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By

Nathan L. Beard

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ABSTRACT

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A matrix method using flexibility influence coefficients was author developed for obtaining the free-free bending and torsional mode shapes, slopes of mode shapes, and natural frequencies of space vehicles. The effects of rotary inertia and shear flexibility are included.

The mode shapes and natural frequencies were determined for a typical space vehicle and compared with those obtained from a modified Stodola method. Twenty mass points were used for the influence coefficient analysis and 201 for the Stodola method.

This report shows that the influence coefficient method will obtain satisfactory mode shapes and frequencies in comparison to a Stodola method.

AUTHOR

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FLUTTER AND VIBRATION SECTION
DYNAMICS ANALYSIS BRANCH
AEROBALLISTICS DIVISION

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

Symbol	Definition
[ ]	square matrix
{ }	column matrix
$\lfloor \; \rfloor$	row matrix
C_{ij}^{FB}	bending deflections due to a unit force
C_{ij}^{FS}	shearing deflections due to a unit force
C_{ij}^{MB}	bending deflections due to a unit moment
C_{ij}^{MS}	shearing deflections due to a unit moment
θ_{ij}^{FB}	bending rotations due to a unit force
θ_{ij}^{FS}	shearing rotations due to a unit force
θ_{ij}^{MB}	bending rotations due to a unit moment
Y_{Ti}	total relative displacement of i^{th} point of mode shape for a cantilever beam
Y'_{Ti}	total slope of cantilever mode shape
Y'_{Bi}	bending slope of mode shape
ω	circular frequency
M_i	mass at i^{th} point on beam
I_{li}	mass moment of inertia of i^{th} beam segment about its midpoint due to its length l.

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
I_{Ri}	mass moment of inertia of i^{th} beam segment about its midpoint due to its radius, R
Y_i	total relative displacement of i^{th} point of mode shape for a free-free beam
Y'_i	slope of mode shape of free-free beam
θ_O	rotation of clamped end of beam when released to vibrate as a free-free beam
Y_O	deflection of clamped end of beam when released to vibrate as a free-free beam
$[C\theta]$	square matrix of influence coefficients 3n x 3n
$[\ mI\ ]$	diagonal matric of masses and mass moment of inertias
$[d]$	dynamic matrix for cantilever beam
$\{u\}$	matrix of unknowns
$[D]$	dynamic matrix for free-free beam
T_{ic}	angle of twist of i^{th} cross section of a cantilever beam
ω_T	natural torsional frequency
R^T_{ij}	cantilever influence coefficients for torsion
J_i	polar mass moment of inertia of i^{th} beam segment
G_i	shear modulus of elasticity at station i
I_{pi}	area polar moment of inertia at station i
X_i	distance from left end of beam to station i

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SUMMARY

Mode shapes and natural frequencies of a uniform and nonuniform beam obtained by a flexibility influence coefficient method were compared with a modified Stodola method. Twenty mass points were used with the influence coefficient method, while 201 were used with the Stodola method for the nonuniform beam analysis. For the uniform beam, 10, 15, and 20 mass points were used, while 145 were used in the Stodola analysis. The results indicate that the flexibility influence coefficient method yields reasonably accurate mode shapes and frequencies in bending and torsion using only twenty mass points. Maximum variation in frequencies between the two methods used was only about 2 percent for the nonuniform beam and 3 percent for the uniform beam. Mode shapes compared very closely with the exception of the 4th modes of the free-free bending and torsion for the nonuniform beam.

I. INTRODUCTION

During the past few years the vibration of various structures and their components has become increasingly important to scientific personnel in many fields. Practically any structure which is subjected to shock or repeated loads experiences vibrations. These vibrations result, in many cases, in structural fatigue, due to repeated stress reversals, or violent structural failure due to a resonant condition.

In the space field, various problems arise in the design of control systems because of the elasticity of the structure. Insulation of sensitive instruments against shock and vibration is a problem which must be considered. Also, acoustical problems arise due to the high energy level of the sound waves emitted by the rocket motors.

This report is concerned specifically with the vibration of non-uniform beams, a problem which is analogous to the structural vibration of a space vehicle airframe. A matrix method using flexibility influence coefficients to determine the mode shapes, slopes of mode shapes, and natural frequencies, both torsional and bending, for a uniform and non-uniform single-beam structure is presented. This analysis includes the effects of rotary inertia and shear flexibility.

The author expresses his appreciation to Mr. C. R. Wells of Chrysler Corporation Space Division for the many helpful suggestions in the preparation of this report.

II. DESCRIPTION

The total linear or angular deflection of any point on a beam can be expressed as the sum of the deflections at that point produced by the individual applied forces and torques. This is the principle of superposition which will be used in writing the deflections and slopes of a vibrating beam. The general equation for the displacements or rotations of points on a beam can be written in the following form:

$$q_i = \sum_{j=1}^{n} C_{ij}\, Q_j \qquad (i = 1, 2, 3, \ldots n) \tag{1}$$

where q_i's are the generalized coordinates, deflection and rotation, C_{ij}'s are flexibility influence coefficients, and Q_j's are generalized forces or torques. The flexibility coefficients can be determined by subdividing a beam into n parts, assuming the mass of each element to be concentrated at the center of the element, applying a unit force and moment separately at each point, and then determining the deflection and slope at each point on the beam for each loading condition. Influence coefficients of this type can be thought of as the reciprocal of the spring constants for each mass point. The generalized forces Q_j are the inertia forces $m_i Y_i \omega^2$ and the inertia torques $I_i Y_i' \omega^2$.

There are seven types of flexibility influence coefficients associated with bending vibration problems. Equations (2), (3), and (4) illustrate their relationship to the total deflection, slope, and bending slope of a beam.

From equations (1) and (2), the total deflection and slope can be obtained for the i^{th} mass point along a vibrating beam.

$$Y_{Ti} = \omega^2 \sum \left(C_{ij}^{FB} + C_{ij}^{FS} \right) m_i \; Y_{Ti} + \omega^2 \sum \left(C_{ij}^{MB} + C_{ij}^{MS} \right) I_{\ell i} \; Y'_{Ti}$$

$$+ \omega^2 \sum \left(C_{ij}^{MB} + C_{ij}^{MS} \right) I_{Ri} \; Y'_{Bi} \qquad (2)$$

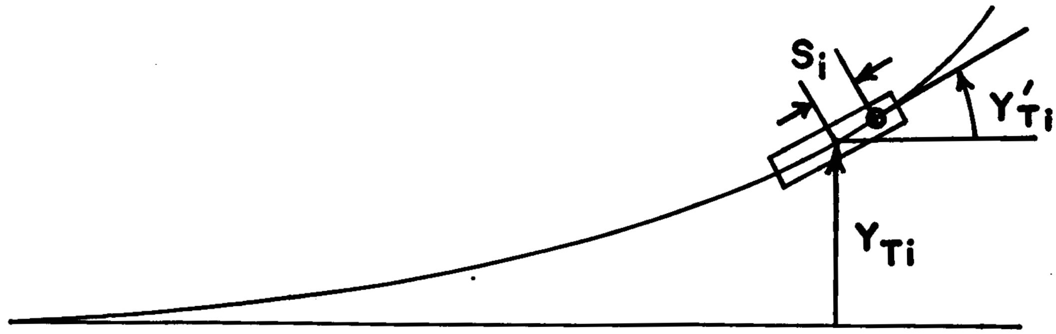
$$Y'_{Ti} = \omega^2 \sum \left(\theta_{ij}^{FB} + \theta_{ij}^{FS} \right) m_i \; Y_{Ti} + \omega^2 \sum \theta_{ij}^{MB} \; J_{\ell i} \; Y'_{Ti} + \omega^2 \sum \theta_{ij}^{MB} \; I_{Ri} \; Y'_{Bi}. \qquad (3)$$

Equations (2) and (3) have 3 unknowns, Y_{Ti}, Y'_{Ti}, and Y'_{Bi}; therefore, an equation for Y'_{Bi} must be written before a solution is possible.

$$Y'_{Bi} = \omega^2 \sum \theta_{ij}^{FB} \; m_i \; Y_{Ti} + \omega^2 \sum \theta_{ij}^{MB} \; I_{\ell i} \; Y'_{Ti} + \omega^2 \sum \theta_{ij}^{MB} \; I_{Ri} \; Y'_{Bi}. \qquad (4)$$

Two different mass moments of inertia are used in equations (2), (3), and (4): $I_{\ell i}$ and I_{Ri}. $I_{\ell i}$ is the inertia due to the length of the section and I_{Ri} is the inertia due to the radius or diameter. In equations (2), (3), and (4) one sees that $I_{\ell i}$ always occurs with Y'_{Ti}, while I_{Ri} occurs with Y'_{Bi}. The reason for this is that the length of an element of beam rotates under both bending and shear loads; therefore, $I_{\ell i}$ is always accompanied by $Y'_{Si} + Y'_{Bi}$ or Y'_{Ti}. In the other case, the diameter or radius of an element rotates only when subjected to a bending load. A shearing load causes sliding of adjacent planes in the vertical direction, but does not produce any rotation of the diameter of the element with respect to the vertical; therefore, I_{Ri} is associated only with Y'_{Bi}.

Since the center of gravity of each element may not coincide with its geometric center, it is necessary to add additional forces and torques due to this unbalance as follows:



(1) A force: $\omega^2 m_i S_i Y'_{Ti}$

(2) A moment: $\omega^2 m_i S_i Y_{Ti}$

(3) Another moment: $\omega^2 m_i S_i^2 Y'_{Ti}$.

Equations (2), (3), and (4) now read:

$$Y_{Ti} = \omega^2 \sum \left(C_{ij}^{FB} + C_{ij}^{FS}\right) m_i \left(Y_{Ti} + S_i Y'_{Ti}\right) + \omega^2 \sum \left(C_{ij}^{MB} + C_{ij}^{MS}\right) \left(J_{\ell i} + m_i S_i^2\right) Y'_{Ti}$$

$$+ \omega^2 \sum \left(C_{ij}^{MB} + C_{ij}^{MS}\right) m_i S_i Y_{Ti} + \omega^2 \sum \left(C_{ij}^{MB} + C_{ij}^{MS}\right) I_{Ri} Y'_{Bi} \qquad (2')$$

$$Y'_{Ti} = \omega^2 \sum \left(\theta_{ij}^{FB} + \theta_{ij}^{FS}\right) m_i \left(Y_{Ti} + S_i Y'_{Ti}\right) + \omega^2 \sum \theta_{ij}^{MB} \left(J_{\ell i} + m_i S_i^2\right) Y'_{Ti}$$

$$+ \omega^2 \sum \theta_{ij}^{MB} m_i S_i Y_{Ti} + \omega^2 \sum \theta_{ij}^{MB} I_{Ri} Y'_{Bi} \qquad (3')$$

$$Y'_{Bi} = \omega^2 \sum \theta_{ij}^{FB} m_i \left(Y_{Ti} + S_i Y'_{Ti}\right) + \omega^2 \sum \theta_{ij}^{MB} \left(J_{\ell i} + m_i S_i^2\right) Y'_{Ti}$$

$$+ \omega^2 \sum \theta_{ij}^{MB} m_i S_i Y_{Ti} + \omega^2 \sum \theta_{ij}^{MB} I_{Ri} Y'_{Bi}. \qquad (4')$$

Equations (2'), (3') and (4') can be written in matrix form as follows:

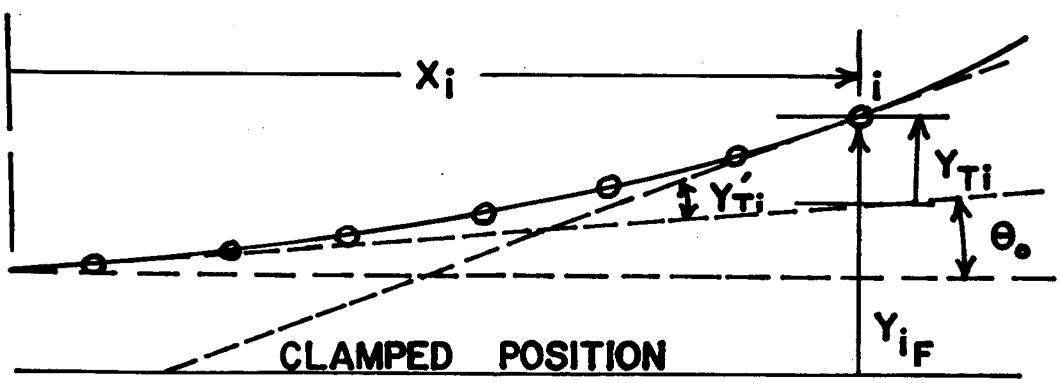
$$\begin{Bmatrix} Y_{Ti} \\ \hline Y'_{Ti} \\ \hline Y'_{Bi} \end{Bmatrix} = \omega^2 \left[\begin{array}{c|c|c} C^{FB} + C^{FS} & C^{MB} + C^{MS} & C^{MB} + C^{MS} \\ \hline \theta^{FB} + \theta^{FS} & \theta^{MB} & \theta^{MB} \\ \hline \theta^{FB} & \theta^{MB} & \theta^{MB} \end{array}\right] \left[\begin{array}{c|c|c} m_i & m_i S_i & 0 \\ \hline m_i S_i & I_{\ell i} + m_i S_i^2 & 0 \\ \hline 0 & 0 & I_{Ri} \end{array}\right] \begin{Bmatrix} Y_{Ti} \\ \hline Y_{Ti} \\ \hline Y'_{Bi} \end{Bmatrix} . \tag{5}$$

Since flexibility influence coefficients are more easily obtained for a cantilever beam than beams with other end conditions, it will be assumed that the above coefficients have been determined for this case.

Equation (5) will be written as

$$\left\{ u \right\} = \omega^2 \left[d \right] \left\{ u \right\} , \tag{6}$$

and upon iteration the mode shapes, slopes of mode shapes, slopes of the bending mode shape, and natural bending frequencies are obtained. To obtain the free-free frequencies, modes shapes, etc., the clamped end of the beam must be allowed to translate and rotate as shown below.



The cantilever deflections and slopes Y_{Ti} and Y'_{Ti} can now be written in terms of the new variables Y_{i_F}, Y'_{i_F}, X_i, θ_o, and Y_o.

$$Y_{Ti} = Y_{i_F} - Y_o - X_i\theta_o \tag{7}$$

$$Y'_{Ti} = Y'_{i_F} - \theta_o \tag{8}$$

$$Y'_{Bi} = Y'_{Bi_F} - \theta_o \tag{9}$$

where Y_{i_F} is the deflection of the free-free beam and Y'_{i_F} is the slope. Matrix equation (5) now reads

$$\left\{ \begin{array}{c} Y_{i_F} - Y_o - X_i\theta_o \\ \vdots \\ Y'_{i_F} - \theta_o \\ \vdots \\ Y'_{Bi_F} - \theta_o \\ \vdots \end{array} \right\} = \omega^2 \left[d \right] \left\{ \begin{array}{c} Y_i \\ \vdots \\ Y'_i \\ \vdots \\ Y'_{Bi} \\ \vdots \end{array} \right\} \tag{10}$$

Writing equation (10) in three separate equations,

$$\left\{ Y_{i_F} \right\} - Y_o \left\{ 1 \right\} - \left\{ X_i \right\} \theta_o = \omega^2 \left[C\theta \right]_{R_1} \left[\diagdown mI \diagdown \right] \left\{ u \right\} \tag{11a}$$

$$\left\{ Y'_{i_F} \right\} - \theta_o \left\{ 1 \right\} = \omega^2 \left[C\theta \right]_{R_2} \left[\diagdown mI \diagdown \right] \left\{ u \right\} \tag{11b}$$

$$\left\{ Y'_{Bi_F} \right\} - \theta_o \left\{ 1 \right\} = \omega^2 \left[C\theta \right]_{R_3} \left[\diagdown mI \diagdown \right] \left\{ u \right\} \tag{11c}$$

where $\left[\, C\theta \,\right]_{R_1}$ is the first partitioned row of the influence coefficient matrix $\left[\, C\theta \,\right]_{R_2}$ the second row and $\left[\, C\theta \,\right]_{R_3}$ the third row. The mass and moment of inertia matrix is $\left\lceil mI \right\rfloor$. The unknowns Y_i, Y'_i, and Y'_{Bi} are designated $\left\{ u \right\}$.

Introducing the boundary conditions that the shear and bending moment are zero at the ends of the beam, we obtain

$$\sum m_i \left(Y_{i_F} + S_i \, Y_{i_F} \right) = 0 \tag{12}$$

$$\sum m_i \left(X_i + S_i \right) \left(Y_{i_F} + S_i \, Y'_{i_F} \right) + \sum I_{CG} \, Y'_{i_F} + \sum I_{Ri} \, Y'_{Bi_F} = 0 \tag{13}$$

where

$$I_{CG} = I_{\ell i} - m_i \, S_i^2 \, .$$

In matrix form, equations (12) and (13) read

$$\left\lfloor m_i \right\rfloor \left\{ Y_{i_F} \right\} \; \left\lfloor m_i \; S_i \right\rfloor \left\{ Y'_{i_F} \right\} = 0 \tag{12'}$$

$$\left\lfloor m_i \;\; X_i + S_i \right\rfloor \left\{ Y_{i_F} \right\} + \Big[\left\lfloor m_i \; X_i \; S_i \right\rfloor + \left\lfloor J_{\ell i} \right\rfloor \left\{ Y'_{i_F} \right\}$$

$$+ \left\lfloor I_{R_i} \right\rfloor \left\{ Y'_{Bi_F} \right\} \Big] = 0. \tag{13'}$$

Solving equations (11a), (11b), and (11c) for Y_{iF}, Y'_{iF}, and Y'_{BiF} and substituting in (12') and (13') yields the following:

$$\left\lfloor m_i \right\rfloor \left\{ Y_o \left\{ 1 \right\} + \left\{ X_i \right\} \theta_o + \omega^2 \left[C\theta \right]_{R_1} \left[mI \right] \left\{ u \right\} \right\}$$
$$+ \left\lfloor m_i \; S_i \right\rfloor \left\{ \theta_o \left\{ 1 \right\} + \omega^2 \left[C\theta \right]_{R_2} \left[mI \right] \left\{ u \right\} \right\} = 0 \tag{14}$$

and

$$\left[m_i \; (X_i + S_i) \right] \left\{ Y_o \left\{ 1 \right\} + \left\{ X_i \right\} \theta_o + \omega^2 \left[C\theta \right]_{R_1} \left[mI \right] \left\{ u \right\} \right\}$$
$$+ \left\lfloor m_i \; X_i \; S_i + I_{\ell i} \right\rfloor \left\{ \theta_o \left\{ 1 \right\} + \omega^2 \left[C\theta \right]_{R_2} \left[mI \right] \left\{ u \right\} \right\}$$
$$+ \left\lfloor I_{Ri} \right\rfloor \left\{ \theta_o \left\{ 1 \right\} + \omega^2 \left[C \right]_{R_3} \left[mI \right] \left\{ u \right\} \right\} = 0. \tag{15}$$

For simplification, the following substitutions are made:

$$\left\lfloor A \right\rfloor = \left\lfloor m_i \right\rfloor \left[C\theta \right]_{R_1} \left[mI \right] + \left\lfloor m_i \; S_i \right\rfloor \cdot \left[C\theta \right]_{R_2} \left[mI \right]$$

$$\left\lfloor B \right\rfloor = \left\lfloor m_i \; X_i \right\rfloor \left[C\theta \right]_{R_1} \left[mI \right] + \left\lfloor m_i \; S_i \right\rfloor \left[C\theta \right]_{R_1} \left[mI \right]$$

$$\left\lfloor D \right\rfloor = \left\lfloor I_{\ell i} \right\rfloor \left[C\theta \right]_{R_2} \left[mI \right] + \left\lfloor m_i \; X_i \; S_i \right\rfloor \left[C\theta \right]_{R_2} \left[mI \right]$$

$$\left\lfloor E \right\rfloor = \left\lfloor I_{Ri.} \right\rfloor \left[C\theta \right]_{R_3} \left[mI \right].$$

Equations (14) and (15) may be written with the above substitutions as follows:

$$Y_o \sum m_i + \theta_o \sum m_i (X_i + S_i) + \omega^2 \left\lfloor A \right\rfloor \left\{ u \right\} = 0. \tag{14'}$$

$$Y_o \sum m_i (X_i + S_i) + \theta_o \left[\sum m_i (X_i^2 + 2X_i S_i) + \sum J_{\ell i} + \sum J_{Ri} \right]$$
$$+ \omega^2 \left\lfloor \left\lfloor B \right\rfloor + \left\lfloor D \right\rfloor + \left\lfloor E \right\rfloor \right\rfloor \left\{ u \right\} = 0. \tag{15'}$$

Making further substitutions:

$$M_o = \Sigma m_i$$

$$I_o = \Sigma m_i (X_i^2 + 2X_i S_i)$$

$$\left\lfloor F \right\rfloor = \left\lfloor B \right\rfloor + \left\lfloor D \right\rfloor + \left\lfloor E \right\rfloor$$

$$S_o = \Sigma m_i (X_i + S_i)$$

$$L_o = \Sigma (J_{\ell i} + J_{Ri}).$$

Equations (14') and (15') are written in the following form:

$$M_o Y_o + S_o \theta_o + \omega^2 \left\lfloor A \right\rfloor \left\{ u \right\} = 0. \tag{14''}$$

$$S_o Y_o + (J_o + L_o) \theta_o + \omega^2 \left\lfloor F \right\rfloor \left\{ u \right\} = 0. \tag{15''}$$

Equations (14") and (15") can be solved simultaneously for θ_o and Y_o.

$$\theta_o = \frac{\omega^2}{K_1}\left\lfloor \frac{S_o}{M_o}\lfloor A \rfloor - \lfloor F \rfloor \right\rfloor \left\{ u \right\}, \tag{16}$$

$$Y_o = -\frac{S_o}{M_o K_1}\,\omega^2 \left\lfloor \left(\frac{S_o}{M_o} + \frac{K_1}{S_o}\right)\lfloor A \rfloor - \lfloor F \rfloor \right\rfloor \left\{ u \right\}, \tag{17}$$

where

$$K_1 = I_o + L_o - \frac{S_o}{M_o}\,.$$

Rewriting equation (10)

$$\left\{ u \right\} = \left\{ \begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 0 \\ \hline 0 \end{array} \right\} Y_o + \left\{ \begin{array}{c} X_i \\ \vdots \\ 1 \\ \hline 1 \\ \hline 1 \end{array} \right\} \theta_o + \omega^2 \left[d \right] \left\{ u \right\}. \tag{18}$$

Substituting for Y_o and θ_o in equation (18) from equations (16) and (17) yields a set of equations in matrix form which can be used to determine the total mode shapes, slopes of mode shapes, slopes of bending mode shapes, and the natural bending frequencies.

$$\left\{ u \right\} = -\frac{S_o}{M_o K_1}\,\omega^2 \left\{ \begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 0 \\ \hline 0 \end{array} \right\} \left\lfloor \left(\frac{S_o}{M_o} + \frac{K}{S_o}\right)\left[A \right] - \lfloor F \rfloor \right\rfloor \left\{ u \right\}$$

$$+ \frac{\omega^2}{K_1} \left\{ \begin{array}{c} X_i \\ \hline 1 \\ \hline 1 \end{array} \right\} \left\lfloor \frac{S_o}{M_o}\lfloor A \rfloor - \lfloor F \rfloor \right\rfloor \left\{ u \right\} + \omega^2 \left[d \right] \left\{ u \right\}. \tag{19}$$

The product of the row times the column matrices yields two square matrices which can be added to $\left[\, d \,\right]$ resulting in

$$\left\{ u \right\} = \omega^2 \left[D \right] \left\{ u \right\}, \tag{20}$$

where $\left[\, D \,\right]$ is the dynamic matrix for a free-free beam experiencing bending vibrations.

The procedure to be used in the development of the influence coefficient matrices for digital computation can be found in Appendix A.

An iteration procedure for obtaining higher modes is given in Appendix C.

Free-Free Torsion Equations

A set of equations for determining torsional frequencies and mode shapes may be written similar to those for free-free bending. The general equation for the angle of twist T_i, of any section i is

$$T_i = \omega_T^2 \sum_{1}^{m} R_{ij}^{T} J_i T_i \qquad (i = 1, 2, \ldots n) \tag{21}$$

where

ω_T is the natural torsional frequency

R_{ij}^{T} is the torsional influence coefficients, and

J_i is the polar mass moment of inertia.

Assuming that the cantilever coefficients can be determined, n equations may be written in matrix form as follows:

$$\left\{ T_{ic} \right\} = \omega^2 \cdot \left[R_{ij}^{TC} \right] \left[\diagdown J_i \diagdown \right] \left\{ T_{ic} \right\}. \tag{22}$$

Next, the clamped end of the beam is released similar to the free-free bending case so that the angle of twist, free-free, may be written as the angle of twist, cantilever, plus some angle of twist, T_o, resulting from the releasing of the clamped end.

$$T_{iF} = T_{ic} + T_o. \tag{23}$$

Solving for T_{ic} above and substituting into equation (22) gives

$$\left\{ T_{iF} \right\} = \left\{ 1 \right\} T_o + \omega_T^2 \left[R_{ij}^{Tc} \right] \left[\diagdown J_i \diagdown \right] \left\{ T_{ic} \right\}. \tag{24}$$

For free-free vibrations, the following equation holds:

$$\sum J_i \; T_{iF} = 0. \tag{25}$$

Or, in matrix notation,

$$\left\lfloor J_i \right\rfloor \left\{ T_{iF} \right\} = 0. \tag{26}$$

Multiplying equation (24) by $\left\lceil J_i \right\rceil$ yields

$$0 = \left\lfloor J_i \right\rfloor \left\{ 1 \right\} T_o + \left\lfloor J_i \right\rfloor \omega_T^2 \left[R_{ij}^{Tc} \right] \left[\diagdown J_i \diagdown \right] \left\{ T_{ic} \right\}. \tag{27}$$

Solving for T_o and substituting into equation (24) gives

$$\left\{ T_{iF} \right\} = - \frac{\omega_T^2}{J_o} \left\{ 1 \right\} \left\lfloor J_i \right\rfloor \left[R_{ij}^{Tc} \right] \left\lceil J_i \right\rfloor \left\{ T_{ic} \right\}$$

$$+ \omega_T^2 \left[R_{ij}^{Tc} \right] \left\lceil J_i \right\rfloor \left\{ T_{ic} \right\}, \tag{28}$$

where

$$J_o = \sum J_i.$$

Equation (28) in simplified form is

$$\left\{ T_{iF} \right\} = \omega_T^2 \left[\left[Z \right] + \left[I \right] \right] \left[d_T \right] \left\{ T_{iF} \right\}, \tag{29}$$

where

$$\left[Z \right] = - \frac{1}{J_o} \left\{ 1 \right\} \left\lfloor J_i \right\rfloor .$$

$\left[I \right]$ is the identity matrix, and

$$\left[d_T \right] = \left[R_{ij}^{Tc} \right] \left\lceil J_i \right\rfloor .$$

In final form

$$\left\{ T_{iF} \right\} = \omega_T^2 \left[D_T \right] \left\{ T_{iF} \right\}. \tag{30}$$

Equation (30) may be iterated on for the mode shapes and natural frequencies.

III. CONCLUSIONS AND RECOMMENDATIONS

Table I illustrates that the influence coefficient method can be used with a reasonably small number of mass points compared to the Stodola method to obtain accurate frequencies and mode shapes. Slopes of mode shapes can also be obtained with this program, but were not included in this report since the mode shapes give an indication of the accuracy one could expect for the slopes. The accuracy of the mode shapes and frequencies is increased with an increase in the number of mass points. The first three mode shapes do not change appreciably, but the fourth mode is sensitive to mass point changes from 10 to 15.

Table II compares the cantilever, free-free bending, and free-free torsional frequencies for the first four modes of a typical space vehicle whose mass and stiffness characteristics are shown in Figures 13 and 14. The EI used in the influence coefficient method for each station was determined by averaging three values taken at 1/4, 1/2, and 3/4 of the length of the mass segment. For EI distributions that vary radically over a particular mass segment, it is recommended that the reciprocal of an average value of the 1/EI distribution be used for the effective EI.

Figures 1 through 12 compare the first four normal modes for the free-free bending, cantilever bending, and the free-free torsion case as obtained by the two methods. Good agreement was obtained in the first three modes of the free-free bending case. The deviation in the fourth mode can possibly be explained by the fact that two extra "sweeping" processes were initiated before this mode was obtained. This arose from the fact that the EI of a section of the nose was small in comparison to the section beginning at station X = -25 (Figure 13). These intermediate modes or "tower modes," as they might be called for a vehicle with an extremely flexible tower on the nose of the vehicle, were not included in this report since the Stodola method did not indicate their existence. Excellent agreement was obtained for the cantilever mode shapes (Figures 5 through 8). Torsional modes were in good agreement through the second mode. A slight deviation occurred in the third mode and a considerable deviation in the fourth. A different method for obtaining cantilever bending and torsional influence coefficients could possibly increase the accuracy of the program. The more difficult variable to evaluate properly for each station appears to be the stiffness (EI); therefore, it is recommended that various ways be tried to determine the proper EI.

TABLE OF RESULTS I

(Uniform Beam)

Free-Free Natural Bending Frequencies (L/sec)	Modes 1	2	3	4
ω_μ^{ST} - (145) MPTS.	48.64	120.52	209.24	305.11
ω_μ^{IC} - (10) MPTS.	48.40	118.15	200.42	283.72
ω_μ^{IC} - (15) MPTS.	48.51	119.20	204.12	292.49
ω_μ^{IC} - (20) MPTS.	48.56	119.58	205.52	295.97

% Variation	Modes 1	2	3	4
ST. ---- I. C. (10)	0.49	1.97	4.22	7.01
ST. ---- I. C. (15)	0.27	1.10	2.45	4.14
ST. ---- I. C. (20)	0.16	0.78	1.80	3.00

ST. ---- Stodola method

I.C. ---- Influence Coefficient method

MPTS. ---- Mass points.

Maximum Normalized Deflections	ST.		I. C. (10)		I. C. (15)		I. C. (20)	
Y_{1L} and Y_{1R}	1.000	1.000	1.000	0.999	1.000	0.999	1.000	0.999
Y_{2L} and Y_{2R}	1.000	-1.000	1.000	-0.997	1.000	-0.997	1.000	-0.998
Y_{3L} and Y_{3R}	1.000	1.000	1.000	0.985	1.000	0.986	1.000	+0.989
Y_{4L} and Y_{4R}	1.000	-1.000	1.000	-0.871	1.000	-0.951	1.000	-0.954

Subscripts L and R denote left and right extremes of beam.

TABLE OF RESULTS II

(Non-Uniform Beam)

Stodola Method Modes

Natural Frequencies Rad/sec	1	2	3	4
(cantilever)	2.37	8.25	18.41	32.93
(free-free)	7.78	18.85	34.58	51.66
(torsion)	36.41	59.15	86.20	117.65

Influence Coefficient Method (20) Pts Modes

Natural Frequencies Rad/sec	1	2	3	4
(cantilever)	2.39	8.40	18.48	32.48
(free-free)	7.95	18.78	33.93	50.90
(torsion)	35.63	58.97	86.19	115.04

Percent Variation Modes

% Var. = $(1 - \frac{\omega_{IC}}{\omega_{ST}})$ 100	1	2	3	4
(cantilever)	0.84	1.82	0.38	-1.36
(free-free)	2.19	-0.42	-1.88	-1.47
(torsion)	-2.14	-0.30	0.00	-2.22

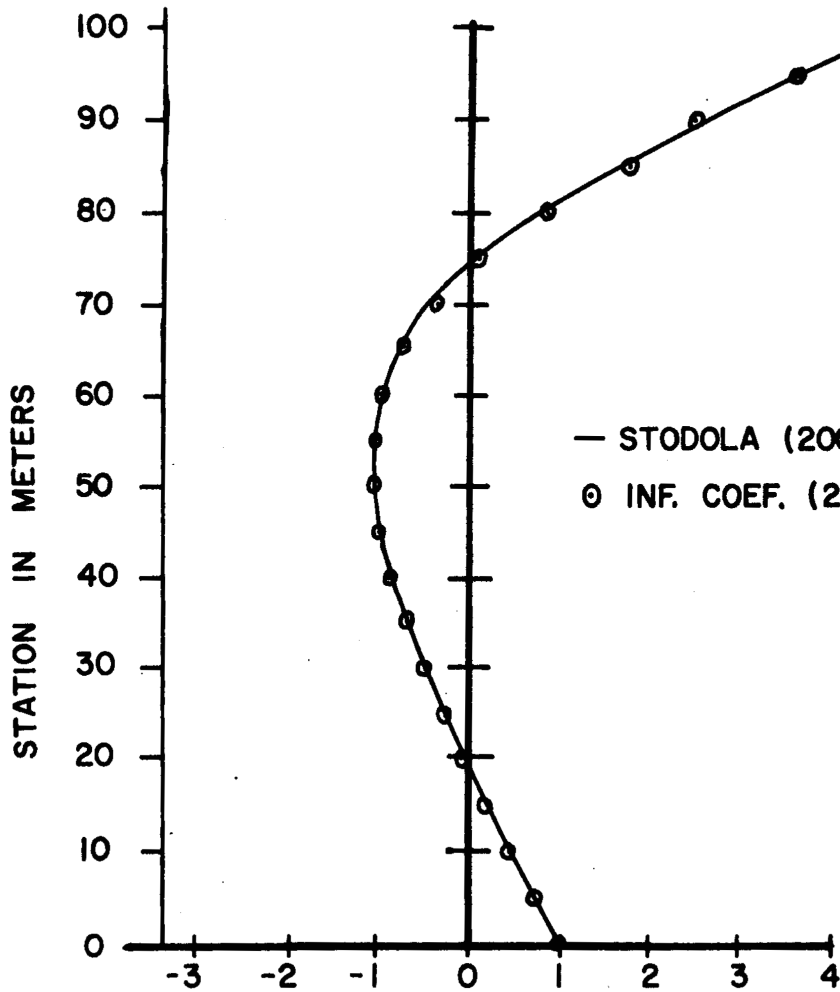


FIG. 1 FIRST FREE - FREE BENDING MODE FOR A NON-UNIFORM BEAM

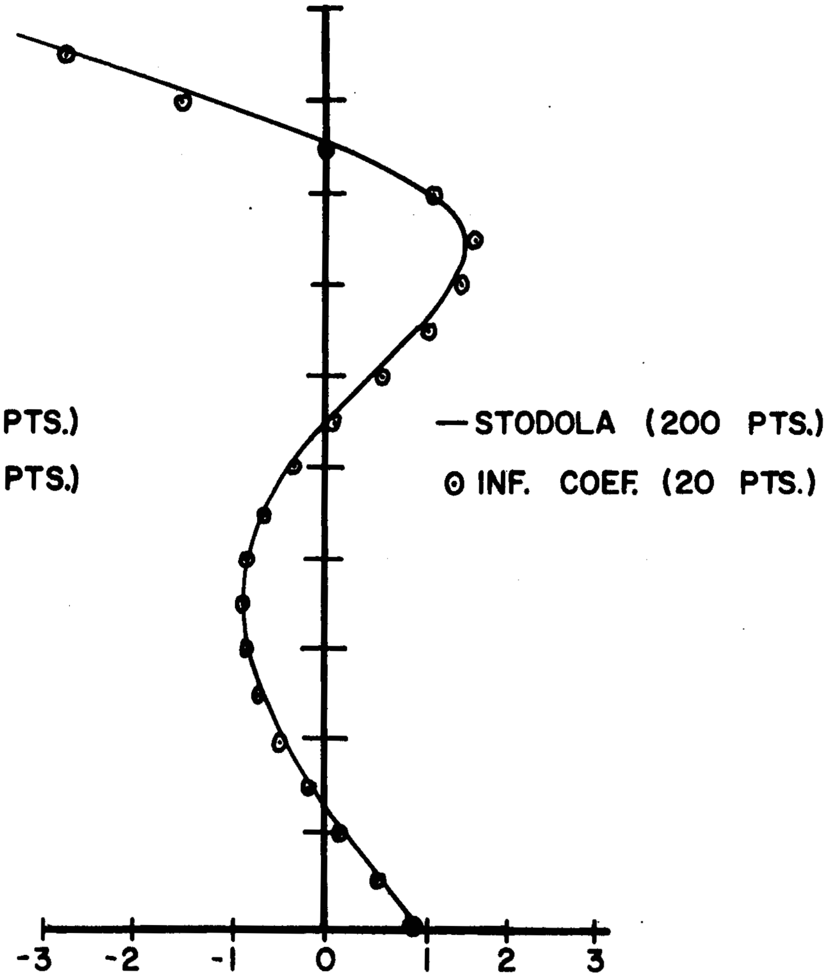


FIG. 2 SECOND FREE - FREE BENDING MODE FOR A NON-UNIFORM BEAM

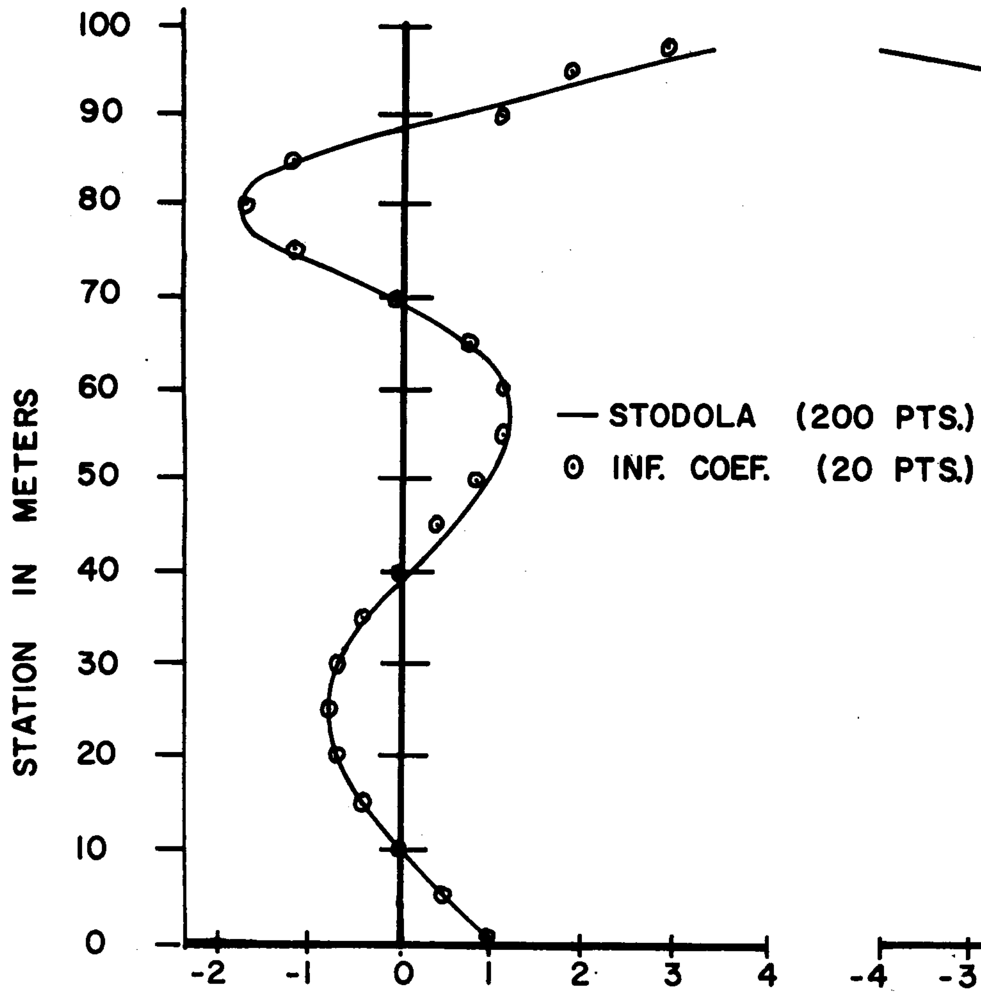


FIG. 3 THIRD FREE-FREE BENDING MODE FOR A NON-UNIFORM BEAM

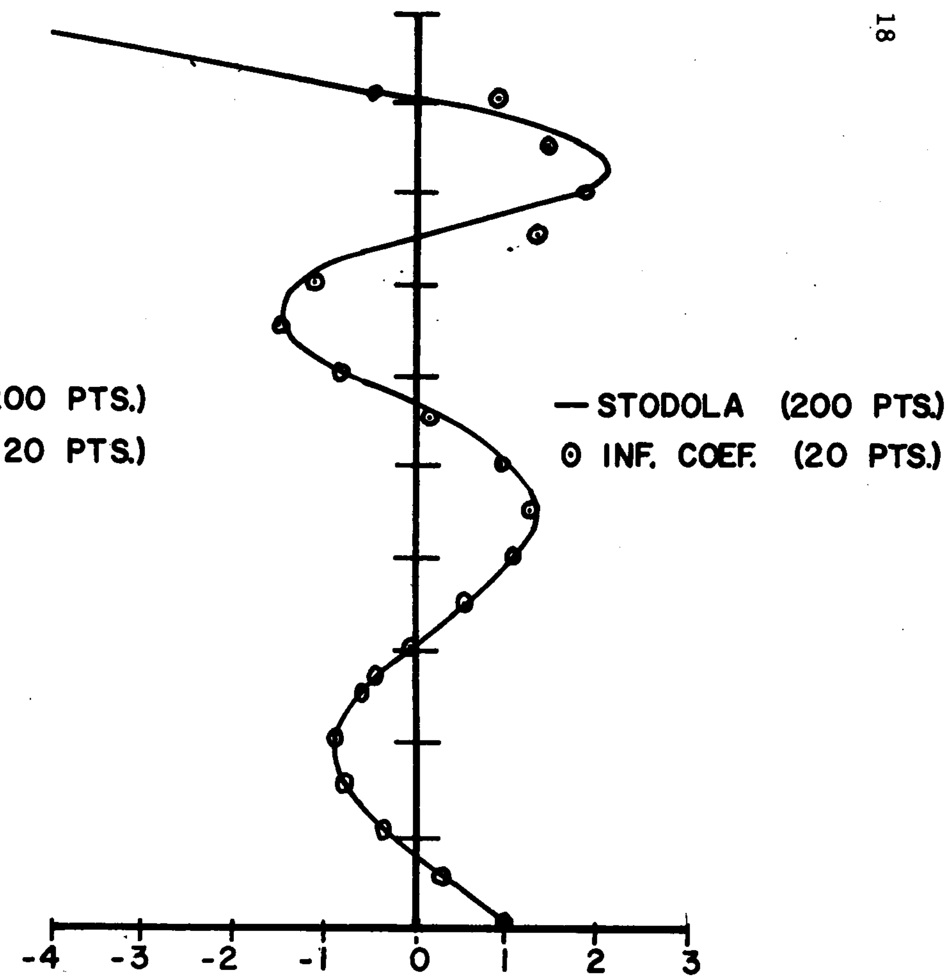


FIG. 4 FOURTH FREE-FREE BENDING MODE FOR A NON-UNIFORM BEAM

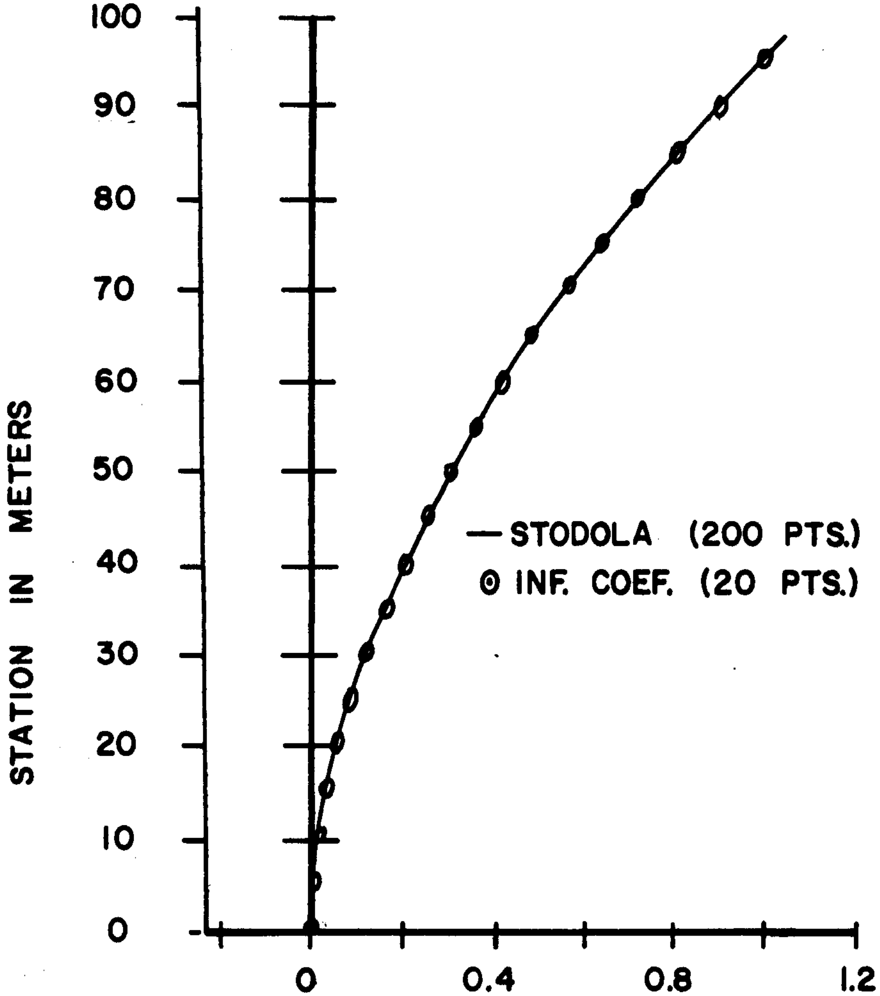


FIG. 5 FIRST CANTILEVER BENDING MODE FOR A NON-UNIFORM BEAM

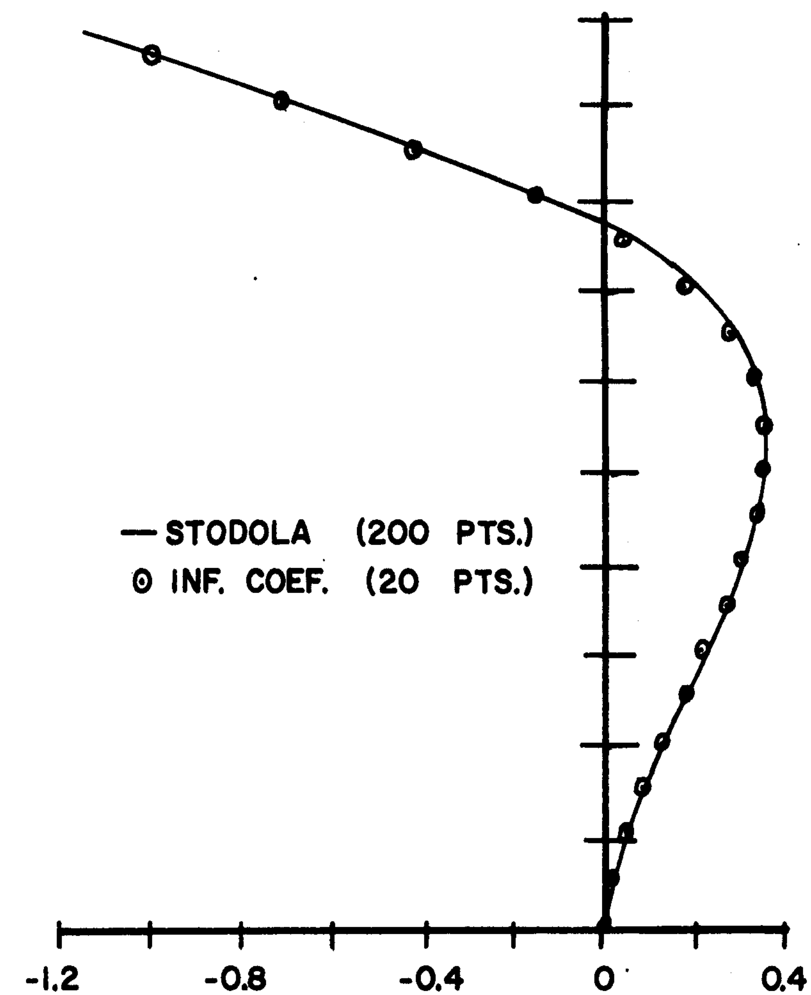


FIG. 6 SECOND CANTILEVER BENDING MODE FOR A NON-UNIFORM BEAM

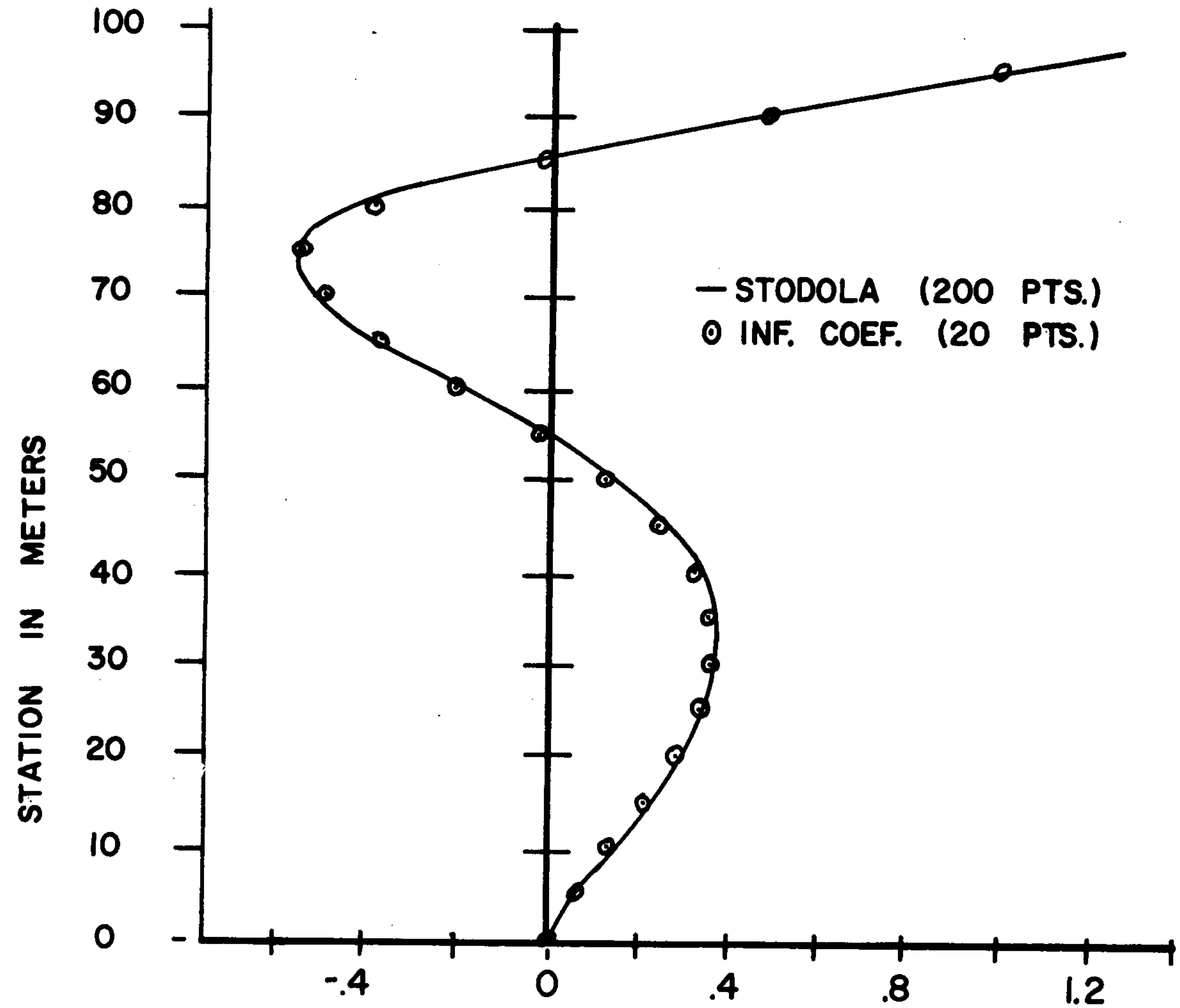


FIG. 7 THIRD CANTILEVER BENDING MODE FOR A NON-UNIFORM BEAM

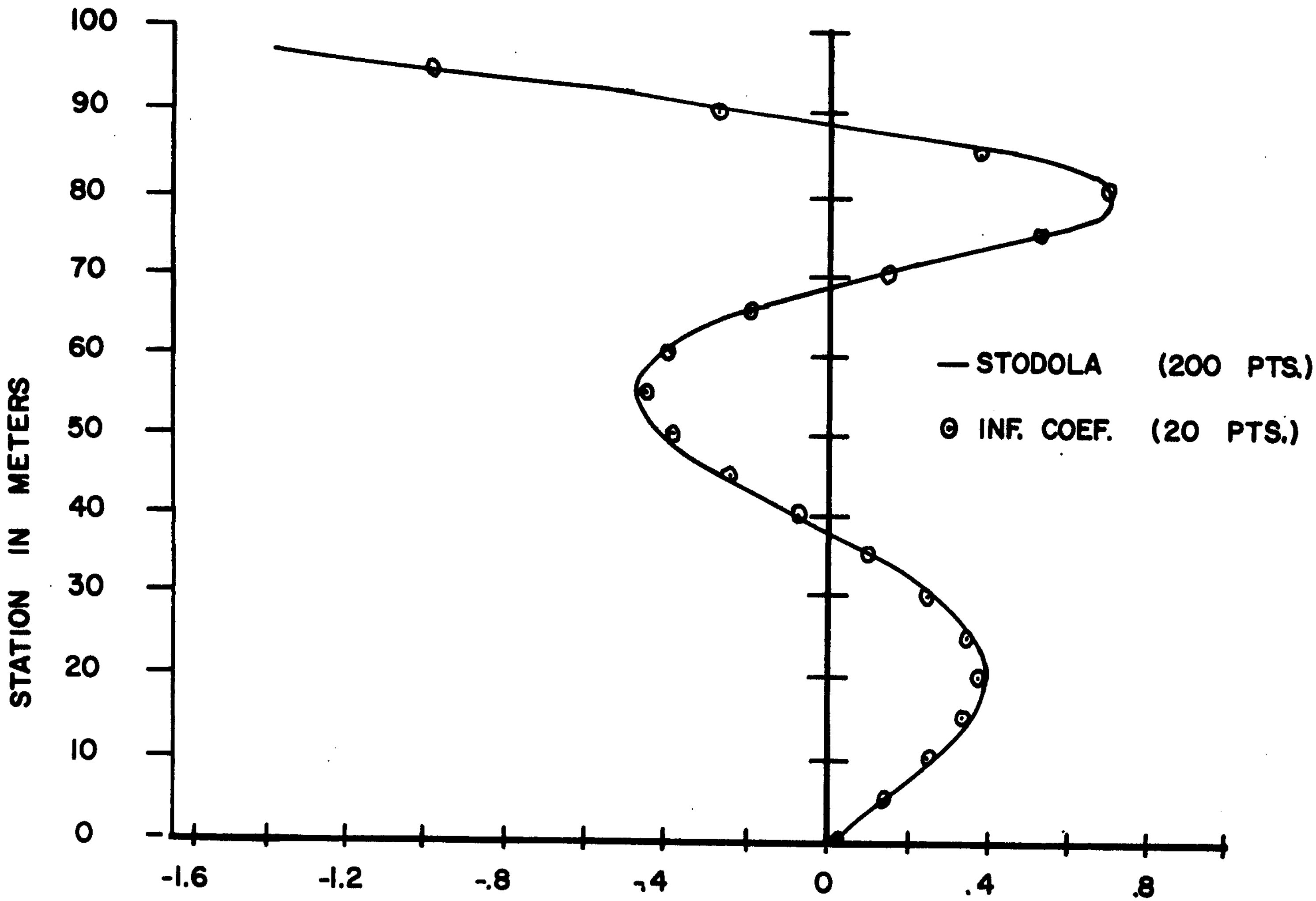


FIG. 8 FOURTH CANTILEVER BENDING MODE FOR A NON-UNIFORM BEAM

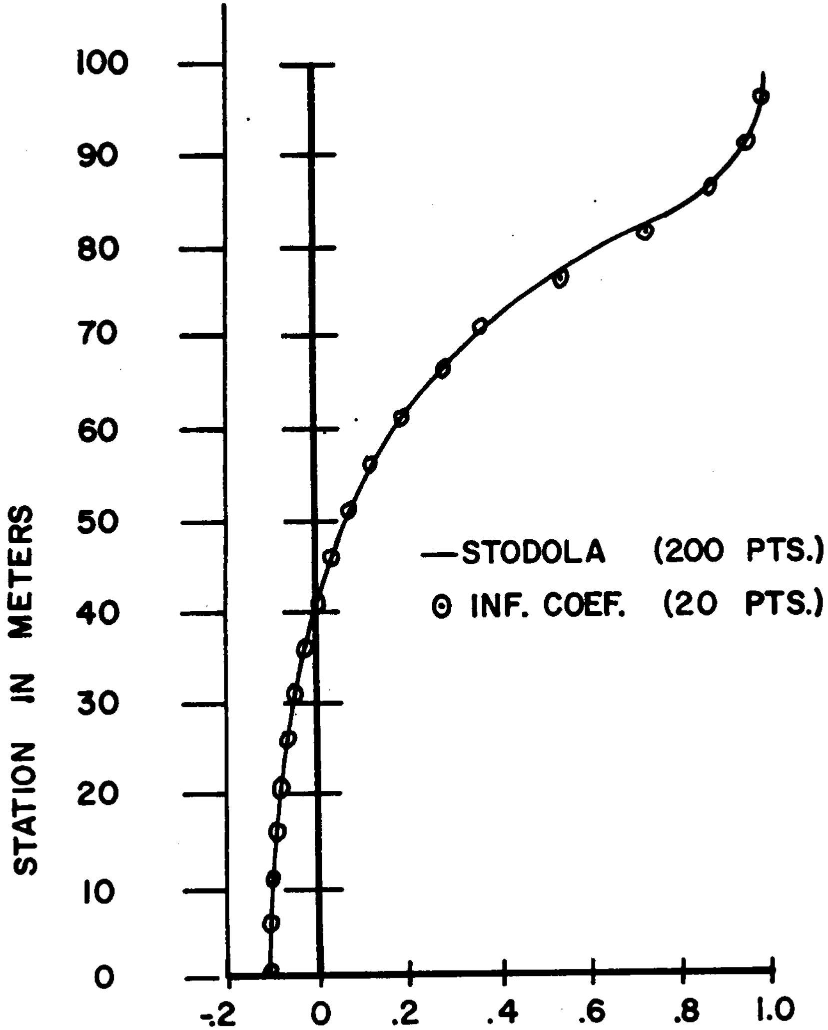


FIG. 9 FIRST FREE - FREE TORSION MODE FOR A NON-UNIFORM BEAM

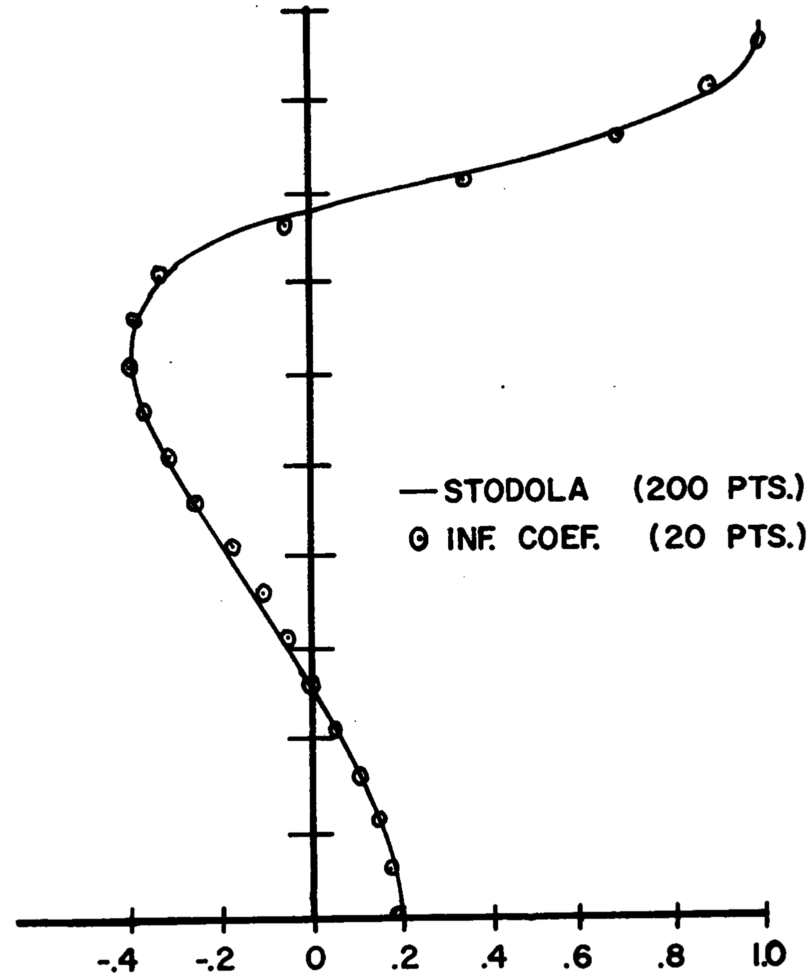


FIG. 10 SECOND FREE - FREE TORSION MODE FOR A NON-UNIFORM BEAM

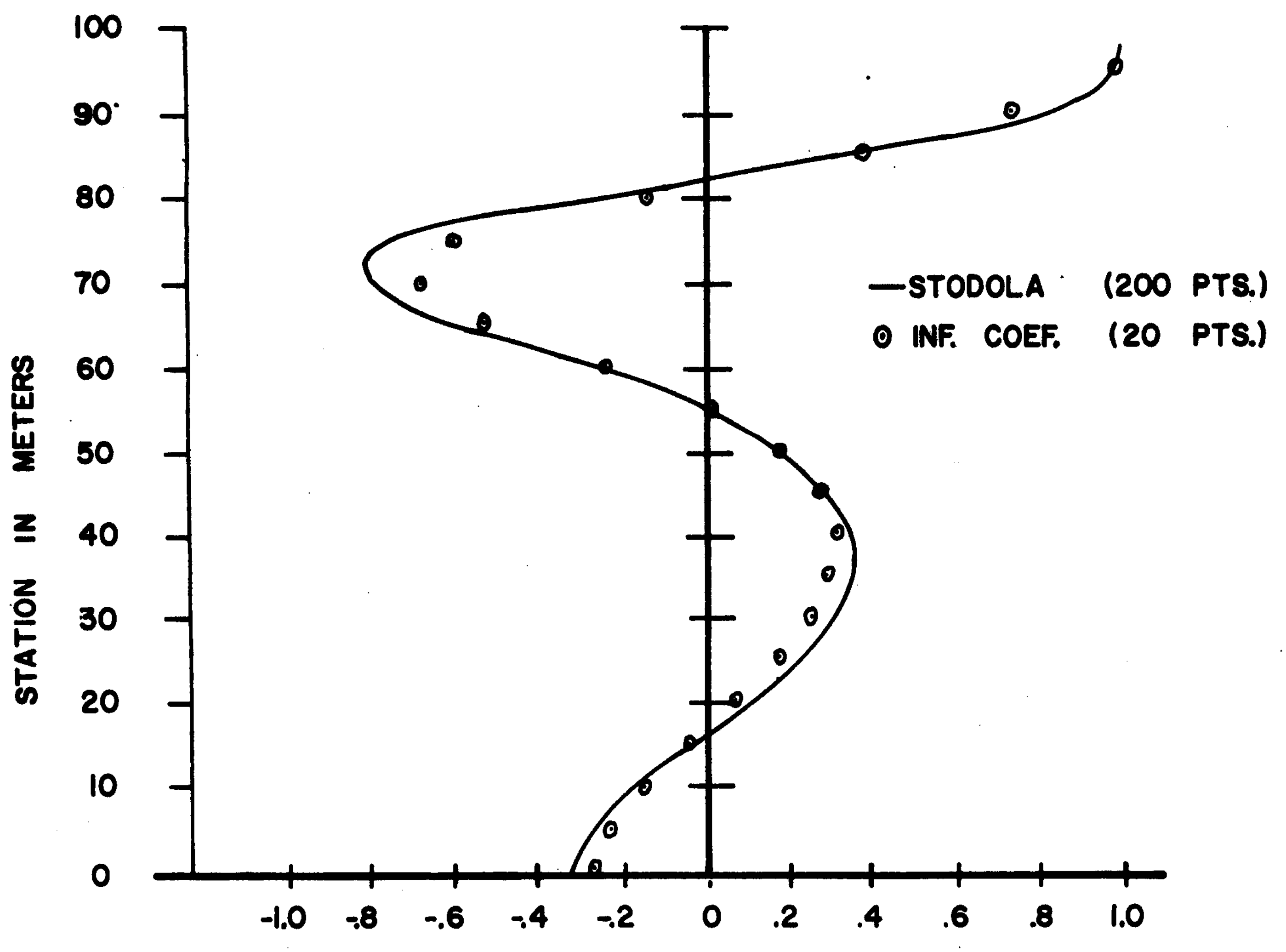


FIG. 11 THIRD FREE-FREE TORSION MODE FOR A NON-UNIFORM BEAM

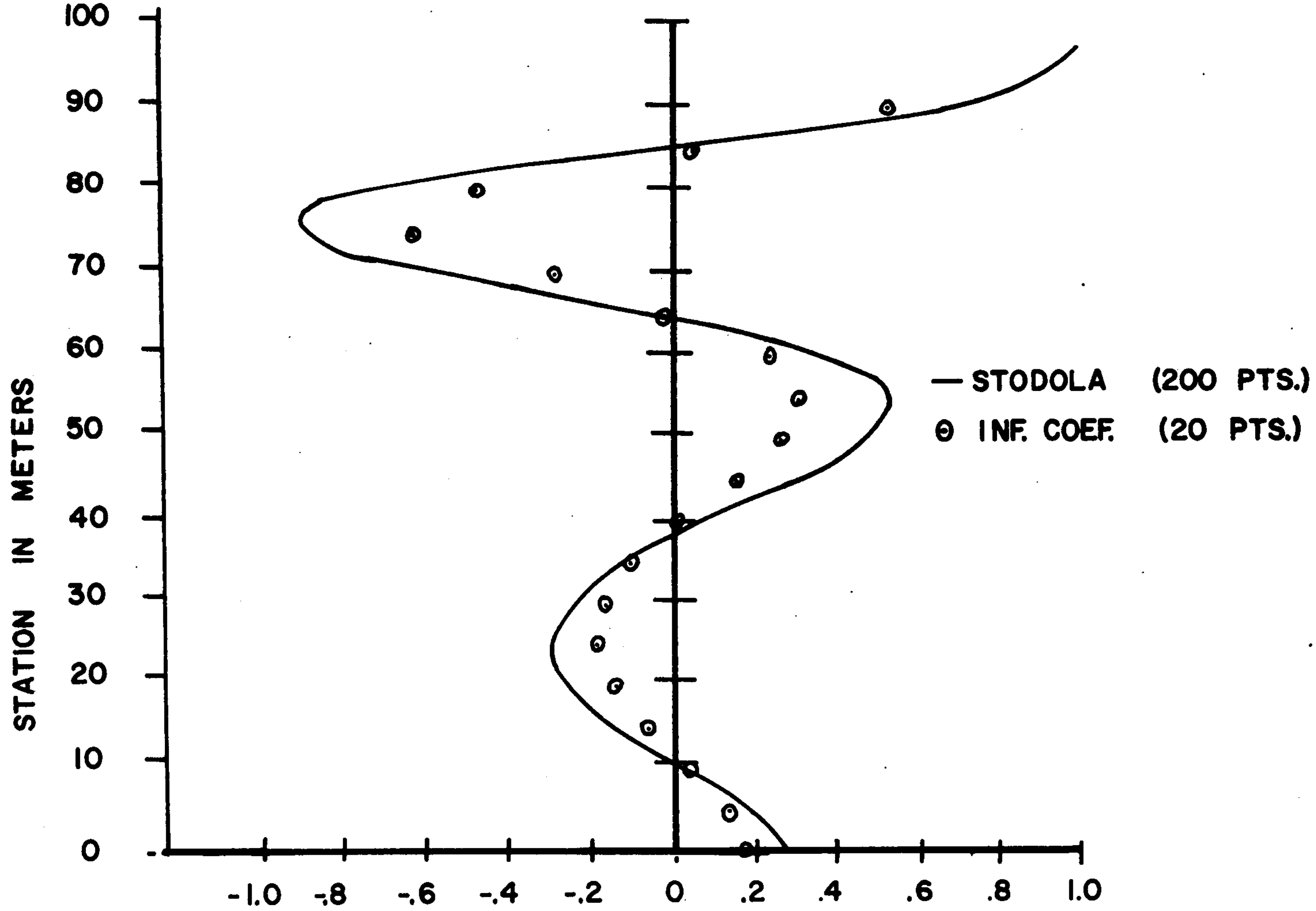


FIG. 12 FOURTH FREE - FREE TORSION MODE FOR A NON-UNIFORM BEAM

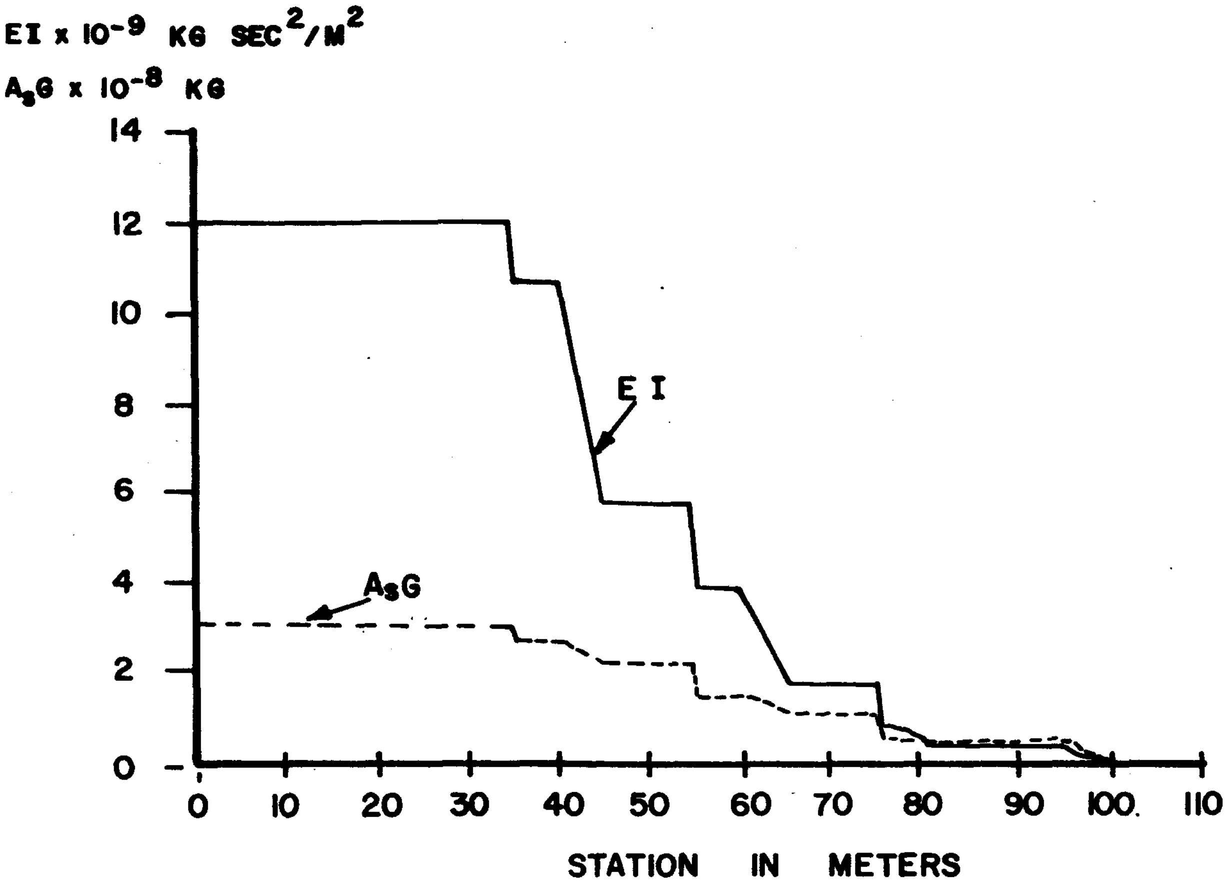
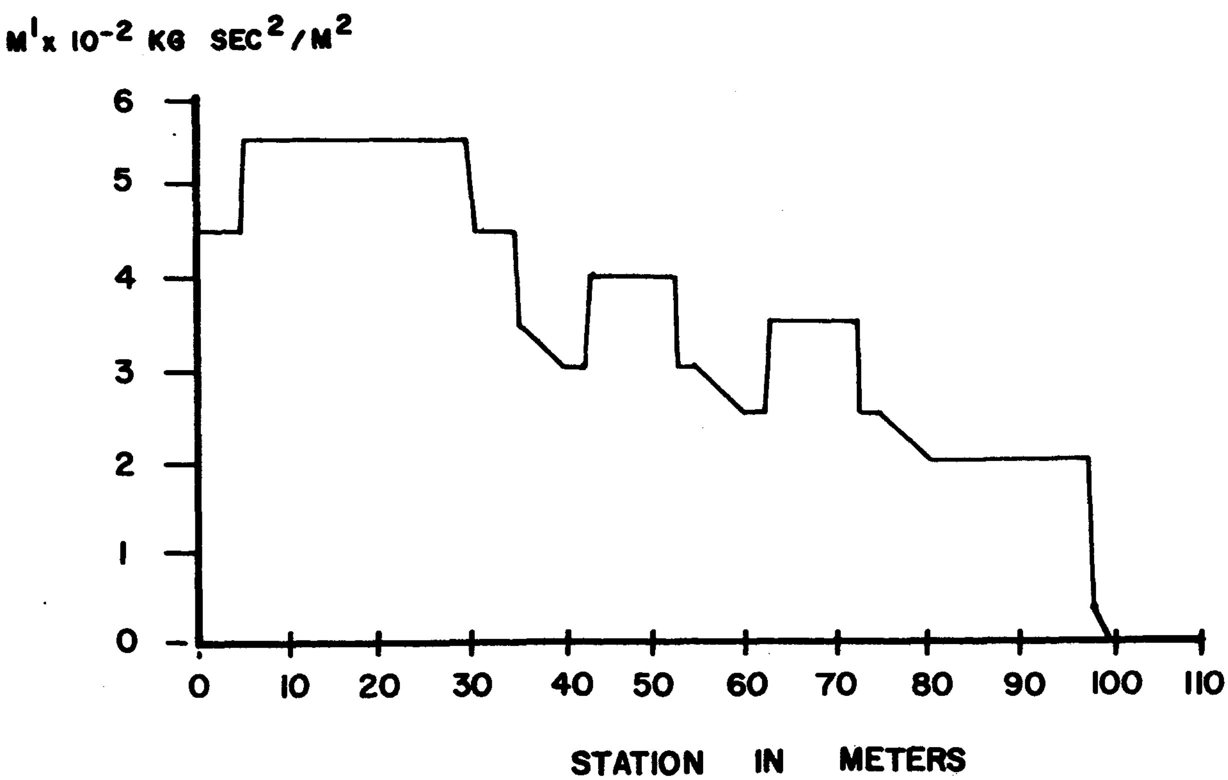


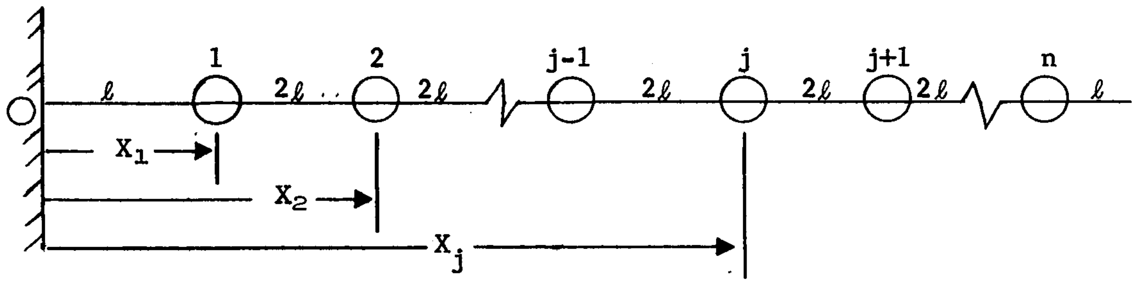
FIG. 13 STIFFNESS DISTRIBUTIONS



APPENDIX A

Cantilever Influence Coefficients for Bending

Consider a beam divided into n equal segments of length 2ℓ with the mass concentrated at the geometric center. Influence coefficients for a unit force may be written as follows:



Using the moment area method, the deflection at station (1) due to a force at (1) is

$$C_{11}^{FB} = \frac{\ell^3}{3E_o\ I_o} .$$

The centroid of the area under the M/EI diagram with respect to station (1) is

$$\bar{X}_1^{FB} = \frac{2\ell}{3}.$$

It then follows that

$$C_{i1}^{FB} = \frac{C_{11}^{FB}}{\bar{X}_1^{FB}} (\bar{X}_1^{FB} + X_i - X_1) \qquad \text{for } i > 1$$

$$C_{22}^{FB} = \left[\frac{5}{2}\frac{(X_2 - X_1)}{E_1\ I_1} + \frac{4}{3}\frac{X_2}{E_o\ I_o}\right]\ell^2 .$$

$$\bar{X}_2^{FB} = \frac{C_{22}^{FB}}{\left[2\frac{(X_2 - X_1)}{E_1\ I_1} + \frac{1}{6}\frac{X_2}{E_o\ I_o}\right]\ell}$$

$$C_{i2}^{FB} = \frac{C_{22}^{FB}}{\bar{X}_2^{FB}} (\bar{X}_2^{FB} + X_i - X_2) \qquad \text{for } i > 2.$$

The following general expressions now can be written:

$$C_{jj}^{FB} = 2\ell \sum_{i=1}^{j} \frac{(X_j - X_i)^2}{E_i \; I_i} \qquad \text{for } j > 2$$

and

$$\bar{X}_j^{FB} = \frac{C_{jj}^{FB}}{2\ell \displaystyle\sum_{i=1}^{j} \frac{(X_j - X_i)}{E_i \; I_i}} \qquad \text{for } j > 2$$

$$C_{ij}^{FB} = \frac{C_{jj}^{FB}}{\bar{X}_j^{FB}} (\bar{X}_j^{FB} + X_i - X_j) \text{ for } i > j > 2$$

$$C_{ij}^{FB} = C_{ji}^{FB}.$$

The influence coefficients C_{ij}^{MB} denote the deflection of station i due to a unit moment at j, which produces a bending deflection and may be written as follows:

$$C_{11}^{MB} = \left(\frac{\ell^2}{6E_1 \; I_1} + \frac{\ell^2}{3E_o \; I_o}\right).$$

The distance from the centroid of the $\frac{M}{EI}$ diagram to station (1) is

$$\bar{X}_1^{MB} = \frac{C_{11}^{MB}}{\dfrac{\ell}{2E_1 \; I_1} + \dfrac{\ell}{2E_o \; I_o}}.$$

For $i > 1$ then

$$C_{i1}^{MB} = \frac{C_{11}^{MB}}{\bar{X}_1^{MB}} (\bar{X}_1^{MB} + X_i - X_1),$$

and for $j > 1$

$$C_{jj}^{MB} = 2\ell \sum_{i=1}^{j} \left[\frac{(X_j - X_i)}{E_i \ I_i}\right] + \frac{\ell^2}{2E_j \ I_j}$$

$$\bar{X}_j^{MB} = \frac{C_{jj}^{MB}}{\ell \sum_{i=1}^{j} \left[\frac{2}{E_i \ I_i} - \frac{1}{E_j \ I_j}\right]} \qquad \text{for } j > 1$$

$$C_{ij}^{MB} = \frac{C_{jj}^{MB}}{\bar{X}_j^{MB}} (\bar{X}_j^{MB} + X_i - X_j) \qquad \text{for } i > j$$

$$C_{ij}^{MB} = C_{ii}^{MB} \qquad \text{for } i < j.$$

The influence coefficients θ_{ij}^{FB} are symmetrical with respect to C_{ij}^{MB} (cf. ref. 1); therefore, the following expressions may be written:

$$C_{ij}^{MB} = \theta_{ji}^{FB}$$

and

$$C_{jj}^{MB} = \theta_{jj}^{FB} .$$

The rotation of station i due to a unit moment at j, which produces bending, yields another set of influence coefficients θ_{ij}^{MB}. Again using the moment area method of reference 3,

$$\theta_{11}^{MB} = \frac{\ell}{E_1\ I_1} + \frac{\ell}{2} \left| \frac{1}{E_o\ J_o} - \frac{1}{E_1\ I_1} \right|$$

$$\theta_{i1}^{MB} = \theta_{11}^{MB} \qquad \text{for } i > 1$$

$$\theta_{jj}^{MB} = 2\ell \sum_{i=1}^{j} \left(\frac{1}{E_i\ I_i} \right) - \frac{\ell}{E_j\ I_j} \qquad \text{for } j > 1$$

$$\theta_{ij}^{MB} = \theta_{jj}^{MB} \qquad \text{for } i > j$$

and

$$\theta_{ij}^{MB} = \theta_{ji}^{MB} \qquad \text{for } i < j.$$

Next, shear deflections due to a unit force will be considered. These coefficients can be determined by multiplying two matrices containing the shear area at each station, A_Si, arranged in the following manner:

$$\begin{bmatrix} \frac{1}{\sqrt{A_{S1}}} & 0 & 0 & 0 & 0 & \cdots\cdots \\ \frac{1}{\sqrt{A_{S1}}} & \frac{1}{\sqrt{A_{S1}}} & \frac{1}{\sqrt{A_{S2}}} & 0 & 0 & \cdots\cdots \\ \frac{1}{\sqrt{A_{S1}}} & \frac{1}{\sqrt{A_{S1}}} & \frac{1}{\sqrt{A_{S2}}} & \frac{1}{\sqrt{A_{S2}}} & 0 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} & 0 \\ \frac{1}{\sqrt{A_{Si}}} & \end{bmatrix} \text{N x 2N-1}$$

$$\left[C_{ij}^{FS} \right] = \frac{\ell}{G} \begin{bmatrix} \ddots & & 0 \\ & \frac{1}{\sqrt{A_{Si}}} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & 0 \\ & \frac{1}{\sqrt{A_{Si}}} & \\ & & \ddots \end{bmatrix}^T .$$

The influence coefficients θ_{ij}^{FS} may be obtained by setting up a diagonal matrix of the $1/A_{si}$ values and multiplying by a triangular matrix as follows:

$$\left[\theta_{ij}^{FS} \right] = \frac{1}{2G} \begin{bmatrix} \ddots & & \\ & \frac{1}{A_{Si}} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & & & \end{bmatrix} = \frac{1}{2G} \begin{bmatrix} \ddots & & \\ & \frac{1}{A_{S_2}} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & 1 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix}$$

To determine the influence coefficients C_{ij}^{MS}, a diagonal matrix of $1/A_{si}$ is multiplied times the transpose of the above triangular matrix.

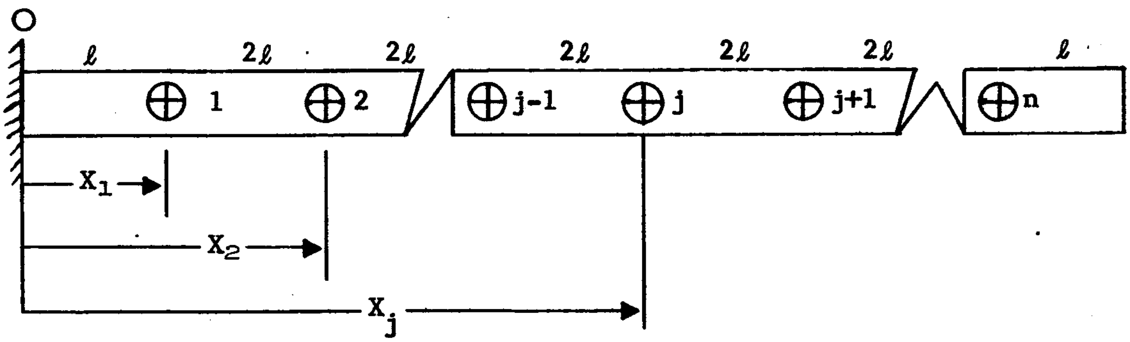
$$\left[C_{ij}^{MS} \right] = \frac{1}{2G} \begin{bmatrix} \ddots & & \\ & \frac{1}{A_{Si}} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & 1 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix}^T .$$

It is assumed that G is constant and therefore can be factored out of each of the matrices used to obtain C_{ij}^{MS}, θ_{ij}^{FS} and C_{ij}^{FS}.

APPENDIX B

Torsional Influence Coefficients

The torsional influence coefficients given in this appendix are written for a cantilever beam. R^T_{ij} denotes the rotation of station i due to a unit torque applied at j.



$$R^T_{11} = \frac{X_1}{G_o\ I_{p_o}}$$

and

$$R^T_{22} = R^T_{11} + \frac{X_2 - X_1}{G\ \ I_{P_1}} .$$

Therefore,

$$R^T_{ii} \text{ or } R^T_{jj} = R^T_{j-1,\ j-1} + \frac{X_j - X_{j-1}}{G_{j-1}\ I_{pj-1}} \qquad j > 1$$

$$R^T_{ij} = R^T_{ii} \quad \text{for } i < j$$

and

$$R^T_{ij} = R^T_{jj} \quad \text{for } i > j.$$

APPENDIX C

Obtaining Higher Modes

After the first mode shape and bending frequency have been determined, the higher modes and frequencies can be obtained as follows:

1. Iterate on the dynamic matrix from the front and obtain a characteristic row.

$$\begin{bmatrix} 1, 1, 1, \ldots \end{bmatrix} \begin{bmatrix} D_1 \end{bmatrix} = \begin{bmatrix} K_1, K_2, K_3, K_r \ldots \end{bmatrix}$$

D_1 is the original dynamic matrix.

2. Normalize to the r^{th} unknown

$$\begin{bmatrix} \frac{K_1}{K_r}, \frac{K_2}{K_r}, \frac{K_3}{K_r}, \ldots 1 \ldots\ldots \end{bmatrix}.$$

3. Form a square matrix with zeros for all elements in every row except the r^{th}. Insert the $\frac{K_i}{K_r}$ normalized row here. This will be called the E_1 matrix.

4. To obtain the new dynamic matrix D_2 for obtaining the second mode, perform the following operations:

$$\begin{bmatrix} D_2 \end{bmatrix} = \begin{bmatrix} D_1 \end{bmatrix} \Big[\begin{bmatrix} I \end{bmatrix} - \begin{bmatrix} E_1 \end{bmatrix} \Big]$$

where $\begin{bmatrix} I \end{bmatrix}$ is an identity matrix.

This same procedure is used to obtain the next mode, etc.

APPENDIX D

Shear Deflections Due to a Pure Moment

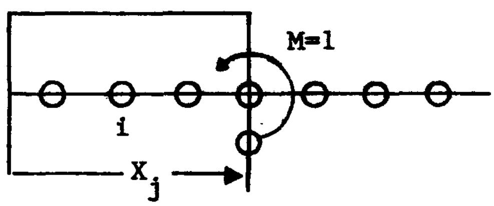
The equation for displacements due to shear deformation is

$$C_{ij}^{S} = \int_{o}^{L} \frac{\psi_s \; S \; d_x}{A \; G} \, ,$$

where s is the shear distribution due to a unit load and ψ is a correction factor used to obtain the proper shear area ($A_s = \frac{1}{\psi} A$). For a unit shear load, S = 1, the above equation can be written as follows:

$$C_{ij}^{S} = \int_{o}^{L} \frac{S \; d \; x}{A_s \; G} \, .$$

Next, consider a beam with a unit moment applied at some point j:



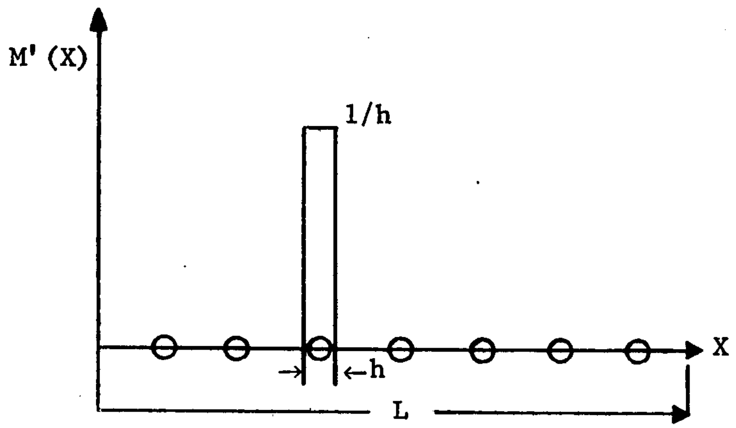
Since $\frac{dM}{dx} = s$ and $M' \; dx = s \; dx$,

$$C_{ij}^{MS} = \int_{o}^{L} \frac{M' \; dx}{A_s G} \, .$$

But $M' = \infty$ at $X = X_j$. However, the above integral can be evaluated by first considering M' as a unit finite impulse function as shown in the following figure.

$$M'(h, X - X_j) = \frac{1}{h} \quad \text{when } X_j < X < X_j + h$$

$$= 0 \quad \text{when } X_j + h < X < X_j$$



Substituting the impulse function for M' in C_{ij}^{MS} and letting $h \rightarrow 0$, we obtain

$$C_{ij}^{MS} = \lim_{h \rightarrow 0} \int_o^L \frac{M'(h, X - X_j)\, dx}{A_s G} = \left(\frac{1}{A_s G}\right)_{X \geq X_j}$$

$$= 0 \quad \text{for } X < X_j \,.$$

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