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GEORGE C. MARSHALL SPACE FLIGHT CENTER

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A DERIVATION OF THE THREE DIMENSIONAL THREE DEGREE OF FREEDOM RE-ENTRY EQUATIONS OF MOTION IN TERRESTRIAL NAVIGATIONAL COORDINATES WITH SUBSEQUENT RESOLUTION INTO INERTIAL PLATFORM COORDINATES

By

Carl L. Colwell 30 Aug. 1963 30 p refs

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ABSTRACT

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The three dimensional three degree of freedom re-entry equations of motion and the input requirements for their solution on the IBM 7090 digital computer are derived. The material presented is the first formulations required to achieve eventually a hardware error analysis capability for three dimensional six degree of freedom closed loop re-entry guidance and control. The presentation consists of the derivation of the three dimensional three degree of freedom re-entry trajectory expressed in terms of terrestrial navigational coordinates. The derivation of the coordinate transformation equations required to express the solution of the above equations in terms of inertial platform coordinates is also presented.

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By Carl L. Colwell

PERFORMANCE ANALYSIS SECTION
GUIDANCE AND CONTROL SYSTEMS BRANCH
ASTRIONICS DIVISION

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DEFINITION OF SYMBOLS

SYMBOL	DEFINITION	UNITS
A_I	Space-fixed azimuth of initial space-fixed velocity	rad
A_W	Earth-referenced azimuth of earth-referenced wind velocity	rad
A_Z	Earth-referenced azimuth of earth-referenced aerodynamic velocity	rad
C_{D_o}	Aerodynamic drag coefficient for zero angle of attack	rad
C_D	Total aerodynamic drag coefficient	rad
C_L	Total aerodynamic lift coefficient	rad
D	Magnitude of the total aerodynamic drag force	kg
$\bar{F}_A$	Total aerodynamic force	kg
$\bar{F}_D$	Total aerodynamic drag force	kg
$\bar{F}_G$	Total gravitational force	kg
$\bar{F}_L$	Total aerodynamic lift force	kg
G	Universal gravitational constant	$m^4/kg\text{-}s^4$
G_r	Magnitude of the gravitational acceleration in the geocentric radial direction	m/s^2
G_λ	Magnitude of the gravitational acceleration in the direction of increasing earth longitude	m/s^2
G_{ϕ_c}	Magnitude of the gravitational acceleration in the direction of increasing geocentric latitude	m/s^2
J	Second order figure constant of the earth	rad
L	Magnitude of the total aerodynamic lift force	kg
M	Mass of the earth	$kg\text{-}s^2/m$
$\bar{R}$	Displacement from the center of the earth	m
S	Vehicle effective aerodynamic surface area	m^2

DEFINITION OF SYMBOLS (Cont'd)

SYMBOL	DEFINITION	UNITS
U	Gravitational potential of the earth	m^2/s^2
$\bar{V}_A$	Aerodynamic velocity	m/s
V_A	Magnitude of aerodynamic velocity	m/s
V_I	Magnitude of initial space-fixed velocity	m/s
V_W	Magnitude of earth-referenced wind velocity	m/s
a_D	Empirical aerodynamic drag coefficient caused by vehicle angle of attack	rad
a_L	Empirical aerodynamic lift coefficient caused by vehicle angle of attack	rad
a_r	Magnitude of space-fixed acceleration in instantaneous direction of increasing geocentric displacement	m/s^2
a_Λ	Magnitude of space-fixed acceleration in instantaneous direction of increasing space-fixed longitude	m/s^2
a_{ϕ_c}	Magnitude of space-fixed acceleration in instantaneous direction of increasing geocentric latitude	m/s^2
b_D	Empirical aerodynamic drag coefficient as a function of angle of attack squared	rad
b_L	Empirical aerodynamic lift coefficient as a function of angle of attack squared	rad
h_o	Initial geocentric altitude	m
h	Geocentric altitude	m
j	Inclination of the initial orbit plane	rad
m	Vehicle mass	$kg\text{-}s^2/m$
q	Dynamic pressure	kg/m^2
r_o	Magnitude of initial radial displacement from the center of the earth	m

DEFINITION OF SYMBOLS (Cont'd)

SYMBOL	DEFINITION	UNITS
r	Magnitude of radial displacement from the center of the earth	m
r_E	Equatorial earth radius	m
r_p	Polar earth radius	m
$\dot{r}_o$	Magnitude of initial radial velocity from the center of the earth	m/s
$\dot{r}$	Magnitude of radial velocity from the center of the earth	m/s
$\ddot{r}$	Magnitude of radial acceleration from the center of the earth	m/s^2
t	Time	s
$\bar{v}_{a_E}$	Space-fixed velocity of air molecule in a stationary atmosphere	m/s
$\bar{v}_{a_w}$	Space-fixed velocity of air molecule in flowing atmosphere	m/s
v_r	Magnitude of space-fixed velocity in the instantaneous direction of increasing geocentric displacement	m/s
v_Λ	Magnitude of space-fixed velocity in the instantaneous direction of increasing space-fixed longitude	m/s
v_{ϕ_c}	Magnitude of space-fixed velocity in the instantaneous direction of increasing geocentric latitude	m/s
x	Reference space-fixed earth-centered cartesian coordinate in equatorial plane	m
x_1	Earth-centered cartesian coordinate generated by first Euler rotation	m
x_2	Earth-centered cartesian coordinate generated by second Euler rotation	m
y	Reference space-fixed earth-centered cartesian coordinate in equatorial plane	m

DEFINITION OF SYMBOLS (Cont'd)

SYMBOLS	DEFINITION	UNITS
y_1	Earth-centered cartesian coordinate generated by first Euler rotation	m
y_2	Earth-centered cartesian coordinate generated by second Euler rotation	m
z	Reference space-fixed earth-centered cartesian coordinate in equatorial plane	m
z_1	Earth-centered cartesian coordinate generated by first Euler rotation	m
z_2	Earth-centered cartesian coordinate generated by second Euler rotation	m
∇	"Del" vector operator, or "Nabla"	1/m
Δt	Numerical integration time step	s
$(\Delta t)_p$	Data printout time step interval	s
Λ	Space-fixed longitude	rad
$\dot{\Lambda}$	Space-fixed longitudinal angular velocity	rad/s
$\ddot{\Lambda}$	Space-fixed longitudinal angular acceleration	rad/s^2
$\bar{\Omega}$	Space-fixed total angular velocity	rad/s
α	Angle of attack	rad
γ	Bank angle: defined as the angle between the aerodynamic lift vector and the vertical geocentric plane containing the aerodynamic velocity vector	rad
ζ	Space-fixed displacement in the initial cross range direction	m
η	Space-fixed displacement in the direction of the initial geocentric altitude	m
η_E	Displacement of sea level surface from the center of the earth in the direction of the initial geocentric altitude	m

DEFINITION OF SYMBOLS (Cont'd)

SYMBOL	DEFINITION	UNITS
η_p	Displacement from sea level surface in the direction of the initial geocentric altitude	m
θ_A	Earth-referenced dive angle of aerodynamic velocity	rad
θ_I	Space-fixed dive angle of initial space-fixed velocity	rad
λ_o	Initial earth longitude	rad
λ	Earth longitude	rad
$\dot{\lambda}_o$	Initial earth longitudinal angular velocity	rad/s
$\dot{\lambda}$	Earth longitudinal angular velocity	rad/s
$\ddot{\lambda}$	Earth longitudinal angular acceleration	rad/s^2
ξ	Space-fixed displacement in initial downrange direction	m
ρ	Atmospheric mass density	kg-s^2/m^4
ϕ_{c_o}	Initial geocentric latitude	rad
ϕ_c	Geocentric latitude	rad
$\dot{\phi}_{c_o}$	Initial geocentric latitudinal angular velocity	rad/s
$\dot{\phi}_c$	Geocentric latitudinal angular velocity	rad/s
$\ddot{\phi}_c$	Geocentric latitudinal angular acceleration	rad/s^2
ω	Angular velocity of earth rotation	rad/s

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SUMMARY

The three dimensional three degree of freedom re-entry equations of motion and input requirements for their solution on the IBM 7090 digital computer are derived. The material presented is the first formulations required to achieve eventually a hardware error analysis capability for three dimensional six degree of freedom closed loop re-entry guidance and control. The presentation consists of the derivation of the three simultaneous differential equations of motion required to solve for the three dimensional three degree of freedom re-entry trajectory expressed in terms of terrestrial navigational coordinates. The derivation of the coordinate transformation equations required to express the solution of the above equations in terms of inertial platform coordinates is also presented.

SECTION I. INTRODUCTION

The problem of hardware error analysis requires that accurate simulation models of various types of mission trajectories be available. These trajectories are used to determine the effects of various instrumental errors on the overall accuracy of the guidance systems.

The general approach to obtaining three degree of freedom equations of motion for a three-dimensional re-entry trajectory is a treatment of classical particle dynamics utilizing vector kinematics for the derivation of the space-fixed acceleration. The dynamical model of the re-entry vehicle is a point mass subjected to the forces of the gravitational field of the earth and aerodynamic lift and drag.

First, the space-fixed acceleration, expressed in terms of a moving coordinate system, is derived by vector kinematics for direct substitution into Newton's Second Law. The gravitational and aerodynamic forces are then expressed in terms of that moving coordinate system and equated to the vehicle mass times the space-fixed acceleration. Subsequent cancellation of the unit vectors from the three component equations then yields the three scalar differential equations to be solved simultaneously for the three dimensional re-entry trajectory.

A judicious sequence of Euler angle rotations of the moving coordinate axes then transforms the trajectory parameters expressed in the moving coordinate system into equivalent parameters in a coordinate system oriented in the direction of an idealized inertial platform coordinate system.

SECTION II. DEFINITION OF COORDINATE SYSTEMS

Two rectangular coordinate systems and one spherical coordinate system are utilized in the treatment of the re-entry problem for which the center of the rotating earth is considered as being space-fixed. These three coordinate systems are illustrated in Figure 1.

The $\overline{1}_x$, $\overline{1}_y$, $\overline{1}_z$ unit vectors represent a right-handed space-fixed rectangular coordinate system which has its origin at the center of the earth. The x and y axes are oriented in the equatorial plane with the x axis oriented in a reference space-fixed meridian plane, and the z axis collinear with the North Pole. This coordinate system serves only as a visual aid in deriving the space-fixed acceleration and is not intended to convey any other physical significance.

The $\overline{1}_{\phi_c}$, $\overline{1}_r$, $\overline{1}_\Lambda$ unit vectors represent a rotating spherical coordinate system which also has its origin at the center of the earth. The unit vectors are oriented in the right-handed sense and represent the instantaneous direction of increasing geocentric latitude, the instantaneous direction of increasing geocentric displacement, and the instantaneous direction of increasing space-fixed longitude, respectively. The three simultaneous differential equations of motion are derived in terms of this coordinate system.

The $\overline{1}_\xi$, $\overline{1}_\eta$, and $\overline{1}_\zeta$ unit vectors represent a space-fixed coordinate system which has its origin on the surface of the earth directly beneath the initial re-entry point. The unit vectors are oriented in the right-handed sense and represent the initial downrange direction, initial geocentric altitude direction, and initial cross range direction, respectively. The solution of the equations of motion is resolved into this coordinate system.

SECTION III. DERIVATION OF SPACE-FIXED ACCELERATIONS

To derive the space-fixed accelerations in terms of the rotating coordinate system $\overline{1}_{\phi_c}$, $\overline{1}_r$, $\overline{1}_\Lambda$, it is convenient to employ the vector differential operator on $\overline{R}$:

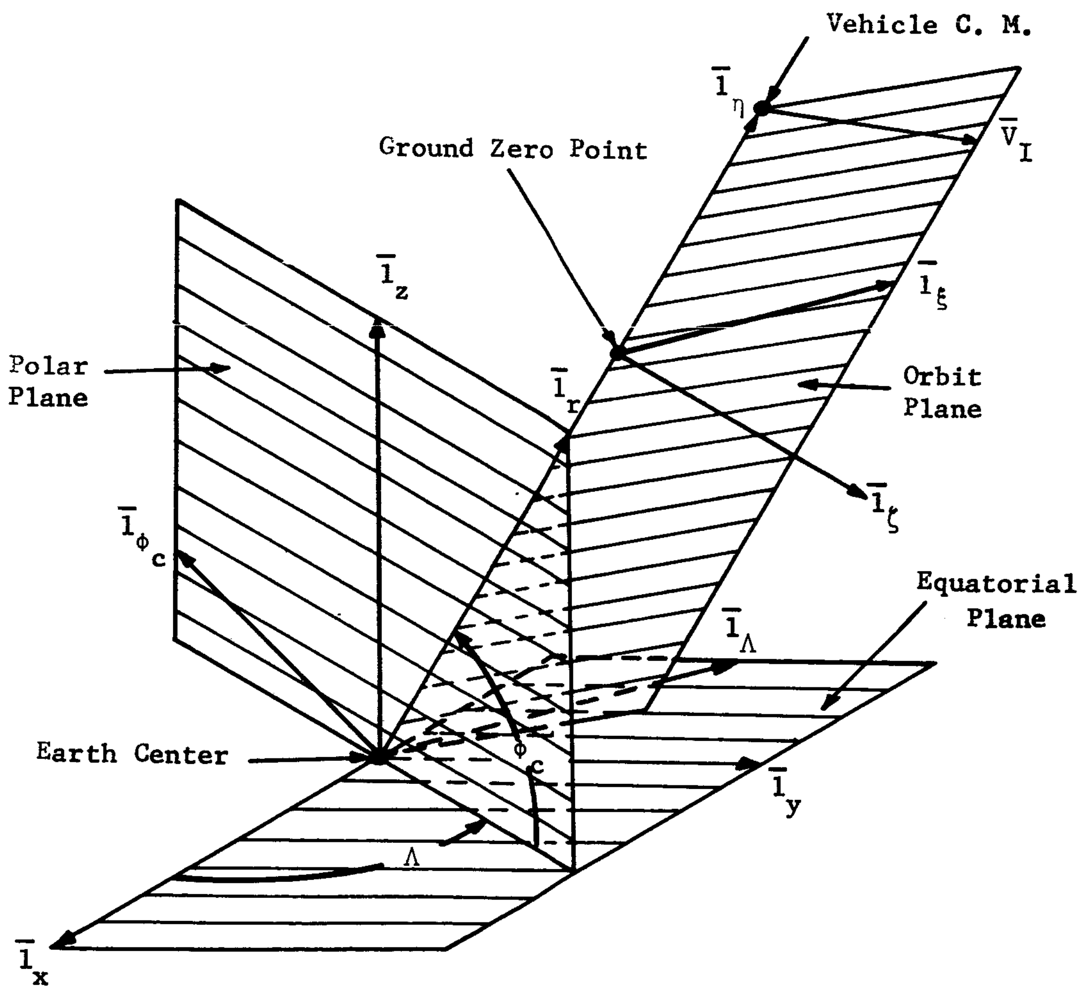


FIGURE 1. RE-ENTRY COORDINATE SYSTEMS

$$\left.\frac{d}{dt}\right)_s \bar{R} = \frac{d}{dt}\bar{R} + \bar{\Omega} \times \bar{R} \tag{1}$$

where the first term is the derivative of the geocentric displacement relative to the space-fixed coordinate system $\bar{1}_x$, $\bar{1}_y$, $\bar{1}_z$, the second term is the derivative of the geocentric displacement relative to the rotating coordinate system $\bar{1}_{\phi_c}$, $\bar{1}_r$, $\bar{1}_\Lambda$, and the third term is the angular velocity vector of the rotating coordinate system relative to the space-fixed coordinate system.

Referring to Figure 1, it is apparent that the total angular velocity vector may be resolved into two component vectors representing angular velocity in the equatorial plane ($\dot{\Lambda}$) and angular velocity in the polar plane ($\dot{\phi}_c$):

$$\bar{\Omega} = \dot{\Lambda}\,\bar{1}z + \dot{\phi}_c\,(-\,\bar{1}_\Lambda) \tag{2}$$

Resolving the vector representing the angular velocity in the equatorial plane still further in the polar plane:

$$\bar{\Omega} = \dot{\Lambda}\cos\phi_c\,\bar{1}_{\phi_c} + \dot{\Lambda}\sin\phi_c\,\bar{1}_r - \dot{\phi}_c\,\bar{1}_\Lambda \tag{3}$$

Substituting equation 3 into equation 1 and expressing $\bar{R}$ as $r\bar{1}_r$ yield the familiar space-fixed velocity:

$$\left.\frac{d}{dt}\right)_s \bar{R} = \frac{d}{dt}(r\bar{1}_r) + (\dot{\Lambda}\cos\phi_c\,\bar{1}_{\phi_c} + \dot{\Lambda}\sin\phi_c\,\bar{1}_r - \dot{\phi}_c\,\bar{1}_\Lambda)$$

$$\times\,(r\bar{1}_r) = \dot{r}\,\bar{1}_r + r\,\dot{\Lambda}\cos\phi_c\,\bar{1}_\Lambda + r\,\dot{\phi}_c\,\bar{1}_{\phi_c} \tag{4}$$

Thus, the space-fixed component velocities are:

$$v_{\phi_c} = r\dot{\phi}_c \tag{5}$$

$$v_r = \dot{r} \tag{6}$$

$$v_\Lambda = r\,\dot{\Lambda}\cos\phi_c \tag{7}$$

A substitution of equation 4 for $\bar{R}$ in equation 1 yields the desired space-fixed accelerations.

$$\left.\frac{d^2}{dt^2}\right)_s \bar{R} = \left.\frac{d}{dt}\right)\left.\frac{d}{dt}\right)_s \bar{R} + \bar{\Omega} \times \left.\frac{d}{dt}\right)_s \bar{R}$$

$$= \left.\frac{d}{dt}\right) (r\ \dot{\phi}_c\ \bar{1}_{\phi_c} + \dot{r}\ \bar{1}_r + r\ \dot{\Lambda}\cos\phi_c\ \bar{1}_\Lambda)$$

$$+ (\dot{\Lambda}\cos\phi_c\ \bar{1}_{\phi_c} + \dot{\Lambda}\sin\phi_c\ \bar{1}_r - \dot{\phi}_c\ \bar{1}_\Lambda)$$

$$\times (r\ \dot{\phi}_c\ \bar{1}_{\phi_c} + \dot{r}\ \bar{1}_r + r\ \dot{\Lambda}\cos\phi_c\ \bar{1}_\Lambda)$$

$$= \dot{r}\ \dot{\phi}_c\ \bar{1}_{\phi_c} + r\ \ddot{\phi}_c\ \bar{1}_{\phi_c} + \ddot{r}\ \bar{1}_r + \dot{r}\ \dot{\Lambda}\cos\phi_c\ \bar{1}_\Lambda$$

$$+ r\ \ddot{\Lambda}\cos\phi_c\ \bar{1}_\Lambda - r\ \dot{\phi}_c\ \dot{\Lambda}\sin\phi_c\ \bar{1}_\Lambda + \dot{r}\ \dot{\Lambda}\cos\phi_c\ \bar{1}_\Lambda$$

$$- r\ \dot{\Lambda}^2\cos^2\phi_c\ \bar{1}_r - r\ \dot{\phi}_c\dot{\Lambda}\sin\phi_c\ \bar{1}_\Lambda$$

$$+ r\ \dot{\Lambda}^2\sin\phi_c\cos\phi_c\ \bar{1}_{\phi_c} - r\ \dot{\phi}_c^{\,2}\ \bar{1}_r + \dot{r}\ \dot{\phi}_c\bar{1}_{\phi_c}$$

$$= (r\ \ddot{\phi}_c + 2\dot{r}\ \dot{\phi}_c + r\ \dot{\Lambda}^2\sin\phi_c\cos\phi_c)\ \bar{1}_{\phi_c}$$

$$+ (\ddot{r} - r\ \dot{\phi}_c^{\,2} - r\ \dot{\Lambda}^2\cos^2\phi_c)\ \bar{1}_r$$

$$+ (r\ \ddot{\Lambda}\cos\phi_c + 2\dot{r}\ \dot{\Lambda}\cos\phi_c - 2r\ \dot{\phi}_c\ \dot{\Lambda}\sin\phi_c)\ \bar{1}_\Lambda \tag{8}$$

Thus, the space-fixed component accelerations are:

$$a_{\phi_c} = r\ \ddot{\phi}_c + 2\dot{r}\ \dot{\phi}_c + r\ \dot{\Lambda}^2\sin\phi_c\cos\phi_c \tag{9}$$

$$a_r = \ddot{r} - r\ \dot{\phi}_c^{\,2} - r\ \dot{\Lambda}^2\cos^2\phi_c \tag{10}$$

$$a_\Lambda = r\ \ddot{\Lambda}\cos\phi_c + 2\dot{r}\ \dot{\Lambda}\cos\phi_c - 2r\ \dot{\phi}_c\ \dot{\Lambda}\sin\phi_c \tag{11}$$

It is convenient to make the substitution of earth longitude for space-fixed longitude so that the equations of motion may then be expressed in terms of geocentric latitude, earth longitude, and geocentric altitude, which are the usual terrestrial navigational parameters. This is accomplished by the angular velocity substitution:

$$\dot{\Lambda} = \dot{\lambda} + \omega \tag{12}$$

where λ is the earth longitude and ω is the angular velocity of the earth. (Note that the new rotating coordinate unit vector $\overline{1}_\lambda$ is collinear with the old rotating coordinate unit vector $\overline{1}_\Lambda$.) The space-fixed velocity and space-fixed acceleration then become:

$$\left.\frac{d}{dt}\right)_s \overline{R} = r\, \dot{\phi}_c\, \overline{1}_{\phi_c} + \dot{r}\, \overline{1}_r + r\, (\dot{\lambda} + \omega) \cos \phi_c\, \overline{1}_\lambda \tag{13}$$

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right)_s \overline{R} = {} & [r\, \ddot{\phi}_c + 2\dot{r}\, \dot{\phi}_c + r\, (\dot{\lambda} + \omega)^2 \sin \phi_c \cos \phi_c]\, \overline{1}_{\phi_c} \\ & + [\ddot{r} - r\, \dot{\phi}_c^{\,2} - r\, (\dot{\lambda} + \omega)^2 \cos^2 \phi_c]\, \overline{1}_r \\ & + [r\, \ddot{\lambda} \cos \phi_c + 2\dot{r}\, (\dot{\lambda} + \omega) \cos \phi_c \\ & - 2r\, \dot{\phi}_c\, (\dot{\lambda} + \omega) \sin \phi_c]\, \overline{1}_\lambda \end{aligned} \tag{14}$$

This acceleration equation is now of the form which can be substituted into Newton's Second Law in order to obtain the equations of motion for a re-entry trajectory.

SECTION IV. DERIVATION OF GRAVITATIONAL FORCES

The gravitational potential external to the surface of an oblate earth is assumed (Ref. 1) to be:

$$U = -\frac{GM}{r}\left[1 + \frac{J}{3}\left(\frac{r_E}{r}\right)^2 (1 - 3 \sin^2 \phi_c)\right] \tag{15}$$

where G is the universal gravitational constant, M is the mass of the earth, r is the geocentric displacement of the point in consideration, J is the second order figure constant for the earth, r_E is the equatorial radius of the earth, and ϕ_c is the geocentric latitude.

The gradient vector operator in spherical coordinates is:

$$\nabla = \frac{1}{r}\frac{\partial}{\partial \phi_c}\overline{1}_{\phi_c} + \frac{\partial}{\partial r}\overline{1}_r + \frac{1}{r \cos \phi_c}\frac{\partial}{\partial \lambda}\overline{1}_\lambda \tag{16}$$

The gravitational force per unit mass may be obtained by taking the negative of the gradient of the gravitational potential:

$$\frac{\bar{F}_G}{m} = - \nabla U$$

$$= - \frac{1}{r} \frac{\partial}{\partial \phi_c} \left\{ - \frac{GM}{r} \left[1 + \frac{J}{3} \left(\frac{r_E}{r} \right)^2 (1 - 3 \sin^2 \phi_c) \right] \right\} \bar{1}_{\phi_c}$$

$$- \frac{\partial}{\partial r} \left\{ - \frac{GM}{r} \left[1 + \frac{J}{3} \left(\frac{r_E}{r} \right)^2 (1 - 3 \sin^2 \phi_c \right] \right\} \bar{1}_r$$

$$- \frac{1}{r \cos \phi_c} \frac{\partial}{\partial \lambda} \left\{ - \frac{GM}{r} \left[1 + \frac{J}{3} \left(\frac{r_E}{r} \right)^2 (1 - 3 \sin^2 \phi_c) \right] \right\} \bar{1}_\lambda$$

$$= - \frac{GM}{r^2} J \left(\frac{r_E}{r} \right)^2 \sin 2\phi_c \, \bar{1}_{\phi_c}$$

$$- \frac{GM}{r^2} \left[1 + J \left(\frac{r_E}{r} \right)^2 (1 - 3 \sin^2 \phi_c) \right] \bar{1}_r - 0 \, \bar{1}_\lambda \tag{17}$$

Thus, the gravitational components of acceleration for the earth are:

$$G_{\phi_c} = - \frac{GM}{r^2} J \left(\frac{r_E}{r} \right)^2 \sin 2\phi_c \tag{18}$$

$$G_r = - \frac{GM}{r^2} \left[1 + J \left(\frac{r_E}{r} \right)^2 (1 - 3 \sin^2 \phi_c) \right] \tag{19}$$

$$G_\lambda = 0 \tag{20}$$

These components are the force per unit mass terms caused by the gravitational acceleration. These terms are to be used in Newton's Second Law.

SECTION V. DERIVATION OF AERODYNAMIC FORCES

For the derivation of the aerodynamic forces on the re-entry vehicle, it is necessary to depart from the particle dynamics approach long enough to adopt a rigid body model of the vehicle by which aerodynamic parameters may be defined. The vehicle is symbolically represented by a blunted conical frustum having a sector of a sphere as its blunt base. The vehicle model has an auxiliary, controlled, moveable lateral surface by which an angle of attack with respect to the vehicle symmetry axis may be generated in the vehicle symmetry plane. The vehicle rigid body dynamics are assumed not to be coupled with the exterior ballistics of the vehicle center of mass, however, and the aerodynamic lift and drag forces are considered as acting through the vehicle center of mass exclusively. Then, the problem remains essentially one of particle dynamics, with the concept of a rigid body vehicle introduced only for the purpose of defining the direction of the lift and drag forces with respect to the aerodynamic velocity vector.

The aerodynamic velocity is the velocity of the vehicle center of mass with respect to the ambient atmosphere. The earth's atmosphere is assumed to be stationary with respect to the rotating earth, with any variations being taken into consideration as wind effects. The aerodynamic velocity is computed as the relative velocity of the vehicle center of mass to a coincident air molecule point mass. The space-fixed velocity of the air particle in the stationary atmosphere is assumed to be:

$$\bar{\bar{V}}_{A_E} = 0\ \bar{1}_{\phi_c} + 0\ \bar{1}_r + r\ \omega \cos \phi_c\ \bar{1}_\lambda \tag{21}$$

Assuming that the wind effects of the air mass are restricted to a mass flow which is laminar with respect to geocentric altitude, the velocity of the air particle then becomes:

$$\bar{\bar{V}}_{A_W} = V_W \cos A_W\ \bar{1}_{\phi_c} + 0\ \bar{1}_r + (r\ \omega \cos \phi_c + V_W \sin A_W)\ \bar{1}_\lambda \tag{22}$$

where V_W and A_W are the speed and azimuth, respectively, of the air flow with respect to stationary air at a geocentric displacement r. The aerodynamic velocity is then obtained by taking the difference of equation 22 from equation 13 to yield the relative velocity:

$$\begin{aligned} \bar{V}_A = {} & (r\ \dot{\phi}_c - V_W \cos A_W)\ \bar{1}_{\phi_c} + \dot{r}\ \bar{1}_r \\ & + (r\ \dot{\lambda} \cos \phi_c - V_W \sin A_W)\ \bar{1}_\lambda \end{aligned} \tag{23}$$

For the resolution of the lift and drag vectors into the rotating coordinate system, it is convenient to define two reference angles to describe the orientation of the aerodynamic velocity vector in that coordinate system. Thus, the dive angle of the aerodynamic velocity vector is defined to be:

$$\theta_A = \cos^{-1}\left\{\frac{\dot{r}}{V_A}\right\} \tag{24}$$

and the azimuth of the aerodynamic velocity vector is defined to be:

$$A_Z = \tan^{-1}\left\{\frac{r\,\dot{\lambda}\cos\phi_c - V_W \sin A_W}{r\,\dot{\phi}_c - V_W \cos A_W}\right\} \tag{25}$$

The aerodynamic drag vector is defined to be collinear with the aerodynamic velocity vector but oppositely directed. The aerodynamic lift vector is defined to be normal to the aerodynamic velocity vector, oriented in the vehicle symmetry plane, and directed oppositely from the vehicle's auxiliary lateral surface.

The resolution of the aerodynamic lift and drag vectors into the rotating coordinate axes is illustrated in Figure 2. The bottom view represents the local geocentric vertical plane which contains the aerodynamic velocity vector, and the top view represents the local geocentric horizontal plane. It should be clarified here that the bank angle γ is defined to be the angle between the lift vector and the vertical geocentric plane containing the aerodynamic velocity vector. The lift and drag resolutions are respectively:

$$\begin{aligned}\overline{F}_L &= L\,(-\sin\gamma\sin A_Z - \cos\gamma\cos\theta_A\cos A_Z)\,\overline{1}_{\phi_c} \\ &+ L\,(\cos\gamma\sin\theta_A)\,\overline{1}_r \\ &+ L\,(\sin\gamma\cos A_Z - \cos\gamma\cos\theta_A\sin A_Z)\,\overline{1}_\lambda\end{aligned} \tag{26}$$

$$\begin{aligned}\overline{F}_D &= D\,(-\sin\theta_A\cos A_Z)\,\overline{1}_{\phi_c} + D\,(-\cos\theta_A)\,\overline{1}_r \\ &+ D\,(-\sin\theta_A\sin A_Z)\,\overline{1}_\lambda\end{aligned} \tag{27}$$

With the lift and drag directions resolved into the rotating coordinate system axes, there remains the magnitudes of those aerodynamic forces which are respectively:

$$L = q\,S\,C_L = \frac{1}{2}\,\rho\,V_A^2\,S\,(a_L\,\alpha + b_L\,\alpha^2) \tag{28}$$

$$D = q\,S\,C_D = \frac{1}{2}\,\rho\,V_A^2\,S\,(C_{D_o} + a_D\,\alpha + b_D\,\alpha^2) \tag{29}$$

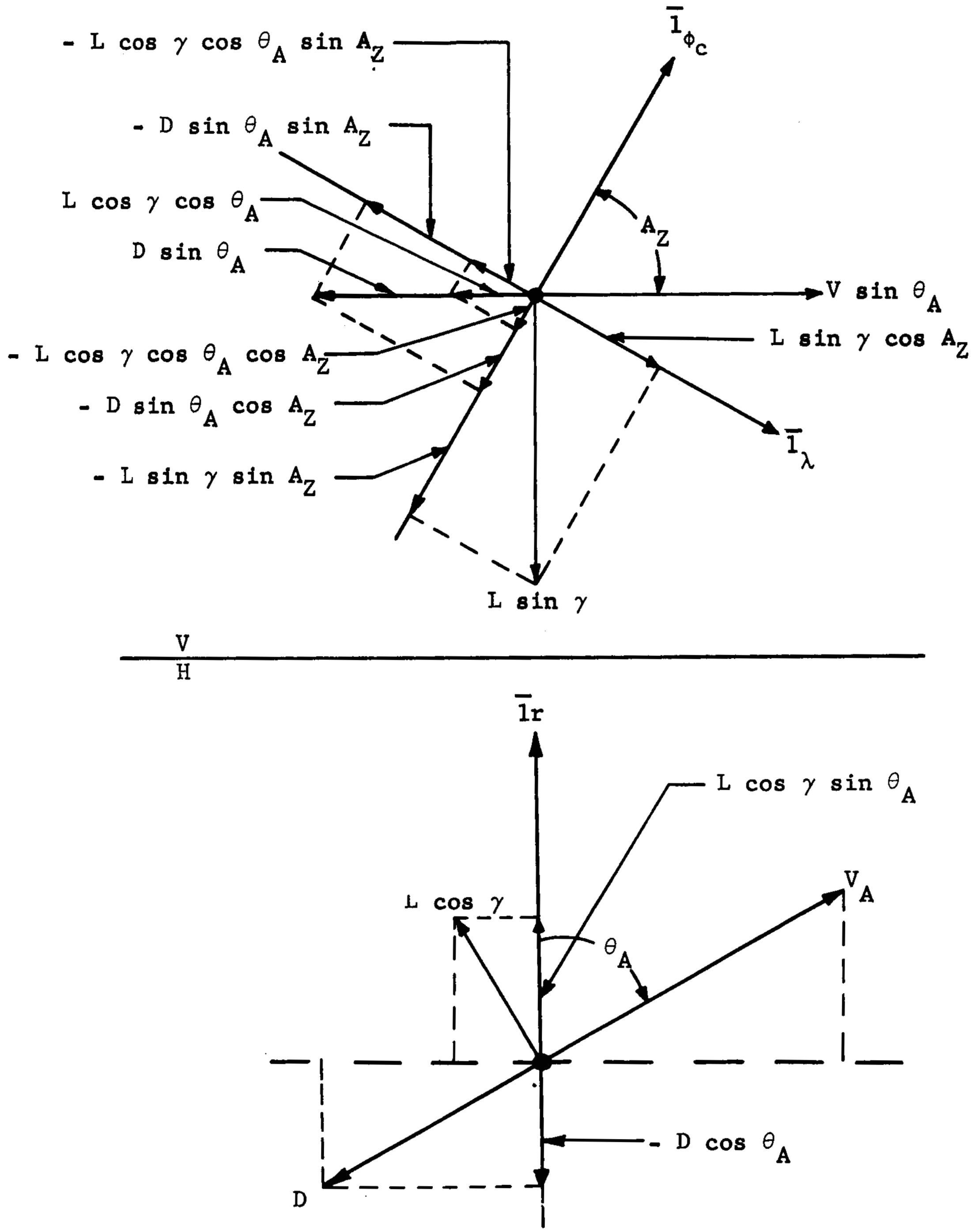


FIGURE 2. AERODYNAMIC FORCE RESOLUTION

where ρ is the atmospheric mass density, S is the vehicle total effective aerodynamic surface area, and a_L, b_L, C_{D_o}, a_D, and b_D are empirical coefficients which determine the total lift and drag coefficients C_L and C_D, respectively, as a function of aerodynamic angle of attack and Mach number.

Equations 28 and 29 yield the aerodynamic forces to be used in Newton's Second Law for the vehicle equations of motion. The aerodynamic force per vehicle mass then becomes:

$$\frac{\overline{F}_A}{m} = \frac{\rho}{2}\frac{V_A^2 S}{m} \{[-\sin\gamma \sin A_Z - \cos\gamma \cos\theta_A \cos A_Z](a_L \alpha + b_L \alpha^2)$$

$$+ [-\sin\theta_A \cos A_Z]\ (C_{D_o} + a_D \alpha + b_D \alpha^2)\}\ \overline{1}_{\phi_c}$$

$$+ \frac{\rho}{2}\frac{V_A^2 S}{m} \{[\cos\gamma \sin\theta_A]\ (a_L \alpha + b_L \alpha^2)$$

$$+ [-\cos\theta_A]\ (C_{D_o} + a_D \alpha + b_D \alpha^2)\}\ \overline{1}_r$$

$$+ \frac{\rho}{2}\frac{V_A^2 S}{m} \{[\sin\gamma \cos A_Z - \cos\gamma \cos\theta_A \sin A_Z](a_L \alpha + b_L \alpha^2)$$

$$+ [-\sin\theta_A \sin A_Z]\ (C_{D_o} + a_D \alpha + b_D \alpha^2)\}\ \overline{1}_\lambda \qquad (30)$$

SECTION VI. DERIVATION OF THE COORDINATE TRANSFORMATION EQUATIONS

During the course of the re-entry trajectory, the orientation of the rotating coordinate axis system in which the equations of motion are expressed is continuously changing with respect to fixed space. It is desired to express certain inertial and space-fixed vector quantities of the moving coordinate system in terms of the inertial platform coordinate system defined in Section II. This resolution may be accomplished by an appropriate sequence of Euler angle rotations of the moving coordinate axes and a subsequent coordinate translation of the origin of the rotated coordinate system from the center of the earth to the surface of the earth.

This Euler angle sequence is illustrated in Figure 3. The first Euler rotation is about the instantaneous $\bar{1}_\lambda$ axis through an angle equal to the instantaneous geocentric latitude. The resolution equations expressed in matrix form are:

$$\begin{bmatrix} \bar{1}_{x_1} \\ \bar{1}_{y_1} \\ \bar{1}_{z_1} \end{bmatrix} = \begin{bmatrix} \cos\phi_c & \sin\phi_c & 0 \\ -\sin\phi_c & \cos\phi_c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{1}_{\phi_c} \\ \bar{1}_r \\ \bar{1}_\lambda \end{bmatrix} \tag{31}$$

The second Euler rotation is about the newly generated $\bar{1}x_1$ axis through an angle equal to the space-fixed longitude. The resolution equations expressed in matrix form are:

$$\begin{bmatrix} \bar{1}_{x_2} \\ \bar{1}_{y_2} \\ \bar{1}_{z_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Lambda & -\sin\Lambda \\ 0 & \sin\Lambda & \cos\Lambda \end{bmatrix} \begin{bmatrix} \bar{1}_{x_1} \\ \bar{1}_{y_1} \\ \bar{1}_{z_1} \end{bmatrix} \tag{32}$$

The third Euler rotation is about the newly generated $\bar{1}z_2$ axis through an angle equal to the initial geocentric latitude. The resolution equations expressed in matrix form are:

$$\begin{bmatrix} \bar{1}_{\phi_{c_o}} \\ \bar{1}_{r_o} \\ \bar{1}_{\lambda_o} \end{bmatrix} = \begin{bmatrix} \cos\phi_{c_o} & -\sin\phi_{c_o} & 0 \\ \sin\phi_{c_o} & \cos\phi_{c_o} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{1}_{x_2} \\ \bar{1}_{y_2} \\ \bar{1}_{z_2} \end{bmatrix} \tag{33}$$

The fourth Euler rotation is about the initial $\bar{1}_r$ axis through an angle equal to the initial space-fixed azimuth. The resolution equations expressed in matrix form are:

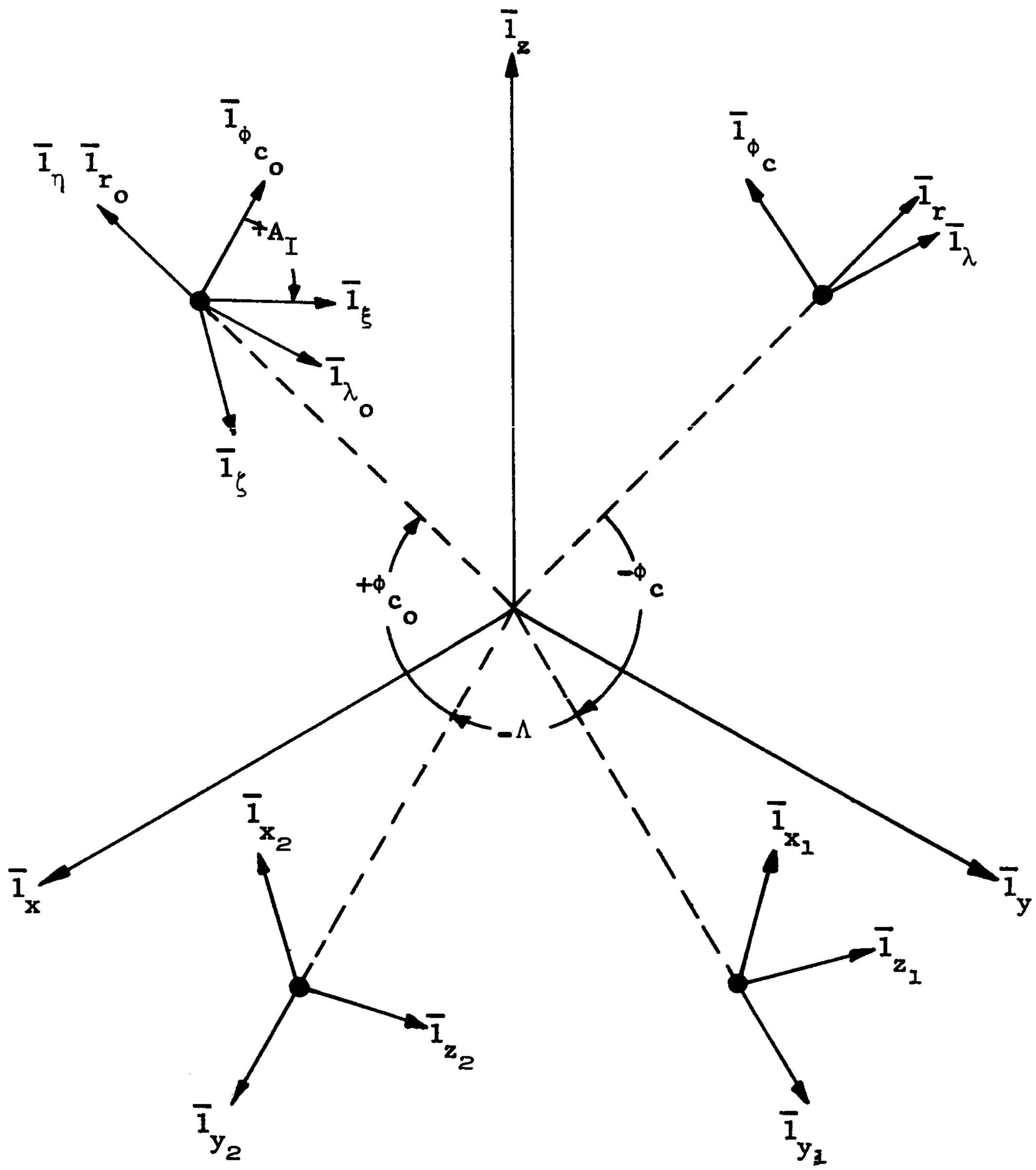


FIGURE 3. COORDINATE AXES EULER ROTATIONS

$$\begin{bmatrix} \bar{1}_\xi \\ \bar{1}_\eta \\ \bar{1}_\zeta \end{bmatrix} = \begin{bmatrix} \cos A_I & 0 & \sin A_I \\ 0 & 1 & 0 \\ -\sin A_I & 0 & \cos A_I \end{bmatrix} \begin{bmatrix} \bar{1}_{\phi_{c_o}} \\ \bar{1}_{r_o} \\ \bar{1}_{\lambda_o} \end{bmatrix} \tag{34}$$

The total vector resolution is then obtained by performing the appropriate matrix multiplications which yield:

$$\begin{bmatrix} \bar{1}_\xi \\ \bar{1}_\eta \\ \bar{1}_\zeta \end{bmatrix} = \begin{bmatrix} \begin{matrix} \cos A_I \cos\phi_{c_o} \cos\phi_c + \\ \cos A_I \sin\phi_{c_o} \cos\Lambda \sin\phi_c \\ -\sin A_I \sin\Lambda \sin\phi_c \end{matrix} & \begin{matrix} \cos A_I \cos\phi_{c_o} \sin\phi_c - \\ \cos A_I \sin\phi_{c_o} \cos\Lambda \cos\phi_c \\ +\sin A_I \sin\Lambda \cos\phi_c \end{matrix} & \begin{matrix} \cos A_I \sin\phi_{c_o} \sin\Lambda \\ +\sin A_I \cos\Lambda \end{matrix} \\[2em] \begin{matrix} \sin\phi_{c_o} \cos\phi_c \\ -\cos\phi_{c_o} \cos\Lambda \sin\phi_c \end{matrix} & \begin{matrix} \sin\phi_{c_o} \sin\phi_c + \\ \cos\phi_{c_o} \cos\Lambda \cos\phi_c \end{matrix} & -\cos\phi_{c_o} \sin\Lambda \\[2em] \begin{matrix} -\sin A_I \cos\phi_{c_o} \cos\phi_c + \\ \sin A_I \sin\phi_{c_o} \cos\Lambda \sin\phi_c \\ -\cos A_I \sin\Lambda \sin\phi_c \end{matrix} & \begin{matrix} -\sin A_I \cos\phi_{c_o} \sin\phi_c + \\ \sin A_I \sin\phi_{c_o} \cos\Lambda \cos\phi_c \\ +\cos A_I \sin\Lambda \cos\phi_c \end{matrix} & \begin{matrix} -\sin A_I \sin\phi_{c_o} \sin\Lambda \\ +\cos A_I \cos\Lambda \end{matrix} \end{bmatrix} \begin{bmatrix} \bar{1}_{\phi_c} \\ \bar{1}_r \\ \bar{1}_\lambda \end{bmatrix} \tag{35}$$

This combined matrix will then transform the space-fixed displacement, velocity, or acceleration vectors from the instantaneous space-fixed orientation of the moving coordinate system into the idealized inertial platform coordinate system orientation. A subsequent translation of the origin along the $\bar{1}_\eta$ axis then yields the final idealized inertial platform coordinate system, i.e.,

$$\eta_P = \eta_E - r_o \tag{36}$$

where η_P is the platform η, η_E is the resolved space-fixed η, and r_o is the radius of the earth at the re-entry point.

SECTION VII. ASSEMBLAGE OF THE FINAL EQUATIONS OF MOTION FOR NUMERICAL SOLUTION

The equations of motion may now be obtained by equating the results of equation 8 to the results of equation 17 plus equation 30. Cancellation of the unit vectors then yields the three simultaneous scalar differential equations:

$$r \ddot{\phi}_c + 2\dot{r} \dot{\phi}_c + r (\dot{\lambda} + \omega)^2 \sin \phi_c \cos \phi_c = - \frac{GM}{r^2} J \left(\frac{r_E}{r}\right)^2 \sin 2\phi_c$$

$$- \frac{L}{m} \sin \gamma \sin A_Z - \frac{L}{m} \cos \gamma \cos \theta_A \cos A_Z - \frac{D}{m} \sin \theta_A \cos A_Z \quad (37)$$

$$\ddot{r} - r \dot{\phi}_c^2 - r (\dot{\lambda} + \omega)^2 \cos^2 \phi_c = - \frac{GM}{r^2} \left[1 + J\left(\frac{r_E}{r}\right)^2 (1 - 3 \sin^2 \phi_c) \right]$$

$$+ \frac{L}{m} \cos \gamma \sin \theta_A - \frac{D}{m} \cos \theta_A \quad (38)$$

$$r \ddot{\lambda} \cos \phi_c + 2\dot{r} (\dot{\lambda} + \omega) \cos \phi_c - 2r \dot{\phi}_c (\dot{\lambda} + \omega) \sin \phi_c$$

$$= \frac{L}{m} \sin \gamma \cos A_Z - \frac{L}{m} \cos \gamma \cos \theta_A \sin A_Z$$

$$- \frac{D}{m} \sin \theta_A \sin A_Z \quad (39)$$

Solving each of the differential equations for the highest derivative then puts them in a form solvable by numerical integration on the IBM 7090 digital computer:

$$\ddot{\phi}_c = - \frac{2\dot{r} \dot{\phi}_c}{r} - (\dot{\lambda} + \omega)^2 \sin \phi_c \cos \phi_c - \frac{GM}{r^3} J \left(\frac{r_E}{r}\right)^2 \sin 2\phi_c$$

$$- \frac{1}{r} \left\{ \frac{L}{m} \sin \gamma \sin A_Z + \frac{L}{m} \cos \gamma \cos \theta_A \cos A_Z \right.$$

$$\left. + \frac{D}{m} \sin \theta_A \cos A_Z \right\} \quad (40)$$

$$\ddot{r} = r\,\dot{\phi}_c^{\,2} + r\,(\dot{\lambda} + \omega)^2 \cos^2 \phi_c$$

$$- \frac{GM}{r^2}\left[1 + J\left(\frac{r_E}{r}\right)^2 (1 - 3\sin^2 \phi_c)\right] + \frac{L}{m}\cos\gamma \sin\theta_A$$

$$- \frac{D}{m}\cos\theta_A \tag{41}$$

$$\ddot{\lambda} = - \frac{2\dot{r}\,(\dot{\lambda} + \omega)}{r} + 2\dot{\phi}_c\,(\dot{\lambda} + \omega)\tan\phi_c$$

$$+ \frac{1}{r\cos\phi_c}\left\{\frac{L}{m}\sin\gamma\cos A_Z - \frac{L}{m}\cos\gamma\cos\theta_A \sin A_Z\right.$$

$$\left. - \frac{D}{m}\sin\theta_A \sin A_Z\right\} \tag{42}$$

To solve this set of differential equations, it is necessary to provide six initial conditions. Three of those initial conditions may be specified by considering the initial position of the vehicle. The initial geocentric latitude and initial earth longitude may be specified directly. The displacement may be derived by specifying the initial geocentric altitude and applying the formula derived in Figure 4.

$$r_o = \frac{r_P}{\sqrt{\left(\frac{r_P}{r_E}\right)^2 \cos^2\phi_c + \sin^2\phi_c}} + h_o \tag{43}$$

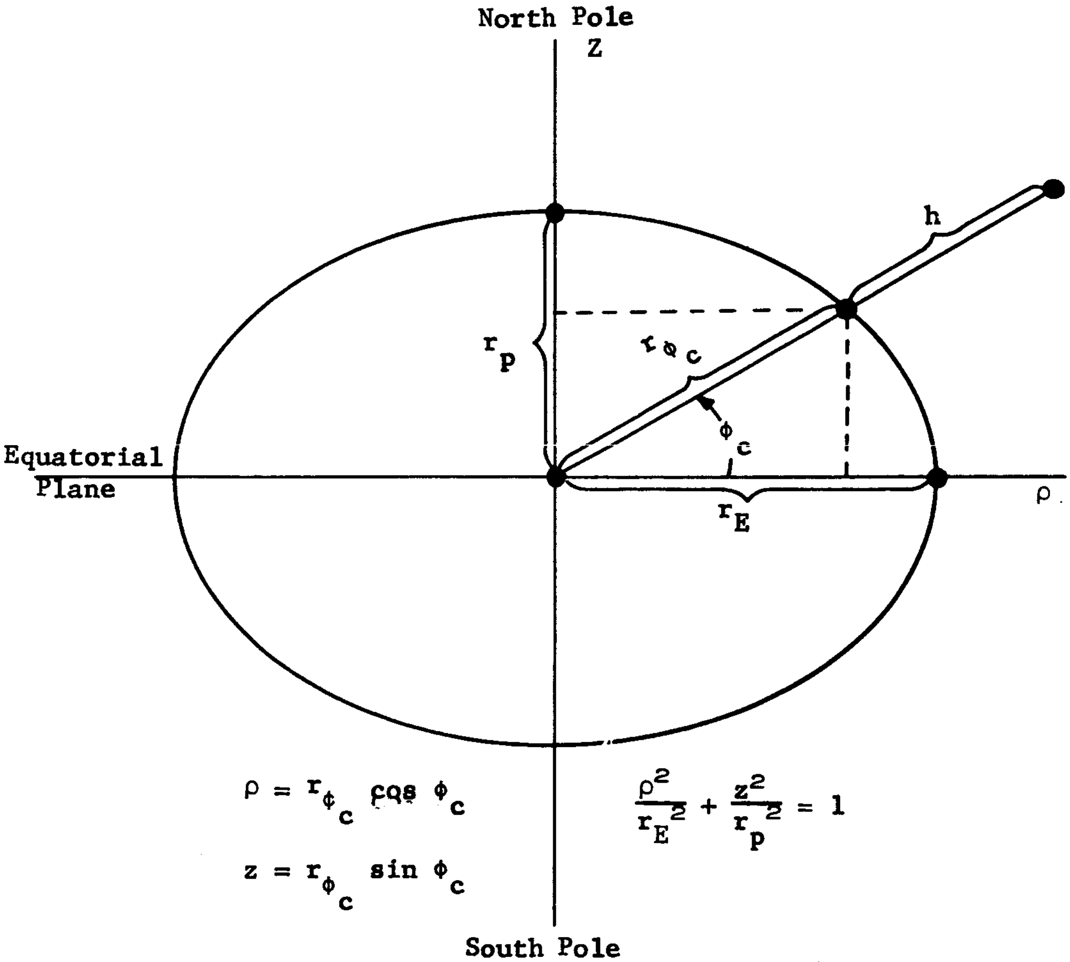
The remaining three initial conditions may be derived by considerations of the initial space-fixed velocity of the vehicle:

$$r_o\,\dot{\phi}_{c_o} = V_I \sin\theta_I \cos A_I \tag{44}$$

$$\dot{r}_o = V_I \cos\theta_I \tag{45}$$

$$r_o\,(\dot{\lambda}_o + \omega)\cos\phi_{c_o} = V_I \sin\theta_I \sin A_I \tag{46}$$

where V_I is the initial space-fixed velocity, θ_I is the initial space-fixed dive angle, and A_I is the initial space-fixed azimuth. The initial space-fixed azimuth might also be expressed in terms of the inclination (j) of the vehicle's instantaneous orbit plane to the equatorial plane:



$$\rho = r_{\phi_c} \cos \phi_c$$

$$z = r_{\phi_c} \sin \phi_c$$

$$\frac{\rho^2}{r_E^2} + \frac{z^2}{r_p^2} = 1$$

$$r_{\phi_c} = \frac{r_E}{\sqrt{\cos^2 \phi_c + (r_E/r_p)^2 \sin^2 \phi_c}} = \frac{r_p}{\sqrt{(r_p/r_E)^2 \cos^2 \phi_c + \sin^2 \phi_c}}$$

$$r = r_{\phi_c} + h$$

FIGURE 4. OBLATE SPHEROID FORMULAE

$$A_I = \sin^{-1}\left\{\frac{\cos j}{\cos \phi_{c_o}}\right\} \tag{47}$$

The initial conditions to be obtained from the initial velocity considerations thus become:

$$\dot{\phi}_{c_o} = \frac{V_I \sin \theta_I \cos A_I}{r_o} \tag{48}$$

$$\dot{r}_o = V_I \cos \theta_I \tag{49}$$

$$\dot{\lambda}_o = \frac{V_I \sin \theta_I \sin A_I}{r_o \cos \phi_{c_o}} \tag{50}$$

SECTION VIII. INPUT REQUIREMENTS FOR THE COMPUTATIONAL PROCEDURE

The input requirements for the IBM 7090 re-entry program fall broadly into three categories: (a) geophysical data (b) aerodynamic data, and (c) initial conditions for the equations of motion.

The geophysical data required are the Gaussian constant (GM), second order figure constant (J), the equatorial radius (r_E), the polar radius (r_p), and the angular velocity of rotation (ω). The first four parameters may be manipulated to produce any oblate spheroid and corresponding gravitational field desired. The values used by the author for those five items can be found in Reference 2.

The aerodynamic data required are atmospheric phenomena and vehicle characteristics. The Computation Division provides 1959 ARDC standard atmospheric mass density and speed of sound data, both as a function of altitude. Provision is made to generate any variation from a standard atmosphere by a density modification factor which is constructed as a curve read function of altitude. A wind velocity profile as a function of altitude and a constant wind azimuth are also available. Provision is made for a constant, effective aerodynamic surface area to vehicle mass rates (S/m), a constant angle of attack (α), and a constant bank angle (γ). The empirical constants (a_L, b_L, c_{D_o}, a_D, and b_D) which determine the lift and drag coefficients may be specified as a curve read function of Mach number.

The initial conditions of the equations of motion may be effected via Section VI by specifying the vehicle's initial space-fixed velocity (V_I), initial space-fixed dive angle (θ_I), initial space-fixed azimuth (A_I) or instantaneous orbital inclination (j), initial earth longitude (λ_o), initial geocentric latitude (ϕ_{c_o}), and initial geocentric altitude (h_o).

The print-out time interval $(\Delta t)_p$ may be chosen to be any multiple of the integration step time interval (Δt) which is also arbitrary according to the computational accuracy desired.

SECTION IX. CONCLUSION

A system of equations has been derived which can be solved numerically on the IBM 7090 digital computer to describe the motion of a vehicle reentering the atmosphere of the earth. Components of acceleration, velocity, and displacement as a function of time can be obtained in a spare-fixed coordinate system which is parallel to the space-fixed platform on board the vehicle. Thus, the trajectory simulation given by these equations is easily adaptable to hardware accuracy studies.

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APPROVAL

A DERIVATION OF THE THREE DIMENSIONAL THREE DEGREE OF FREEDOM RE-ENTRY EQUATIONS OF MOTION IN TERRESTRIAL NAVIGATIONAL COORDINATES WITH SUBSEQUENT RESOLUTION INTO INERTIAL PLATFORM COORDINATES

By Carl L. Colwell

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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