Analysis of Elastic-Plastic Shells of Revolution Containing Discontinuities

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The common junction of three dissimilar general shells of revolution is analyzed. Axisymmetric loading may be in the form of surface forces, concentrated forces and moments at the junction, and arbitrary thermal gradients. Basic differential equations available for elastic shells are extended for application in the elastic-plastic regime. The von Mises-Hencky yield criterion, deformation theory of plasticity, and successive approximations are used to determine plastic strains. Postyield material behavior is arbitrary. Linearized finite-difference equations are solved directly using coefficient matrices that incorporate conditions of equilibrium and compatibility at the junction discontinuity. The solution to a sample problem is given.

Nomenclature

C = extensional rigidity, $\int_h E\, d\zeta$, lb/in.
D = flexural rigidity, $\frac{1}{(1-\nu^2)}\int_h E\zeta^2 d\zeta$, lb-in.
E = modulus of elasticity, psi
H,N = radial and tangential stress resultants, lb/in.
h = thickness of shell wall, in.
M = couple, lb-in./in.
p = distributed surface load, psi
r,s,z = radial, meridional, and axial coordinates to reference surface, in.
T = temperature, °F
u = radial deflection, in.
V = axial stress resultant, $-\frac{1}{r}\int_\xi r\alpha_0 p_V\, d\xi$, lb/in.
α = coefficient of thermal expansion, in./in.-°F
α_0 = reference length, in.
β = meridional rotation, rad
$\Delta(\ )$ = increment of ( )
ϵ = normal strain, in./in.
ζ = transverse coordinate, satisfies $\int_h E\zeta\, d\zeta = 0$, in.
κ = change in curvature, (in.)$^{-1}$
ν = Poisson's ratio
ξ = s/α_0
σ = normal stress, psi

Subscripts

a,b,c = junction, shells a, b, and c, respectively
d = boundary, shell c
e,t,p = effective, total, and plastic
H,V = radial and axial
i,j = station indices
θ,ξ,ζ = circumferential, meridional, and transverse

Superscripts

e = external
($\wedge$) = reference surface

Introduction

IN certain missile and space vehicle structures, small plastic strains may be permitted in an effort to use material more efficiently. In addition to increasing allowable loads on continuous shell regions, a small amount of yielding may have the beneficial effect of relieving maximum stresses in regions of discontinuity. The purpose of this paper is to present a shell analysis with which both this reduction in stress concentration and the corresponding increase in strain concentration may be determined quickly and accurately.

Many analyses have been presented for elastic shells of revolution containing discontinuities. One of the most general and convenient is that proposed by Radkowski et al.[1] In this reference, basic differential equations derived by Reissner[2] are solved by a very direct finite-difference technique, which employs coefficient matrices. Continuous shell regions are analyzed separately for five independent sets of boundary conditions and are then joined at discontinuities using the well-known method of simultaneous junction equations and superposition.

Mendelson and Manson[3] presented a method for analyzing thermal stresses in a uniform cylindrical shell in the elastic-plastic regime. This method was later applied to general shells of revolution by Stern,[4] using the earlier Reissner-Meissner differential equations. Both of these references are applicable only to continuous shell regions. In addition, the numerical method employed in Ref. 4 requires the simultaneous solution of a set of equations. Because each finite difference station along the shell meridian adds two equations to this set, this method is more limited than that of Ref. 1.

References 1 and 2 were used by Wilson and Spier[5] to develop a solution for the small finite elastic deformations of continuous shells of revolution. This problem is similar to that of elastic-plastic deformation in that nonlinear terms occur in the basic differential equations of each. Also, in both Refs. 3 and 5 the equations are linearized by treating the nonlinear terms as knowns and correcting their values by successive approximations.

The present analysis is much more general than previous work because it is applicable to both continuous and discontinuous shell regions in the elastic-plastic regime. A wide range of shell geometries, loadings, and material properties may be specified. For elastic discontinuity problems, multiple solutions and simultaneous junction equations are eliminated. In addition, a great measure of the generality, directness, and convenience of the shell analysis of Ref. 1 is preserved in the presence of discontinuities and plastic flow.

Method

The common junction of three dissimilar general shells of revolution is used as a general discontinuity model (Fig. 1). Wall thicknesses, material properties, distributed loads, and temperatures may vary along the several shell meridians. In addition, material properties and temperatures may vary arbitrarily through the shell walls. These data may

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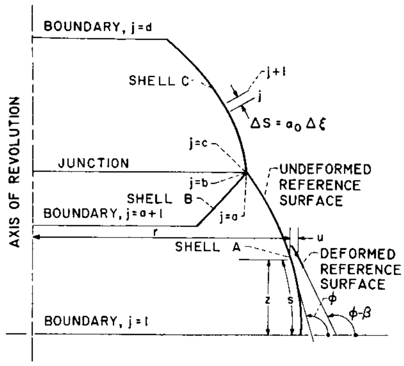
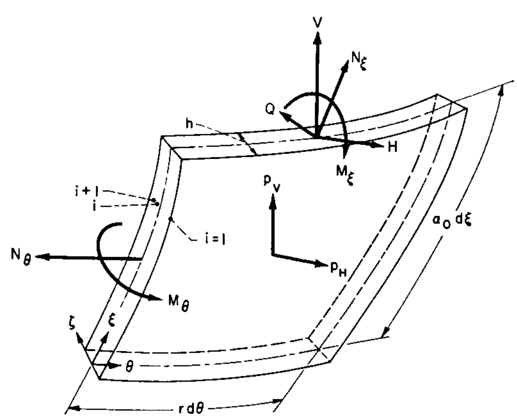


Fig. 1 Meridional geometry of general shell discontinuity structure.

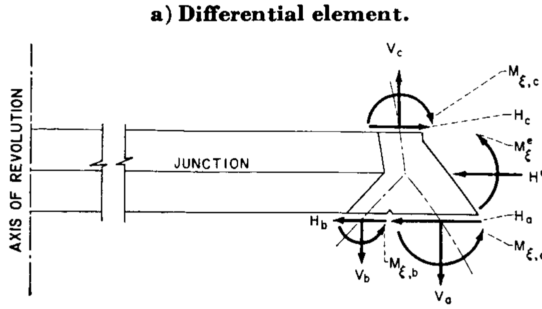
be specified at discrete points, if desired. An external axisymmetric force and couple are applied at the junction for further generality.

The analysis of the three continuous shell regions is considered first. Basic differential equations of Ref. 2 are extended for application in the elastic-plastic regime in accordance with Ref. 3. These modified linearized equations are then solved by the finite-difference technique of Ref. 1 for two quantities, (rH) and β, from which all stresses and strains in the shell may be computed. The shell meridians are traversed twice: first, to compute coefficient matrices at each station using geometrical and load data; and, second, to calculate (rH) and β.

After the continuous regions are analyzed, junction equations are derived which express the structural continuity that exists at the intersection of the three shells. Since discontinuity forces and couples are not linear with applied loads in the elastic-plastic regime, standard methods of analysis which use linear edge influence coefficients[1,7] are not applicable. Instead, the junction equations are satisfied by incorporating them directly into the previously mentioned coefficient matrices, which preserves the direct character of the finite-difference technique. Finally, the numerical technique is summarized, and a sample problem is solved for both strain-hardening and perfectly-plastic material behavior.



a) Differential element.



b). Junction element.

Fig. 2 Stress resultants, couples, and loads.

In addition to the usual assumptions of thin shell theory, the von Mises-Hencky yield criterion and the deformation theory of plasticity are assumed to be valid. Postyield material behavior is arbitrary.

Analysis

A. Shell Equations

1. Basic differential equations

In accordance with the procedure developed by Reissner,[2] equilibrium, compatibility, stress-strain, and strain-displacement equations in a thin elastic shell of revolution may be combined and reduced to two second-order coupled differential equations. The dependent variables are a representative deformation β and a representative stress function (rH). The independent variable is ξ, the meridional coordinate. If extended to include plastic as well as elastic and thermal deformations, these two basic equations become nonlinear and have the following form:

$$(rH)'' + \Gamma(rH)' + \Theta(rH) + \Lambda\beta = \lambda_1 + \tau_1 + \pi_1 \quad \text{(1a)}$$

$$\beta'' + \Upsilon\beta' + \Phi\beta + \psi(rH) = \lambda_2 + \tau_2 + \pi_2 \quad \text{(1b)}$$

Primes denote differentiation with respect to ξ, and the coefficients are

$$\left.\begin{aligned}
\Gamma &= \frac{(r/C\alpha_0)'}{r/C\alpha_0} \qquad \Upsilon = \frac{(rD/\alpha_0)'}{rD/\alpha_0} \\
\Theta &= -\left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/C\alpha_0)'}{r/C\alpha_0}\right] \\
\Phi &= -\left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'D/\alpha_0)'}{rD/\alpha_0}\right] \\
\Lambda &= -\frac{z'}{r/C\alpha_0} \qquad \psi = \frac{z'}{rD/\alpha_0} \\
\lambda_1 &= -\left\{\left[\frac{(r/C\alpha_0)'}{r/C\alpha_0} + \nu\frac{r'}{r}\right](r\alpha_0 p_H) + (r\alpha_0 p_H)'\right\} + \\
&\quad \left\{\left[\frac{z'r'}{r^2} + \nu\frac{(z'/C\alpha_0)'}{r/C\alpha_0}\right](rV) + \nu\left(\frac{z'}{r}\right)(rV)'\right\} \\
\lambda_2 &= \frac{1}{rD/\alpha_0}\,[r'(rV)] \\
\tau_1 &= -\alpha_0 C\left(\frac{1}{C}\int_h E\alpha T\,d\zeta\right)' \\
\tau_2 &= \frac{\alpha_0}{(1-\nu)D}\left(\int_h E\alpha T\zeta\,d\zeta\right)' \\
\pi_1 &= \frac{1}{r/C\alpha_0}\left[\frac{r'}{C}\int_h E\epsilon_{\xi p}\,d\zeta - \left(\frac{r}{C}\int_h E\epsilon_{\theta p}\,d\zeta\right)'\right] \\
\pi_2 &= \frac{1}{rD/\alpha_0}\left\{\left[\frac{r}{1-\nu^2}\int_h E(\epsilon_{\xi p} + \nu\epsilon_{\theta p})\zeta\,d\zeta\right]' - \right. \\
&\quad \left.\frac{r'}{1-\nu^2}\int_h E(\epsilon_{\theta p} + \nu\epsilon_{\xi p})\zeta\,d\zeta\right\}
\end{aligned}\right\} \quad \text{(1c)}$$

The integrals through the shell walls may be evaluated analytically or numerically.

The left-hand sides of Eqs. (1a) and (1b) and the loading terms λ_1 and λ_2 are identical with those derived in Ref. 2 for elastic shells. The thermal terms τ_1 and τ_2 are developed in Ref. 1. The plasticity terms π_1 and π_2 are derived in a similar manner, by using the elastic-plastic stress-strain relations of Ref. 3:

$$\left.\begin{aligned}
\sigma_\theta &= [E/(1-\nu^2)][(\epsilon_\theta + \nu\epsilon_\xi) - \\
&\qquad (\epsilon_{\theta p} + \nu\epsilon_{\xi p}) - (1+\nu)\alpha T] \\
\sigma_\xi &= [E/(1-\nu^2)][(\epsilon_\xi + \nu\epsilon_\theta) - \\
&\qquad (\epsilon_{\xi p} + \nu\epsilon_{\theta p}) - (1+\nu)\alpha T]
\end{aligned}\right\} \quad \text{(1d)}$$

Although these last two terms contain nonlinear plastic strains, it is convenient to treat them not as unknown but as known quantities that may be determined in an iterative manner from previous, less accurate solutions. For example, π_1 and π_2 may be assumed to be equal to zero to obtain an elastic solution to Eqs. (1). Values of $\epsilon_{\xi p}$ and $\epsilon_{\theta p}$ may be calculated from these elastic results and used to obtain somewhat more accurate approximations for π_1 and π_2. This procedure is repeated until additional solutions produce negligible changes in the plastic strain terms.

General boundary conditions on Eqs. (1) can be expressed in terms of the quantities (rH), $(rH)'$, β, and β'. Because these equations will be solved using the method of finite differences, it is more convenient to express these boundary conditions directly in difference notation. This is done in the following section.

2. *Basic difference equations*

The shell reference meridians are divided into equal increments (Fig. 1). Using central differences, the derivatives in Eqs. (1) may be expressed as

$$(rH)_i'' = \frac{(rH)_{i+1} - 2(rH)_i + (rH)_{i-1}}{(\Delta\xi)^2}, \text{ etc.}$$

In matrix form, the basic shell difference equations then become

$$A_j y_{j+1} + B_j y_j + F_j y_{j-1} = g_j \qquad j = \begin{cases} 2, a-1 \\ a+2, b-1 \\ c+1, d-1 \end{cases} \tag{2a}$$

Capital letters represent 2 × 2 and lower-case letters 2 × 1 matrices.

These are

$$\left.\begin{aligned} A &= \begin{bmatrix} 1 + (\Delta\xi/2)\Gamma & 0 \\ 0 & 1 + (\Delta\xi/2)\Upsilon \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \\ B &= \begin{bmatrix} -2 + (\Delta\xi)^2\Theta & (\Delta\xi)^2\Lambda \\ (\Delta\xi)^2\Psi & -2 + (\Delta\xi)^2\Phi \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ F &= \begin{bmatrix} 1 - (\Delta\xi/2)\Gamma & 0 \\ 0 & 1 - \Delta\xi/2\, \Upsilon \end{bmatrix} = \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix} \\ g &= \begin{bmatrix} (\Delta\xi)^2(\lambda_1 + \tau_1 + \pi_1) \\ (\Delta\xi)^2(\lambda_2 + \tau_2 + \pi_2) \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \\ y &= \begin{bmatrix} (rH) \\ \beta \end{bmatrix} \end{aligned}\right\} \tag{2b}$$

Boundary conditions that may involve first derivatives of (rH) and β require the use of forward and backward differences. A boundary condition may then be expressed as a general linear combination of the unknowns at the boundary and at the adjacent station.

In matrix notation, the boundary conditions become

$$\left.\begin{aligned} y_1 &= k - Jy_2 \\ y_{a+1} &= m - Ly_{a+2} \\ y_d &= t - Ry_{d-1} \end{aligned}\right\} \tag{3}$$

3. *Numerical integration*

Following the method of Ref. 1, the equation matrix (2a) is solved as follows. Set

$$y_j = q_j - P_j y_{j+1} \tag{4a}$$

Then

$$\left.\begin{aligned} y_{j-1} &= q_{j-1} - P_{j-1} y_j \\ y_{j+1} &= P_j^{-1}(q_j - y_j) \end{aligned}\right\} \tag{4b}$$

Substituting Eqs. (4b) into (2a) gives

$$(-A_jP_j^{-1} + B_j - F_jP_{j-1})y_j + (A_jP_j^{-1}q_j + F_jq_{j-1} - g_j) = 0 \qquad j = \begin{cases} 2, a-1 \\ a+2, b-1 \\ c+1, d-1 \end{cases}$$

This equation is satisfied independently of y_j by setting

$$-A_jP_j^{-1} + B_j - F_jP_{j-1} = A_jP_j^{-1}q_j + F_jq_{j-1} - g_j = 0 \tag{4c}$$

Therefore,

$$P_j = (B_j - F_jP_{j-1})^{-1}A_j \tag{5a}$$

$$j = \begin{cases} 2, a-1 \\ a+2, b-1 \\ c+1, d-1 \end{cases}$$

$$q_j = P_jA_j^{-1}(g_j - F_jq_{j-1}) \tag{5b}$$

From Eqs. (3) and (4a),

$$P_1 = J \qquad q_1 = k \qquad P_{a+1} = L \qquad q_{a+1} = m \tag{5c}$$

Using the last of Eqs. (3) and Eq. (4a) results in

$$y_d = (I - RP_{d-1})^{-1}(t - Rq_{d-1}) \tag{6}$$

in which I equals the 2 × 2 identity matrix.

Equations for P_c, q_c, y_b, and y_a are needed to complete this numerical integration. They will be derived in the next section from the junction conditions. With these additional equations, consecutive calculations of the y matrices may be made using Eqs. (6) and (4a), after the P and q matrices are calculated by Eqs. (5). For convenience, matrix equations (3, 4a, 5, and 6) are expanded in Ref. 6.

B. Junction Equations

1. *Derivation*

Certain conditions of compatibility and equilibrium must be fulfilled at the junction of the three shells to maintain structural continuity. These are 1) the junction is hingeless; 2) the reference surfaces of the three shells experience equal radial deflections at their junction; and 3) a differential ring element that includes the junction section is in static equilibrium.

The first condition leads directly to the equations

$$\beta_a = \beta_b = \beta_c \tag{7a}$$

If the superscript (^) is used to denote a reference surface, the radial deflection of this surface at any station is $r_j\epsilon_{\theta,j}$. Therefore, the second continuity condition may be expressed by the equations

$$\epsilon_{\theta,a} = \epsilon_{\theta,b} = \epsilon_{\theta,c} \tag{7b}$$

Referring to Fig. 2, the junction equilibrium conditions are

$$(rH)_a + (rH)_b = (rH)_c - \bar{r}H^* \tag{7c}$$

in which $\bar{r}$ is the nominal radial coordinate to the junction, and

$$M_{\xi,a} + M_{\xi,b} = M_{\xi,c} - M_\xi^* \tag{7d}$$

Conditions (7b) and (7d) must be expressed in terms of (rH) and β. Following closely the derivation in Ref. 1, modified slightly by the stress-strain equations (1d),

$$\left.\begin{aligned} \epsilon_\theta &= \frac{1}{C}\left[\frac{(rH)'}{\alpha_0} - \frac{\nu}{\alpha_0}\frac{r'}{r}(rH)\right] + \lambda_3 + \tau_3 + \pi_3 \\ \epsilon_\xi &= \frac{1}{C}\left[-\nu\frac{(rH)'}{\alpha_0} + \frac{r'}{\alpha_0 r}(rH)\right] + \lambda_4 + \tau_4 + \pi_4 \end{aligned}\right\} \tag{8a}$$

in which

$$\left.\begin{aligned}
\lambda_3 &= \frac{1}{C}\left[rp_H - \frac{\nu z'}{\alpha_0 r}(rV)\right] \\
\lambda_4 &= \frac{1}{C}\left[-\nu rp_H + \frac{z'}{\alpha_0 r}(rV)\right] \\
\tau_3 &= \frac{1}{C}\int_h E\alpha T\, d\zeta \qquad \tau_4 = \tau_3 \\
\pi_3 &= \frac{1}{C}\int_h E\epsilon_{\theta p}\, d\zeta \qquad \pi_4 = \frac{1}{C}\int_h E\epsilon_{\xi p}\, d\zeta
\end{aligned}\right\} \quad (8b)$$

and

$$M_\xi = D(\kappa_\xi + \nu\kappa_\theta) + \tau_5 + \pi_5 \quad (8c)$$

in which

$$\left.\begin{aligned}
\kappa_\xi &= \frac{\beta'}{\alpha_0} \qquad \tau_5 = \frac{-1}{1-\nu}\int_h E\alpha T\zeta\, d\zeta \\
\kappa_\theta &= \frac{r'}{r}\frac{\beta}{\alpha_0} \qquad \pi_5 = \frac{-1}{1-\nu^2}\int_h (\epsilon_{\xi p} + \nu\,\epsilon_{\theta p})E\zeta\, d\zeta
\end{aligned}\right\} \quad (8d)$$

The rotationally symmetric external junction loads H^e and M_ξ^e may arise through the action of an elastic constraint such as a reinforcing ring, or they may be independent of the shell deformations. Therefore, let

$$\left.\begin{aligned}
H^e &= s_{11}(\bar{r}\epsilon_{\theta,c}) + s_{12}\beta_c + \bar{H} \\
M_\xi^e &= s_{21}(\bar{r}_{\epsilon\theta,c}) + s_{22}\beta_c + \bar{M}_\xi
\end{aligned}\right\} \quad (8e)$$

The coefficients s_{11} to s_{22} are spring constants, which can be determined from the geometry and material properties of the elastic constraint. They define the linear dependence of H^e and M_ξ^e on the deformation of shell *c;* $\bar{M}_\xi$ may result from nonconcurrence of the three reference meridians.

When only two shells are involved, Eqs. (7) may be easily specialized by deleting the terms with the subscript *b*. The numerical solution of these modified equations will then be similar to that of the following section.

2. *Numerical solution*

Before substituting Eqs. (8) into Eqs. (7), an expression relating y_i' to y_i is derived. In central difference notation, using (4b),

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2(\Delta\xi)} = \frac{(P_{i-1} - P_i^{-1})y_i + (P_i^{-1}q_i - q_{i-1})}{2(\Delta\xi)}$$

or

$$\left.\begin{aligned}
y_i' &= W_i y_i + x_i \\
&\text{Using Eqs. (4c),} \\
W_i &= \frac{(P_{i-1} - P_i^{-1})}{2(\Delta\xi)} = \frac{(I + A_i^{-1}F_i)P_{i-1} - A_i^{-1}B_i}{2(\Delta\xi)} \\
x_i &= \frac{(P_i^{-1}q_i - q_{i-1})}{2(\Delta\xi)} = -\frac{(I + A_i^{-1}F_i)q_{i-1} - A_i^{-1}g_i}{2(\Delta\xi)}
\end{aligned}\right\} \quad (9a)$$

Expanding the first of Eqs. (9a) gives

$$\left.\begin{aligned}
(rH)_i' &= w_{11,i}(rH)_i + w_{12,i}\beta_i + x_{1,i} \\
\beta_i' &= w_{21,i}(rH)_i + w_{22,i}\beta_i + x_{2,i}
\end{aligned}\right\} \quad (9b)$$

After Eqs. (7a) and (9) are substituted into Eqs. (8), the junction equations become

$$(rH)_a = \frac{1}{\delta_{1,a}}[\delta_{1,c}(rH)_c + (\delta_{2,c} - \delta_{2,a})\beta_c + (\delta_{3,c} - \delta_{3,a})] \quad (10a)$$

$$(rH)_b = \frac{1}{\delta_{1,b}}[\delta_{1,c}(rH)_c + (\delta_{2,c} - \delta_{2,b})\beta_c + (\delta_{3,c} - \delta_{3,b})] \quad (10b)$$

$$(rH)_a + (rH)_b = (1 - \bar{r}^2 s_{11}\delta_{1,c})(rH)_c - \bar{r}(\bar{r}s_{11}\delta_{2,c} + s_{12})\beta_c - \bar{r}(\bar{r}s_{11}\delta_{3,c} + \bar{H}) \quad (10c)$$

and

$$\delta_{4,a}(rH)_a + \delta_{4,b}(rH)_b = (\delta_{4,c} - \bar{r}s_{21}\delta_{1,c})(rH)_c + (\delta_{5,c} - \delta_{5,a} - \delta_{5,b} - \bar{r}s_{21}\delta_{2,c} - s_{22})\beta_c + (\delta_{6,c} - \delta_{6,a} - \delta_{6,b} - \bar{r}s_{21}\delta_{3,c} - \bar{M}_\xi) \quad (10d$$

in which

$$\left.\begin{aligned}
\delta_{1,j} &= \left(\frac{w_{11} - \nu(r'/r)}{\alpha_0 C}\right)_j \qquad \delta_{4,j} = \left(\frac{w_{21}}{\alpha_0/D}\right)_j \\
\delta_{2,j} &= \left(\frac{w_{12}}{\alpha_0 C}\right)_j \qquad \delta_{5,j} = \left(\frac{w_{22} - \nu(r'/r)}{\alpha_0/D}\right)_j \\
\delta_{3,j} &= \left(\frac{x_1}{\alpha_0 C} + \lambda_3 + \tau_3 + \pi_3\right)_j \\
\delta_{6,j} &= \left(\frac{x_2}{\alpha_0/D} + \tau_5 + \pi_5\right)_j \qquad j = \begin{cases} a \\ b \\ c \end{cases}
\end{aligned}\right\} \quad (10e)$$

Substituting $(rH)_a$ and $(rH)_b$ from Eqs. (10a) and (10b) into Eqs. (10c) and (10d) yields

$$\gamma_1(rH)_c + \gamma_2\beta_c = \gamma_3 \quad (11a)$$

and

$$\gamma_4(rH)_c + \gamma_5\beta_c = \gamma_6 \quad (11b)$$

Equations (11) will be satisfied independently of $(rH)_c$ and β_c if

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0$$

or

$$\left.\begin{aligned}
\delta_{1,c} &= \frac{\delta_{1,a}\delta_{1,b}}{\delta_{1,a} + \delta_{1,b} + \bar{r}^2 s_{11}\delta_{1,a}\delta_{1,b}} \\
\delta_{2,c} &= \frac{\delta_{1,a}\delta_{2,b} + \delta_{2,a}\delta_{1,b} + \bar{r}s_{12}\delta_{1,a}\delta_{1,b}}{\bar{\delta}_{1,a} + \tilde{\delta}_{1,b} + \bar{r}^2 s_{111}\bar{\delta}_{a,}\tilde{\delta}_{1b}} \\
\delta_{3,c} &= \frac{\delta_{1,a}\delta_{3,b} + \delta_{3,a}\delta_{1,b} + \bar{r}\bar{H}\delta_{1,a}\delta_{1,b}}{\delta_{1,a} + \delta_{1,b} + \bar{r}^2 s_{11}\delta_{1,a}\delta_{1,b}} \\
\delta_{4,c} &= \delta_{1,c}\left(\frac{\delta_{4,a}}{\delta_{1,a}} + \frac{\delta_{4,b}}{\delta_{1,b}} + \bar{r}s_{21}\right) \\
\delta_{5,c} &= \delta_{2,c}\left(\frac{\delta_{4,a}}{\delta_{1,a}} + \frac{\delta_{4,b}}{\delta_{1,b}} + \bar{r}s_{21}\right) - \left(\frac{\delta_{2,a}\delta_{4,a}}{\delta_{1,a}} + \frac{\delta_{2,b}\delta_{4,b}}{\delta_{1,b}} - \delta_{5,a} - \delta_{5,b} - s_{22}\right) \\
\delta_{6,c} &= \delta_{3,c}\left(\frac{\delta_{4,a}}{\delta_{1,a}} + \frac{\delta_{4,b}}{\delta_{1,b}} + \bar{r}s_{21}\right) - \left(\frac{\delta_{3,a}\delta_{4,a}}{\delta_{1,a}} + \frac{\delta_{3,b}\delta_{4,b}}{\delta_{1,b}} - \delta_{6,a} - \delta_{6,b} - \bar{M}_\xi\right)
\end{aligned}\right\} \quad (12)$$

Using Eqs. (12), Eqs. (10e) may be solved for the components of the W_c and x_c matrices. Finally, Eqs. (9a) may be inverted to give

$$\left.\begin{aligned}
P_{c-1} &= \{(A + F)^{-1}[2(\Delta\xi)AW + B]\}_c \\
q_{c-1} &= -\{(A + F)^{-1}[2(\Delta\xi)Ax - g]\}_c
\end{aligned}\right\} \quad (13)$$

Table 1 Summary of numerical analysis procedure

Step	Quantities to be calculated	Meridional stations	Equation
1	J,L,R		(3)
2	$\Gamma,\Theta,\Lambda,\Upsilon,\Phi,\Psi,\lambda_1$ to λ_4,τ_1 to τ_4; π_1 and π_2 assumed (usually based on $\epsilon_{\theta p}$ and $\epsilon_{\xi p}$ equal to zero)	2 to a; $a+2$ to b; c to $d-1$	(1),(8)
3	A,B,F	Same as above	(2)
4	g	Same as above	(2)
5	$\lambda_3,\lambda_4,\tau_3$ to τ_5; π_3 to π_6	$1,a,a+1,b,c,d$	(8)
6	k,m,t		(3)
7	P,q	1 to $a-1$; $a+1$ to $b-1$	(5)
8	W,x } junction calculations	a and b	(9)
9	$\delta_{1,a}$ to $\delta_{6,a},\delta_{1,b}$ to $\delta_{6,b}$ } junction calculations		(10)
10	$\delta_{1,c}$ to $\delta_{6,c}$ } junction calculations		(12)
11	W,x } junction calculations	c	(10)
12	P,q } junction calculations	$c-1$	(13)
13	P,q	c to $d-1$	(5)
14	$(rH),\beta$	d to c	(6),(4)
15	$(rH),\beta$ — junction calculations	b and a	(7),(10)
16	$(rH),\beta$	$b-1$ to $a+1$; $a-1$ to 1	(4)
17	π_3,π_4	All	(8)
18	$\hat{\epsilon}_\theta,\hat{\epsilon}_\xi,\kappa_\theta,\kappa_\xi$	All	(8)
19	$\epsilon_\theta,\epsilon_\xi,\epsilon_\zeta,\epsilon_{et}$	All	(14)
20	$\epsilon_{ep},\epsilon_{\theta p},\epsilon_{\xi p}$	All	(15)
21	π_1,π_2 Repeat steps 4 to 19 until convergence of $\epsilon_{\theta p}$ and $\epsilon_{\xi p}$	2 to a; $a+2$ to b; c to $d-1$	(1)
22	$\sigma_\theta,\sigma_\xi,\sigma_e$	All	(1),(16)

Although no station $j = c - 1$ actually exists in shell c, it is convenient to assume its fictitious presence and then to use Eqs. (5) to calculate the elements of P_c, q_c, and the remaining coefficient matrices in the structure. The terms (rH) and β are then calculated with Eqs. (6) and (4a) for shell c, Eqs. (7a, 10a, and 4a) for shell a, and Eqs. (7a, 10b, and 4a) for shell b. Stresses and strains throughout the structure may then be calculated from these basic shell variables as shown in the following section. Equations (9) and (13) are expanded in Ref. 6.

C. Elastic-Plastic Strain Equations

Derivation of the following elastic-plastic strain equations is given in detail in Ref. 3, in accordance with the von Mises-Hencky yield criterion and the deformation theory of plasticity. After reference surface strains and curvature changes have been computed throughout the structure by Eqs. (8), strains through the shell walls are calculated by the equations

$$\epsilon_\theta = \hat{\epsilon}_\theta + \zeta\kappa_\theta \qquad \epsilon_\xi = \hat{\epsilon}_\xi + \zeta\kappa_\xi \tag{14a}$$

Assuming $\sigma_\xi = 0$ and all volume changes are elastic,

$$\epsilon_\zeta = -\frac{\nu}{1-\nu}(\epsilon_\theta + \epsilon_\xi) + \frac{1+\nu}{1-\nu}\alpha T - \frac{1-2\nu}{1-\nu}(\epsilon_{\xi p} + \epsilon_{\theta p}) \tag{14b}$$

By definition,

$$\epsilon_{et} = \frac{2^{1/2}}{3}[(\epsilon_\xi - \epsilon_\theta)^2 + (\epsilon_\theta - \epsilon_\zeta)^2 + (\epsilon_\zeta - \epsilon_\xi)^2]^{1/2} \tag{14c}$$

$$\epsilon_{ep} = \epsilon_u - \frac{\sigma_u}{E} = \begin{Bmatrix} 0 & \sigma_e \leq \sigma \text{ yield} \\ \epsilon_{et} - \dfrac{2(1+\nu)}{3}\dfrac{\sigma_e}{E} & \sigma_e > \sigma \text{ yield} \end{Bmatrix} \tag{15a}$$

in which the subscript u indicates the results of a uniaxial tensile test. A plot of ϵ_{ep} vs ϵ_{et} may be easily constructed using Eqs. (15a). Finally, from the deformation theory of plasticity,

$$\epsilon_{\theta p} = \frac{\epsilon_{ep}}{3\epsilon_{et}}(2\epsilon_\theta - \epsilon_\xi - \epsilon_\zeta) \qquad \epsilon_{\xi p} = \frac{\epsilon_{ep}}{3\epsilon_{et}}(2\epsilon_\xi - \epsilon_\theta - \epsilon_\zeta) \tag{15b}$$

Plastic strains computed by Eqs. (15) are used to improve the level of approximation in the π terms of Eqs. (1c) and (8), establishing the iterative nature of this elastic-plastic solution. After convergence of these plastic strains, stresses are calculated throughout the structure by Eqs. (1d) and

$$\sigma_e = (\sigma_\theta^2 - \sigma_\theta\sigma_\xi + \sigma_\xi^2)^{1/2} \tag{16}$$

Results

A. Summary of Numerical Analysis Procedure

The method of analysis developed in this report is summarized in Table 1. This table serves as a convenient flow chart for computer programming of the method. The calculation procedure is as follows:

1) Express boundary conditions in finite difference notation and determine elements of matrices J, L, and R from Eqs. (3).

2) Calculate geometry terms ($\Gamma,\Theta,\Lambda,\Upsilon,\Phi$, and Ψ), pressure load terms (λ_1 to λ_4), and thermal load terms (τ_1 to τ_4) for all interior shell stations, using Eqs. (1) and (8). Assume convenient values (usually zero) for plasticity terms π_1 and π_2.

3) Calculate elements of matrices A, B, and F for all interior stations using Eqs. (2). Steps 1 to 3 do not have to be repeated during iteration that follows.

4) Compute elements of g matrix at all interior stations by Eqs. (2).

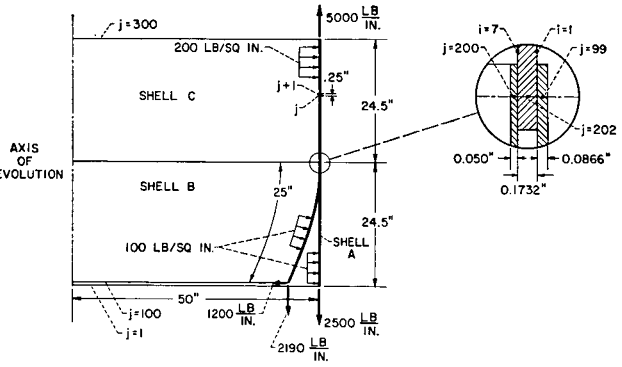
5) Calculate pressure, thermal, and plasticity terms required for strain and moment calculations at boundary and junction stations, using Eqs. (8).

6) Determine elements of remaining boundary condition matrices k, m, and t by Eqs. (3).

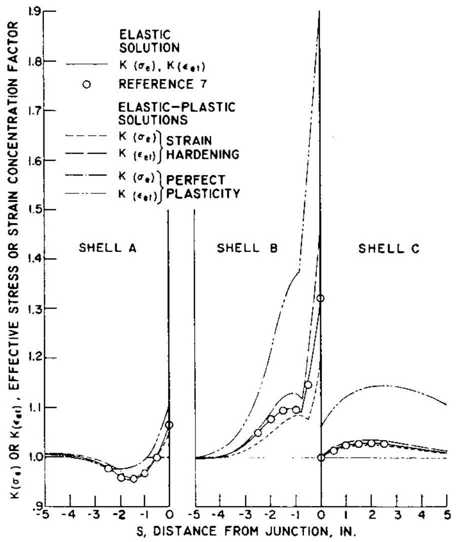
7) Calculate elements of coefficient matrices P and q for shells a and b, using Eqs. (5); P matrices for these two shells do not change during iteration.

8) To begin junction calculations, determine elements of W and x matrices at the junction stations in shells a and b, from Eqs. (9).

9) Calculate δ terms for shells a and b using Eqs. (10).



a) Geometry and loads.



b) Effective stress and strain concentration factors.

Fig. 3 Sample problem.

10) Calculate δ terms for shell c using Eqs. (12).

11) Calculate elements of W and x matrices at junction station in shell c by inverting Eqs. (10).

12) Complete junction calculations by computing elements of P and q matrices at fictitious station $j = c - 1$.

13) As a continuation of step 7, calculate elements of P and q matrices in shell c.

14) To begin the calculation of (rH) and β terms, compute their values in shell c, using Eqs. (6) and (4).

15) Calculate (rH) and β at the junction stations of shells a and b by Eqs. (7) and (10).

16) Continue step 14 in shells a and b.

17) Calculate plasticity terms π_3 and π_4 for all shell stations using Eqs. (8).

18) Calculate reference surface strains and curvature changes using Eqs. (8).

19) Calculate total strains throughout the structure by Eqs. (14).

20) Calculate plastic components of strain by Eqs. (15) and an equivalent plastic strain vs equivalent total strain diagram.

21) Compute plasticity terms π_1 and π_2 for all interior stations, using Eqs. (1) and the results of step 20. Repeat steps 4 to 19 until the plastic components of strain have converged sufficiently or for a fixed number of iterations.

22) Compute stresses throughout the structure, using Eqs. (1) and (16).

For an elastic shell structure, equate all plasticity terms to zero and proceed directly from step 19 to step 22.

B. Sample Problem

The geometry and loading of the sample problem structure are illustrated in Fig. 3. The structure consists of two cylinders and a portion of a sphere joined at a common section. Each shell has constant wall thickness and internal pressure. The following dimensions, loads, and material properties are selected so that yielding is incipient in the membrane regions of the structure: $\bar{r} = 50.0$ in., $h_a = 0.0866$ in., $h_b = 0.0500$ in., $h_c = 0.1732$ in., $E = 10^7$ psi, $p_a = 100$ psi, $p_b = 100$ psi, $p_c = 200$ psi, $\nu = 0.333$, and $\sigma_{\text{yield}} = 50{,}000$ psi.

Postyield material behavior is assumed to follow the relationship

$$\epsilon_{ep} \begin{cases} = C_1 + C_2\epsilon_{et} + C_3(\epsilon_{et})^2 & \epsilon_{et} > C_4 \\ = 0 & \epsilon_{et} \leq C_4 \end{cases}$$

The sample problem is solved for both a strain-hardening and a perfectly plastic material. For these two cases, the material coefficients become 1) for strain-hardening, $C_1 = 1.287 \times 10^{-3}$, $C_2 = -0.8655$, $C_3 = 129.5$, and $C_4 = 4.445 \times 10^{-3}$; and 2) for perfect plasticity, $C_1 = -4.445 \times 10^{-3}$, $C_2 = 1.00$, $C_3 = 0$, and $C_4 = 4.445 \times 10^{-3}$.

The shells are subdivided into meridional and transverse stations with $(\Delta s) = 0.25$ in., $a = 99$, $b = 200$, $c = 202$, $d = 300$, and $i_{\max} = 7$.

By referring to Fig. 2b and the junction detail of Fig. 3a, the external junction moment becomes

$$\bar{M} = 0.1116\ V_b - 0.1299\ V_a = -45.75 \text{ lb-in./in.}$$

Finally, if membrane deformation is assumed at all three shell boundaries, the boundary condition matrices are

$$J = L = R = k = t = 0$$

and

$$m = \begin{bmatrix} (rN_\xi \cos\varphi)_{a+1} \\ 0 \end{bmatrix} = \begin{bmatrix} 52{,}592 \text{ lb} \\ 0 \end{bmatrix}$$

The solution to this problem is presented in Fig. 3b in the form of effective stress and strain concentration factors. These factors are

$$K(\sigma_e) = \frac{(\sigma_e)_{\max}}{(\sigma_e)_{\text{membrane}}} = \left(\frac{\sigma_e}{50{,}000 \text{ psi}}\right)_{\max}$$

$$K(\epsilon_{et}) = \frac{(\epsilon_{et})_{\max}}{(\epsilon_{et})_{\text{membrane}}} = \left(\frac{\epsilon_{et}}{0.004445 \text{ in./in.}}\right)_{\max}$$

The variation of these concentration factors with meridional distance s, measured from the junction, is given for the three cases considered. Referring to Fig. 3a, negative values of s represent distances measured downward from the junction, in shells a and b.

In Fig. 3b, the solid lines represent the solution for a perfectly elastic material. Stress and strain concentration factors are equal with a maximum value of 1.31 occurring at the junction in shell b. Circles indicate the results of a standard elastic discontinuity solution that uses edge influence coefficients. The dashed lines present results for the strain-hardening material. Decreases in stress concentration are accompanied by corresponding increases in strain concentration, which reaches a maximum value of 1.44. Finally, the dashed-dotted lines show the elastic-perfectly plastic solution. This includes the upper limit of strain concentration and lower limit of stress concentration for various types of postyield behavior. Maximum effective strain concentration is 1.89, again occurring at the junction in the hemispherical bulkhead.

The elastic solution is clearly verified by the standard discontinuity analysis presented in Ref. 7. The two elastic-plastic solutions may be evaluated by determining the extent to which they satisfy the junction continuity conditions given in Eqs. (7). The quantities that are compared in these equations are presented in Table 2.

Table 2 Junction deformations and forces

	Strain-hardening solution			Perfectly plastic solution		
	Station			Station		
Quantity	*a*	*b*	*c*	*a*	*b*	*c*
β, rad	0.01439	0.01439	0.01439	0.01671	0.01671	0.01671
$\bar{\epsilon}_\theta$, in./in.	0.004522	0.004573	0.004540	0.004795	0.004819	0.004814
(rH), in.-lb/in.	−724.42	536.27	−188.15	−404.45	77.91	−326.54
M, lb-in./in.	16.97	−5.09	−34.30	12.22	−0.93	−31.75

The meridional couples M_ξ were determined from the meridional stresses σ_ξ by numerical integration. With these quantities, it can be shown that Eqs. (7) are satisfied very closely for both cases.

For further verification, the present method was applied to a continuous shell problem for which an elastic-plastic solution had already been obtained. This problem, which was presented in detail in Ref. 3, consisted of the deformation of a uniform cylindrical shell by an axial thermal gradient. Excellent agreement between the results of the two methods was found, which shows that the numerical analysis presented in this report is equally applicable to continuous and discontinuous shell regions.

Conclusions

A numerical method has been presented for solving problems involving the axisymmetric elastic-plastic deformation of general shells of revolution which may contain discontinuities in geometry, material properties, or loads. The method was used to determine effective stress and strain concentration factors at the common junction of three dissimilar shells in the elastic-plastic regime. The following conclusions were found:

1) This method produced an elastic solution that agreed very well with that obtained by a standard discontinuity analysis.

2) By using successive approximations, the method produced a convergent elastic-plastic solution.

3) In this elastic-plastic solution, conditions of equilibrium and compatibility at the discontinuity were satisfied.

4) When applied to the analysis of a continuous shell region, close agreement with known elastic-plastic results was found.

5) Elastic-plastic shells with discontinuities may be analyzed with almost the same directness and ease as completely elastic continuous shells.

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