned}
x_k &= a (\cos E - e) \\
y_k &= a \sqrt{1 - e^2} \sin E \\
r_k &= a (1 - e \cos E) \\
\dot{x}_k &= - \frac{\sqrt{\mu a}}{r_k} \sin E \\
\dot{y}_k &= \frac{\sqrt{\mu a (1 - e)^2}}{r_k} \cos E
\end{aligned}
\right] \quad \text{Elliptic}
$$

for the ellipse and

$$\left.\begin{aligned} x_k &= a\,(\cosh E' - e) \\ y_k &= -a\sqrt{e^2 - 1}\,\sinh E \\ r_k &= a\,(1 - e\cosh E) \\ \dot{x}_k &= -\frac{\sqrt{-\mu a}}{r_k}\,\sinh E \\ \dot{y}_k &= \frac{\sqrt{\mu a\,(1 - e^2)}}{r_k}\,\cosh E \end{aligned}\right\}\ \text{Hyperbolic} \qquad (9)$$

for the hyperbola.

The eccentric anomaly E is obtained as a function of time from

$$\begin{aligned} E - e\sin E - (E_o - e\sin E_o) &= n\,(t - t_o) \qquad \text{Elliptic} \\ e\sinh E - E - (e\sinh E_o - E_o) &= n\,(t - t_o) \qquad \text{Hyperbolic} \end{aligned} \qquad (10)$$

The elements a, e, P, Q are given in terms of the initial conditions by

$$\begin{aligned} r_o &= (R_o \cdot R_o)^{\frac{1}{2}} \\ d_o &= R_o \cdot \dot{R}_o \\ v_o^2 &= \dot{R}_o \cdot \dot{R}_o \\ a &= \left(\frac{2}{r_o} - \frac{v_o^2}{\mu}\right)^{-1} \end{aligned} \qquad (11)$$

$$n = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}$$

$$e \sin E_0 = \frac{d_0}{\sqrt{\mu a}}$$

$$e \cos E_0 = \frac{r_0 v_0^2}{\mu} - 1 = 1 - \frac{r_0}{a}$$

Elliptic (11)

$$n = \frac{\mu^{\frac{1}{2}}}{(-a)^{3/2}}$$

$$e \sinh E_0 = \frac{d_0}{\sqrt{-\mu a}}$$

$$e \cosh E_0 = 1 - \frac{r_0}{a} = \frac{r_0 v_0^2}{\mu} - 1$$

Hyperbolic

$$e = \left[\left(1 - \frac{r_0}{a}\right)^2 + \frac{d_0^2}{\mu a} \right]^{\frac{1}{2}}$$

$$P = \frac{\cos E_0}{r_0} R_0 - \sqrt{\frac{a}{\mu}} \sin E_0 \dot{R}_0$$

$$\sqrt{1 - e^2}\, Q = \frac{\sin E_0}{r_0} R_0 + \sqrt{\frac{a}{\mu}} (\cos E_0 - e) \dot{R}_0$$

Elliptic

$$P = \frac{\cosh E_0}{r_0} R_0 - \sqrt{\frac{-a}{\mu}} \sinh E_0 \dot{R}_0$$

$$\sqrt{e^2 - 1}\, Q = \frac{\sinh E_0}{r_0} R_0 - \sqrt{\frac{-a}{\mu}} (\cosh E_0 - e) \dot{R}_0$$

Hyperbolic

While this formulation of the solution of the two-body problem is attractive from a computational point of view, the scheme will have difficulties with circular and near circular orbits (since P and Q are ill defined), a somewhat paradoxial situation. However, these difficulties can be completely removed by reformulating the solution to be independent of the vectors P and Q and the eccentric anomaly E.

2. Development of NASA Two Body Equations

It is noticed that Eqs. (9) give R_k and $\dot{R}_k$ as linear combinations of P and Q and Eqs. (11) give P and Q in turn as linear combinations of R_0 and $\dot{R}_0$. It is clear, therefore, that R_k and $\dot{R}_k$ may be expressed as linear combinations of R_0 and $\dot{R}_0$.

$$\begin{aligned} R_k &= f\,R_0 + g\,\dot{R}_0 \\ \dot{R}_k &= \dot{f}\,R_0 + \dot{g}\,\dot{R}_0 \end{aligned} \qquad (12)$$

The functions f and g can be expressed completely in terms of the incremental eccentric anomaly $\theta = E - E_0$ and the terms $e \sin E_0$ and $e \cos E_0$, which can be unambiguously expressed in terms of R_0 and $\dot{R}_0$. The following expressions for f and g result:

$$\left.\begin{aligned} f &= -\frac{a}{r_0}\left[-\cos\theta + 1\right] + 1 \\ g &= \sqrt{\frac{a}{\mu}}\,r_0 \sin\theta + \frac{a d_0}{\mu}(-\cos\theta + 1) = \frac{\theta - \sin\theta}{-n} + t - t_0 \\ \frac{r}{a} &= (1 - \cos\theta) + \frac{r_0}{a}\cos\theta + \frac{d_0}{\sqrt{\mu a}}\sin\theta \\ \dot{f} &= -\frac{\sqrt{\mu a}}{r r_0}\sin\theta \\ \dot{g} &= -\frac{a}{r}\left[1 - \cos\theta\right] + 1 \end{aligned}\right\} \text{Elliptic} \qquad (13)$$

$$f = \frac{a}{r_0}\left[(\cosh\theta - 1)\right] + 1$$

$$g = \sqrt{\frac{-a}{\mu}}\; r_0 \sinh\theta - \frac{a d_0}{\mu}(\cosh\theta - 1) = -\frac{\sinh\theta - \theta}{n} + t - t_0$$

$$\frac{r}{a} = -(\cosh\theta - 1) + \frac{R_0}{a}\cosh\theta - \frac{d_0}{\sqrt{-\mu a}}\sinh\theta \qquad (13)$$

$$\dot{f} = \frac{\sqrt{-\mu a}}{r r_0}\sinh\theta$$

$$\dot{g} = \frac{a}{r}(\cosh\theta - 1) + 1 \qquad \text{Hyperbolic}$$

A similar technique applied to Kepler's Eqs. (10) results in the following equation, which furnishes θ as a function of the time.

$$n(t-t_0) = \theta - \sin\theta + \frac{r_0}{a}\sin\theta + \frac{d_0}{\sqrt{\mu a}}(1 - \cos\theta) \qquad \text{Elliptic}$$

$$(14)$$

$$n(t-t_0) = \sinh\theta - \theta - \frac{r_0}{a}\sinh\theta + \frac{d_0}{\sqrt{-\mu a}}(\cosh\theta - 1) \qquad \text{Hyperbolic}$$

If the function f_1, f_2, f_3, f_4 are defined as

	Ellipse	Hyperbola
f_1 (θ)	= θ - sin θ	= sinh θ - θ
f_2 (θ)	= 1 - cos θ	= cosh θ - 1
f_3 (θ)	= sin θ	= sinh θ
f_4 (θ)	= cos θ	= cosh θ

The solution of the two-body problem for both elliptic and hyperbolic orbits is given by:

$$
\begin{aligned}
f &= \frac{-|a|}{r_0} f_2 + 1 \\
g &= -\frac{1}{n} f_1 + (t - t_0) \\
\frac{r}{|a|} &= f_2 + \frac{r_0}{|a|} f_4 + \frac{d_0}{\sqrt{\mu |a|}} f_3 \\
\dot{f} &= -\sqrt{\frac{\mu}{|a|}} \frac{1}{r_0} \frac{|a|}{r} - f_3 \\
\dot{g} &= \frac{-|a|}{r} f_2 + 1 \\
n(t-t_0) &= f_1 + \frac{r_0}{|a|} f_3 + \frac{d_0}{\sqrt{\mu |a|}} f_2
\end{aligned}
\qquad (16)
$$

D. Numerical Procedures

It will be noted that the only distinction between elliptic and hyperbolic paths is completely contained in the definitions of f_1, f_2, f_3, f_4 (Eqs. 15). If these functions are considered as power series expansions in θ^2 as given in Eqs. (17), then the switch from elliptic to hyperbolic cases is achieved by replacing $-\theta^2$ by θ^2, permitting compact and efficient programming.

Considering the first 14 terms of a power series expansion for sin θ, cos θ, sinh θ and cosh θ, the power series expansion for f_1 and f_2 may be expressed as:

$$f_1 = \theta - \sin\theta = -\theta\left\{\left(\left(\left(\frac{-\theta^2}{27.26} + 1\right)\frac{-\theta^2}{25.24} + 1\right) \ldots + 1\right)\frac{-\theta^2}{3.2}\right\}$$

$$f_2 = 1 - \cos\theta = -\left\{\left(\left(\left(\frac{-\theta^2}{26.25} + 1\right)\frac{-\theta^2}{24.23} + 1\right) \ldots + 1\right)\frac{-\theta^2}{3.2}\right\}$$

Elliptic

(17)

$$f_1 = \sinh\theta - \theta = \theta\left\{\left(\left(\left(\frac{\theta^2}{27.26} + 1\right)\frac{\theta^2}{25.24} + 1\right) \ldots + 1\right)\frac{\theta^2}{3.2}\right\}$$

$$f_2 = \cosh\theta - 1 = \left(\left(\left(\frac{\theta^2}{26.25} + 1\right)\frac{\theta^2}{24.23} + 1\right) \ldots + 1\right)\frac{\theta^2}{2.1}$$

Hyperbolic

The use of Chebyshev polynomials allows the calculation of f_1 and f_2 to the same accuracy with fewer terms. They are currently being programmed.

In order to insure a minimum loss of accuracy, the method of computation of f_1, f_2, f_3, f_4 will depend upon the magnitude of θ. The following tests are made to determine the method of computation of f_1, f_2, f_3, f_4.

<u>Hyperbolic Case</u>

(a) If $|\theta| < 1$,

Then f_1 and f_2 are computed by Eqs. (17)

$$f_3 = \theta + f_1$$

$$f_4 = 1 + f_2$$

(b) If $|\theta| > 1$,

Then compute $f_0 = e^{\theta}$

$$f_3 = 1/2\left(f_0 - \frac{1}{f_0}\right)$$

$$f_4 = 1/2\left(f_0 + \frac{1}{f_0}\right) \qquad (18)$$

$$f_1 = f_3 - \theta$$

$$f_2 = f_4 - 1$$

Elliptic Case

(c) If $|\theta| < \frac{\pi}{3}$

Then f_1 and f_2 are computed by Eqs. (17)

$f_3 = \theta - f_1$

$f_4 = 1 - f_2$

(d) If $\theta > \frac{\pi}{3}$

Then f_4 is computed by means of Rand polynomials

$f_2 = 1 - f_4$

Also if $|\theta - \sin\theta| < |\sin\theta|$

(or approximately if $\theta <$ ~~109~~ 1.9°)

then f_1 is computed by Eqs. (17)

$f_3 = \theta - f_1$

Otherwise if $|\theta - \sin\theta| > |\sin\theta|$

(or approximately if $\theta >$ ~~109~~ 1.9°)

then f_3 is computed by means of Rand polynomials

$f_1 = \theta - f_3$.

A special problem arises in the computation of the terms involved in Eqs. (6) due to the loss of accuracy in subtracting the nearly equal terms involved. An expression based on the binomial expansion removes this difficulty. This method supplies results more accurate than the straightforward computation for terms of the type of:

$$\frac{R}{r^3} - \frac{R_0}{r_0^3}$$

if $|R - R_0|$ is small compared to r and is known more accurately than can be computed by taking the difference between R and R_0. It should be emphasized that this is the case for the Encke term

$$\frac{R_{vc}}{r_{vc}^3} - \frac{R_k}{r_k^3}$$

since R_{vc} is computed from

$$R_{vc} = R_{vc} + \xi$$

and ξ is small and known more accurately than can be computed from $R_{vc} - R_k$.

Another example will be the sun's and planets' perturbations in Earth reference and the moon's perturbations for a satellite within 3 radii of the Earth. For example

$$R_{vs} = R_{es} + R_{ve}$$

R_{ve} is small and known more accurately than can be computed from $R_{vs} - R_{es}$.

Special Computation of Perturbation Terms

$$R = R_0 + \Delta$$

$$\frac{R}{r^3} - \frac{R_0}{r_0^3} = \frac{R}{r^3} - \frac{R}{r_0^3} + \frac{R}{r_0^3} - \frac{R_0}{r_0^3}$$

$$= \frac{R}{r^3}\left[1 - \left(\frac{R}{r_0}\right)^3\right] + \frac{\Delta}{r_0^3}$$

$$= \frac{\Delta}{r_0^3} + \frac{R}{r^3}\left[1 - (1 + u)^{3/2}\right]$$

or finally:

$$\frac{R}{r^3} - \frac{R_0}{r_0^3} = \frac{\Delta}{r_0^3} + \frac{R}{r^3}\sum_{i=1}^{6} a_i u^i$$

where

$$u = \frac{2}{r_0^2} \left(R_0 + \frac{1}{2} \Delta\right) . \Delta$$

$$a_1 = -\frac{3}{2}, \; a_2 = -\frac{3}{8}, \; a_3 = \frac{1}{16}, \; a_4 = -\frac{3}{128}, \; a_5 = \frac{3}{256}, \; a_6 = \frac{-7}{1024}$$

The six terms are adequate for $|u| < .1$. For larger values of u straightforward computation is adequate.

IV. Status of Orbit Determination Program To Date.

The orbit determination program, as it currently exists, is considered to be the nucleus of what is needed for ground based lunar missions. In its final form the program will be a complex, general purpose orbit determination program, requiring the use of a large computer of the IBM 7094 variety. It is, however, conceivable that diminutive versions could be evolved for special purposes, e.g., small computers at individual tracking stations, and/or spacecraft on-board computers.

The current program has been under development for slightly more than one year, but is based on experience gained by the personnel involved since 1958. At that time, the Interplanetary Trajectory Program, based on a modified Encke method, was initiated. That program has been widely accepted and utilized by MSC and many Apollo bidders and contractors.

The Encke method is particularly well suited for computing trajectories involving the effect of small perturbations acting over long time arcs. Since only the departure from the local two body orbit is integrated, it is possible to compute small perturbations to the full digit accuracy of the machine capacity. Moreover, round-off error due to a large number of integration steps, can be effectively eliminated. The Encke method divides the desired solution into two parts; the first, and larger, is an algebraically determined exact solution of the local Kepler orbit; the second, and smaller portion, is the integration of the deviations from this Kepler orbit and contains the accumulation of round-off error. By suitably rectifying the Kepler orbit whenever the integrated portion becomes too large, it is possible to control the round-off error.

In the process of working with the Interplanetary Trajectory Program, the variation of perigee height for the Echo Satellite was computed. (Fig. 1). The observed variation, indicated by the circles, is compared with the computed variations for a period of more than 380 days. This single computation carried out for 380 days shows remarkable agreement for a numerical program over such an extended number of orbits. This was carried out with a fairly crude approximation for the atmospheric model. In another example, the inclination as a function of time (Fig. 2) in orbit for IMP or Interplanetary Monitoring Probe, computed by two different programs is shown. One program developed by Halphen and improved by Dr. Musen, and the other the Interplanetary Trajectory Program. This satell is in a highly eccentric orbit and is strongly perturbed by the Sun and the Moon. Halphen's program is designed only for the study of the long period terms of satellites. Again, the interplanetary Program has shown its ability to retain a reasonable orbit for extended periods of time. However, Halphen's method is not useful in computing real orbits, because the short period terms are neglected. These results indicate that the Interplanetary Program would be quite capable of producing an orbit determination program. The current program utilized many of the features of the earlier program and is known as SPAT-DC for Space Trajectory Differential Corrections.

SPAT-DC utilizes the following from the earlier programs:

A. The modified Encke method--but incorporating a new two body solution which eliminates the singularities due to zero eccentricity and zero inclination.

B. The sixth order backward difference integration (due to Cowell).

C. Planetary Ephemerides from the Naval Observatory, including the Sun, Moon, Jupiter, Mars and Venus. The other planets can be readily added.

The current program is described in Reference 2. It contains a new set of differential correction parameters which permits the utilization of observation data over long time arcs without encountering the danger of a singular differential correction matrix. Reference 3 shows that as more than one parameter depends on the energy, the correction matrix approaches singularity more rapidly. By way of illustration, if one uses the components of the initial position and velocity vectors, R_0 and $\dot{R}_0$, as orbit differential correction parameters, the solution would not hold up for more than about 10 to 12 orbits. The best that one can hope to do in this regard is to select a set of variables in which only one depends on the energy. Such a set is the conventional orbit elements (a, e, i, ω, Ω, t_p). However, these exhibit the singularities of zero eccentricity, zero inclination, etc. The present program eliminates these difficulties, but will require special treatment for the parabola and the rectilinear orbit, but with these two exceptions (which are simply added to the existing program) any orbit that may occur can be handled.

To generate the correction matrix, the program integrates the required number of separate trajectories simultaneously. It can accomodate up to twenty variables. A conventional least-squares technique is then employed to obtain the new initial R_0 and $\dot{R}_0$.

V. Variational Parameters

To discuss the parameters used for variational equations in the Differential Correction Program, it is first necessary to review briefly the algebra of rigid vector rotations. The significant parameters involved in rotating vectors are shown in Figures 3 and 4. Small rotations are considered, for example R_2 and R_1 in such a manner as to keep the scalar product equal to a constant.

Two views of the vector rotation are shown. The side view is the plane of the vectors R_1 and R_2 and Ψ is the angle through which R_2 is rotated.

A coordinate system is defined such that e_1 is a unit vector normal to R_1 and in the plane of R_1 and R_2, e_3 is the unit vector in the direction of R_1 and e_2 is the unit vector which defines the right handed coordinate system. It can be shown that the new vector R'_2 is given by the equation in Figure 4.

The initial version of the NASA SPAT-DC Program uses seven basic variational parameters. See Figure 5. The first of these, $\delta\alpha_1$ is a small rotation of the position vector about the velocity vector with the resultant position and velocity given by the equations. This particular rotation has the effect of changing the argument of perigee plus a small changes in the inclination of the orbit and the right ascension of the ascending node.

The second variation, $\delta\alpha_2$ is a small rotation of the velocity about the position with the resulting equations shown, and has the effect of changing the inclination alone, plus small changes in the argument of perigee and the right ascension of the ascending node.

The third variational parameter, $\delta_{\sigma 3}$ is a rotation of the position and the velocity simultaneously about the angular momentum. Thus, the entire ellipse is rotated about its focus in its own plane. This has the effect of changing the argument of perigee. These first three rotations leave the orbit invariant, and they only change its orientation in space. These effects are analogous to changing the inclination, the argument of perigee and the right ascension of the ascending node, without, however, displaying the singularities of these latter parameters.

The fourth variation $\delta\alpha_4$, is a rotation of the position vector about the angular momentum vector.

The fifth parameter, $\delta\alpha_5$ is the sole parameter which affects the energy in the orbit and is a change in the reciprocal of the semi-major axis. This is done by also changing R and $\dot{R}$ in such a way as to keep the eccentricity equal to a constant.

The sixth variation, $\delta\alpha_6$ is a change in the magnitude of the position vector. As a result, there must be a corresponding change in the magnitude of the velocity vector to keep the energy invariant plus a rotation of the position vector about the angular momentum vector to keep $\delta\alpha_4$ invariant.

The seventh and final variable, $\delta\alpha_7$ is a small change in the drag coefficient. The partials are then obtained by integrating seven different trajectories each corresponding to a particular $\Delta\alpha$ and the coordinates of the computed position subtracted from the coordinates of the reference trajectory and divided by the variation $\Delta\alpha$ in a secant approximation to the derivatives. These then are the basic constituents of the program.

To reiterate, the main advantages are that the parameters are chosen so that only one depends on the energy and all singularities have been eliminated.

VI. Differential Correction Methods

Since the orbit position and velocity is not directly observable, it is necessary to infer these variables from a sequence of observations which are functions of the trajectory. In the conventional methods, a linear relationship is assumed between the deviations in the observations and the corresponding deviations in the orbit variables. Thus, an error in the orbit position will correspond to some predictable error in the observation. A large number of observations are made, overdetermining the linear system of equations. A least square technique is used to obtain the best value of the orbit errors to fit the known observation errors. Since the equations of motion are essentially nonlinear, this linear region becomes more and more constrictive about the nominal trajectory the longer the time period over which the prediction is made. Thus, the least square technique often produces a result, fitting data over a long time arc, which is outside the linear range. This produces problems in convergence and consumes machine time. Reducing the number of observations to a shorter time arc helps avoid this difficulty. The weighted least squares is often used in this manner. However, a large number of observations is always needed in order to properly evaluate the effect of the random instrument errors.

A. Minimum Variance

The Schmidt-Kalman (References 4, 5) minimum variance technique avoids the difficulties mentioned above. This procedure permits a complete optimum estimate of the orbit variables and the observation errors from each single observation. It has been adequately shown that the two methods converge to the same answer eventually for the same total set of data (Reference 6). Moreover, the variance technique always converges more rapidly since it requires less data at each stage than the least square procedure. At best, the least square technique can be said to be as good as the minimum variance.

It is worth briefly stating some of the important formulae for the minimum variance technique. The nominal trajectory may be obtained by integrating a system of differential equations with certain unknown parameters,

$$\dot{x}_i = f_i \left(x_j , \alpha , t \right) . \tag{20}$$

Since these parameters are in general not observable, it is necessary to make observations whose predicted value is given as a function of the trajectory coordinates

$$y = y \left(x , t \right) . \tag{21}$$

The deviation of a vehicle from its nominal instantaneous position, due to a change in the orbit parameters, may be obtained from the linear term of the Taylor series

$$\Delta x_i (t) = \sum \frac{\partial x_i (t)}{\partial x_j (0)} \Delta x_j (0) . \tag{22}$$

The matrix of partial derivatives (Φ) is called the state transition matrix and relates the expected deviation from the nominal trajectory at some time (t) with the deviation at some previous time (t_o)

$$\Delta x_i (t) = \Phi \left(t , t_o \right) \Delta x_i \left(t_o \right) . \tag{23}$$

An estimate of the predicted deviation in the observation corresponding to the deviation from the nominal trajectory, including observation errors, is given by the linear equation

$$\Delta y (t) = \sum \frac{\partial y (t)}{\partial x_i (t)} \Delta x_i (t) + \epsilon \quad . \tag{24}$$

The covariance matrix of the squares of the expected deviations of the orbit variables may be derived from Eq. (23)

$$P (t) = \Phi (t , t_o) P (t_o) \Phi^* (t , t_o) \quad . \tag{25}$$

This equation predicts the change in expected error in the orbit as time increases. The covariance matrix of the observations as derived from Eq. (24) is given by

$$Y (t) = M P (t) M^* + \bar{\epsilon}^2$$

$$\text{where } M = M (t) = \left(\frac{\partial y (t)}{\partial x (t)} \right) . \tag{26}$$

It is required to find the optimum estimate of the relationship between the orbit errors and the observation errors in a form inverse to Eq. (24).

The optimum estimate for the solution of Eq. (24) from a single observation is given

$$\Delta x (t) = K (t) \Delta y (t) \quad . \tag{29}$$

Estimating Δy from its expected value in Eq. (24)

$$\Delta \dot{x} (t) \sim K (t) \left[M \Delta x + \epsilon \right] . \tag{30}$$

The equivalent covariance equality for Eq. (30) is given by

$$P M^{*} = K (M P M^{*} + \overline{\epsilon^{2}}) . \tag{31}$$

In the above, it is assumed that the observation errors are uncorrelated with the orbit position errors. Solving the Eq. (31) for the optimum estimator K (t) and substituting into Eq. (29)

$$\Delta x (t) = P (t) M^{*} (t) Y^{-1} (t) \Delta y (t) \tag{32}$$

This solution may be used to correct the orbit positions at the end of each observation. Furthermore, the covariance matrix of the orbit errors may be reduced to a smaller uncertainty following each observation

$$P^{+} (t) = P^{-} (t) - P^{-} (t) M^{*} (t) Y^{-1} (t) M (t) P^{-} (t) . \tag{33}$$

where $P^{-} (t)$ is the value of P (t) obtained from Eq. (25)

Using the best prediction of the orbit position, a new trajectory may be integrated following each observation. In addition, a new state transition matrix and covariance matrix may be computed to predict the next observation. In this manner, the complete orbit information is accumulated about a nominal trajectory of higher and higher accuracy.

B. Weighted Least Squares

The method of least squares may also be used to obtain a solution of the deviations in the state variables corresponding to the actual deviations in the observations. Equation (24) may be rewritten to relate the deviations in the observations to the deviations in the state variables at the initial time.

$$\Delta y (t) = M (t) \phi (t , t_o) \Delta x (t_o) + \epsilon \quad . \tag{34}$$

This equation may be written for an entire sequence of observations in order to form an overdetermined system of linear equations for the unknown $\Delta x (t_o)$.

The solution of Eq. (34) by the method of least squares is given by

$$\Delta x_i (t_o) = (A^* , A)^{-1} A^* \Delta \hat{y} \tag{35}$$

where $A = M (t) \phi (t , t_o)$ and where $\Delta \hat{y}$ is the actual deviation in the observations.

In this solution no a priori knowledge of the expected deviations in the observation is required or assumed known.

The value of the initial parameter obtained from Eq. (35) may be used to derive a new nominal trajectory and a new set of residuals, to obtain an improved estimate of the initial state variables, $\Delta x (t_o)$. The process may be repeated until no further improvement is obtainable from the observation data.

In order to speed up the rate of convergence, it is necessary to include estimates of the expected deviations in the observation through the use of covariance matrices and the method of weighted least squares. The solution of the initial state variables by the method of weighted least squares is given by

$$\Delta x_i(t_o) = [A^* W A + P_{i-1}^{-1}]^{-1} A^* W \Delta \hat{y} \qquad (36)$$

where $W = (\bar{\sigma}^2)^{-1}$

$$P_{i-1} = \mathrm{Cov}[\Delta x_{i-1}(t_o), \Delta x_{i-1}(t_o)] \qquad (37)$$

The essential difference between the method of minimum variance, least squares and weighted least squares is contained in a comparison of equations (32), (35), and (36).

The method of least squares and weighted least squares both relate the estimate of the initial parameters to an entire sequence of observational residuals spread over an extended time arc. In contrast, the method of minimum variance relates the present estimate of the state variable deviation to the prese actual deviations in the observations. The linear assumptions required for the updating theory are violated to a much less degree in the method of minimum variance than in the method of weighted least squares.

VII. Evaluation of the Partial Derivatives

It is seen that both the least square technique and the minimum variance technique require the computation of the partial derivatives of the observations with respect to the variational parameters. As one approach these may be obtained by integrating additional trajectories and forming the differences using the secant method. However, the program recommended in this report obtains the matrices of partial derivatives analytically in terms of the instantaneous two body orbital coordinates. Thus, the complete orbit prediction and partial derivative matrices may be obtained in essentially the same computing time as that of the nominal trajectory.

The partial derivatives are obtained from the product of the M matrix, discussed in Section VI, and a state transition matrix. The method of obtaining the state transistion matrix is based on a generalization of an Encke method applied to linear prediction theory. It is assumed that the equations of motion may be decomposed into two factors

$$\dot{x} = g\,(x\,,\,t) + h\,(x\,,\,t) \tag{38}$$

where

$$|\,g\,| >> |\,h\,| \quad . \tag{39}$$

It is further assumed that a closed form solution of the differential equations is known for the case where $h = 0$,

$$\dot{s} = g\,(s\,,\,t) \quad . \tag{40}$$

Furthermore, the state transition matrix for the approximating solution is known in closed form

$$\Delta s\,(t) = \phi\,(t\,,\,t_o)\,\Delta s\,(t_o) \tag{41}$$

Let the deviation between the state variable and its approximation be given by

$$p(t) = x(t) - s(t) \tag{41}$$

The perturbation equations of motion may now be written in the generalized Encke form

$$\dot{p} = g(x, t) - g(s, t) + h(x, t) \quad . \tag{42}$$

In order to guarantee that the deviation, p, is never permitted to grow too large, the process of rectification is introduced. Whenever a predetermined value of p is exceeded, the integration is terminated at time t_r. A new set of initial conditions are introduced, setting $p(t_r)$ equal to zero. Integration proceeds again about this new nominal approximate solution.

Since the deviation between x and s is never permitted to exceed the given value, the partial derivatives of the state variables from their nominal value may also be limited. Thus it is possible to write an approximate state transition matrix

$$\Phi(t, t_o) \simeq \phi(t, t_o) \quad \text{for } t \leq t_r \tag{43}$$

Moreover, the approximate state transition matrix is known in closed form from the knowledge of the known solution of the nominal trajectory for Eq. (41). Following each rectification, it is necessary to relate the state transition matrix at time t to the initial time. This may be accomplished by multiplying the approximate state transition matrix for times within each rectification interval by its value at the last rectification.

$$\phi (t, t_o) \simeq \phi (t, t_o) = \phi (t, t_{r_1}) \phi (t_{r_1}, t_o) \quad (44)$$

In this manner, a closed form expression for the state transition matrix may be obtained without integrating large quantities of differential equations. Moreover, the error in the computation may be limited by the proper use of the rectification technique.

As has been stated, the use of the conventional state variables, namely initial position and velocity, leads to a transition matrix which is poorly conditioned and which contains rapidly varying functions of time for its elements. In the procedure recommended here, the first six variational parameters described in Section V have only one secular term, namely that due to a.

In addition they have the characteristic that they completely determine the orbit independently of the orbit orientation or shape and never break down.

A complete modification of the Schmidt-Kalman equations in terms of these new parameters has been derived and is now available for incorporation in the prediction scheme.

The modified Schmidt-Kalman equations are listed below. Deviation of the orbit variables in terms of the new parameters are given by

$$\begin{aligned} \Delta x (t) &= \left(\frac{\partial x}{\partial \alpha} \right) \Delta \alpha (t) = S (t, t) \Delta\alpha(t) \\ \Delta y (t) &= M S \Delta\alpha (t) = N (t) \Delta\alpha (t) \quad . \end{aligned} \quad (45)$$

The new transition matrix is given by

$$\Delta\alpha (t) = \left(\frac{\partial\alpha(t)}{\partial\alpha(0)} \right) \Delta\alpha(0) = \alpha(t, t_o) \Delta\alpha (0) \quad . \quad (46)$$

The observation errors in terms of the new parameters are given by

$$\Delta y\ (t) = N(t)\ \Delta\alpha\ (t) + \epsilon \tag{47}$$

where $N(t) = M(t)\ S\ (t,\ t)$.

The corresponding covariance matrices are given by

$$Q\ (t) = \Omega\ Q\ (0)\ \Omega^*$$

$$P\ (t) = S\ Q\ (t)\ S^* \tag{48}$$

$$Y\ (t) = N\ Q\ (t)\ N^* + \overline{\epsilon^2}$$

The inverse relationship between the orbit parameters and the observational errors is given by

$$\Delta\alpha(t) = L\ (t)\ \Delta y\ (t) \quad . \tag{49}$$

The optimum estimator L (t) is given by

$$L\ (t) = Q\ N^*\ Y^{-1} \quad . \tag{50}$$

The corrected estimator after each observation is given by

$$Q^+ = Q^- - Q^-\ N^*\ Y^{-1}\ N\ Q^- \quad . \tag{51}$$

Using these modified equations, it is now possible to use the Schmidt-Kalman scheme for both short and long term predictions. Moreover, the computing time need not exceed that necessary for a single nominal trajectory.

It is sometimes necessary to obtain an estimate of the instrument errors from the accumulated experience of many observations. Let the observed error between the predicted and actual observations be given by $(\overline{Y})$. Then, an estimate of the instrument errors, after many observations, may be obtained by

$$\overline{\epsilon^2}\ (t) = (I - N\ L)\ (\overline{Y})\ (I - N\ L)^* \quad . \tag{52}$$

To illustrate the simplicity and elegant beauty of these equations, the following matrices of partial derivatives are presented

$$
S = \begin{vmatrix}
\frac{-H}{v} & 0 & \frac{H \times R}{h} & \frac{n a^2}{h^2} H \times R - a R & -\frac{a R}{r} & + \frac{\mu^2 |a|}{a h^2 r^2 v^2 n} \alpha_6 \alpha_4 H \times R \\
0 & \frac{H}{r} & \frac{H \times \dot{R}}{h} & 0 & a\frac{\dot{R}}{2} & \frac{\mu a R}{r^2 v^2}
\end{vmatrix} \qquad (53)
$$

$$
\Omega = \begin{vmatrix}
\frac{f\, v(t)}{v_0} & -\frac{g\, v(t)}{r_0} & 0 & 0 & 0 & 0 \\
\frac{-\dot{f}\, r(t)}{v_0} & \frac{\dot{g}\, r(t)}{r_0} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \Omega_{34} & +\frac{3}{2} \frac{h \mu a}{r^3(t)\, r^2} (t-t_0) & \Omega_{36} \\
0 & 0 & 0 & \dot{g} & -\frac{3}{2} \frac{\mu}{n a r(t)} \alpha_6 (t-t_0) & \frac{-r_0 \dot{f}}{a\, n} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{-n a g}{r(t)} & -\frac{3}{2} \frac{a^2 n}{r(t)} \alpha_4 (t-t_0) & \frac{r_0 f}{r(t)}
\end{vmatrix}
$$

where

$$\Omega_{34} = \mu a^2 n \frac{1}{r^2} \left\{ \frac{(\dot{g}-1)}{r^2(t)} - \frac{\mu r_0 \dot{f}^2}{\mu} - \frac{\dot{f}}{h^2} \left(\frac{\mu}{r(t)} \quad g - d_0 \right) \right\}$$

and

$$\Omega_{38} = -\frac{ah}{r_0 \mu^2} \dot{r} \left\{ \frac{r_0}{r(t)} \left[f_4(\theta) + \frac{r_0}{r(t)} \right] - \frac{\mu}{r_0 v_0^2} \dot{g} \right.$$

$$\left. + \frac{d_0}{h^2} \left(1 - \frac{r_0}{a}\right) \frac{\mu}{r_0 v_0^2} \left[\frac{-\mu}{r(t)} \quad g - d_0 \right] \right\}$$

These serve to indicate how simply all the required matrices may be computed as functions of the nominal trajectory. The quantities α_4 and α_8 which appear in the matrix Ω are defined as

$$\alpha_4 = \frac{R \cdot \dot{R}}{\sqrt{\mu a}} \quad \text{and } \alpha_8 = 1 - \frac{r}{a}$$

VIII. Types of Observations

One of the advantages of the Schmidt-Kalman scheme is the ability to determine the optimum observation to be made at each orbit position in order to have the greatest decrease in uncertainty. The method permits a predicted estimate of the decrease in uncertainty for each observation prior to having made it. In this manner, a choice may be made to obtain more rapid convergence to the proper solution. It is possible to use range and/or range rate at each position of the orbit in order to obtain the maximum information to be gained by using each or both. In addition, observations made from the vehicle during mid-course, from accelerometers. optical measurements, etc. may be used in an interspersed manner to optimum advantage. It has been known for some time that the simplicity in designing, constructing and using c w Doppler (Reference 7) range and range rate instrumentation greatly favors these types of observations.

The Schmidt-Kalman technique, by processing the data within the linear range of the prediction theory, and through the use of the recommended orbit correction parameters, makes it possible to rely on these instrumentation techniques over large portions of the tracking complex.

A. Transformation Equations for Tracking Data.

It is sometimes desirable to transform the computed data into a topocentric coordinate system. The computed data is obtained in a cartesian coordinate system (x,y,z) defined with the origin at the center of the Earth, the x-axis in the direction of the vernal equinox and the x-y plane in the equatorial plane of the Earth. The series of coordinate transformations required is as follows:

A rotation about the z axis through angle θ' so that the x' -z' plane is in the meridian plane of the station:

Then:

$$
\begin{aligned}
x' &= x \cos \theta' + y \sin \theta' \\
y' &= -x \sin \theta + y \cos \theta' \\
z' &= z
\end{aligned}
\qquad (54)
$$

where

x', y', z' are the rotated coordinates and θ' is the right ascension of the meridian at the time of observation. θ' is computed by adding the longitude of the station to the right ascension of Greenwich at time of the observation. The right ascension of Greenwich for the beginning of the year in question is obtained by Newcomb's formula, updated to a 1960 origin. The instantaneous hour angle is then obtained by a linear formula, separating daily and hourly rates.

A translation of the x', y', z' axes from the center of the Earth to the station in question

$$
\begin{aligned}
x'' &= x' -(C + h) \cos \varphi \\
y'' &= y' \\
z'' &= z' -(S + h) \sin \varphi
\end{aligned}
\qquad (55)
$$

where φ is the geodetic latitude and h the height above sea level of the station in question. C and S are parameters which account for the polar oblateness of the Earth and are given by

$$C = (1 - e^2 \sin^2 \varphi)^{-\frac{1}{2}}$$
$$S = (1 - e^2)\, C \tag{56}$$

A rotation about the y" axis through an angle 90° $-\varphi$ so that the z" axis is in the zenith

$$x''' = x'' \sin \varphi - z'' \cos \varphi$$
$$y''' = y'' \tag{57}$$
$$z''' = x'' \cos \varphi + z'' \sin \varphi$$

This series of transformations will transform the geocentric coordinates of a point into the topocentric system. Transformation of topocentric to geocentric coordinates is achieved by the following set:

$$x'' = x''' \sin \varphi + z''' \cos \varphi$$
$$y'' = y \tag{58}$$
$$z'' = -x \cos \varphi + z''' \sin \varphi$$

$$x' = x'' + (C + h) \cos \varphi$$
$$y' = y'' \tag{59}$$
$$z' = z'' + (S + h) \sin \varphi$$

$$x = x' \cos \theta' - y' \sin \theta'$$

$$y = x' \sin \theta' + y' \cos \theta' \qquad (60)$$

$$z = z'$$

It is clear, that application of these transformations requires the complete knowledge of the topocentric displacement vector. The following paragraphs describe the treatment of various sets of incomplete data that may be available.

B. Formulae for observed variables and their Partial Derivatives.

The program will accept the following types of observational data, singly or in combination:

1. Range
2. Range rate
3. Right ascension and declination
4. Azimuth and elevation and Minitrack observations

In order to generate the differential corrections described in Section VI, it is necessary to compute residuals which consist of the difference between computed values of the observables and the observation data. In addition, it is necessary to compute partial derivatives of the observables with respect to the orbit parameters. The range, range rate, right ascension and declination can be expressed directly in terms of the geocentric state variables and the required partial derivatives may be obtained as follows:

Range: The computed value of the range is given by

$$\rho = \left[(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2 \right]^{\frac{1}{2}} \qquad (61)$$

and the matrix of partial derivatives for the range is given by

$$M(t) = \left[\frac{x - x_s}{\rho}, \frac{y - y_s}{\rho}, \frac{z - z_s}{\rho}, 0, 0, 0 \right] \qquad (62)$$

Range Rate: The computed range rate is given by

$$\dot{\rho} = \frac{1}{\rho} \left[(x - x_s) (\dot{x} + \omega_e y_s) + (y - y_s) (\dot{y} - \omega_e x_s) + (z - z_s) \dot{z} \right] \qquad (63)$$

The matrix of partial derivatives of the range rate with respect to the state variables is given by

$$M(t) = \left[\frac{\dot{x} + \omega_e y_s}{\rho} - \dot{\rho} \frac{(x-x_s)}{\rho^2}, \frac{\dot{y} - \omega_e x_s}{\rho} - \frac{\dot{\rho}(y-y_s)}{\rho^2}, \frac{\dot{z}}{\rho} - \frac{\dot{\rho}(z-z_s)}{\rho^2}, \right.$$
$$\left. \frac{x - x_s}{\rho}, \frac{y - y_s}{\rho}, \frac{z - z_s}{\rho} \right] \qquad (64)$$

Right Ascension and Declination: The expressions for the right ascension and the declination may be written as

$$\sin D = \frac{z - z_s}{\rho}$$

$$\tan RA = \frac{y - y_s}{x - x_s} \qquad (64\text{-a})$$

The matrix of partial derivatives is given for D by

$$M(t) = \frac{-(z-z_s)}{\rho^2 \cos D} \left[\frac{x - x_s}{\rho}, \frac{y - y_s}{\rho}, \frac{(z-z_s)^2 - \rho^2}{\rho(z-z_s)}, \quad 0, 0, 0 \right]$$

and for R A by (64-b)

$$M(t) = \frac{1}{\sec^2 RA} \left[-\frac{y - y_s}{(x - x_s)^2}, \frac{1}{x - x_s}, 0, 0, 0, 0 \right]$$

Azimuth and elevation and the Minitrack observations are most conveniently expressed in a topocentric, local horizon coordinate system and to treat them it is useful to introduce the following relation between the topocentric and geocentric coordinates:

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = \begin{bmatrix} \sin\varphi \cos\theta' & \sin\varphi \sin\theta' & -\cos\varphi \\ -\sin\theta' & \cos\theta' & 0 \\ \cos\varphi \cos\theta' & \cos\varphi \sin\theta' & \sin\varphi \end{bmatrix} \begin{Bmatrix} x-x_s \\ y-y_s \\ z-z_s \end{Bmatrix} \quad (65)$$

This relation is used in developing the required partial derivatives for these angular observations.

Azimuth and Elevations: The expressions for azimuth and elevation are

$$A = \tan^{-1} \frac{y''}{-x''} \quad (66)$$

and

$$E = \tan^{-1} \frac{z''}{(\rho^2 - z''^2)^{\frac{1}{2}}}$$

The corresponding matrices of partial derivatives are

for A

$$M(t) = \frac{-1}{\rho^2 - z'''^2} \Big[-x''' \sin \theta' - y''' \cos \theta' \sin \varphi, \tag{67}$$

$$x''' \cos \theta' - y''' \sin \theta' \sin \varphi,\ y''' \cos \varphi,\ 0,\ 0,\ 0 \Big]$$

and for E

$$M(t) = \frac{1}{(\rho^2 - z'''^2)^{\frac{1}{2}}} \Big[\cos \theta' \cos \varphi - \frac{z'''}{\rho^2} (x - x_s),$$

$$\sin \theta' \cos \varphi - \frac{z'''}{\rho^2} (y - y_s), \tag{68}$$

$$\sin \varphi - \frac{z'''}{\rho^2} (z - z_s),\ 0,\ 0,\ 0 \Big]$$

Minitrack Observations: The Minitrack system direction cosines are expressed in terms of the topocentric coordinates as

$$\ell = -\frac{x'''}{\rho}$$

$$m = \frac{y'''}{\rho} \tag{69}$$

$$n = \frac{z'''}{\rho}$$

the corresponding matrices of partial derivatives are for ℓ

$$M(t) = \frac{1}{\rho}\Big[- \sin\varphi\cos\theta' + \frac{x'''(x-x_s)}{\rho^2},\ -\sin\varphi\sin\theta' + \frac{x'''(y-y_s)}{\rho^2},$$

$$\cos\varphi + \frac{x'''(z-z_s)}{\rho^2},\ 0, 0, 0\Big]$$

for m

$$M(t) = \frac{1}{\rho}\Big[- \sin\theta' - \frac{y'''(x-x_s)}{\rho^2},\ \cos\theta' - \frac{y'''(y-y_s)}{\rho^2},$$

$$- \frac{y'''(z-z_s)}{\rho^2},\ 0, 0, 0\Big] \qquad (70$$

and for n

$$M(t) = \frac{1}{\rho}\Big[\cos\varphi\cos\theta' - \frac{z'''(x-x_s)}{\rho^2},\ \cos\varphi\sin\theta' - \frac{z'''(y-y_s)}{\rho^2},$$

$$\sin\varphi - \frac{z'''(z-z_s)}{\rho^2},\ 0, 0, 0\Big]$$

IX. Current Work

The discussion which follows outlines some of the work to be done over the next 3-4 year period.

1. The current program uses the conventional least squares correction technique. It works, and has been time tested, but there are better and more modern techniques, such as the Kalman-Schmidt which has shown promise of being far superior. The major reason for this is that both techniques are using linear theory to correct non-linear effects. However, the conventional least squares requires an abundantly over-determined solution which in real life means waiting to obtain a large number of observations during which time the non-linear effects build up. Conversely, the Kalman-Schmidt uses each observation as it occurs and thus is much closer to the linear range on which both theories are based. Another is the one currently in use at JPL, which briefly is a weighted least squares technique. Therefore, one of the first problems is to develop the Kalman-Schmidt, and the weighted least squares techniques for the present program, to thoroughly test them, and compare the techniques.

2. During the checkout of the present program, the problem inherent in many orbit determination programs to date has also occurred. Namely, the problem of ill-conditioned data. Furthermore, the problem varies with the type of orbit involved, e.g., for highly eccentric orbits angle data alone does not determine the energy very well, but for nearly circular orbits they do not determine the orbit orientation in space very well. For range and range-rate data, the converse is true.. As a result, the problem of increasing the rate of convergence or pursuing several other possible avenues to overcome the ill-conditioned data problem requires extensive research and development.

3. Many time consuming details have not been completely accomplished, e.g., atmospheric models, ionospheric refraction, mutation, lunar libration, choice of reference equinox, selection of optimum time standards, adequate conversion from ephemeris time to the appropriate time standard to name just a few.

4. It is recognized that integrating the number of trajectories corresponding to the number of variables is a poor and time consuming way of obtaining the correction matrix of partial derivatives. It is felt that here again the Encke techniques will offer great advantages over other orbit determination techniques, if one can make use of the analytic two body orbit to determine the derivatives. If this works, we will be back to integrating only one trajectory. There are, however, problems -- and research and develop nt in this area is required.

5. Before the program can approach anything like a real time program, the entire program must be examined w.r.t. time consumed in each operation, each sequence of operations and each subroutine within the confines of the required or desired accuracy. In addition, the program should automatically select its own optimum integration interval -- but here, again, is a research and development study since there is a trade-off on how long it takes to change integration intervals, and how much is gained by affecting the change. These selections may also be orbit type sensitive.

6. The final program should also make use of spacecraft on-board observations, optionally, alone or in conjunction with ground observations. Therefore, the compatibility of the two syrtems should be coordinated. In addition, the output from this program should be useable, when desired, to up-date the on-board information.

7. There are a myriad of detailed arrangements to standardize incoming observational data to keyed formats as well as outputs. Suitable connections to the vast network of stations etc.

8. The program should be adaptable to the rendezvous mission and it may be desirable not to commit to a pre-designed translunar trajectory, but rather to determine the appropriate one after the rendezvous has been affected. In fact, in light of the limitless number of possible lunar trajectories, it may be desirable to select the translunar trajectory, even for a single booster launch, while the spacecraft is in the parking orbit.

9. There are positive indications that it may be possible, through the Kalman-Schmidt techniques to design a program capable of differentially correcting from the ground up, i.e., during boost. If this were so, one of the largest difficulties of all orbit determination programs would be eliminated, i.e., the error in beginning with a preconceived (nominal) R and V. This error is largely responsible for such tracking errors as wrong lobe identification from tracking stations etc.

10. Finally, and certainly one of the most important, is the problem of building in the decision making into the program wherever possible. This problem is somewhat interdependent with number 9, since the type of decision which must be made depends on how large an error must be accounted for. This is particularly true for linear theories such as these because any malfunction can cause the solutions to be outside the linear range. The present program rejects all observations which are outside the 3σ deviation for each iteration, but this only scratches the surface of handling the problem of bad data. JPL has a very extensive data editing program. Related to this is a study of maximizing information returned from various types of tracking data, e.g., range data, which may very well determine the energy of highly eccentric

orbits. It may be possible to neglect other variables and get a first order approximation of drag correction, similarly refraction corrections etc.

11. In all this, new concepts, variables and programs are being introduced. It should go without saying that a continuous and exhaustive shakedown of all of these under all conceivable conditions should be a constant part of this total effort, e.g., determining the characteristics of the differential parameters under all types of orbits, at various positions in each of these orbits, with ideal observational data, noisy generated observational data and wherever possible, actual satellite or space probe data.

It also goes without saying that all this work shall be properly documented with progress reports, interim and final reports. Numerous problems will also arise during the course of this work for which solutions must be found to the mutual satisfaction of all.

X. Refraction Correction

A. Refraction Correction For Pulse Radar

In the event that the observation signal is an actual measurement of a time delay in transmitting a radar pulse, the assumption is usually made that the range is obtained by simply multiplying the transit time by the vacuum speed of light, c, as follows (see figure 6)

$$\rho_a = c\,t = \int_{r_o}^{r_v} \frac{c}{v}\, ds \quad . \tag{71}$$

In terms of the index of refraction, n, the apparent range is

$$\rho_a = \int_{r_o}^{r_v} n\, ds \quad . \tag{72}$$

The actual path traversed by the signal is given by the rule that the transit time is a minimum. From the calculus of variations, the solution of the path is characterized by the fact that

$$k = \text{constant} = n(r)\, r \cos\gamma = n(o)\, r_o \cos\lambda_o \quad . \tag{73}$$

Referring to fig. 7 the relation between the arc length, ds, and the path angle, γ, is given by

$$ds = \frac{n\, r\, dr}{\sqrt{n^2\, r^2 - k^2}} \quad . \tag{74}$$

Equation (72) may be written

$$\rho_a = \int_{r_o}^{r_v} \frac{n^2\, r\, dr}{\sqrt{n^2\, r^2 - k^2}} \tag{75}$$

Eq. (75) may be integrated in closed form provided n(r) is a constant.

$$\rho_{i+1} - \rho_i = \sqrt{n_i^2 r_{i-1}^2 - k^2} - \sqrt{n_i^2 r_i^2 - k^2} \quad . \tag{76}$$

Assuming that n(r) is piecewise constant over several layers in the atmosphere, it is possible to obtain the apparent range by summing the successive solutions of Eq. (76) until the computed geocentric distance r_v is obtained. Since the constant, k, is given in terms of the initial sight angle, $k = n(o)\, r(o) \cos \lambda_o$, the apparent range is seen to be a function of the geocentric distance, r_v, and the ground station sight angle, λ_o.

$$\rho_a = \sum_{i=1}^{v} (\rho_i - \rho_{i-1}) \tag{77}$$

B. Refraction Correction for Range or Range Rate of CW Doppler Radar Data.

When the signal observations for range or range rate are obtained from a CW carrier measuring either phase shift or frequency shift, it is necessary to correct for the additional change in frequency or phase due to the effect of the refraction on the phase velocity of the signal. A first order estimate of this correction may be obtained by using the concept of the average index of refraction over the signal path. The Doppler shift is given by

$$\dot{\rho} = - \frac{\Delta f}{f} \frac{c}{\bar{n}} \tag{78}$$

where Δf is the difference in frequency between the emitted and returned signals, and $\bar{n}$ is the value of the average index of refraction defined as follows

$$\bar{n} = \frac{\int_{h_o}^{h_v} n(h)\, dh}{h_v - h_o} \tag{79}$$

Eq. (79) may be integrated using the assumption that the index of refraction is piecewise constant

$$\bar{n} = \frac{\Sigma\, n_i\, (h_{i+1} - h_i)}{h_v - h_o} \tag{80}$$

The altitude of the vehicle may be obtained from the assumed nominal computed position of the vehicle above the geoid. To apply the refraction correction for CW phase range and range rate observations it is only necessary to replace the value of the speed of light in vacuum by c/N in the normal equations used to obtain range and range rate estimates from the signal.

XI. Recommendations

It is recommended that a general purpose orbit prediction and tracking program be generated incorporating the features outlined in this report.

It is further recommended that specific attention be paid to the requirements for the Apollo space mission so that this program may be used in the Apollo tracking complex, or as a back-up for whatever system is finally decided upon.

It is recommended that a detailed comparison between the methods suggested herein and the least square procedure, using other parameters, as well as the weighted least square procedure recommended by J.P.L. be extensively explored so that the best features of each may be incorporated in the final acceptable scheme.

XII. REFERENCES

1. Pines, S., Payne, M., Wolf, H., "Comparison of Special Perturbation Methods in Celestial Mechanics," Aeronautical Research Laboratories, Wright-Patterson Air Force Base, Ohio, August 1960.

2. Pines, S., Wolf, H., Richman, J., "Orbit Determination and General Purpose Differential Correction Program," NASA NAS 5-293, Republic Aviation Corporation.

3. Pines, S., Wolf, H., Payne, M., "Asymptotically Singular Differential Correction Matrices," NASA NAS 5-293, Republic Aviation Corporation.

4. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," Journal of Basic Engineering Vol. 82, No. 1, March, 1960, Pages 35-50.

5. Smith, G. L., Schmidt, S. F., and McGee, L. A., "Application of Statistical Filter Theory to the Optimal Estimation of Position and Velocity on Board a Circumlunar Vehicle," NASA TN D-1205, 1962.

6. Magness, T. A., McGuire, J. B., "Comparison of Least Squares and Minimum Variant Estimates," Annals of Mathematical Statistics, Vol. 33, June, 1962.

7. Vonbun, F., "Analysis of the Range and Range-Rate Tracking System," NASA TN D-1178, 1962.

8. Pfister, W., Keneshea, T. J., "Ionospheric Effects on Positioning of Vehicles at High Altitudes," Air Force Cambridge Research Center, AFCRC TN-56-203, March 1956.

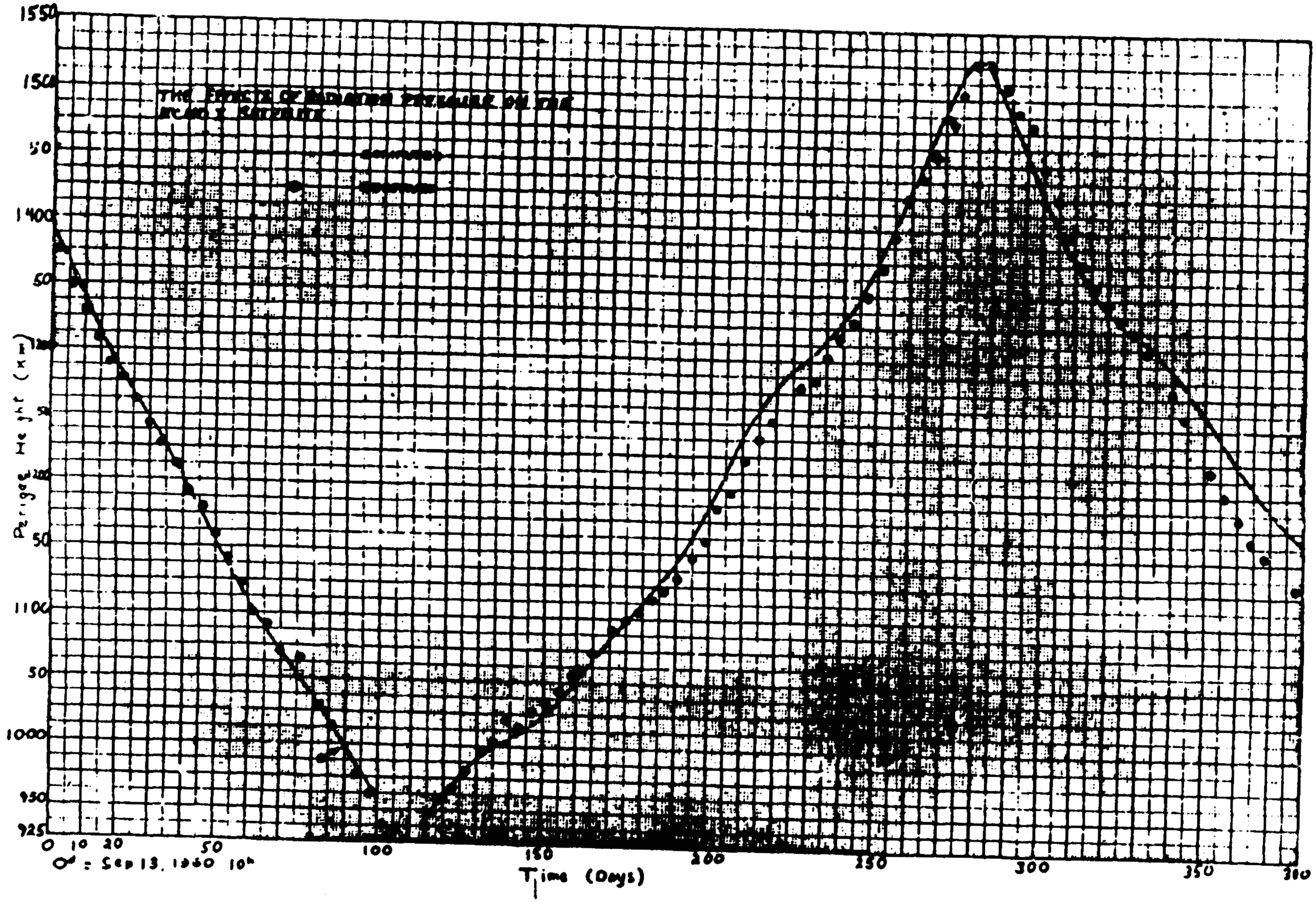

THE EFFECTS OF RADIATION PRESSURE ON THE ECHO I SATELLITE
Perigee Height (km)
1550
1500
50
1400
50
1300
50
1200
50
1100
50
1000
950
925
0 10 20 50 100 150 200 250 300 350 310
0 = Sep 13, 1960 10h
Time (Days)

Figure 1

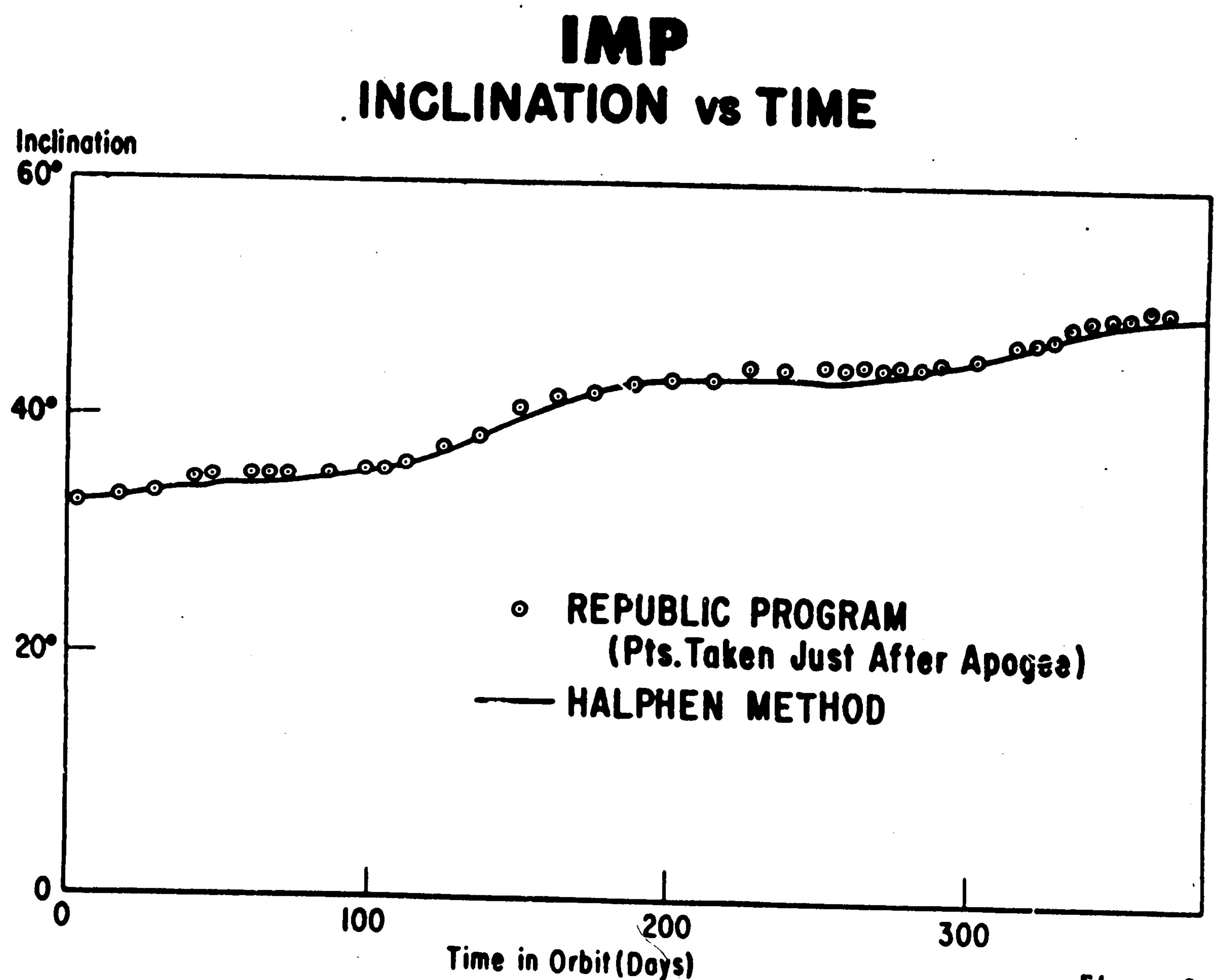
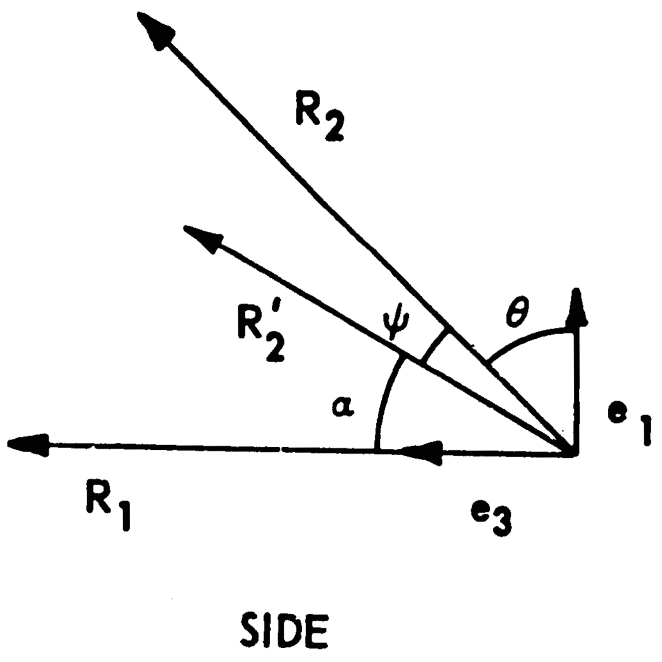

IMP
INCLINATION vs TIME
Inclination
60°
40°
20°
0
0
100
200
300
Time in Orbit (Days)
REPUBLIC PROGRAM
(Pts. Taken Just After Apogee)
HALPHEN METHOD

Figure 2

RIGID VECTOR ROTATIONS (1)

A small rotation of R_2 about R_1



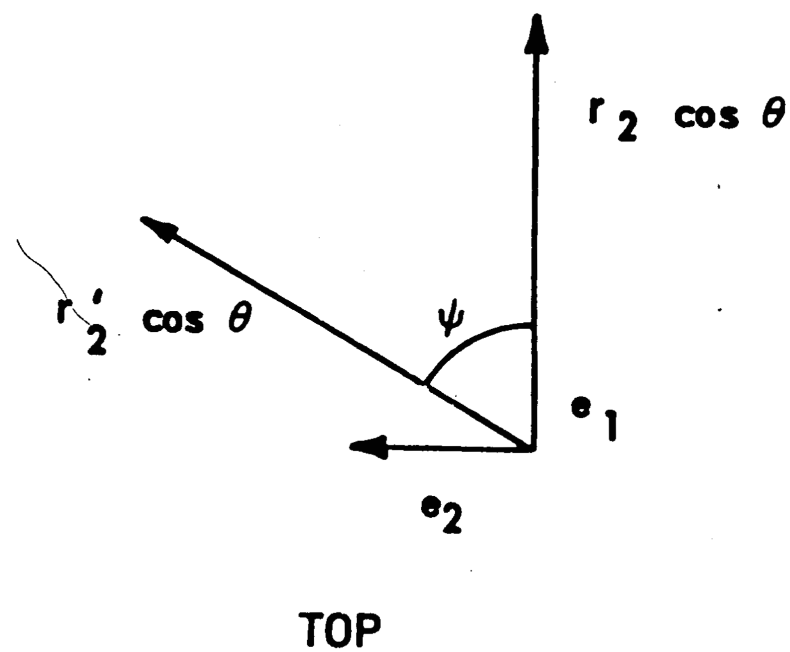


Figure 3

RIGID VECTOR ROTATIONS (2)

$$r_2' = r_2$$

$$e_3 = \frac{R_1}{r_1}, \quad e_2 = \frac{R_1 \times R_2}{|R_1 \times R_2|}, \quad e_1 = e_2 \times e_3$$

$$R_2' = \frac{(R_1 \cdot R_2) R_1}{r_1^2} + \frac{R_1 \times R_2 \sin \psi}{r_1} + \frac{(R_1 \times R_2) \times R_1 \cos \psi}{r_1^2}$$

Figure 4

$$\Delta\alpha_1 \begin{cases} R'_{o_1} = \dfrac{\dot R_o(\dot R_o \cdot R_o)}{\dot r_o^{2}}(1-\cos\Delta\alpha_1) + R_o\cos\Delta\alpha_1 + \dfrac{(\dot R_o \times R_o)}{\dot r_o}\sin\Delta\alpha_1 \\ \dot R'_{o_1} = \dot R_o \end{cases}$$

$$\Delta\alpha_2 \begin{cases} R'_{o_2} = R'_{o_1} \\ \dot R'_{o_2} = \dfrac{R'_{o_1}(R'_{o_1} \cdot \dot R_o)}{r'^{2}_{o_1}}(1-\cos\Delta\alpha_2) + \dot R_o\cos\Delta\alpha_2 + \dfrac{(R'_{o_1} \times \dot R_o)}{r'_{o_1}}\sin\Delta\alpha_2 \end{cases}$$

$$\Delta\alpha_3 \begin{cases} R'_{o_3} = R'_{o_1}\cos\Delta\alpha_3 + \dfrac{\dot R'_{o_2}(r_{o_1}^{2}) - R'_{o_1} \cdot \dot R'_{o_2})}{|R'_{o_1} \times \dot R'_{o_2}|}\sin\Delta\alpha_3 \\ \dot R'_{o_3} = \dot R'_{o_2}\cos\Delta\alpha_3 + \dfrac{\dot R'_{o_2}(\dot R'_{o_2} \cdot R'_{o_1}) - R'_{o_1}(\dot r_{o_2}^{2})}{|R'_{o_1} \times \dot R'_{o_2}|}\sin\Delta\alpha_3 \end{cases}$$

(Figure 5)

$$\frac{R'_{o_3} \cdot \dot{R}'_{o_3}}{\sqrt{\mu|a|}} + \Delta\alpha_4 = \frac{R' \cdot \dot{R}'}{\sqrt{\mu|a'|}}$$

$$\frac{1}{a} + \Delta\alpha_5 = \frac{1}{a'}$$

$$1 - \frac{r'_{o_3}}{a} + \Delta\alpha_6 = 1 - \frac{r'}{a'}$$

Simultaneous solution of the three preceding equations yields

$$r' = r'_{o_3} \left(\frac{1 - \frac{a}{r'_{o_3}} \Delta\alpha_6}{1 + a\,\Delta\alpha_5} \right)$$

$$v'^2 = \mu \left(\frac{2}{r'} - \frac{1}{a'} \right)$$

$$\Delta\gamma = \arccos \frac{R'_{o_3} \cdot \dot{R}'_{o_3}}{r'_{o_3} v'_{o_3}} - \arccos \frac{1}{r'v'} \sqrt{\left| \frac{a'}{a} \right|} \left(R'_{o_3} \cdot \dot{R}'_{o_3} + \sqrt{\mu|a|}\,\Delta\alpha_4 \right)$$

$$R'_{o_4} = R'_{o_3} \cos\Delta\gamma + \frac{H'_{o_3} \times R'_{o_3}}{h'_{o_3}} \sin\Delta\gamma$$

$$\dot{R}'_{o_4} = \dot{R}'_{o_3}$$

Figure 5 (2)

The final R and $\dot{R}$ for all six $\Delta\alpha$'s is then

$$R_{O_7} = R'_{O_6} \frac{r'}{r'_{O_5}}$$

$$\dot{R}_{O_7} = \dot{R}'_{O_6} \frac{v'}{v'_{O_5}}$$

The last parameter change does not affect the initial conditions but enters into the acceleration computation.

$$\Delta\alpha_7 \qquad \left\{ \left(\frac{C_D A}{W} \right) = \frac{C_D A}{W} + \Delta\alpha_7 \right.$$

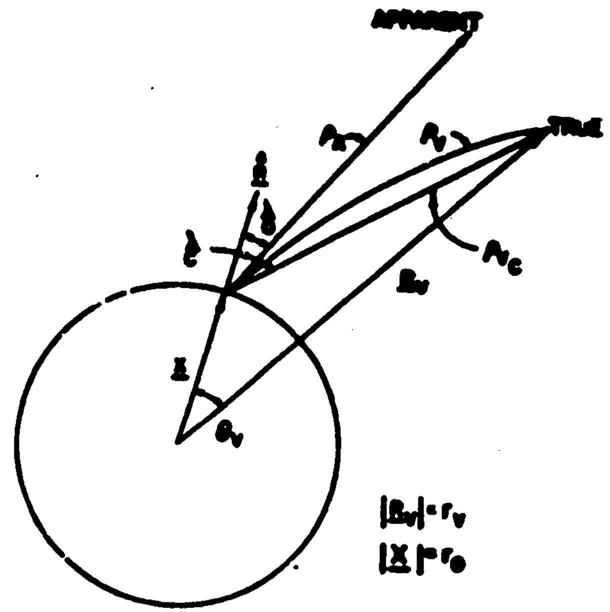


Figure 6—Typical ray path

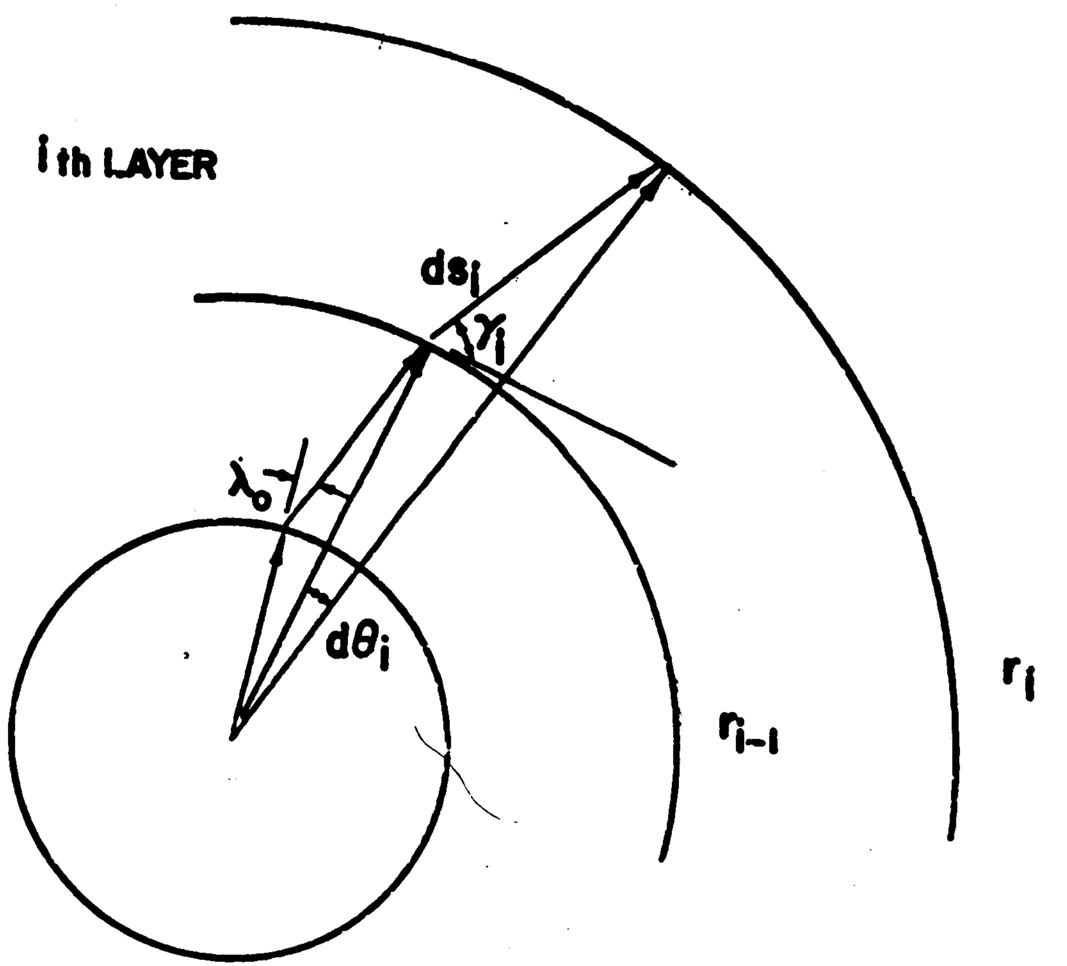


Figure 7—Geometry for determining beam angle curvature