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Determination of the Earth's Gravitational Field

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Abstract. In all three principal methods of measuring the earth's gravitational field—gravimetry, satellite orbit perturbations, and astrogeodetic networks—both mathematical theory and observational techniques have been developed in recent years to more than sufficient accuracy to define and determine the irregularities of the field significant on a planetary scale. The common defect of all three methods is inadequate distribution of observations. At present, the different methods agree only for major features such as the Indian Ocean minimum, and the normalized spherical harmonic coefficients of the potential $\bar{C}_{nm}$, $\bar{S}_{nm}$ are known to be of an order of magnitude of about $\pm 1.3 \times 10^{-6}/(n-1)$ with an uncertainty of about $\pm 0.6 \times 10^{-6}$ for the tesseral terms. Gradual improvement is anticipated with additional observations and the application of better statistical techniques made possible by modern computers. Author

INTRODUCTION

The subject of this review is the determination of the earth's gravitational field, given measurements that are adequate in accuracy but incomplete in their distribution. The emphasis will be on the wide-scale variations of the field rather than on local variations. In accordance with this emphasis, we shall usually employ the spectral representation of the gravitational field rather than the spatial.

Detailed discussion will be attempted only for developments since the advent of artificial satellites in 1957. Earlier developments are described adequately in publications generally available: the texts by *Heiskanen and Vening-Meinesz* [1958], *Jeffreys* [1959], and *Bomford* [1962]; the *Handbuch der Physik* articles by *Garland* [1956] and *Jung* [1956]; and, recently translated into English, the important treatise by *Molodenskiy, Yeremeyev, and Yurkina* [1960].

MATHEMATICAL EXPRESSION OF THE GRAVITY FIELD

The reference figure. The physically logical reference figure is that of a rotating fluid in equilibrium. The principal variable involved in such a model is the radial density distribution. The theoretical investigation of rotating fluid models of the earth is still being pursued, principally by *Ledersteger* [1962; see this reference for earlier papers].

The more mathematically tractable ellipsoid of revolution, which differs by quantities of the order of 10^{-6} from a rotating fluid, is generally used in practice for a reference figure. A closed expression for the gravity (i.e., gravitational attraction plus centrifugal force), first obtained by *Pizzetti* [1894], requires the use of ellipsoidal harmonics. For practical application, the present accuracy of determination of the gravity field makes it more convenient to use series developments in spherical harmonics. For an ellipsoidal figure specified by an equatorial semimajor axis a, an equatorial gravity γ_e, a polar semiaxis b, and a rate of rotation ω, formulas of sufficient precision for the subsequent discussions in this review are [*Jung*, 1956, pp. 542–562; *Heiskanen and Vening-Meinesz*, 1958, pp. 51–53; *Bomford*, 1962, pp. 414–416]

$$f = (a - b)/a \tag{1}$$

$$m = \omega^2 a/\gamma_e \tag{2}$$

$$kM = a^2\gamma_e[1 - f + \tfrac{3}{2}m - \tfrac{15}{14}mf + 0(f^3)] \tag{3}$$

$$\gamma = \gamma_e[1 + (\tfrac{5}{2}m - f - \tfrac{17}{14}mf)\sin^2\phi_g + (\tfrac{1}{8}f^2 - \tfrac{5}{8}mf)\sin^2 2\phi_g + 0(f^3)] \tag{4}$$

$$r_e = a[1 - (f + \tfrac{3}{2}f^2)\sin^2\phi_g + \tfrac{3}{2}f^2\sin^4\phi_g + 0(f^3)] \tag{5}$$

$$J_2 = \tfrac{2}{3}f(1 - \tfrac{1}{2}f) - \tfrac{1}{3}m[1 - \tfrac{3}{2}m - \tfrac{2}{7}f] + 0(f^3) \tag{6}$$

$$J_4 = -\tfrac{4}{35}f(7f - 5m) + 0(f^3) \tag{7}$$

$$U = \frac{kM}{r}\left[1 - J_2\left(\frac{a}{r}\right)^2 P_2(\sin\phi) - J_4\left(\frac{a}{r}\right)^4 P_4(\sin\phi) + 0(f^3)\right] \tag{8}$$

where k is the gravitational constant, M is the mass, ϕ is the geocentric latitude, ϕ_g is the geodetic latitude, r_e is the radial spherical coordinate of the surface of the ellipsoid, U is the gravitational potential, and the P_n are Legendre polynomials. Equations 3 through 7 are essentially those given by *Helmert* [1884] and are used by most geodesists. Another set derived by *DeSitter* [1938] is preferred by most astronomers.

The expression of the external field of an ellipsoid of revolution has recently been developed in series to higher-order terms by *Cook* [1959], *Hirvonen* [1960], and *Lambert* [1961] and expressed in closed formulas by *Caputo* [1963].

Variations in the field: first approximation. To accommodate departures of the gravitational potential of the earth from the reference figure, the field can be expressed as a sum of spherical harmonics:

$$V = \frac{kM}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} \bar{P}_{nm}(\sin\phi)\{\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\}\right] \tag{9}$$

The gravitational potential V is positive, following the sign convention of astronomy and geodesy. The absence of $n = 1$ terms from (9) means that the coordinate origin coincides with the center of mass.

$\bar{P}_{nm}$ (sin ϕ) is the Legendre associated polynomial. The overbar signifies a normalization; the one we shall apply in this review makes

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\left[\bar{P}_{nm}(\sin\phi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\lambda\right]^2 \cos\phi\, d\phi\, d\lambda = 4\pi \tag{10}$$

so that

$$\bar{P}_{nm}(\sin\phi) = \left[\frac{(2n+1)(n-m)!\,(2-\delta_{0m})}{(n+m)!}\right]^{1/2}\frac{\cos^m\phi}{2^n} \cdot \sum_{t=0}^{k}\frac{(-1)^t(2n-2t)!}{(n-m-2t)!\,t!\,(n-t)!}\sin^{n-m-2t}\phi \tag{11}$$

where δ_{0m} is the Kronecker delta, and k is the integer part of $(n - m)/2$.

The coefficients $\bar{C}_{nm}$, $\bar{S}_{nm}$ are independent parameters

$$\bar{C}_{20} = -J_2/\sqrt{5} \approx -0.0004842 \tag{12}$$

All other $\bar{C}_{nm}$, $\bar{S}_{nm}$ are $0(10^{-6})$. The magnitude of f (or J_2 or $\bar{C}_{20}$) means that variations of the field referred to the ellipsoidal surface r_e expressed by (5) can be expressed by the $\bar{C}_{nm}$, $\bar{S}_{nm}$ with a proportionate error of only 10^{-3}. The variation most commonly referred to the ellipsoid is the geoid, i.e., the equipotential surface of the family defined by (9) for which V is equal to the U of (8) on the ellipsoidal surface in (5). Expressed in this manner, the geoid height is

$$N = (V - U)/\gamma = T/\gamma \tag{13}$$

The ellipsoidal surface r_e is an equipotential for the gravitational potential U *plus* the centrifugal force potential ψ:

$$\psi = \frac{\omega^2}{2} r^2 \cos^2 \phi \tag{14}$$

Measurements of the gravity acceleration can be referred to the geoid through leveling observations. To compare the observed acceleration g with the acceleration γ on the reference model, we must therefore allow for the height difference N:

$$\begin{aligned}\Delta g &= g(r_e + N) - \gamma(r_e) \\ &= -\left[\frac{\partial(V + \psi)}{\partial n}\right]_{r=r_e+N} + \left[\frac{\partial^2(U + \psi)}{\partial n^2} N + \frac{\partial(U + \psi)}{\partial n}\right]_{r=r_e}\end{aligned} \tag{15}$$

where the derivatives are normal to the ellipsoid.

Using (13) and taking the spherical approximation $-2\gamma/r$ for $\partial\gamma/\partial r$, we obtain

$$\Delta g = (\partial T/\partial n) - 2T/r \tag{16}$$

The last term in (16) is usually referred to as Bruns' term.

For the coefficient $\bar{A}_{nm}$, $\bar{B}_{nm}$ of a particular term in the spherical harmonic expression of Δg, we obtain from (9), (15), and (16)

$$\bar{A}_{nm}, \bar{B}_{nm} = \gamma(n - 1)\{\bar{C}_{nm}, \bar{S}_{nm}\} \tag{17}$$

To relate the geoid height N at a particular point to the anomalies Δg over the surface of the earth, we develop the anomalies in spherical harmonics about the point as a pole and use the orthogonality relationships between spherical harmonics to obtain:

$$N = \frac{R}{4\pi\gamma} \int_{\text{sphere}} S(\cos\theta)\, \Delta g\, d\sigma \tag{18}$$

where

$$S(\cos\theta) = \sum_{n=2}^{\infty} \frac{\sqrt{2n+1}}{n-1} \bar{P}_n(\cos\theta) \tag{19}$$

$\bar{P}_n(\cos\theta)$ is the normalized zonal harmonic. The closed form of $S(\cos\theta)$ is Stokes' function [*Stokes*, 1849]:

$$S(\cos\theta) = \csc\frac{\theta}{2} - 3\cos\theta \ln\left(\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) - 6\sin\frac{\theta}{2} + 1 - 5\cos\theta \tag{20}$$

The similar function for the slope of the geoid, or deflection of the vertical, was obtained by *Vening-Meinesz* [1928] and *de Graaff-Hunter* [1935].

If the coordinate system were rotated and the generalized additional theorem applied to the spherical harmonics, the coefficients $\bar{C}_{nm}$, $\bar{S}_{nm}$ referred to the new axes would be expressed entirely as functions of the coefficients of degree n referred to the old axes. Furthermore, the quantity

$$\sigma_n^2 = \sum_m \{\bar{C}_{nm}^2 + \bar{S}_{nm}^2\} \tag{21}$$

remains invariant under rotation [*Kaula*, 1959*b*]. Thus, the 'degree variance' σ_n^2 is a measure of the amount of variability in a certain wavelength that is independent of the coordinate axes' directions. By summary σ_n^2 over n from 2 to infinity, we obtain the mean square amplitude.

The foregoing relationships were all derived on the assumption that the functions were harmonic, which requires that the geoid be external to all masses. Since the actual geoid is not external to all masses, hypothetical transfers of mass must be made before 'reduction to sea level' of gravity observations and application of Stokes' function. This problem has historically occupied most of the attention of physical geodesy [*Heiskanen and Vening-Meinesz*, 1958, pp. 147–186; *Bomford*, 1962, pp. 416–446]. Advocates can still be found for most of the reductions that have been proposed in the past. The free air anomaly, which implies that the external topography is condensed in a thin layer just below the geoid; smoothed modifications of the free air anomaly [*de Graaff-Hunter*, 1960]; the isostatic anomaly, which implies that topographic excesses and deficiencies are redistributed uniformly through the crust (Pratt-Hayford compensation) or transferred to the crust-mantle interface (Airy-Heiskanen compensation); and even the Rudzki anomaly, which implies that the external masses are replaced by internal masses yielding the same geoid [*Tengstrom*, 1962] are all still applied for geodetic purposes, while the Bouguer anomaly, which implies complete removal of the mass excess and deficiencies, is most often applied in geologic interpretation.

The free air anomaly is the most often used, mainly for the practical reason that it does not require detailed evaluation of the topography (except for the local area). As *Levallois* [1962] emphasized, all the methods of reduction can be considered as equivalent provided that the 'indirect effect' (the effect on Δg of the geoid shift that occurs because of the implied mass transfer) of each method is properly taken into account. Hence the practical distinction becomes which type of anomaly can be considered as most 'representative'; i.e., for which type can the observed point values be interpolated most easily.

Isostatic anomalies are generally regarded as the most representative, but they are also the most laborious, since it is difficult to automate estimation of topographic elevations. Work is continuing to facilitate Airy-Heiskanen isostatic reductions, such as, e.g., the reduction maps for the distant zones of the Isostatic Institute [*Kärki et al.*, 1961]. For analytic application of the Airy-Heiskanen isostatic reduction to spherical harmonics, formulas of *Jung* [1952] are convenient. The isostatic reduction can be expressed as a power series in $h(\phi, \lambda)$, the height of the solid surface, and $w(\phi, \lambda)$, the depth of the water. By expressing w as a surface coating, Jung develops the isostatic correction in terms of the spherical harmonic expansion of the topography plus water to quadratic terms in the topography of the form:

$$\Delta g_{i.} = \sum_{n=0}^{\infty} \{e_n[E]_n + f_n[E^2]_n\} \tag{22}$$

where $[E]_n$ and $[E^2]_n$ are the sums of terms of degree n in the harmonic expression of

$$E = h - (1.03/\sigma_0)w \tag{23}$$

and its square, and e_n and f_n are functions of the depth of compensation and the crustal and mantle densities. The functions e_n and f_n contain terms of the form $\{1 - (1 - t)^{2n+1}\}$, where t is the ratio of the compensation depth to the earth's radius, and so (22) is probably imprecise for higher values of n for this reason as well as because of the lack of higher powers of E. For the purposes of this review, (22) can be considered a means of estimating Δg where E is known but not g.

Variations in the field: higher approximations. As in any problem, further elaboration can be made either in the reference model or in the expressions of departures from the model. The first type of elaboration was made by *Sagrebin* [1956] who refined Stokes' formula (equation 18) to an accuracy of order f^2 by using elipsoidal harmonics; his solution has since been improved and corrected by *Molodenskiy et al.* [1960] and by *Bjerhammar* [1962b]. These investigators found that the error in Stokes' formula was indeed of the order of f, i.e., 20 cm or less in N.

The principal theoretical developments in physical geodesy in recent years have been in nonlinear expression of the departures from the model by considering the problem at the physical surface of the earth rather than at the geoid, thus eliminating the inaccuracies due to ignorance of the density within the earth. These developments were initiated in a paper by *Jeffreys* [1931], and have been carried forward principally by *Molodenskiy* [1945, 1948]. Full details are given by *Molodenskiy et al.* [1960], and a brief summary by *Molodenskiy* [1962]. The derivation of the method currently applied in the USSR starts with the expression of the disturbed potential T in the form of a surface layer φ on the physical surface of the earth. Taking the derivative of T normal to the physical surface, using the Bruns' equation 16, and allowing for the slope of the surface with respect to the equipotential, we obtain an expression for the free air anomaly Δg_F in terms of integrals over the equipotential surface S. When we assume that the projection of the surface layer φ onto a sphere $\bar{S}$, of radius R, can be expressed in terms of a power series over the discrepancy between $\bar{S}$ and S, the solution is obtained as a successive approximation in four cycles, $n = 0$ through 3, over (setting negatively subscripted variables zero)

$$G_n = R \int \frac{h - h_0}{r_0^3} \left[R\chi_{n-1} - \tfrac{3}{4}(h - h_0)\chi_{n-2} - \tfrac{3}{2}R\left(\frac{h - h_0}{r_0}\right)^2 \chi_{n-3} \right] d\sigma + 2\pi\chi_{n-2} \tan^2 \alpha + \delta_{0n}\Delta g_F \tag{24}$$

$$\chi_n = \frac{G_n}{2\pi} + \frac{3}{(4\pi)^2} \int G_n[S(\cos \theta) - \tfrac{1}{2}] \, d\sigma \tag{25}$$

where h_0 is the 'normal' height at the point where the quantity on the left is computed, h is the 'normal' height at the location of the quantities within the integral,

r_0 is the chord distance of the sphere between the two points, α is the slope of the surface at the point of computation, and the integration is over the unit sphere. The 'normal' height differs from the orthometric height in that the normal gravity is used in place of the true gravity g. Then to obtain T, the disturbance of the potential:

$$T_n = \frac{R}{4\pi} \int G_n[S(\cos \theta) - \tfrac{1}{2}]\, d\sigma - \frac{R^2}{2} \int \frac{(h - h_0)^2}{r_0^3} \chi_{n-2}\, d\sigma \tag{26}$$

$$T = \sum_n T_n \tag{27}$$

Similar formulas are derived for the deflection of the vertical.

Theoretical developments similar to Molodenskiy's have been made by *Levallois* [1957], *Arnold* [1959*a*, *b*], *de Graaff-Hunter* [1960], *Hirvonen* [1960], *Moritz* [1961], *Bjerhammar* [1962*a*], and *Cook* [1963*b*]. Summaries and comparisons are given by *Arnold* [1960*a*], *Hirvonen* [1961], *Tengstrom* [1961], and *Molodenskiy et al.* [1962], who criticize the approximations made in some of the other theories.

Examples of the numerical significance of the improved theory are given by *Molodenskiy et al.* [1960], *Arnold* [1960*b*], *Tengstrom* [1961], and *Pellinen* [1962]. The conclusion is that the difference in geoid height is always small, of the order of tens of centimeters at most, but that the effect on the deflection of the vertical can be appreciable, several seconds of arc, in extreme situations. Of particular interest for the determination of the long-wave components of the field is the computation by *Pellinen* [1962] of the effect of the G_1 term from (24) on the low-degree harmonics. He found that the rms effect up through degree 3 was ± 0.20 mgal on the normalized Δg coefficients, or about $\pm 0.15 \times 10^{-6}$ for the potential coefficients $\bar{C}_{nm}$, $\bar{S}_{nm}$.

STATISTICAL CONSIDERATIONS

Random processes referred to two types of continuums are of concern: time series, which comprise the perturbations of a satellite orbit, and distributions over a spherical surface, which comprise the variations of the earth's gravity field.

We shall first discuss heuristically the general questions of spectral analysis and estimation by quadratic sum minimization, and then apply the conclusions to the particular cases of time series and distributions over a spherical surface.

Spectral analysis and quadratic sum minimization. Discrepancies $f(s)$ of observations from a mathematical model that are small enough to be considered as linear can be represented as a linear transformation of a set of parameters $\mathbf{x}$:

$$f(s) = \mathbf{c}^T(s)\mathbf{x} \qquad s \in S \tag{28}$$

The superscript T denotes the transpose. The coordinates s may be of any number of dimensions. S is a subspace, not necessarily connected, of a total space T. Some of, or all, the parameters $\mathbf{x}$ are members of an enumerable infinite set which are orthogonal over T, i.e., the square array

$$\mathbf{A} = \int_T \mathbf{c}(t)\mathbf{c}^T(t)\, d\tau \tag{29}$$

is diagonal ($d\tau$ is an element of volume in T). The functions $\mathbf{c}(t)$ are usually normalized so that the diagonal elements of $\mathbf{A}$ are either 1 or a constant times $\int_T d\tau$. $\mathbf{x}$ can be divided into subsets $\mathbf{x}_n$ such that $\mathbf{c}_n^T(t)\mathbf{x}_n$ is invariant under rotation of the coordinate systems. The quadratic sum

$$\sigma_n^2 = \mathbf{x}_n^T\mathbf{x}_n \tag{30}$$

over one of these subsets is the degree variance, already mentioned in (21) for the case of T which is a spherical surface. For many phenomena, only the σ_n^2 are of interest, and not the individual components of $\mathbf{x}_n$. Usually it is assumed that the random process is isotropic, i.e. the covariance

$$K(r, s) = E\{f(r)f(s)\} \tag{31}$$

where E denotes the mean value over T and is a function of only the distance (the length along the geodesic) between r and s. Under isotropy, the covariance can be expressed as

$$K(r, s) = \sum_n k_n c_{n0}(d_{rs})\sigma_n^2 \tag{32}$$

where d_{rs} is the distance between r and s, c_{n0} is a member of $\mathbf{c}_n$, and k_n is constant. The process of spectral analysis is that of forming numerical estimates of $K(r, s)$ and then using the orthogonality expressed by (29) to determine the σ_n^2. The principal problem in this process is sampling: (1) the distribution of sample points, considering the wavelengths corresponding to the highest degree σ_n^2 anticipated to be significant (the 'aliasing' problem); and (2) the weighting, if any, applied to take into account the incomplete extent of the observed subspace S with respect to the total space T (the 'window' problem).

If the individual components of $\mathbf{x}_n$ are wanted, or if some of the parameters of $\mathbf{x}$ are not coefficients of orthogonal functions, then spectral analysis is insufficient and quadratic sum minimization must be applied. In this case, the estimates $K(r, s)$ are still made, but the $\mathbf{x}$ are determined so that, subject to (28),

$$\int_S \int_S \mathbf{x}^T\mathbf{c}(r)K^{-1}(r, s)\mathbf{c}^T(s)\mathbf{x}\, dr\, ds = \text{minimum} \tag{33}$$

where the inverse $K^{-1}(r, s)$ is a function such that

$$\int_S K^{-1}(r, s)K(s, t)\, ds = \delta(r, t) \tag{34}$$

where $\delta(r, t)$ is the Dirac delta function. The solution is

$$\mathbf{x} = \mathbf{W} \int_S \int_S \mathbf{c}(r)K^{-1}(r, s)f(s)\, dr\, ds \tag{35}$$

where $\mathbf{W}$ is a symmetric matrix whose elements are the estimated variances and covariances of the χ's: e.g., σ_n^2/N_n for the orthogonal function coefficients, where N_n is the number of terms of degree n.

The minimization of (33) is most generally proved as the determination of the projection $\mathbf{c}^T\mathbf{x}$ in the subspace specified by the chosen elements of $\mathbf{x}$ of the vector $f(r)$ in a Hilbert space with metric $K^{-1}(r, s)$, or measure $K(r, s)$.

More familiar is the form of (28) and (33) where (1) $f(s)$ is evaluated at a finite set of points, and (2) for some of the parameters in $\mathbf{x}$ there is no a priori estimate of the variances. Also the covariance matrix $\mathbf{W}$ of the $\mathbf{x}$ may be known rather than $\mathbf{K}$, the matrix equivalent of $K(r, s)$. In this case, the set of points can be expressed as a vector, and the varianceless part of $\mathbf{x}$ broken off as a separate vector $\mathbf{z}$:

$$\mathbf{Cx} + \mathbf{Mz} = \mathbf{f} \tag{36}$$

$$(\mathbf{f} - \mathbf{Mz})^T\mathbf{K}^{-1}(\mathbf{f} - \mathbf{Mz}) = \mathbf{x}^T\mathbf{W}^{-1}\mathbf{x} = \text{minimum} \tag{37}$$

At least one element is nonzero in each row of $\mathbf{C}$.

Solution by the method of Lagrangian multipliers obtains:

$$\mathbf{x} = \mathbf{WC}^T\mathbf{K}^{-1}(\mathbf{I} - \mathbf{M}(\mathbf{M}^T\mathbf{K}^{-1}\mathbf{M})^{-1}\mathbf{M}^T\mathbf{K}^{-1})\mathbf{f} \tag{38}$$

$$\mathbf{z} = (\mathbf{M}^T\mathbf{K}^{-1}\mathbf{M})^{-1}\mathbf{M}^T\mathbf{K}^{-1}\mathbf{f} \tag{39}$$

where

$$\mathbf{K} = \mathbf{CWC}^T \tag{40}$$

and $\mathbf{I}$ is the identity matrix. The covariance matrix of $\mathbf{z}$ is

$$\mathbf{V} = (\mathbf{M}^T\mathbf{K}^{-1}\mathbf{M})^{-1} \tag{41}$$

and of the corrected $\mathbf{x}$

$$\mathbf{U} = \mathbf{W} - \mathbf{WC}^T\mathbf{K}^{-1}(\mathbf{I} - \mathbf{M}(\mathbf{M}^T\mathbf{K}^{-1}\mathbf{M})^{-1}\mathbf{M}^T\mathbf{K}^{-1})\mathbf{CW} \tag{42}$$

The rows of $\mathbf{WC}^T$ (and columns of $\mathbf{CW}$) in (38) and (42) need be only those of the elements of $\mathbf{x}$ of interest. If $\mathbf{K}$ is calculated by (40), however, the $\mathbf{W}$ must include all parameters of appreciable effect. Conventionally, the elements of $\mathbf{x}$ are sometimes referred to as 'observations' and the elements of $\mathbf{z}$ as 'parameters'; mathematically, the distinction is whether or not covariance $\mathbf{W}$ can be preassigned.

Sometimes there are added to the system of (36) and (37) 'side conditions':

$$\mathbf{Nz} = \mathbf{k} \tag{43}$$

is a minor complication that can be removed by eliminating a corresponding number of elements in $\mathbf{z}$.

Ordinary least squares is the case in which

$$\mathbf{C} = \mathbf{W} = \mathbf{I} \tag{44}$$

Prediction and interpolation by linear regression is the case in which it is desired to estimate f in the subspace $T - S$:

$$f(r) = \mathbf{c}^T(r)\mathbf{x} \qquad r \varepsilon T - S \tag{45}$$

or, letting $\mathbf{g}$ be a set of values of f in $T - S$ and $\mathbf{D}$ be the corresponding array of $\mathbf{c}^T$'s, and using (38) (assuming no z)

$$E\{\mathbf{g}\} = \mathbf{DWC}^T\mathbf{K}^{-1}\mathbf{f} = \mathbf{Bf} \tag{46}$$

A result that is also arrived at through solving the Wiener-Hopf equations for the regression coefficients $\mathbf{B}$ is

$$\mathbf{DWC}^T = \mathbf{BCWC}^T$$

or

$$\mathbf{K}_{gf} = \mathbf{B}\mathbf{K}_{ff} \tag{47}$$

Sometimes the solution for $\mathbf{x}$ or $\mathbf{z}$ is made in stages, with different condition equations 36 in each stage, either because the nature of the process is evolutionary (e.g., orbit predictions) or because a single stage computation would be too unwieldy. In this case, the vector $\mathbf{x}$ for a particular stage can be considered as consisting of three parts: (1) new observations $\mathbf{x}_x$, with associated coefficient matrix $\mathbf{C}_x$ and covariance matrix $\mathbf{W}_x$; (2) old observations $\mathbf{x}_y$, with associated coefficient matrix $\mathbf{C}_y$ and covariance matrix $\mathbf{W}_y$:

$$\mathbf{W}_y = \mathbf{P}_y\mathbf{U}_0\mathbf{P}_y^T \tag{48}$$

where $\mathbf{U}_0$ is the $\mathbf{U}$ computed by (42) in the previous stage, or a submatrix thereof, and $\mathbf{P}_y$ is a propagation or transition matrix taking into account change of the parameters (or 'state variables') specifying the reference model; and (3) previously estimated parameters $\mathbf{x}_z$, with associated coefficient matrix $\mathbf{C}_z$ and covariance matrix $\mathbf{W}_z$:

$$\mathbf{W}_z = \mathbf{P}_z\mathbf{V}_0\mathbf{P}_z^T \tag{49}$$

where $\mathbf{V}_0$ is the $\mathbf{V}$ of (41) or a submatrix thereof. The $\mathbf{C}_x$, $\mathbf{C}_y$, $\mathbf{C}_z$ are combined to form a new $\mathbf{C}$ and the $\mathbf{W}_x$, $\mathbf{W}_y$, $\mathbf{W}_z$ are combined to form a new $\mathbf{W}$ for another solution according to (36) through (41). In calculating the residuals $\mathbf{f}$ for each new stage, the corrections $\mathbf{x}$, $\mathbf{z}$ from previous stages are incorporated in the reference model.

The foregoing discussion is a consequence of attempts to combine ideas of time series analysis of *Bartlett* [1956] and *Parzen* [1961] with least squares and its generalization as given by *Arley and Buch* [1950] and *Brown* [1955, 1957]. The results can probably be found to be subsumed in the general treatments of *Bochner* [1955] and *Yaglom* [1961] by those able to cope with the recondite mathematics therein. The derivation of (36) through (42) has recently been discussed in detail by *Stearn and Richardson* [1962]. The linear regression prediction of (45) through (47) is applied by *Moritz* [1962*b*] to observations on a plane and by *Parzen* [1961] to time series in continuous, rather than discrete, form. Special cases of the staged computation of $\mathbf{x}_x$, $\mathbf{x}_y$, $\mathbf{x}_z$ with $\mathbf{W}_y$, $\mathbf{W}_z$ calculated as in (48) and (49) are: (1) the optimal prediction of *Kalman* [1960]: $\mathbf{W}_z$, $\mathbf{C}_z$, and $\mathbf{M}$ are $\mathbf{0}$; and (2) the Bayes estimation of *Parzen* [1962*a*] and the preassigned covariance of *Kaula* [1961*c*]: $\mathbf{W}_y$ and $\mathbf{C}_y$ are $\mathbf{0}$; $\mathbf{P}_z$ is $\mathbf{I}$; and $\mathbf{C}_x$, $\mathbf{C}_z$, $\mathbf{M}$, $\mathbf{f}$ have the forms

$$\mathbf{C}_x = \begin{Bmatrix} \mathbf{C}_x \\ \hline \mathbf{0} \end{Bmatrix} \qquad \mathbf{C}_z = \begin{Bmatrix} \mathbf{0} \\ \hline -\mathbf{I} \end{Bmatrix} \qquad \mathbf{M} = \begin{Bmatrix} \mathbf{M}_x \\ \hline \mathbf{I} \end{Bmatrix} \qquad \mathbf{f} = \begin{Bmatrix} \mathbf{f}_x \\ \hline \mathbf{0} \end{Bmatrix} \tag{50}$$

Another variation of estimation by quadratic sum minimization is Gram-Schmidt orthogonalization [e.g., *Robinson*, 1959]. The function $f(s)$ of (28) is represented as

$$f(s) = \mathbf{d}^T\mathbf{y} \qquad s \in S \tag{51}$$

where the functions $\mathbf{d}$ are orthogonal over S, not T; i.e.,

$$\int_S \mathbf{d}(s)\mathbf{d}^T(s)\, d\tau = \mathbf{I} \tag{52}$$

the identity matrix. Setting

$$\mathbf{Bd} = \mathbf{c} \tag{53}$$

we get

$$\mathbf{BB}^T = \int_S \mathbf{cc}^T\, d\tau \tag{54}$$

$\mathbf{B}$ is thus not unique. The Gram-Schmidt orthogonalization process finds a unique $\mathbf{B}$ by requiring that, for the arbitrarily selected sequence of functions in $\mathbf{c}$, $\mathbf{B}$ is triangular. Letting the subscripts on d_i, c_i denote ordering in the arbitrary sequence Gram-Schmidt orthogonalization yields

$$d_1(s) = c_1(s) \tag{55}$$

$$d_i(s) = c_i(s) - \sum_{k=1}^{i-1} \frac{d_k(s) \int_S d_k c_k\, d\tau}{\int_S d_k^2\, d\tau} \qquad i > 1 \tag{56}$$

If the $\mathbf{x}$ of (28) has a finite number of members, $\mathbf{x}$ can be determined by simple least squares; i.e., in (39) $\mathbf{z}$ represents $\mathbf{x}$, $\mathbf{K}$ is $\mathbf{I}$, and $\mathbf{M}$ is the array of values of $\mathbf{c}$ and $\mathbf{f}$ is the array of values of f for a set of points or volume element means. In practice, $\mathbf{x}$ has an infinite number of members, and it is not known beforehand which are significant; also, the disposition of S within T may be such that $\mathbf{M}^T\mathbf{M}$ approaches singularity, so that it is computationally impracticable to invert. If Gram-Schmidt orthogonalization is applied, then by (52) $\mathbf{M}^T\mathbf{M}$ is diagonal and (39) degenerates to a simple Fourier analysis in which the coefficients are determined one by one until it is decided that nothing more of significance is being found. Gram-Schmidt orthogonalization thus is a systematic method of estimation that avoids the inversion $(\mathbf{M}^T\mathbf{M})^{-1}$ of simple least squares or the inversion $\mathbf{K}^{-1}$ of quadratic sum minimization (equation 35 or 38 with $\mathbf{M} = \mathbf{0}$). It does not, however, abolish the 'window' effect—the distortion of estimates of lower-degree coefficients by higher-degree variations—and by avoiding the assumptions of stationarity and isotropy of spectral analysis and quadratic-sum minimization it is accordingly a less effective predictor of the variations in the subspace $T - S$.

A technique similar to Gram-Schmidt orthogonalization in principle is stepwise least squares: the parameters in $\mathbf{x}$ are determined one each or one subset at a time, and the contribution of a particular set is subtracted from $\mathbf{f}$ before determining those of the next set. This method is commonly employed in excessively large problems when the parameters are not coefficients of orthogonal functions. It could perhaps be considered as a special case of the staged least squares (equation 50) in which $\mathbf{W}_Z$ is set zero, but differs in that the same condition equations 36 are used repeatedly with different sections of the $\mathbf{M}$ matrix set zero, which implies that the situation must approach orthogonality for the procedure to be justifiable.

Time series. Probability models of time series, and the analysis of continuous or uniformly spaced observations thereof, have been developed in considerable detail [*Bartlett,* 1956; *Grenander and Rosenblatt,* 1957; *Blackman and Tukey,* 1959; *Parzen,* 1961]. Applications to geophysics have been discussed by *Holloway* [1958], *Munk and MacDonald* [1960], and *Van Isacker* [1961].

The time series of significance in the present problem are vector quantities characterized by a discrete spectrum of fixed phase and amplitude plus a continuous spectrum of noise [*Kaula,* 1963*a*]:

$$x_f(t) = \mathcal{R}\left\{\sum_{n=1}^{N} \alpha_{fn} \exp\,[i\lambda_n t] + \int_0^\infty \beta_f(\omega) \exp\,[i\omega t]\, d\omega\right\} \tag{57}$$

where the subscript f denotes a vector component and n denotes one of a set of frequencies; $\mathcal{R}$ denotes the real part, i is $\sqrt{-1}$, and the vectors α_{fn}, $\beta_f(\omega)$ are complex. The λ_n are all known; the α_{fn} are linear functions of a set of parameters p_l, less than N in number, which it is our problem to determine. The duration T of observation of the time series is such that the discrete spectrum stands out above the noise; i.e.,

$$\beta_f(\omega)(2\pi/\omega T) \ll \alpha_{fn} \qquad \omega \geq \lambda_n \tag{58}$$

but

$$\beta_f(\omega)(2\pi/\omega T) \gg \alpha_{fn} \qquad \text{some} \quad \omega \ll \lambda_n \tag{59}$$

If the $x_f(t)$ were continuously observed with observational errors random, the condition expressed by (58) would assure that the p_l could be accurately determined. In the problem of interest, however, the $x_f(t)$ are incompletely and intermittently observed, and furthermore the observations are affected by unknown biases; i.e., $x_f(t)$ must be replaced by

$$x_f(t) \rightarrow I(t)\, \frac{\partial x_f}{\partial y_j}\left[y_j(t) - \frac{\partial y_j(t)}{\partial q_m}\, q_m\right] \tag{60}$$

where $I(t)$ is unity during observation and zero at other times, and the number of components denoted by j is less than those denoted by f. Under these circumstances, the effective orthogonality expressed by (58) is destroyed and the condition expressed by (59) greatly increases the length of record required to estimate the parameters p_l, q_m by ordinary methods such as simple least squares, as well as increasing the chance that sources of systematic distortion could be hidden in the noise $\beta_f(\omega)$.

The problem of spectral analysis of time series with missing observations has been treated by *Jones* [1962] and *Parzen* [1962*b*]. These discussions treat $I(t)$ as an 'amplification factor':

$$x(t) = I(t)y(t) \tag{61}$$

and show that for covariance estimates for any lag v

$$\hat{R}(v) = R_x(v)/R_I(v) \tag{62}$$

The condition expressed by (59) suggests, however, that the time series itself would be useful to determine only the covariance due to the low-frequency continuous part of the spectrum, and that since the $\partial\alpha_{fn}/\partial p_l$ are known it would be better to obtain estimates of the high-frequency discrete spectrum effects from independent estimates of the variances of the p_l's. The practical problem then becomes the large dimension of the covariance matrix which must be manipulated in computation.

Distributions over a spherical surface. Spectral representation on a spherical surface is discussed by *Schoenberg* [1942], *Obukhov* [1947], *Kaula* [1959*b*], and *Jones* [1963]. In this case, the coordinates s of (28) are latitude and longitude (ϕ, λ) and the functions $c^T(s)$ are surface spherical harmonics. With the normalization specified by (10) and (11), and the assumption of isotropy, the covariance of (32) becomes

$$K(r, s) = \sum_n \frac{\sigma_n^2}{\sqrt{2n+1}} \bar{P}_{n0}(\cos \theta_{rs}) \tag{63}$$

The distributions over a sphere significant in the present problem are scalar quantities characterized by spectrums with significant contributions from wave numbers as high as 4000. Hence the sample to determine the covariances $K(\theta_{rs})$, and thence to estimate the σ_n^2's for small n's should be large and randomly distributed to avoid aliasing [*Shapiro and Silverman*, 1960]. Furthermore, in (38) and (39) the matrix $\mathbf{K}$ employed must be calculated directly from the estimated covariances $K(\theta_{rs})$ and not from $\mathbf{W}$, as stated in (40).

The cross covariance between two different variables f, h on a spherical surface, assuming isotropy, will be

$$K_{xy}(\theta_{rs}) = \sum_n \frac{\sigma_n(fh)}{\sqrt{2n+1}} \bar{P}_n(\cos \theta_{rs}) \tag{64}$$

where

$$|\sigma_n(fh)| \leq \sqrt{\sigma_n^2(f)\sigma_n^2(h)} \tag{65}$$

If f is known over part (S) of the sphere, and h is known over all (T) the sphere, the best estimate of a coefficient $\bar{C}_{nm}(f)$ will be the weighted mean of the estimate obtained from the f in S by (35) and the estimate obtained using $\bar{C}_{nm}(h)$ and the cross-degree variance $\sigma_n(fh)$:

$$E\{\bar{C}_{nm}(f)\} = \frac{1}{\rho_s + \rho_h}\left[\rho_s \frac{\sigma_n^2(f)}{2n+1} \int_S \int_S c_{nm}(r) K^{-1}(r, s) f(s)\, dr\, ds + \rho_h \frac{\sigma_n(fh)}{\sigma_n^2(h)} \bar{C}_{nm}(h)\right] \tag{66}$$

$$\rho_s = \frac{1}{\sigma^2\{\bar{C}_{nm}(f) \mid f \in S\}} = \frac{1}{\left[\dfrac{\sigma_n^2(f)}{2n+1}\left\{1 - \dfrac{\sigma_n^2(f)}{2n+1} \displaystyle\int_S \int_S c_{nm}(r) K^{-1}(r, s) c_{nm}(s)\, dr\, ds\right\}\right]} \tag{67}$$

$$\rho_h = \frac{1}{\sigma^2\{\bar{C}_{nm}(f) \mid \bar{C}_{nm}(h)\}} = \frac{1}{\left[\dfrac{1}{2n+1}\left\{\sigma_n^2(f) - \dfrac{[\sigma_n(fh)]^2}{\sigma_n^2(h)}\right\}\right]} \tag{68}$$

Similarly, for the optimum estimate of f at a set of points in $T - S$ a modification of (46), (47) can be used:

$$E\{\mathbf{g}\} = [\mathbf{V}_{gf}^{-1} + \mathbf{V}_{gh}^{-1}]^{-1}[\mathbf{V}_{gf}^{-1}\mathbf{K}_{gf}\mathbf{K}_{ff}^{-1}\mathbf{f} + \mathbf{V}_{gh}^{-1}\mathbf{K}_{gh}\mathbf{K}_{hh}^{-1}\mathbf{h}] \tag{69}$$

where

$$\mathbf{V}_{gf} = \mathbf{U}_{gg} - \mathbf{K}_{gf}\mathbf{K}_{ff}^{-1}\mathbf{K}_{fg} \tag{70}$$

$$\mathbf{V}_{gh} = \mathbf{U}_{gg} - \mathbf{K}_{gh}\mathbf{K}_{hh}^{-1}\mathbf{K}_{hg} \tag{71}$$

in which

$$\mathbf{U}_{gg} = \mathbf{I}\sigma^2\{f\} \tag{72}$$

If it is assumed that f can be estimated from h by some physical rule such as the isostatic rule of (22), then the foregoing statistical treatment should be applied to f less that part of f accounted for by the physical rule. This treatment is to be emphasized if the parameters of the rule were determined from samples of extreme, rather than average, characteristics, as were those of the isostatic reductions [*Heiskanen and Vening-Meinesz*, 1958, pp. 187–221].

Spherical harmonic analysis is most common in geomagnetism, where the standard methods have been to interpolate to a latitude and longitude grid, to fit Fourier series to parallels, and then to fit Legendre functions to the parallel coefficients [*Chapman and Bartels*, 1940]. *Fougere* [1963] determines polynomials by Gram-Schmidt orthogonalization and then determines the spherical harmonics corresponding to those polynomials found significant. The methods applied in geomagnetism, bypassing the covariance matrix, seem possible, however, only because of the relatively small contribution of higher harmonics.

USE OF GRAVIMETRY

Observing system. A review of gravimetric techniques will appear shortly [*LaCoste and Harrison*, 1964]; see also *Harrison* [1962]. Of principal concern in the present review are the distribution of gravimetric data and the magnitude of possible systematic errors.

The distribution of available gravimetry is shown in Figure 1, which is based on the map of *Uotila* [1961] updated by information from *Worzel et al.* [1963] and *McCahan* [1963]. The principal defect of the distribution has always been, of course, the correlation with topography, both on the wide scale shown by Figure 1 and locally, as, for example, in observations being conducted mainly in the valleys of mountainous areas and on the islands in archipelagoes. The principal difficulty in improving worldwide coverage is the development of economic airborne gravimeters with accurate velocity determination or seaborne gravimeters capable of obtaining accurate results in a reasonable rough sea.

Systematic errors possibly affecting series of observations which either are the principal data for an appreciable area or provide the reference or calibration net for other observations include: (1) connection of national and other gravimetric systems to the world reference network; (2) calibration: the scale factors utilized for gravimeters; (3) referral of seaborne measurements to land reference

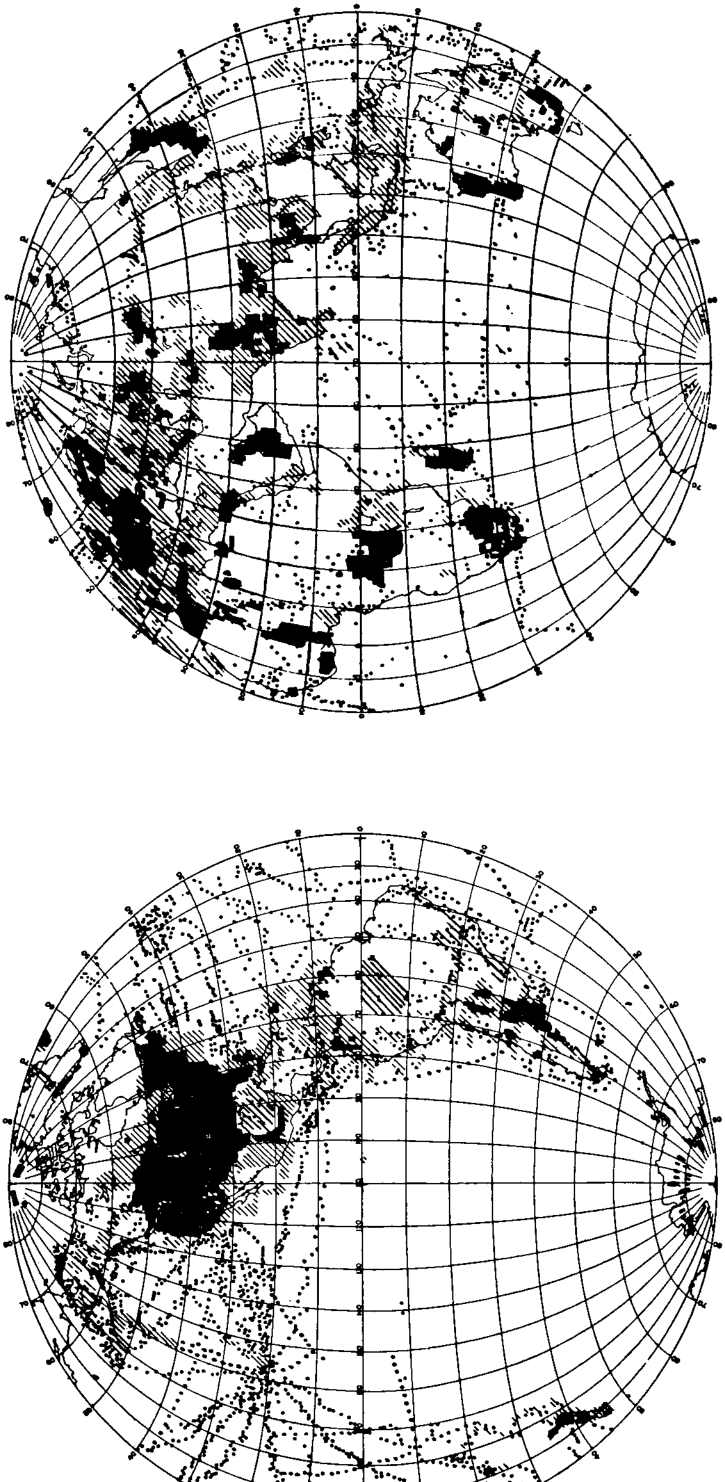

Fig. 1. Distribution of available gravimetry.

stations; (4) navigation errors affecting the position and Eötvos correction for airborne and seaborne gravimetry; and (5) horizontal acceleration and level error effects on airborne and seaborne gravimetry. Errors in categories (1), (2), and (3) are discussed by *Woollard* [1961; *Woollard and Rose*, 1963]. *Woollard* [1961] concludes that the principal gravimetric systems are connected with an average error of about ±0.4 mgal on the basis of gravimeter connections since 1948 and pendulum connections since 1953. *Morelli* [1963] is more pessimistic, citing as an example a range of 3.2 mgal in values obtained for the Rome reference station by various observers since 1960. The principal improvement needed is more reliable north-south pendulum lines to calibrate gravimeters.

The referral of measurements at sea to land reference systems appears to be poorer: *Woollard* [1961] concludes that Vening-Meinesz's pendulum values are probably reliable to ±2 mgal through measurements made on land with the pendulums, even though 13 of 24 comparisons with Vening-Meinesz's harbor measurements differ by 5 or more mgal. Comparisons from track crossings of different cruises are somewhat better, usually averaging ±5 mgal, which includes other effects. Navigation error is generally considered to be the principal source of error in air and sea measurements, limiting their random accuracy to ±3 to ±5 mgal [*Harrison*, 1962]. There does not seem to be any estimate of systematic navigation error. Horizontal acceleration and level errors dependent on the heading of the ship relative to the sea of several milligals have been reported [*Allan et al.*, 1962]. Serious systematic effects of this sort, however, must be less than the ±5 mgal of crossing comparisons with submarine measurements avoiding sea swell effects.

It appears that the most likely source of significant systematic error in gravimetry is in the connection to the reference system of a cruise of marine measurements which constitute the sole data in an appreciable area, but that this error is less than 5 mgal.

The absolute value of gravity is needed for comparison of results obtained from gravimetry with those from satellite orbits. The correction to the Potsdam standard is currently estimated to be about -13 ± 1 mgal. Improvement is anticipated from several determinations in progress [*Cook*, 1963*a*].

An important practical task is the collection and processing of the millions of gravimetric observations that have been made. The leader in accomplishing this task has been W. A. Heiskanen, who established the gravity centers at the Isostatic Institute in Helsinki and the Ohio State University in Columbus. Descriptions of the collection and processing are given by *Uotila* [1960] and *Heiskanen* [1962]. Other collections of gravity data exist at the U. S. Naval Oceanographic Office in Washington and the Bureau Gravimetrique Internationale in Paris. Standardization of punched card format for single observations is being worked out by these centers. To exploit fully the existing gravimetry for determination of the worldwide field there is still needed agreement on a standardization of the local treatment to obtain area means for, say, 1° by 1° or 100-km by 100-km blocks, as described in the next section, and of the recording thereof on punched cards or other automated form.

Local treatment. Gravity anomalies calculated at observation points cannot be used directly in the determination of covariances, harmonic coefficients, and geoid heights as described in preceding sections of this review because of the needs to smooth out the high amount of local variability; to remove, as much as possible, the effect of correlation of observation distribution with local topography; and to keep the computations to a manageable size. The first two of these desiderata indicate the use of a gravity reduction which takes into account the topography. However, only *Tanni* [1948] and *Uotila* [1962] have utilized isostatic anomalies. All other calculations of the geoid from gravimetry have placed greater emphasis on the third of the desiderata and have applied at most a simple linear correlation formula to reduce free air gravity anomalies to the mean elevation of the area they represent:

$$\Delta g_m = a_0 + b\bar{h} \tag{73}$$

where $\bar{h}$ is the mean elevation. *Hirvonen* [1934] and Dubovskiy [*Molodenskiy*, 1945] did not apply any such correction in their early determinations; *Jeffreys* [1943] and *Zhonglovich* [1952] applied the correction to the anomalies for 10° by 10° squares, which were the direct mean of observed values in 1° by 1° squares; *Heiskanen* [1957], *Kaula* [1959*a*, *b*], and *Uotila* [1962] used mean anomalies of 1° by 1° squares corrected to mean elevation and then applied step-by-step, or Markov, extrapolation and interpolation to obtain the mean values of 5° by 5° squares. The formation of the mean anomalies for 1° by 1° squares was done largely by *Uotila* [1960], who found mean values of b of 0.118 mgal/meter on land and 0.069 mgal/meter at sea. In cases where the distribution of observations with respect to elevation in a 1° by 1° square was insufficient to determine a gradient, Uotila used the gradients from nearby squares or the mean gradients of all squares.

Calculations of isostatic anomalies for much of the available gravimetry have been made at the Ohio State University, the Isostatic Institute [*Heiskanen*, 1962], and the Bureau Gravimetrique Internationale, but published maps of the reduced anomalies have been limited to a few areas such as Europe and North Africa [*Coron*, 1962].

The development of computers encourages the application of analytic techniques to replace graphical methods for smoothing, averaging, etc. There have been several papers on use of Fourier series [*Tsuboi*, 1959; *Bullard and Cooper*, 1948; *Tomoda and Aki*, 1955; *Dean*, 1958] and polynomials [*Oldham and Sutherland*, 1955; *Brown*, 1956; *Grant*, 1957; *Krumbein*, 1959; *Grant and Elsaharty*, 1962; *Mandelbaum*, 1963]. However, most of these have been devoted to the problem of 'continuation' [*Peters*, 1949; *Hirvonen*, 1952; *Tengstrom*, 1959; *Orlin*, 1959; *Strakhov*, 1962] (extrapolation upward, to match airborne measurements, or downward, to determine crustal densities) and assume a fairly good distribution of observations. The most extensive application of Fourier methods has been by *Kivioja* [1962], who applied them to gravity anomalies over areas up to 10° by 35°. He found distance over which Fourier predictions could be extrapolated to be less than 4°.

The most important application of gravity anomalies has been, and probably will continue to be, the determination of crustal structure in conjunction with seismic and geologic data [e.g., *Woollard*, 1959; *Woollard et al.*, 1960; *Press*, 1960; *Talwani et al.*, 1961; *Oliver et al.*, 1961]. This circumstance suggests that the geology and seismology of an area can be used to predict gravity anomalies where gravimetry is lacking. These geologic methods have been applied most extensively by *Durbin* [1961] and *Woollard* [1962]. For the south central United States, where the rms anomaly is ± 21 mgal, *Durbin* [1961] reports startlingly good results: the rms error of such predictions is only ± 10 mgal. The situation does not appear to be so clear-cut in the more voluminous study by *Woollard* [1962], particularly for less stable regions. Also, one suspects that in many areas lacking gravimetry the geologic mapping will also be inadequate.

Statistical analysis has been applied to local variations of gravity anomalies by *de Graaf-Hunter* [1935], *Hirvonen* [1956, 1962], *Kaula* [1957, 1959*a*, *b*], *Baussus* [1960, 1961], *Moritz* [1962*a*, *b*, *c*], and *Rapp* [1962]. Fundamental to these analyses is the covariance function, as defined by (31). The principal defect of these analyses is the lack of adequate samples (as well as an adequate sampling theory) to obtain numerical estimates of covariance. The largest sample yet analyzed was by *Kaula* [1959*a*, *b*], consisting of nine areas each about 220 km square in size and containing 52 to 140 members. Taking the data in blocks, rather than long lines, severely limits the number of independent estimates, because of the high correlation that will exist between two product pairs which are approximately parallel. The results obtained are shown in Figure 2, together with those obtained by *Kaula* [1957] from profiles across the United States and *Hirvonen* [1962] and *Rapp* [1962] from samples in Ohio and Finland, which unfortunately appear to be areas of smaller than average anomalies. Hirvonen finds from his sample that the formula for covariance over distance d,

$$K(d) = K_0/(1 + c^2d^2) \tag{74}$$

where K_0 and c are arbitrary constants, fits quite well, which is convenient for analytical development of other functions. However, a negative exponential form $K_0 \exp\{-bd\}$ or lower exponent on d in (74) appears to fit better to the larger sample of *Kaula* [1959*a*].

More commonly used than the covariance in earlier studies is the mean square anomaly for a square of side length s, related to the covariance K_d by [*Hirvonen*, 1962]:

$$G_s^2 = \int_0^{\sqrt{2}} WK_d\, dr \tag{75}$$

where

$$r = d/s$$

$$W = (2\pi - 8r + 2r^2)r \qquad 0 < r < 1 \tag{76}$$

$$W = (2\pi - 4 - 2r + 8\sqrt{r^2 - 1} - 8\tan^{-1}\sqrt{r^2 - 1})r \qquad r > 1 \tag{77}$$

If a single observation at the center is taken as representative of a square of side length s, it will have a mean square error of representation:

$$E_s^2 = G_0^2 - G_s^2 \tag{78}$$

The rigorous method of extrapolating and interpolating anomalies is to use the linear regression coefficients as specified by (46) and (47) [*Baussus*, 1960; *Moritz*,

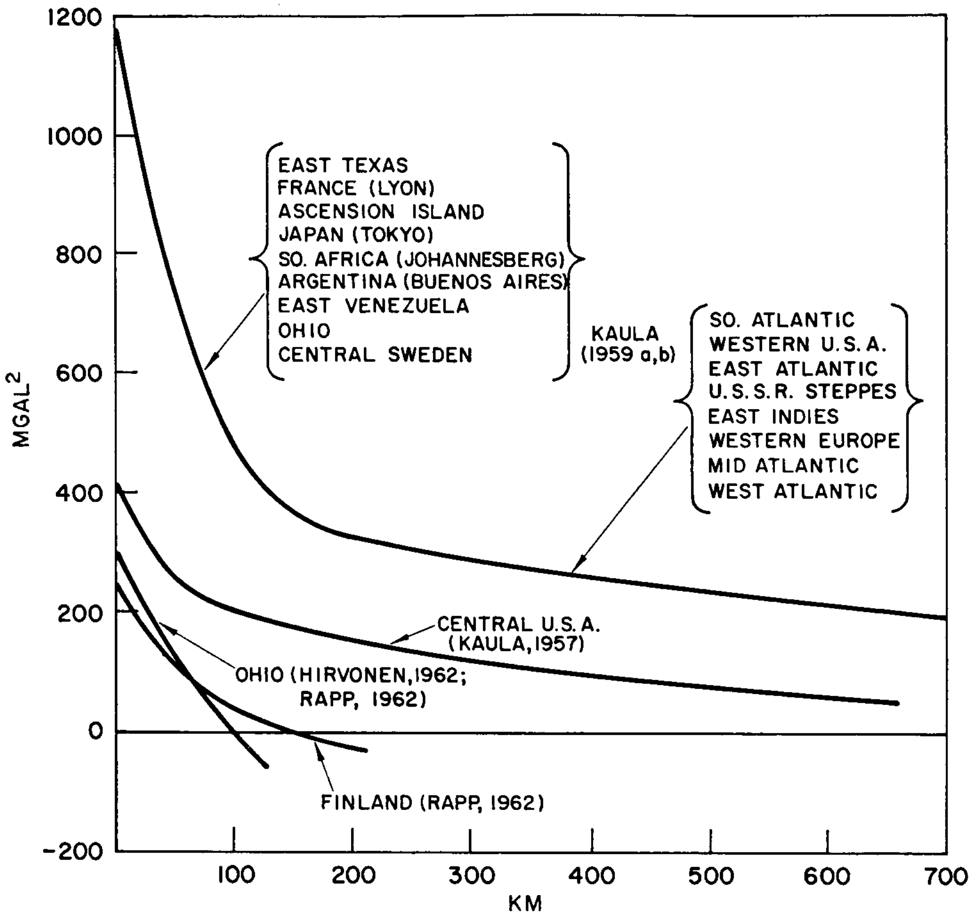

Fig. 2. Estimates of local and regional covariance of gravity anomalies.

1962*b*]. Moritz also generalizes the linear regression to include the topography and anomalies corrected for linear correlation by equation 73. *Kaula* [1959*a*, *b*] applied extensively to 1° by 1° square means the method of Markov estimation in terms of gravity and topography, which estimates for the $(i + 1)$th member of a series of step-by-step extrapolations

$$E\{\Delta g_{i+1} \mid h_{i+1}, \Delta g_i, h_i\} = \frac{\int_{-\infty}^{\infty} xP\{\Delta g_{i+1} = x, h_{i+1} \mid h_i, \Delta g_i\}\, dx}{P\{h_{i+1} \mid h_i, \Delta g_i\}} \tag{79}$$

As *Baussus* [1960] pointed out, this Markov estimation assumes that the covariance function $K(d)$ can be represented as a negative exponential of d. If $K(d)$ is better represented by Hirvonen's form (equation 74), appreciable improvement is obtainable when we use suitably spaced 'next-to-nearest' as well as nearest neighbors in estimation [*Moritz*, 1962*b*].

The statistics of other functions can be derived as linear transformations of those for gravity anomalies, either with or without the intermediary of a spectral representation: anomalies at higher altitude [*Kaula*, 1959*a*; *Hirvonen*, 1962; *Moritz*, 1962*a*, *c*], geoid heights [*Kaula*, 1959*a*], and deflections of the vertical [*Kaula*, 1959*a*; *Kaula and Fischer*, 1959].

Worldwide treatment. The statistical analysis of local variations described above has been extended to larger areas by *Hirvonen* [1956] and *Kaula* [1959*a*, *b*]. For covariance estimates, Kaula used, in addition to the local samples, eight regional samples covering an area about 10° by 10° with 56 to 115 members each (see Figure 2), plus a single world sample consisting of 569 5° by 5° mean anomalies based on observations in 18 per cent or more of their 1° by 1° squares. The covariances obtained are shown in Figures 2 and 3. Table 1 gives the spherical

TABLE 1. Degree Variances of Free Air Gravity Anomalies [*Kaula*, 1959*b*]

n	σ_n^2, mgal²	n	σ_n^2, mgal²	n	σ_n^2, mgal²	n	σ_n^2, mgal²
2	7	9	22	17	12	25	9
3	44	10	15	18	19	26	11
4	30	11	18	19	10	27	4
5	10	12	7	20	7	28	8
6	24	13	15	21	14	29	5
7	3	14	23	22	10	30	−2
8	23	15	22	23	9	31	1
		16	6	24	11	32	2

harmonic degree variances $\sigma_n^2\{\Delta g\}$ derived from the covariances by spectral analysis, as described by (32) and (29). These results are certainly of the correct order of magnitude, but the analysis needs to be redone with a properly randomized sample

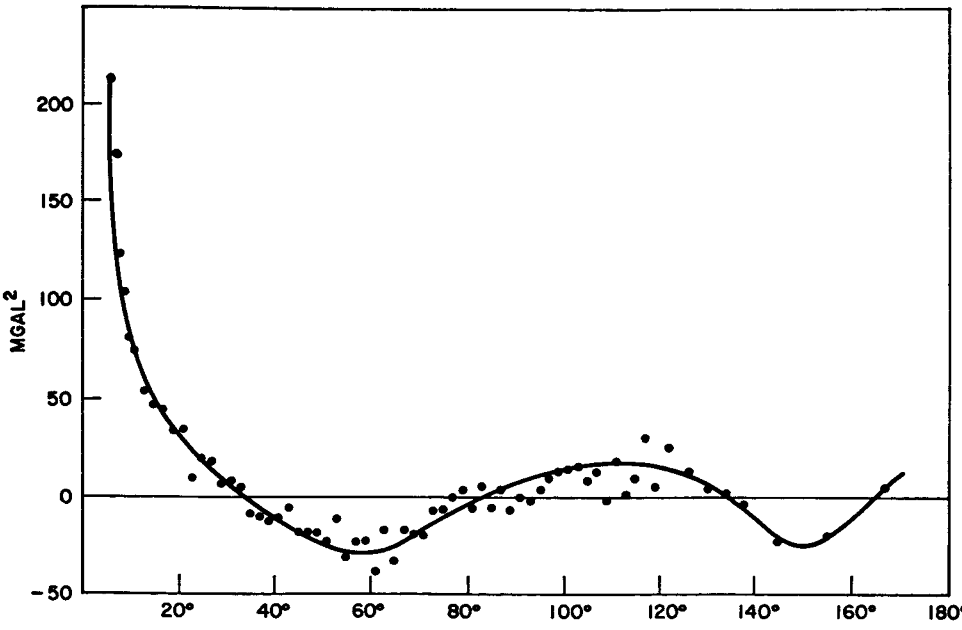

Fig. 3. Estimated worldwide covariance of gravity anomalies.

[cf. *Shapiro and Silverman,* 1960]. *Hirvonen* [1956] did not have the computer facilities necessary to make an extensive estimate of covariance, and so he used mainly mean square mean G_s^2 and error of representation E_s^2, based on five regions averaging 8° by 8° plus the data of *Tanni* [1948]. His results imply a more gently varying field than Kaula's.

These authors and *Cook* [1950, 1951], *Kaula* [1957], and *Molodenskiy et al.* [1960] devote considerable attention to estimating the accuracy of determination of the geoid height and slope at specific points, given either the actual data distribution or a hypothetical dense net within a given distance of the point. Hirvonen's and Kaula's studies yield uncertainties that are probably too low, owing to the samples used and the assumption of randomness of 30° by 30° square means; Cook's and Molodenskiy's studies yield uncertainties that are probably too high because of use of the least-squares-determined spherical harmonic coefficients of *Jeffreys* [1943] and *Zhonglovich* [1952], respectively.

The principal point of difference in various attempts to determine the external field from the incomplete information shown in Figure 1 has been on whether or not, and how, to utilize the topography as a means of interpolating and extrapolating over great distances. Dubovskiy [*Molodenskiy,* 1945], and *Tanni* [1948] used the topography by assuming zero isostatic anomaly in the unsurveyed areas. *Kaula* [1959*b*], using the topography by applying the Markov extrapolation technique of equation 79 to 5° by 5° means worldwide, probably obtained results similar to isostatic assumption although less susceptible to bias because of its broader statistical basis. In applying numerical integration, *Heiskanen* [1957] assumed zero free air anomaly with respect to the International Formula and *Zhonglovitch* [1952] assumed zero anomaly with respect to his least-squares-determined third degree figure. Analyses limited to best fits to the observed gravimetry are those of *Jeffreys* [1943], *Zhonglovich* [1952] for harmonic degrees two, three, and four, and *Uotila* [1962]. All these solutions employed simple least squares—i.e., the harmonic coefficients were obtained as parameters $\mathbf{z}$ in (39) with coefficients $\mathbf{C}$ and covariance matrix $\mathbf{W}$ of the diagonal form specified by (44). To overcome the ill-conditioning due to neglect of covariance, *Jeffreys* [1943] grouped his data heavily into 30° by 30° square means and omitted all harmonic coefficients that the normal equation diagonal coefficient and constant indicated as making a small contribution before determining the remaining coefficients through degree 3 by least squares. *Zhonglovich* [1952] used the 10° by 10° square means and tried several least-squares solutions for different harmonics up through degree 4 before choosing as best a stepwise solution in which the coefficients for each successive degree 2, 3, 4 were determined from the residuals with respect to previously determined degrees. *Uotila* [1962], benefiting from considerably more data, determined harmonic coefficients up through degree 4 by direct least-squares fit to 5° by 5° mean anomalies, holding the harmonics $(n, m) = (1, 0), (1, 1)$, and $(2, 1)$ zero and $(2, 0), (3, 0), (4, 0)$ fixed at satellite-determined values.

Jeffreys [1959, 1961] rejects use of the topography for long-range estimation; it is not clear whether he does so because the earlier determinations of gravity at sea obtained mostly positive anomalies, because the long-wave correlation is small (i.e., $\sigma_n(fh)$ small for small n in equation 64), because the applications of the topography have involved the questionable isostatic or Markovian assumptions, or because, as a geophysicist, he is interested in estimating the amplitude, rather than the phase, of the variations. In any case it seems clear that (1) a solution optimum in the sense of quadratic sum minimization will be the one using the most information—i.e., the estimate by (72) will be better than that by (35); (2) the optimum estimates will be smaller in the mean square than the true values as a consequence of the smoothing effect of any prediction procedure; and (3) the appropriate measures of amplitude alone are the degree variances σ_n^2.

It is difficult to characterize any of the existing solutions as approximations to the solutions described by (35) and (66), so that it is not clear to what extent they fall short of fully exploiting the available data. In principle, the general solution up to about 12th degree harmonics should be easy with modern computing facilities: the half wavelength for the 12th degree is 15°, there are about 160 15° by 15° squares with observations, and the 160 by 160 covariance matrix can be stored all at once in the core and inverted in a couple of minutes in an IBM 7094 computer. The topography can also be incorporated in a more effective manner than before by using a new development thereof in spherical harmonics by Bruins [*Vening-Meinesz*, 1959].

USE OF SATELLITE ORBITS

Observing system. The principal types of satellite tracking that have produced results useful for geodesy are essentially those described in the review by *Kaula* [1962*a*]: the 500-mm focal length $f/1$ Baker-Nunn tracking cameras; the 1000-mm focal length, $f/5$, modified aerial reconnaissance cameras, fixed and equatorially mounted; the Minitrack 108- and 136-Mc/s radio interferometers; and the Transit 324- to 162-Mc/s radio Doppler trackers. The most important recent instrumental development has been the Anna geodetic satellite [*Macomber*, 1963] which incorporates magnetically oriented xenon-gas-discharge lamps capable of generating a flash of 8800 candle seconds along the axis, in addition to radio transponders and beacons.

The distribution of the tracking stations of some of the principal systems used for geodesy is shown in Figure 4. The nonuniformity of distribution gives rise to some statistical problems because of the limited coverage of an orbit by a limited number of stations. Solving the spherical triangle formed by the orbital plane, the equator, and the meridian of a particular longitude λ, we obtain

$$\omega + f = \sin^{-1}(\sin\phi/\sin i) \tag{80}$$

$$\Omega - \theta = \lambda - \cot^{-1}(\pm\sqrt{\tan^2 i \csc^2\phi - \sec^2 i}) \tag{81}$$

where i, ω, f, and Ω are, respectively, the inclination, argument of perigee, true anomaly, and longitude of the node of the orbital ellipse, and θ is the Greenwich sidereal time. Since the inclination i varies but slightly, a station of position (ϕ, λ) can observe the satellite only near certain values of $\omega + f$ and $\Omega - \theta$. For fairly well-distributed networks of about twelve stations, such as the Smithsonian Astrophysical Observatory Baker-Nunn cameras and the Applied Physics Laboratory transit Doppler stations, the distortion due to this effect is probably slight except for series of camera observations dependent on solar illumination over dura-

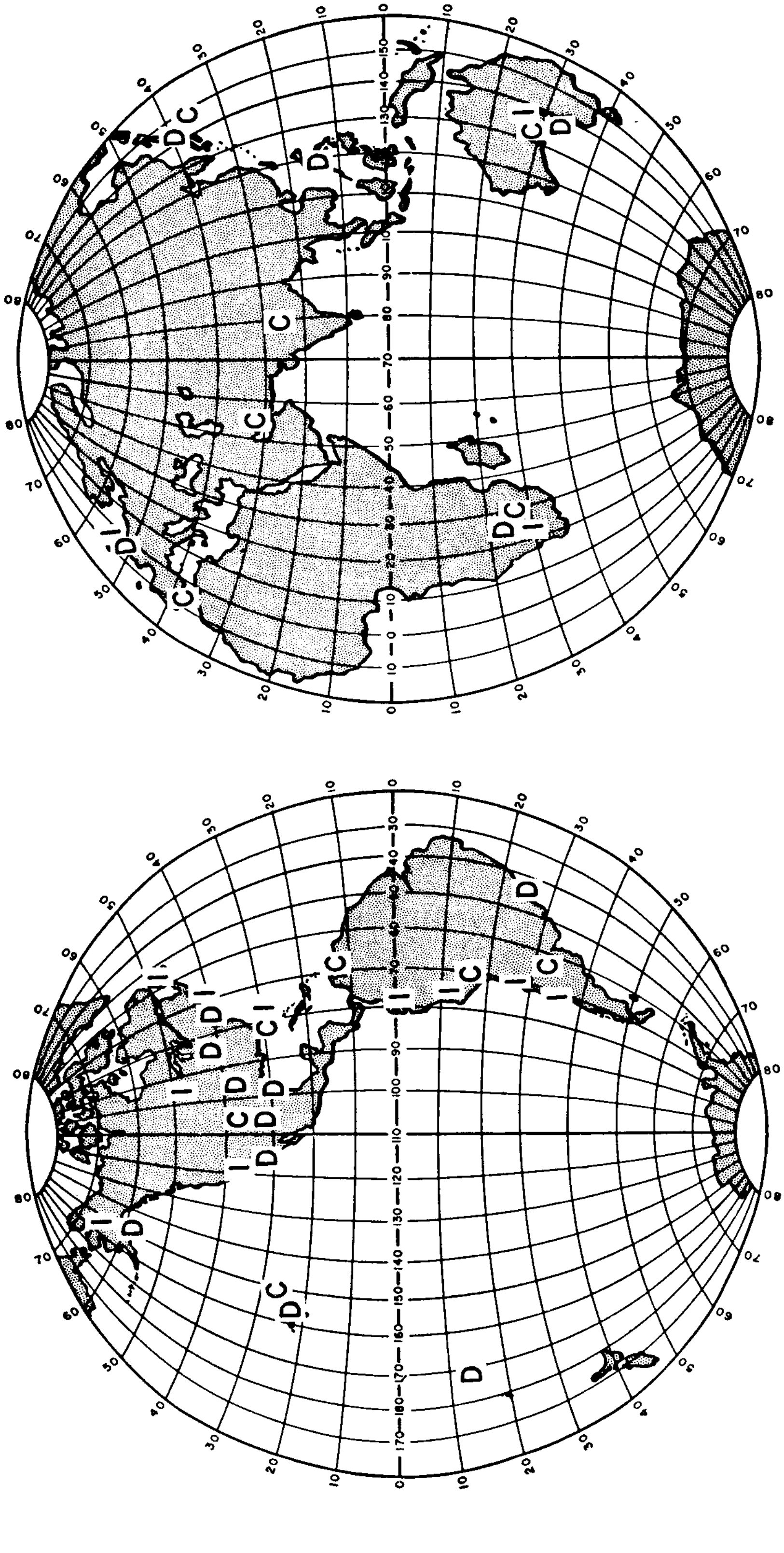

Fig. 4. Some satellite tracking systems used for geodesy: *C*, Baker-Nunn cameras; *D*, transit Doppler receivers; *I*, Minitrack radio interferometers and Mots cameras.

tions short compared to a full cycle of nodal motion with respect to the sun—i.e., less than a few months.

For the purpose of determining orbital variations, the random errors are quite satisfactorily small for all the systems mentioned above except the Minitrack interferometers, for which the effect of ionospheric refraction irregularities on observed directions are of the order of ± 0.0005 to ± 0.001, or $\pm 100''$ to $\pm 200''$. Ionospheric refraction is also probably a source of some systematic error in the Minitrack system. In the transit Doppler system, ionospheric refraction is believed to be largely eliminated by simultaneous transmission on two frequencies, and the most probable source of systematic error is shifting of the reference frequency provided by the oscillator in the satellite, to the extent that the frequency is often taken as a separate unknown for each observed pass. The principal likely systematic error in camera observations is in the timing of observations of sunlit satellites, caused by difficulties of synchronization of the camera shutter with the clock and by error in the propagation delay time, possibly of the order of 0.001 sec.

From the point of view of determining the gravitational field, the greatest systematic error common to all observational systems is error in tracking station positions. Error of positions of stations within the same triangulation system with respect to each other should be less than 20 or 30 meters; the principal possible exceptions are the positions of stations in South America and South Africa with respect to those in the same system in the northern hemisphere. Errors of position of triangulation systems with respect to each other and to the center of mass of the earth should be less than 100 meters for continental systems and less than 500 meters for isolated stations. In addition to these errors that should exist, there have been cases where mistakes of as much as a kilometer have existed in station positions.

Dynamics. The aspects of celestial mechanics important to understanding of close satellite orbits are explained in recent texts such as those of *Baker and Makemson* [1960] and *Brouwer and Clemence* [1961]. Satellite orbit dynamics with emphasis on geodetic applications is discussed by *Kaula* [1962] and *Mueller* [1963].

The theoretical problem of satellite orbits is to solve the equations of motion:

$$\ddot{\mathbf{r}} = \nabla(V + R_s) + \nabla_v R_d \tag{82}$$

where ∇ is the gradient with respect to position, ∇_v is the gradient with respect to velocity, V is the earth's gravitational potential as given by (9), R_s is the gravitational plus radiation pressure potentials of third bodies (sun, moon, etc.), and R_d is the atmospheric drag potential, a function of both position and velocity of the satellite. The information we wish to extract from satellite orbits rests in V, and so for the moment we neglect the other two potentials. The three second-order equations (82) are generally reduced to six first-order equations by change of variables:

$$\dot{s}_i = \sum_j C_{ij}(\mathbf{s}) \frac{\partial V}{\partial s_j} \tag{83}$$

The simplest set of six variables $\mathbf{s}$ would, of course, be the position vector components $\{x, y, z\}$ and the velocity vector components $\{\dot{x}, \dot{y}, \dot{z}\}$, referred to inertial space with origin at the earth's center. These six variables can be transformed to the six parameters of a Kepler ellipse with one focus at the origin: $\{a, e, i, \Omega, \omega, f\}$. The relationships between these parameters and the earth-fixed coordinates $\{u, v, w\}$ are shown in Figure 5. In the angle $\omega + f$, ω is the argument of perigee, the angle from the ascending node Ω to perigee, the point of closest approach of the ellipse to the origin, and f is the true anomaly, the angle from perigee to the satellite. Alternate ways of expressing the anomaly of the satellite are [*Kaula*, 1962, p. 194] the eccentric anomaly E:

$$\tan \tfrac{1}{2}E = [(1 - e)/(1 + e)]^{1/2} \tan \tfrac{1}{2}f \tag{84}$$

and the mean anomaly M:

$$M = E - e \sin E \tag{85}$$

The form of (83) in which $\mathbf{s}$ is the six Keplerian elements is advantageous because in the case of a purely central field, i.e., V is kM/r, the only nonzero rate of change is that of the anomaly; furthermore, for the mean anomaly M it is constant (Kepler's third law):

$$n = \dot{M} = (kM)^{1/2}a^{-3/2} \tag{86}$$

where M on the right is the mass of the central body.

In the case of a satellite moving in the actual potential field of the earth, by (9), for which the departures from a central field are $0(10^{-3})$, the Keplerian elements are still convenient because the rates of variation $\dot{s}_i$ in (83) will be small except

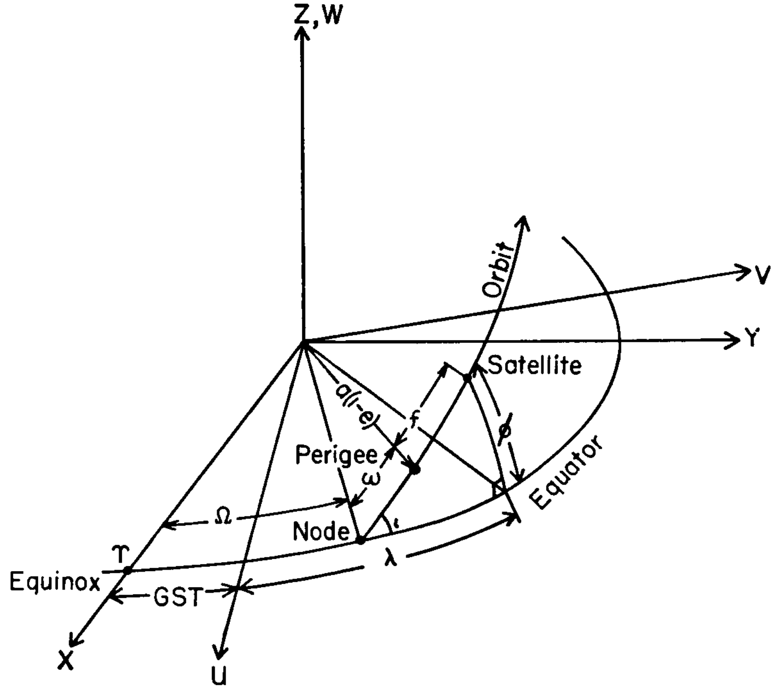

Fig. 5. Orbit and coordinate systems.

for $\dot{M}$, and hence representable as perturbations of the elliptic motion. The form of (83) for Keplerian elements is [*Brouwer and Clemence*, 1961, p. 289; *Kaula*, 1962, p. 198]:

$$
\begin{aligned}
\dot{a} &= (2/na)\cdot(\partial V/\partial M) \\
\dot{e} &= \frac{1-e^2}{na^2e}\cdot\frac{\partial V}{\partial M} - \frac{(1-e^2)^{1/2}}{na^2e}\cdot\frac{\partial V}{\partial \omega} \\
\dot{\omega} &= -\frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial V}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2e}\frac{\partial V}{\partial e} \\
di/dt &= \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial V}{\partial \omega} - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial V}{\partial \Omega} \\
\dot{\Omega} &= \frac{1}{na^2(1-e^2)^{1/2}\sin i}\cdot\frac{\partial V}{\partial i} \\
\dot{M} &= n - \frac{1-e^2}{na^2e}\frac{\partial V}{\partial e} - \frac{2}{na}\frac{\partial V}{\partial a}
\end{aligned}
\tag{87}
$$

As is indicated by Figure 5, it is a purely geometrical problem to transform $V(r, \phi, \lambda)$ as given by (9) to a form $V(\Omega, i, \omega, a, e, M)$ suitable for taking derivatives in (83). The solution which is given by *Kaula* [1961*a*]:

$$
V = kM\left[\frac{1}{r} + \frac{1}{a}\sum_{n=2}^{\infty}\left(\frac{a_e}{a}\right)^n \sum_{m=0}^{n} N_{nm} \sum_{p=0}^{n} F_{nmp}(i)\cdot \sum_{q=-\infty}^{\infty} G_{npq}(e) S_{nmpq}(\omega, M, \Omega, \theta)\right] \tag{88}
$$

where the normalization factor

$$
N_{nm} = \left[\frac{(2n+1)(n-m)!\,(2-\delta_{0m})}{(n+m)!}\right]^{1/2} \tag{89}
$$

$$
\begin{aligned}
F_{nmp}(i) = \sum_t \frac{(2n-2t)!}{t!\,(n-t)!\,(n-m-2t)2^{2n-2t}} \sin^{n-m-2t} i \\
\cdot \sum_s \binom{m}{s} \cos^s i \sum_c \binom{n-m-2t+s}{c}\binom{m-s}{p-t-c}(-1)^{c-k}
\end{aligned}
\tag{90}
$$

in which t is summed from 0 to p or k (defined after equation 11), whichever is less; s, from 0 to m; and c, over all values making the two binomial coefficients nonzero;

$$
G_{np(2p-n)}(e) = (1-e^2)^{(1/2)-n} \sum_{d=0}^{p'-1} \binom{n-1}{2d+n-2p'}\binom{2d+n-2p'}{d}\left(\frac{e}{2}\right)^{2d+n-2p'} \tag{91}
$$

where $p' = p$, $p \leq n/2$, $p' = n - p$, $p \leq n/2$, for $q = 2p - n$; for $q \neq 2p - n$, more complicated infinite series are required [*Kaula*, 1961*a*, equations 24–26]; and

$$
\begin{aligned}
&S_{nmpq}(\omega, M, \Omega, \theta) \\
&= \begin{bmatrix} \bar{C}_{nm} \\ -\bar{S}_{nm} \end{bmatrix}_{(n-m)\text{ odd}}^{(n-m)\text{ even}} \cos\{(n-2)\omega + (n-2p+q)M + m(\Omega-\theta)\} \\
&\quad + \begin{bmatrix} \bar{S}_{nm} \\ \bar{C}_{nm} \end{bmatrix}_{(n-m)\text{ odd}}^{(n-m)\text{ even}} \sin\{(n-2p)\omega + (n-2p+q)M + m(\Omega-\theta)\}
\end{aligned}
\tag{92}
$$

Other expressions are given by *Musen* [1960] and *Groves* [1960]. If V as expressed by equations 88–92 is differentiated and the derivatives are placed in (87), the form of S_{nmpq} in (92) indicates that $\dot{a}$, $\dot{e}$, di/dt must be sinusoidal. If we assume that these sinusoidal variations are small enough that a, e, i can be considered as constant on the right of (87), then (92) further indicates that only $\dot{\omega}$, $\dot{\Omega}$, and $\dot{M}$ have constant terms, and that these terms arise from cases where n is even, m is zero, p is $n/2$, and q is zero. Hence to a linear approximation, (87) can be integrated assuming a, e, i, n constant (except for n in $\dot{M}$, the variation of which is obtainable from equation 86 and $\dot{a}$) and M, ω, Ω secularly changing.

However, since $\bar{C}_{2,0}$ is $0(10^{-3})$ while all the other $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are $0(10^{-6})$, a linear approximation does not suffice for $\bar{C}_{2,0}$: terms with coefficient $\bar{C}_{2,0}{}^2$ are required. As usual when a problem becomes nonlinear, it becomes appreciably more complicated, and care must be taken in defining the constants of integration. The obvious choices are the values of the Keplerian elements at a particular instant of time, i.e. the osculating elements. The osculating elements are the constants of integration when the problem is solved in a purely numerical manner.

When the problem is developed further for an iterative or analytic solution, however, it is usually found more convenient to define the constants as elements of a fictitious reference orbit or an intermediate orbit. In some theories this intermediary is defined geometrically, as, e.g., representing all constant and secular terms of the actual orbit; in other theories it can be defined dynamically as corresponding to a constant part of the potential V. Other aspects in which theories may differ are the coordinate system; the independent variable; the orbital elements—often a canonical set is used, i.e. one such that the C_{ij} in (83) consists of only one nonzero constant per row; and the point at which the development of the problem changes from algebraic to numerical. This large variety of possibilities gave rise to a large number of theoretical papers in the period 1957–1961 on the orbit of a close satellite of an oblate planet to at least $0(J_2{}^2)$ in secular motions (J_2 is $-\sqrt{5}\ \bar{C}_{2,0}$). The theories that probably have been used the most are those of *Musen* [1959], who adapted the Hansen lunar theory to a form suitable for solution by iteration; *Brouwer* [1959], who applied Von Zeipel's method of canonical transformation with a purely Keplerian intermediary; *Kozai* [1959*a*, *b*], who extended the Lagrangian equations 87 to higher-order terms; *Vinti* [1959, 1961] and *Izsak* [1960], who both separated the equations of motion by using ellipsoidal coordinates; *King-Hele* [1958], who employed a Keplerian ellipse of fixed inclination and perigee argument as intermediary and solved in successive approximations according to powers of J_2 and e; and *Merson* [1961], who made a development similar to King-Hele's starting from osculating elements when the satellite is at the node. (See *Kaula* [1962] for more description and comparison.)

Since 1960, more theoretical attention has turned to the problems of resonance associated with close satellites. The problem of critical inclination in the vicinity of $\sec^{-1}\sqrt{5}$, or 63°26′, at which perigee motion is zero, is of relatively little geodetic interest; the most complete solution is probably that of *Izsak* [1962], as extended by *Aoki* [1963]. Of much more geodetic significance are the 24-hour orbits (semimajor axis 42,000 km), which resonate with tesseral harmonics $\bar{C}_{nm}$, $\bar{S}_{nm}$ for which $n - m$ is even, since

$$\dot{\Omega} + \dot{\omega} + \dot{M} - \dot{\theta} \approx 0 \tag{93}$$

The problem of orbits to which (93) applies with potential

$$V = \frac{kM}{r} + R = kM\left[\frac{1}{r} + \frac{a_e^2}{a^3} N_{22}F_{220}G_{200}S_{2200}\right] \tag{94}$$

has been analyzed by *Blitzer et al.* [1962, 1963], who use a linearized system of differential equations; by *Musen and Bailie* [1962], who isolate $(\Omega + \omega + M - \theta)$ as a canonical element and develop the Hamiltonian in powers of the ratio $(\partial R/\partial L)/(\partial^2 R/\partial L^2)$, where $L = \sqrt{kMa}$; and by *Morando* [1963], who applies Von Zeipel's method and develops the determining function in powers of $(C_{22}^2 + S_{22}^2)^{1/4}$.

Elaborately developed theories such as those of *Musen* [1959] and *Brouwer* [1959] are advantageous to analyze secular changes for zonal harmonics and to conserve computer time when observations are infrequent (say, an average of less than one per hour). Computation of orbital arcs of a few days or less using frequent radio tracking is still done mostly by numerical integration.

As necessary as a correct dynamical theory, and often more laborious, are differential correction systems to determine from observations the numerical values to use in the theories. Examples of differential correction programs for close satellites are those of *Veis and Moore* [1960] and *Merson* [1963]. The geometrical aspects of differential correction schemes are fairly complicated but straightforward (see *Veis* [1960] and *Kaula* [1961*a*, 1962] for details). The statistical aspects are not so complicated, but are much less satisfactorily treated, partly because a rigorous treatment would require excessive computer storage and time, and partly because adequate statistical models are lacking. Satellite orbit observations always have an appreciable amount of serial correlation, principally because of the inability to account for air drag. However, the statistical analysis of satellite orbit accelerations has received relatively little attention, the only active worker thereon being *Moe* [1963]. No differential correction method takes into account correlation between observations of different passes, as suggested in the section of this review on time series. The most elaborate treatment existing [*Kaula*, 1963*b*] allows for correlation within a pass and applies various other devices such as giving greater weight to the across-track than to the along-track component of an observation; preassigning variances and covariances of parameters being determined in the analysis, so that mathematically they become 'observations'; and weighting observations according to their distribution with respect to phase angles believed critical.

Determination of zonal harmonics. For the principal secular or long-period effect of a zonal harmonic, $m = 0$, the disturbing terms in (88) can be taken as:

$$\text{Long period } R_{no} = -kMJ_n \frac{a_e^n}{a^{n+1}} \cdot F_{nok}(i)G_{nk(2k-n)}(e)\begin{cases} 1, & n \text{ even} \\ 2\sin\omega, & n \text{ odd}\end{cases} \tag{95}$$

where J_n is $-\sqrt{2n+1}\,\bar{C}_{n,0}$. The linear perturbations can be obtained by using this disturbing function in the Lagrangian equations of motion 87; in addition, nonlinear terms J_2^2 and J_2J_n, n odd, need to be taken into account.

The customary method of determining the J_n's from satellite orbits is to analyze the long-term variation of orbital elements determined by a differential correction fit to observations over a few days at a time. Precautions that must be taken in analyzing secular changes of the node Ω and perigee ω to determine even degree zonal harmonics include:

1. The set of satellite orbits used should have a variety of inclinations sufficient to separate the different harmonics.
2. The constants of integration determined by analyzing observations must be consistent with the algebraic form of the terms containing J_2^2 ($\bar{C}_{2,0}^2$ in our notation).
3. The mean value of the constants of integration must be accurately determined for the duration from which the secular rates $\dot{\Omega}$, $\dot{\omega}$ are determined. It is also to be emphasized that mean values of the constants of integration will differ from mean values of osculating elements for some theories (e.g., the inclination in the theory of *King-Hele* [1958], or the mean motion in the theory of *Brouwer* [1959]). Averaging of the elements a and e is important in order to remove secular drag effects; if the a, e used are not average values, but rather values for some epoch noncentral to the duration used, about the best that can be done is to use the perturbation in mean anomaly to correct node and perigee assuming the perigee height to remain fixed [*O'Keefe et al.*, 1959; *Kozai*, 1962, 1963*a*]:

$$\Delta(\Omega, \omega) = \frac{(\dot{\Omega}, \dot{\omega})}{3n} \frac{7 - e}{1 + e} \Delta M \tag{96}$$

The effects of errors Δe, Δi, Δa in the orbital elements on determination of the secular rate of the node are [*Kozai*, 1962]:

$$\Delta\dot{\Omega} = \dot{\Omega}\left[\frac{4e}{1 - e^2} \Delta e - \tan i \, \Delta i - \frac{7}{2a} \Delta a\right] \tag{97}$$

For some satellites of high inclination, error in determining zonal harmonics will thus come more from error in the mean inclination than in $\dot{\Omega}$ itself.

4. If luni-solar attraction, radiation pressure, and other perturbations are not removed in determining the mean values of the constants of integration, they can distort determination of the rates $\dot{\Omega}$ and $\dot{\omega}$ not only through purely secular effects but also through periodic perturbations. A periodic perturbation $\Delta(\Omega, \omega) \sin \{\kappa t - \lambda\}$ will affect the apparent secular rate from observations lasting from t_1 to t_2 by an amount $\delta(\dot{\Omega}, \dot{\omega})$:

$$\delta(\dot{\Omega}, \dot{\omega}) = \frac{\Delta(\Omega, \omega)[\sin \{\kappa t_2 - \lambda\} - \sin \{\kappa t_1 - \lambda\}]}{t_2 - t_1} \tag{98}$$

5. If the perturbations are removed in determining the constants of integration, in addition to direct effects $\Delta_1(\Omega, \omega)$, the interaction of perturbations Δe, Δi with the secular effect of J_2 may cause an indirect effect $\Delta_2(\Omega, \omega)$ large enough that it should be taken into account:

$$\Delta_2(\Omega, \omega) = \frac{\partial(\dot{\Omega}, \dot{\omega})}{\partial e} \int \Delta e \, dt + \frac{\partial(\dot{\Omega}, \dot{\omega})}{\partial i} \int \Delta i \, dt \tag{99}$$

6. Satellites with low perigee, nonspherical shape, and large area-to-mass ratio should be avoided because of the difficulty in calculating drag and radiation pressure effects on the mean elements and the secular rates, particularly on $\dot{\omega}$.

7. Use of orbits provided by routine prediction services should be avoided, because of possible imprecision in the definition of orbital elements and inaccuracy in the determination thereof.

The most important recent analyses of secular motions have been by *Kozai* [1962] and *King-Hele et al.* [1963]. Kozai uses $\dot{\Omega}$ and $\dot{\omega}$ of thirteen satellites ranging from 32.9° to 66.8° in inclination. However, he weights the data inversely proportional to the squares of the standard deviations, which vary greatly, so that the result is almost entirely determined by the $\dot{\Omega}$ of 1960ι_2 (inclination 47.2°) and the $\dot{\Omega}$ and $\dot{\omega}$ of 1959α_1 and 1959η (inclinations 32.9°, 33.4°). *King-Hele et al.* [1963] used only the nodal motion $\dot{\Omega}$ of seven satellites at widely spaced intervals of inclination from 32.9° to 97.4° and weighted each satellite about equally. However, to obtain inclinations above 53.8°, they used elements provided by routine prediction services. The disagreement between the results of *Kozai* [1962] and *King-Hele et al.* [1963] in Table 2 at this late date is disappointing. What are required are accurately determined orbits of high inclination. A limited amount of the requisite data are now available in the form of 552 precisely reduced Baker-Nunn camera observations over 54 days of 1961$\alpha\delta_1$, which has inclination 95.9°, perigee height 3500 km. The observed nodal motion $\dot{\Omega}$ determined as a byproduct of analysis for tesseral harmonics [*Kaula,* 1963*d*] is +0.21037° ± 0.00010/day. Including a luni-solar effect of −0.00006°/day, the calculated motion $\dot{\Omega}$ is +0.21034°/day using the J_2 through J_8 of *Kozai* [1962], and 0.21056°/day using the J_2 through J_{12} of *King-Hele et al.* [1963].

The determination of the odd degree zonal harmonics is somewhat easier, since there do not appear to be any other significant effects of period $2\pi/\dot{\omega}$. The principal precautions are to include the J_2J_n terms, following (99), and the drag correction of (96) [*Kozai,* 1959*a*]. Results by the principal investigators are given in Table 2.

TABLE 2. Zonal Harmonic Coefficients of the Gravitational Field

Coefficient*	*Newton et al.* [1961]	*Smith* [1961, 1963]	*Kozai* [1962]	*Shelkey* [1962]	*King-Hele et al.* [1963]	*Anderle and Oesterwinter* [1963]
$J_2 \times 10^6$		1083.15	1082.48	1082.61	1082.86	1082.47
$J_3 \times 10^6$	−2.36	−2.44	−2.56	−1.94		−2.48
$J_4 \times 10^6$		−1.4	−1.84	−1.52	−1.03	−1.40
$J_5 \times 10^6$	−0.19	−0.18	−0.06	−0.41		−0.14
$J_6 \times 10^6$		0.7	0.39	0.73	0.72	
$J_7 \times 10^6$	−0.28	0.30	−0.47			
$J_8 \times 10^6$			−0.02		0.34	
$J_9 \times 10^6$			0.12			
$J_{10} \times 10^6$					0.50	
$J_{12} \times 10^6$					0.44	

* $J_n = -\sqrt{2n+1}\,\bar{C}_{n0}$

Determination of GM. The rapid motion of artificial satellites suggests that their mean motion may serve as an accurate method of determining GM, the product of the gravitational constant and the earth's mass, through Kepler's equation 86.

Drag and radiation pressure have their greatest effects on the mean anomaly, however, and so it is particularly desirable to have a satellite for which these effects are minimized and readily calculable: spherical, low area-to-mass ratio, small eccentricity, high inclination, perigee height above 1000 km. Determinations of GM from such satellites have been made as byproducts of tesseral harmonic analyses by *Kaula* [1963*b*, *d*], using camera observations, and by *Anderle and Oesterwinter* [1964], using Doppler observations. The determination from camera data depends on the geodetic triangulation connecting tracking stations to establish scale; that from Doppler data may be affected by ionospheric refraction effects. A more distant satellite would reduce these effects: in this category are the use of a lunar probe [*Hamilton et al.*, 1963] and those methods measuring the distance of the moon by radar [*Yaplee et al.*, 1963] and by triangulation [*Fischer*, 1962]. The radar method now obtains an internal accuracy of ± 200 meters for the distance of the moon by correcting for the variations in topography on the moon's surface. Both the radar and triangulation methods are affected by error in the radius in the moon in the direction of the earth, for which the scatter of different determinations is about 2 km, equivalent to a variation of 0.00007×10^{14} m^3/sec^2 in GM. Another possible source of error is the ratio μ of the moon's mass to the earth's, appearing in the modified Kepler equation

$$GM = n^2 a^3 \frac{(1 + \beta)^3}{(1 + \mu)} \tag{100}$$

where β is the effect of the sun on the mean distance. When the astronomical unit found by radar measurements to Venus is used, the values of μ^{-1} deduced from the

TABLE 3. Determinations of GM

Method	Reference	Sources of Error	GM 10^{20} cm^3 sec^{-2}
Terrestrial geodesy	*Fischer* [1962]	Triangulation	3.986040
	Kaula [1961*b*]	Gravimetry	3.986020 ± 0.000028
	a_e from *Kaula* [1961*b*]		3.986043
	γ_e from *Uotila* [1962]		
Lunar motion and radar distance	*Yaplee et al.* [1963]*	Lunar radius	3.986057
Lunar motion and triangulated distance	*Fischer* [1962]*	Lunar radius	
	Crommelin *a*	Triangulation	3.986451
	Fischer [1962]*		3.986078
	O'Keefe and Anderson *a*		
Lunar probe and Doppler	*Hamilton et al.* [1963]	Observational Station position	3.986016 ± 0.000025
Close satellite and Doppler	*Anderle and Oesterwinter* [1963]	Observational Orbit perturbations	3.985889
Close satellite and camera	*Kaula* [1963*b*], $1960\iota_2$	Triangulation	3.986037 ± 0.000012
	Kaula [1963*d*], $1961\alpha\delta_1$	Orbit perturbations	3.985993 ± 0.000011

* These references are sources for only the distance to the nearest point of the moon; a lunar radius of 1738.7 km and a lunar mass of 1/81.3015 have been used to calculate the values of GM given in the last column.

lunar inequality (the monthly revolution of the earth about the center of mass of the earth and moon) based on observations of Eros range from 81.26 to 81.36. A new determination of the inequality from Doppler observations of the Mariner 2 Venus probe yields a preliminary value of 81.3015 ± 0.0033 [*Hamilton et al.*, 1963].

Determination of tesseral harmonics. The determination of tesseral harmonics from satellite orbits depends principally on the terms in the disturbing function, (88), for which the argument is $\{(n - 2p)\omega + m(\Omega - \theta)\}$, and for which $|q|$ is small. Because of the unavoidable inclusion of the earth's rotation rate $\dot{\theta}$, the frequencies involved differ by orders of magnitude from those involved in analysis for zonal harmonics, and so the problems are quite different. On the one hand, long-term-drag, luni-solar, and radiation pressure effects are of little influence; on the other hand, errors in station position and nonuniform distribution of observations become important. Usually the frequency of observation is not much higher than the frequencies of the orbital perturbations caused by the tesseral harmonics; hence it is not possible to employ the method of first determining smoothed osculating elements from the observations and then analyzing variations in these elements. The analysis must be made directly in terms of the observations themselves; i.e., there must be formed a partial derivative of each observation with respect to each of the parameters sought. A rigorous solution will then follow that of the times series of equations 57–60, in which $x_f(t)$ are the Keplerian elements, p_l are the gravitational coefficients, y_i are the observations, and q_m are the corrections to station position coordinates. The difficulty in the rigorous solution, as stated, is taking into account the covariance between observations, since this entails arrays of dimension comparable to the number of observations. Furthermore, the statistical analysis of drag effects has not been developed sufficiently [*Kaula*, 1961*c*; *Moe*, 1963]. Consequently, in practice all analyses have neglected covariance between observations of different passes and have applied various devices to minimize the effect of this neglect:

1. Higher weighting of the across-track than of the along-track component of an observation is used, because a proportionately much greater part of the drag effect is in the mean anomaly than is true for the gravitational effects [*Kaula*, 1963*b*, *Izsak*, 1962].
2. The duration for which a set of reference elements, or constants of integration, are determined is limited to one to four weeks. This measure is also desirable to keep the residuals down to not more than a small multiple of the anticipated gravitational effects.
3. Arbitrary polynomials are used to represent some of the variation in mean anomaly.
4. The observations are weighted inversely to their density with respect to the angle $(\Omega - \theta)$ in an attempt to restore some of the orthogonality of gravitationally caused variations to the variations caused by drag [*Kaula*, 1963*b*].

The nonuniform distribution of observations arises from the geometrical effect described by (75) plus, in the case of camera observations, from dependence on clear weather and solar illumination of the satellites. This nonuniform distribution, the large number of parameters involved, and the similarity of the perturbations by different gravitational terms (l, m) and (n, m) for which $(l - n)$ is even, all combine to produce an ill-conditioned least-squares solution from observations of a single satellite. The desirable remedy is to combine several arcs of several satellites of widely varying inclination in one solution. However, such a solution would involve well over 100 unknowns, so further compromises must be made. These compromises have been of two principal types:

1. Stepwise least squares. This method is employed by *Izsak* [1963], *Anderle and Oesterwinter* [1964], and others. For camera observations there are two steps: in the first step, the reference elements for each orbital arc are determined holding fixed the gravitational coefficients and station positions; in the second step, the orbital elements are held fixed, while the gravitational coefficients and station positions are determined from the residuals of the observations with respect to these reference orbits. For Doppler observations, there are three steps: the above two steps are preceded by a step in which frequency constant and frequency drift corrections are determined for each pass. The likely defect of these methods is that the parameters determined in the earlier steps will absorb some of the effects of the gravitational coefficients and station position shifts. This defect may be particularly severe in the case of orbital arcs with relatively few camera observations. *Izsak* [1963] omits the worst of such arcs, but this measure introduces further chance of bias because the arcs so omitted will tend to be those for which the apogee is in the southern hemisphere, away from most of the stations. Both *Izsak* [1963] and *Anderle and Oesterwinter* [1964] make solutions in which all station positions are allowed to move freely with respect to each other; however, computer program limitations do not permit them to determine all the gravitational coefficients compatible with the accuracy implied by this method.
2. Preassigned covariance matrix [*Kaula*, 1963*b*, *d*]. For each orbital arc, the gravitational coefficients and station positions are determined at the same time as the reference elements. To keep the solution from 'blowing up' because of ill-conditioning, a covariance matrix $\mathbf{W}_s$ is preassigned to the gravity and position parameters in accordance with (50). The variances in $\mathbf{W}_s$ for the gravitational coefficients are based on the degree variances in Table 1; those for the positions of major datums are based on the results of the world geodetic system solution of *Kaula* [1961*b*], and those for the positions of isolated stations on statistical analyses of deflections of the vertical. Stations connected by geodetic triangulation are assumed to translate together, consistent with an accuracy of about ± 20 meters. The principal defect of the preassigned variance and covariance method is that it may influence the relative magnitude of the results, particularly for harmonics causing similar frequency perturbations in the orbit.

Earlier estimates of tesseral harmonics from satellite orbits [*Izsak*, 1961*a*; *Kaula*, 1961*c*; *Kozai*, 1961; *Newton*, 1962] yielded a wide scatter of results and are of interest now only for some discussion of methods. Table 4 gives the most recent results of most of the principal investigators; those by *Kozai* [1963*b*], *Izsak* [1963], and *Kaula* [1963*d*] are based on camera data; those by *Anderle and Oesterwinter* [1964] and *Guier* [1963] are based on Doppler data. In addition to the gravitational coefficients given in the table, *Kaula* [1963*d*] determined 10 tesseral coefficients of degrees 5 and 6, plus 18 datum coordinate shifts; *Anderle and Oesterwinter*

[1964] also determined 54 station coordinate shifts. The most recent determinations from terrestrial data [*Kaula*, 1961*b*; *Uotila*, 1962] are also given in Table 4. Figure 6 is the geoid map corresponding to the solution by *Kaula* [1963*d*].

TABLE 4. Harmonic Coefficients of the Gravitational Field

Coefficient	*Kaula* [1961*b*]	*Uotila* [1962]	*Kozai* [1963*b*]	*Izsak* [1963]	*Anderle and Oesterwinter* [1964]	*Kaula* 1963*d*]	*Guier* [1963]
$\bar{C}_{20} \times 10^6$	−484.23	(−484.10)	(−484.10)	(−484.10)	−484.09	−484.08	
$\bar{C}_{22} \times 10^6$	0.75	0.69	1.11	1.50	2.85	1.88	2.60
$\bar{S}_{22} \times 10^6$	−0.38	−2.25	−1.47	−0.62	−1.53	−1.38	−0.99
$\bar{C}_{30} \times 10^6$	0.78	(0.91)	(0.97)	(0.97)	0.94	0.97	
$\bar{C}_{31} \times 10^6$	1.03	0.10	1.75	1.04		1.52	1.64
$\bar{S}_{31} \times 10^6$	0.39	−0.63	0.14	0.06		0.14	0.18
$\bar{C}_{32} \times 10^6$	0.97	1.19	0.34	0.27		−0.02	0.84
$\bar{S}_{32} \times 10^6$	−0.03	−0.22	−0.24	−0.53		0.42	−0.07
$\bar{C}_{33} \times 10^6$	0.57	1.50	−0.45	0.51		0.70	1.06
$\bar{S}_{33} \times 10^6$	1.40	2.44	0.60	0.89		0.76	1.01
$\bar{C}_{40} \times 10^6$	0.25	(0.57)	(0.61)	(0.61)	0.47	0.67	
$\bar{C}_{41} \times 10^6$	−0.63	−0.15	−0.30	−0.30	−0.71	−0.33	−0.60
$\bar{S}_{41} \times 10^6$	−0.15	−0.20	−0.46	−0.34	−0.40	0.37	−0.49
$\bar{C}_{42} \times 10^6$	0.46	0.82	−0.18	0.16	0.45	0.01	0.27
$\bar{S}_{42} \times 10^6$	0.42	0.46	0.21	0.55	1.20	0.35	1.19
$\bar{C}_{43} \times 10^6$	0.51	1.20	0.59	0.36	2.64	0.17	1.33
$\bar{S}_{43} \times 10^6$	−0.01	−0.63	0.02	0.25	−0.60	0.41	−0.05
$\bar{C}_{44} \times 10^6$	−0.10	0.66	0.78	0.46		−0.01	−0.37
$\bar{S}_{44} \times 10^6$	0.36	−0.12	1.26	0.77		0.18	0.31

USE OF ASTROGEODESY

The oldest sources of information about the variations in the gravitational field are the slopes of the geoid with respect to the surface of a reference ellipsoid determined from the differences between the astronomic and geodetic positions:

$$\begin{aligned} \xi &= \phi_a - \phi_g \\ \eta &= (\lambda_a - \lambda_g) \cos \phi \end{aligned} \tag{101}$$

Along a line of geodetic triangulation or traverse in azimuth A the slope will be:

$$t = \eta \sin A + \xi \cos A \tag{102}$$

and for the change in geoid height over an arc distance Ψ there is conventionally applied the simple integration

$$N_B = N_A - R \int_0^{\psi} t\, dl \tag{103}$$

where R is the radius of the earth.

However, if the triangulation from which the geodetic positions (ϕ_g, λ_g) were calculated with a scale correction $R/(R + h)$ for reduction to sea level altitude h (the 'development method'), then an error will accumulate because of neglect of the further scale correction $R/(R + N)$ for reduction to ellipsoid

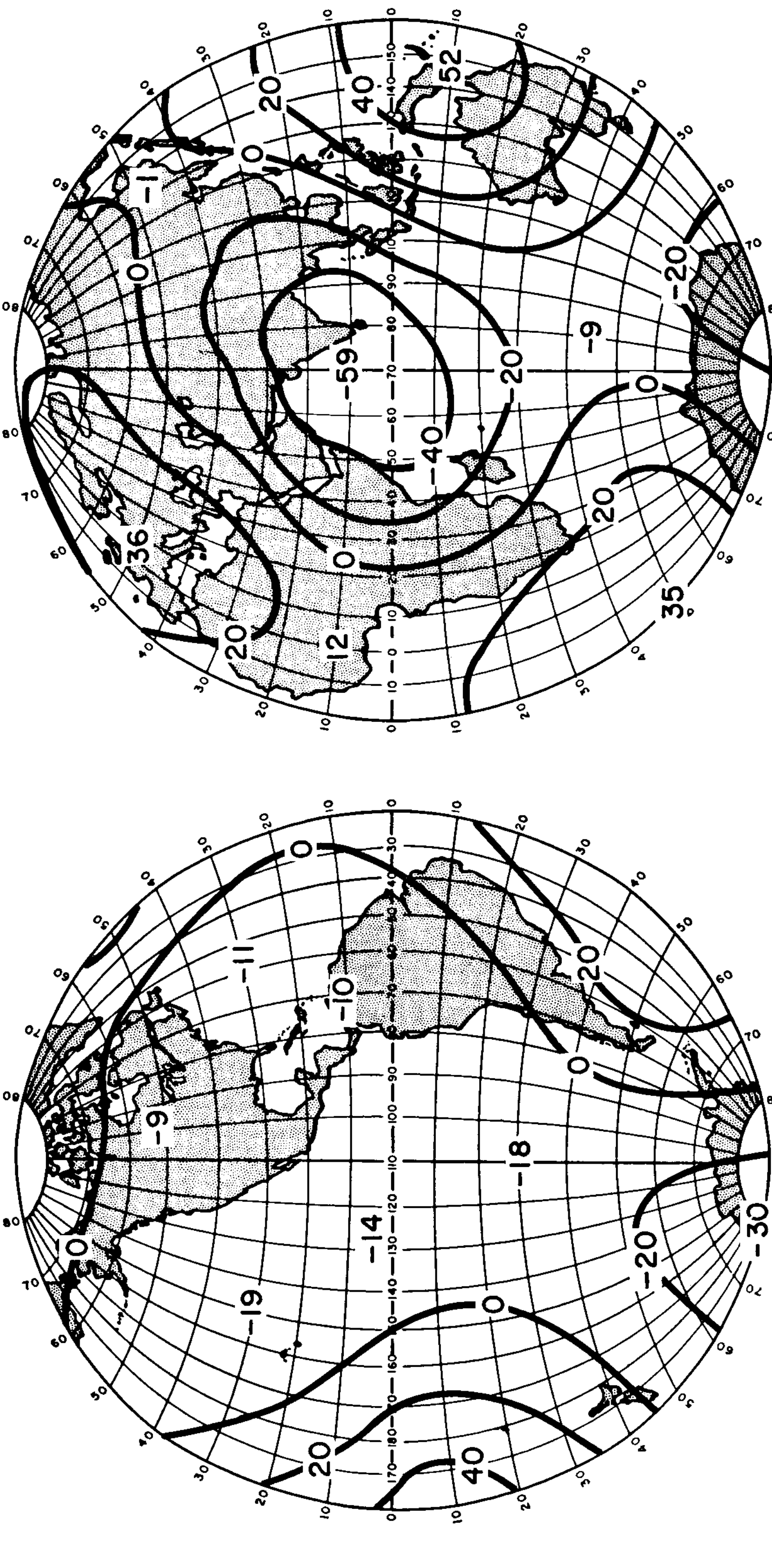

Fig. 6. Geoid heights referred to an ellipsoid of flattening 1/298.24, based on camera observations of satellites 1959α_1, 1959η, 1960ι_2, 1961δ_1, 1961$\alpha\delta_1$ [*Kaula*, 1963*d*].

(the 'projection method'). After a few thousand kilometers, this error will make a difference of several tens of meters in N_B calculated by (103). The elimination of this discrepancy is known as the Molodenskiy correction; given t, N calculated by (102) and (103) from development-computed ϕ_g, λ_g, the correct slope θ and height ζ will be [*Molodenskiy et al.*, 1960, p. 33]

$$\begin{aligned} \theta_B &= t_B + (\theta_A - t_A)\cos\psi + (\zeta_A - N_A)\sin\psi + \frac{1}{R}\int_0^\psi N\cos(l-\psi)\,dl \\ \zeta_B &= N_B + R(t_A - \theta_A)\sin\psi + (\zeta_A - N_A)\cos\psi + \int_0^\psi N\sin(l-\psi)\,dl \end{aligned} \tag{104}$$

If geodetic control computed by the development method is adjusted forcing conditions on large circuits properly applicable only to projection-computed control, then (104) will not remove all error. However, circuits large enough for this distortion to be significant are rare.

The most extensive application of (101) through (104) to astrogeodetic data has been by *Fischer* [1959*a*, *b*; 1960*a*, *b*; 1961]. Figure 7 shows the distribution of the data she has applied. The limited geographic extent of the astrogeodetic data makes it primarily of value as an independent check on the results obtained by gravimetric and satellite means. The significance of the various spacings of the astronomic stations in Figure 7 is that the principal source of error in determining astrogeodetic geoid height differences is the error in determining the continuously varying geoid slope by interpolation between the observed point values. For station spacings S in kilometers, the mean square expected error in the geoid height $E_s\{\epsilon_i^2\}$ derived from the degree variances in Table 1 (extended to higher degrees) is approximated by [*Kaula*, 1961*b*]:

$$E_s\{\epsilon_i^2\} \approx (0.019S - 1.4)^2 \tag{105}$$

This rule probably gives an underestimate for the geoid height error resulting from arcs carried through rugged topography such as that down the west coast of South America. However, it seems safe to conclude that the rms error for the relative location with respect to each other of points in the major systems shown in Figure 7 is of the order of ± 15 meters in the radial coordinate, as well as the two horizontal. This estimate is confirmed by misclosures of less than 25 meters for 10,000-km loops around the Caribbean [*Fischer*, 1959] and around the Black and Caspian seas and through Turkestan [*Fischer*, 1961]. (See *Bomford* [1960; 1962, pp. 143–159, 325–327] for more detailed discussion of triangulation and geoid height accuracy, and *Rice* [1962] for an example of an optimum geoidal section survey.)

There are several ways of combining astrogeodetic and gravimetric geoid data. In the USSR, gravimetry is used to interpolate deflections between astrogeodetic stations [*Molodenskiy et al.*, 1960, pp. 125–141]. The traditionally advocated method of combination is to compare the astrogeodetic and gravimetric geoid heights at a few carefully selected points and then to minimize the discrepancy between the two by adjusting the datum position and ellipsoid parameters [*Heiskanen and Vening-Meinesz*, 1958, pp. 299–310]. This method has been applied extensively by *Rice* [1952] to sixteen stations in the United States and by *Szabo* [1962] to six stations in the United States and twenty-three stations in

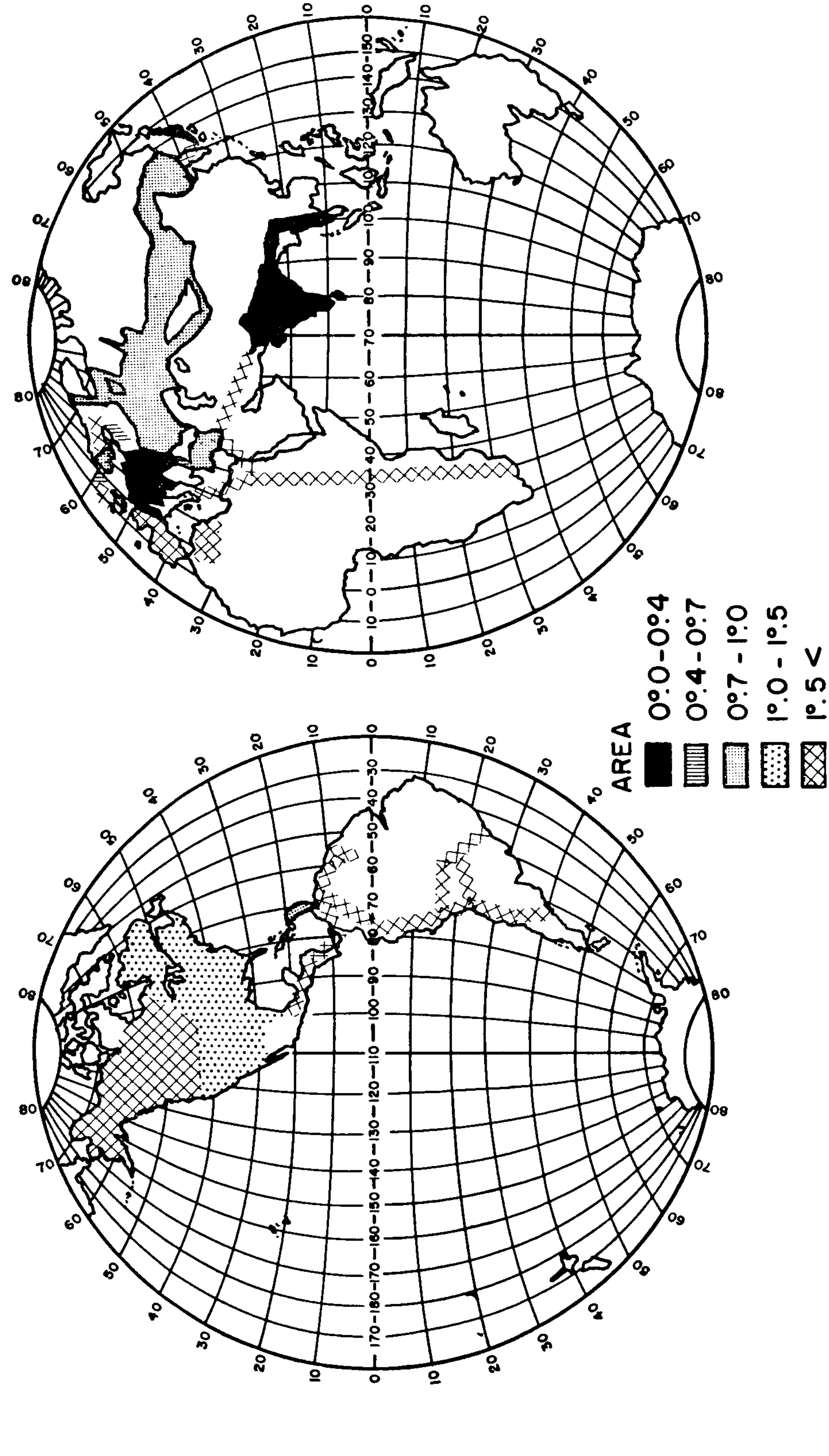

Fig. 7. Distribution of astrogeodetic observations.

Eurasia. However, since there is still appreciable error in the gravimetrically computed deflection at even the best points (probably at least $\pm 1.5''$) the logical conclusion is to compare astrogeodetic and gravimetric geoids wherever the former is available. *Zhonglovich* [1956] made such a comparison by minimizing the sum $\sum \{(\eta_a - \eta_g)^2 + (\xi_a - \xi_g)^2\}$ for the mean values of ninety-six 4° by 4° squares covering North America, using his geoid [*Zhonglovich,* 1952] for η_g, ξ_g. The mean curvature of the ellipsoid he believed best fitting yields an equatorial radius of 6,378,104 ± 42 meters when the flattening of 1/298.3 is enforced. *Fischer* [1960, 1961] minimized the sum $\sum (N_a - N_g)^2$ at 301 points at 5° intervals throughout the area pictured in Figure 7, using N_g calculated by *Kaula* [1959*a*, *b*]. Fixing the flattening at 1/298.3, the equatorial radius obtained varied from 6,378,160 to 6,378,166 meters on different assumptions. *Kaula* [1961*b*] used the same data as Fischer, but applied a much more detailed statistical treatment, in which 106 condition equations were written requiring that the corrected differences in astrogeodetic geoid height between mean values for 10° by 10° squares agree with the difference calculated from the corrected harmonic coefficients of the gravitational field. The quadratic sum minimized was

$$\text{minimum} = \mathbf{x}_A\mathbf{W}_A^{-1}\mathbf{x}_A + \mathbf{x}_G\mathbf{W}_G^{-1}\mathbf{x}_G + \mathbf{x}_s\mathbf{W}_s^{-1}\mathbf{x}_s \tag{106}$$

where $\mathbf{x}_A$ comprises the corrections to the 106 astrogeodetic geoid height differences; $\mathbf{W}_A$ is the associated covariance matrix, taking into account interpolation error, error of representation, and the discrepancy between the actual geoid and that represented by harmonics to the 8th degree; $\mathbf{x}_G$ comprises the 81 spherical harmonic coefficients through the 8th degree; $\mathbf{W}_G$ is the covariance matrix produced by the analysis of gravimetry by *Kaula* [1959*a*, *b*]; and $\mathbf{x}_s$ and $\mathbf{W}_s$ pertain to supplementary measurements of secular and long periodic satellite motions and geoid height matching between adjacent but unconnected datums. A measure of the agreement between astrogeodetic and gravimetric geoids was the value obtained for the quadratic sum of (106); it was 44 per cent higher than the mean χ square expectancy. Increasing the standard deviations accordingly, 6,378,163 ± 15 meters and 1/298.24 ± 0.01 were obtained for the ellipsoid parameters. The values of the harmonic coefficients obtained through the 4th degree are given in Table 4. The standard deviations obtained for the tesseral harmonic coefficients of degree n averaged about $\pm 0.9 \times 10^{-6}/(n - 1)$. The rms discrepancies from the satellite solution of *Kaula* [1963*d*] are $\pm 1.0 \times 10^{-6}$ for degree 2, $\pm 0.6 \times 10^{-6}$ for degree 3, and $\pm 0.3 \times 10^{-6}$ for degree 4, so that the disagreement is one which can be reasonably expected.

CONCLUSIONS

The application of more elaborate statistical techniques made possible by modern computers could very probably extract more information about the gravity field from existing data, both gravimetric and satellite, and would facilitate the planning of programs of additional observations. However, these elaborations would be added to what is already a rather complicated task of measurement, data processing, and theoretical analysis, such that both the benefits and penalties of any modifications are difficult to predict. Also, this review has not touched on the geophysical application of the gravity field. It can be argued that an appreciable effort to determine the gravity field is not worth while because knowledge is lacking of the long-term rheology of the earth's interior as well as the mathematical techniques to apply all but the simplest rheologies. Certainly much of what has been discussed in this review is not necessary for the application of gravimetry to study local and regional variations in the crust. What part of it is useful to study variations on a continental or oceanic scale and the implications thereof as to the state of the mantle is perhaps worthy of consideration: To what harmonic degree should the field be developed to apply to ideas of a weak upper mantle or to convection currents? Are the degree variances a useful tool? Should coefficients be derived by techniques which yield mean square values smaller than those of the actual coefficients? The answer is probably the customary one that we cannot predict what the future will want, and hence should strive for the most accurate representation possible.

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