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DYNAMIC SATELLITE GEODESY

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Introduction - Many characteristics of the artificial satellite orbits useful to geodesy are similar to those of the planetary and natural satellite orbits which have long been the object of attention of celestial mechanics: slowly varying Keplerian ellipses of moderate eccentricity. Hence many of the classical methods can be applied. Characteristics peculiar to close satellites, and hence requiring special attention, are the capability of a wide variety of inclinations, the dominant perturbation by the earth's oblateness; the many other small perturbations by variations of the earth's gravitational field, which are the chief reason why satellites are of geodetic interest; the importance of atmospheric drag; and the irregular distribution of observations around the orbit. These characteristics, together with the availability of large high-speed computers, create some interesting new problems for satellite geodesy, despite the basic similarity to classical problems.

Celestial Mechanics - For purely gravitational effects on the orbit of a particle, the differential equations can be written:

$$\ddot{\mathbf{r}} = \nabla V(\mathbf{r}) \tag{1}$$

when $\mathbf{r}$ is the position vector, V is a scalar potential, and ∇ denotes the gradient. The three second order equations (1) can easily be converted to six first order equations by taking as variables in addition to the three position components $\mathbf{r}$ the three velocity components $\dot{\mathbf{r}}$. Then given a law for V and position and velocity at a particular time, $\underset{\sim}{r}_0$ and $\dot{\underset{\sim}{r}}_0$, the equations of motion can be readily solved by numerical integration.

However, other developments may be more conducive to insight or more economical of computer time. In the first approximation for the earth,

$$V = kM/r \tag{2}$$

where k is the gravitational constant, M is the earth's mass, and r is the radius vector. Use of (2) in (1) and conversion of (1) to spherical coordinates obtains as a first integral

$$r^2\dot{f} = h, \tag{3}$$

where $\dot{f}$ is the rate of angular motion about the origin in the plane defined by the position and velocity vectors of the satellite, and h is a constant. (3) is known as the integral of areas. Replacement of r by its inverse and use of (3) to eliminate dt in favor of df results in an equation which can be integrated to obtain 1/r in terms of f:

$$\frac{1}{r} = \frac{1 + e \cos f}{a(1-e^2)} \tag{4}$$

the equation for the radius vector of an ellipse with a semi-major axis a, an eccentricity e, the origin at a focus, and f, called the true anomaly, measured from the point of closest approach to the focus. We also get

$$h = [kMa\,(1-e^2)]^{\frac{1}{2}} \tag{5}$$

To completely specify the position of the particle, in addition to a, e, and f three more variables are required to express the orientation of the ellipse in the plane of motion: ω, the argument of perigee; and to express the orientation of the plane with respect to an inertial reference: i, the inclination, and Ω, the longitude of the node. See Figure 1 for the definition of these variables, which are identical with the Eulerian angles.

To integrate (3) to obtain the position in terms of time, we replace the true anomaly f by the eccentric anomaly E as follows: define the major axis as the abscissa; take a circle tangent to the ellipse at the ends of the major axis; and define the eccentric anomaly E as the angle subtended at the center of the circle by the point on the circle whose abscissa is the same as the point on the ellipse of true anomaly f. Then

$$r = a(1 - e \cos E) \tag{6}$$

Differentiating (4) and (6) and equating the two expressions for dr, df can be eliminated for dE and a form of (3) is obtained which can be integrated to:

$$E - e \sin E = kM^{\frac{1}{2}} a^{-3/2} (t - t_0)$$

$$= n(t - t_0) = M \tag{7}$$

where t_0 is the time of passing the point of closest approach to the origin, called perigee; (7) is called Kepler's equation. n is called the mean motion; and M is called the mean anomaly.

For cases where the potential V in (1) does not have the central form (2), but is instead:

$$V = \frac{kM}{r} + R, \tag{8}$$

where R is known as the disturbing function, the Keplerian ellipse and its orientation can still be regarded as a coordinate system: at any instant the situation of a satellite in inertially fixed geocentric coordinates can be expressed by the six numbers $\{a,e,i,M,\omega,\Omega\}$ instead of the six numbers $\{x,y,z,\dot{x},\dot{y},\dot{z}\}$. By using the partial derivatives of $\underset{\sim}{r}$, $\dot{\underset{\sim}{r}}$ with respect to the Keplerian elements, the six differential equations in terms of $\underset{\sim}{r}, \dot{\underset{\sim}{r}}$ can be converted, after considerable algebra, into:

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{de}{dt} &= \frac{1-e^2}{na^2e}\frac{\partial R}{\partial M} - \frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\cdot\frac{\partial R}{\partial \omega} \\
\frac{d\omega}{dt} &= -\frac{\cos i}{na^2(1-e)^{\frac{1}{2}}\sin i}\cdot\frac{\partial R}{\partial i} + \frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\cdot\frac{\partial R}{\partial e} \\
\frac{di}{dt} &= \frac{\cos i}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\cdot\frac{\partial R}{\partial \omega} - \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\cdot\frac{\partial R}{\partial \Omega} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\cdot\frac{\partial R}{\partial i} \\
\frac{dM}{dt} &= n - \frac{1-e^2}{Na^2e}\cdot\frac{\partial R}{\partial e} - \frac{2}{na}\cdot\frac{\partial R}{\partial a}
\end{aligned}\right\} \tag{9}$$

This form of the equations of motion is known as the Lagrangian planetary equations.

<u>The Earth's Gravitational Field</u>. To use equation (9), the potential V, including the disturbing function R, must be expressed in terms of Keplerian

elements. Because of the strong smoothing effects of the extrapolation to altitude and the integration to obtain satellite position, the most convenient representation of the Earth's gravitational field for application to orbits is in spherical harmonics (see "Use of Spherical Harmonics in Physical Geodesy" by U. A. Uotila and "The Figure of the Earth" by W. M. Kaula):

$$V = \frac{kM}{r}\left[1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n P_{nm}(\sin \Psi)\,(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\right] \quad (10)$$

where a_e is the equatorial radius, Ψ is the geocentic latitude, $P_{nm}(\sin \Psi)$ is the Legendre Associated Polynomial, λ is the longitude, and C_{nm}, S_{nm} are independent coefficients. The interest of geodesy in close satellite orbits is to use their perturbations to determine these coefficients C_{nm}, S_{nm}. By a purely geometrical transformation, treating the Keplerian ellipse as an alternative to spherical polar coordinates, there is obtained from (10) a disturbing function (Kaula, 1961):

$$R = \frac{kM}{a} \sum_{n=2}^{\infty} \left(\frac{a_e}{a}\right)^n \sum_{m=0}^{n} \sum_{p=0}^{n} F_{nmp}(i) \sum_{q=-\infty}^{\infty} G_{npq}(e) \times$$

$$\times \left[\begin{Bmatrix} C_{nm} \\ -S_{nm} \end{Bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \cos \{(n-2p)\,\omega + (n-2p+q)M + m(\Omega-\theta)\} \right.$$

$$\left. + \begin{Bmatrix} S_{nm} \\ C_{nm} \end{Bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \sin \{(n-2p)\,\omega + (n-2p+q)\,M + m(\Omega-\theta)\} \right] \quad (11)$$

where θ is the Greenwich Sidereal Time, and

$$F_{nmp}(i) = \sum_{t} \frac{(2n-2t)!}{t!(n-t)!(n-m-2t)!2^{2n-2t}} \sin^{n-m-2t} i \times$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^{s} i \sum_{c} \binom{n-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \qquad (12)$$

In (12), k is the integer part of $(n-m)/2$; t is summed from 0 to the lesser of p or k, and c is summed over all values making the binomial coefficients non-zero. $G_{npq}(e)$ is always $O(e^{|q|})$. The general expression for $G_{npq}(e)$ is rather complicated. There is, however, a simpler form for the more important long period terms for which M is absent from the argument; i.e., for which q is $(2p-n)$:

$$G_{np(2p-n)}(e) = \frac{1}{(1-e^{2})^{n-\frac{1}{2}}} \sum_{d=0}^{p'-1} \binom{n-1}{2d+n-2p'} \binom{2d+n-2p'}{d} \left(\frac{e}{2}\right)^{2d+n-2p'} \qquad (13)$$

where $p' = p$ for $p \leq n/2$, and

$p' = p$ for $p \geq n/2$.

The form of (11) indicates that the elements a,e,i appear in every term of the disturbing function R, but that by an appropriate combination of subscripts n,m,p,q the elements ω, M, Ω and the sidereal time θ may be absent from certain terms. One such combination is nmpq = 2010, for which the coefficient C_{20} is the dominant coefficient in R, expressing the earth's oblateness. Use of this term as the disturbing function in the equations of motion (9) results in no perturbation of a,e, or i, but in constant changes in ω, Ω, and M:

$$\left.\begin{aligned}
\frac{d\omega}{dt} &= \frac{3nC_{20}}{4(1-e^2)^2}\left(\frac{a_e}{a}\right)^2 (1-5\cos^2 i) \approx 3.35\,(5\cos^2 i-1)\ \text{degrees/day} \\
\frac{d\Omega}{dt} &= \frac{3nC_{20}}{2(1-e^2)^2}\left(\frac{a_e}{a}\right)^2 \cos i \approx -6.70\cos i\ \text{degrees/day} \\
\frac{dM}{dt} &= n - \frac{3nC_{20}}{4(1-e^2)^{3/2}}\left(\frac{a_e}{a}\right)^2 (3\cos^2 i-1) \approx 14.37 + .0093\,(3\cos^2 i-1)\ \text{revs/day}
\end{aligned}\right\} \quad (14)$$

where (12) and (13) have been used to obtain $F_{201}(i)$ and $G_{210}(e)$, and the numerical examples are based on a C_{20} of -.000484 and typical orbit specifications of .01 eccentricity and 1.12 a_e semi-major axis.

Rates of secular motion of the perigee and node as specified by (14) are the dominant characteristic observed in satellites which are high enough not to be in a state of rapid decay due to atmospheric drag. This characteristic is so dominant that for the effect of other terms in the earth's field, as given by (11), the integration can be performed assuming that a,e, and i are constant and ω,Ω, and M have a constant rate of change on the right hand side of (9). For example, the perturbation of the node by a term of subscripts n,m,p,q

$$\Delta\Omega_{nmpq} = \frac{kMa_e^n\,(\partial F_{nmp}/\partial i)\,G_{npq}}{na^{n+3}(1-e^2)^{1/2}\sin i\,\{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})\}}$$

$$\left[\begin{Bmatrix} C_{nm} \\ -S_{nm}\end{Bmatrix}_{n-m\ \text{odd}}^{n-m\ \text{even}} \sin\,\{(n-2p)\omega + (n-2p+q)M + m(\Omega-\theta)\}\right.$$

$$\left. - \begin{Bmatrix} S_{nm} \\ C_{nm}\end{Bmatrix}_{n-m\ \text{odd}}^{n-m\ \text{even}} \cos\,\{(n-2p)\omega + (n-2p+q)M + m(\Omega-\theta)\}\right] \qquad (15)$$

A particular term which has been the object of considerable orbit analysis is $\Delta\Omega_{2210}$, the semi-daily perturbation of the node by the equatorial ellipticity:

$$\Delta\Omega_{2210} = \frac{kMa_e^2(\partial F_{221}/\partial i)G_{210}}{2na^3(1-e^2)^{\frac{1}{2}}\sin i(\dot{\Omega}-\dot{\theta})} [C_{22} \sin 2(\Omega-\theta) - S_{22} \cos 2(\Omega-\theta)]$$

$$= \frac{3kMa_e^2 \cos i}{2na^3(1-e^2)^2(\dot{\Omega}-\dot{\theta})} [C_{22} \sin 2(\Omega-\theta) - S_{22} \cos 2(\Omega-\theta)] \qquad (16)$$

The appearance of the rate $\{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})\}$ in the denominator of (15) and the rates given by (14) indicate that the largest perturbations should be those for which $(n-2p+q)$ and m both vanish: i.e., long period or secular effects of the zonal harmonics C_{no}, and that the largest effect of a tesseral harmonic C_{nm}, S_{nm} (for which $m \neq 0$) will normally be one for which $(n-2p+q)$ vanishes.

Non-linear Perturbations. Th description of the orbit as a Keplerian ellipse described by equations (6) and (7) plus secular motions (14) and linear periodic perturbations such as (15) is inadequate for geodetic purposes. Because the oblateness C_{20} is about 1000 times larger than the other coefficients C_{nm}, S_{nm}, there are to be expected non-linear effects of coefficient C_{20}^2 which are of comparable magnitude to the linear effects of coefficients C_{nm}, S_{nm}. As in any non-linear problem, there are many ways to proceed. Some theories develop (9) or similar equations to the next higher order by substituting the linear approximation on the right (e.g., Kozai, 1959b). The algebra entailed in such a development is considerable, however. An alternative frequently adopted is to transform

(9) to a simpler form by a change of variables; the most popular such set are the Delaunay variables:

$$\begin{aligned} L &= (\mu a)^{\frac{1}{2}}, & \ell &= M \\ G &= [\mu a(1-e^2)]^{\frac{1}{2}}, & g &= \omega \\ H &= [\mu a(1-e^2)]^{\frac{1}{2}}\cos i, & h &= \Omega \end{aligned} \tag{17}$$

If we set

$$F = \frac{\mu}{2a} + R, \tag{18}$$

known as the force function, then (9) becomes:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial \ell}, & \frac{d\ell}{dt} &= -\frac{\partial F}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial F}{\partial \omega}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G} \\ \frac{dH}{dt} &= \frac{\partial F}{\partial \Omega}, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H} \end{aligned} \tag{19}$$

Normally a closed solution to (19) with F as in (18) cannot be found. The solution customarily adopted is to take the known solution for another force function F' close to F and express the solution for F as Taylor series for the small differences $L-L'$, $G-G'$, etc. Continuity conditions require that the terms of these Taylor series all be expressible as derivatives of a single scalar function, called the determining function. The solution for F' is called the intermediate orbit. Different theories usually differ in their selection of the intermediate orbit. A theory which has a simple intermediary will have complications in developing the determining function and its derivatives (e.g., Brouwer, 1959; Kozai, 1962); while a theory which has an intermediary which accounts for nearly all the motion will entail considerable algebra in expressing position and velocity in the intermediary (e.g., Vinti, 1959, 1961; Izsak, 1960).

Some theories have intermediaries which are not defined by a force function F', but rather by geometrical conditions, such as those of Merson (1961) and Musen (1959, 1961). The theory of Musen (1959) also differs in that it is designed for solution by numerical iteration, rather than by analytical series development.

Most standard close satellite theories break down for orbits which have an eccentricity or inclination around zero. The remedy is to use orbital elements which do not require a distinct perigee or node for their precise numerical definition.

Another cause of breakdown is that for some combination of subscripts n,m,p,q the rate appearing in the denominator of the perturbation, such as (15), approaches zero:

$$(n-2p)\,\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta}) \approx 0, \tag{20}$$

so that a resonant situation occurs. The one which has received the most attention is that of the critical inclination, for which $\dot{\omega}$ is zero (e.g., Izsak, 1963). Resonant orbits of more interest to geodesy are those of synchronous communication satellites, for which $(\dot{\omega} + \dot{M} + \dot{\Omega} - \dot{\theta})$ is zero. These orbits will resonate with any term in the gravitational field for which (n-m) is even, since it will have the near-zero rate for a subscript p of (n-m)/2 and q zero. However, the semi-major axis of such an orbit is so large--about 6.6 earth radii-- that it probably would be a sensitive determiner of only the lowest degree such harmonic: C_{22}, S_{22}.

The stable point for the resonance will be the longitude of the minimum for the equatorial ellipticity, $[\tan^{-1}(S_{22}/C_{22}) \pm \pi]/2$. In the first approximation, the solution is similar to that for a pendulum, with the longitude of the satellite expressed by an elliptic integral. If the initial longitude is sufficiently close to the stability point, the longitude will librate about the point; if it is outside these limits, the satellite will drift. The rate of motion will be proportionate to $(C_{22}^2 + S_{22}^2)^{\frac{1}{4}}$. See Musen and Bailie (1962) and Morando (1963).

Other Perturbations. The most important other effect on close satellites of geodetic interest is atmospheric drag. Because of the much greater density in vicinity of perigee, the principal effect of drag is a slow decrease in the eccentricity and the semi-major axis. The rate of this decrease will vary with the rotation of the perigee with respect to the bulge of the atmosphere toward the sun, and with the intensity of the solar flux. In addition, there will be an effect on the rates of motion of the node and perigee due to interaction between the drag and the earth's oblateness, which must be considered in analyzing secular changes to determine the even degree zonal harmonics. All these variations make it impracticable to develop an analytical theory of drag. Numerical integration or numerical harmonic analysis is necessary to apply any realistic atmospheric model. Furthermore, there are incalculable irregular variations with respect to even the best of atmospheric models; as a consequence arbitrary polynomials are more frequently used. For geodetic satellites it is desirable to keep the perigee high enough and area-to-mass ratio low enough, so that irregular variations whose wave lengths are not long with respect to the spacing of observations are negligible. For perigees above 800 or 1000 kilometer altitude and area-to-mass ratios less than 0.2 $cm^2\ gm^{-1}$, the amplitude of irregular variations of less than monthly period are generally less than 5×10^{-5}

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FIGURES

1. Geometry of a Keplerian orbit.

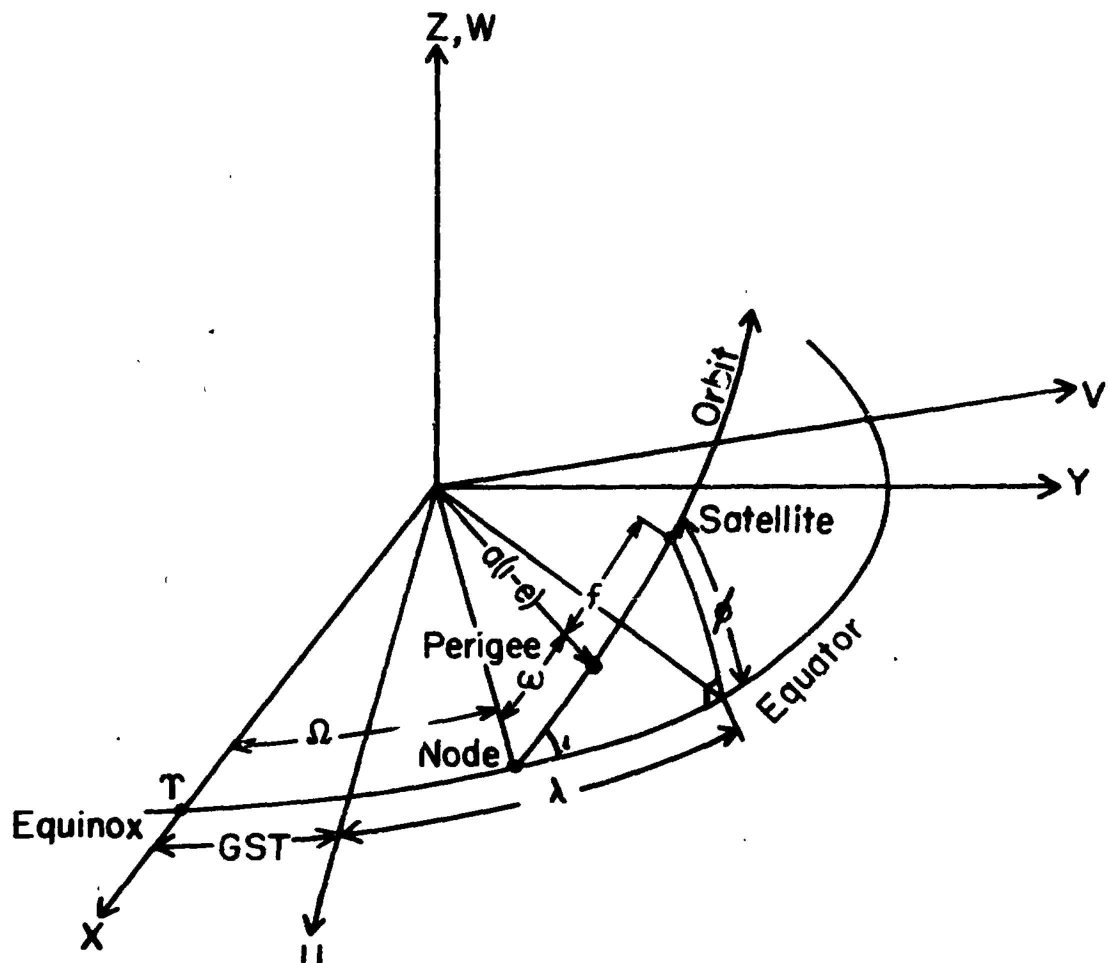
Z,W
V
Y
Orbit
Satellite
f
a(1-e)
Perigee
ω
Ω
Node
ι
φ
Equator
λ
♈
Equinox
GST
X
U

Fig. 1