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THEORETICAL STUDY OF ORBITAL ELEMENT CONTROL WITH POTENTIAL APPLICATIONS TO MANNED SPACE MISSIONS

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Moffett Field, Calif.

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SUMMARY

To facilitate the modification of established lunar or planetary orbits, a simple method has been developed for determining the impulsive velocity increments required to make prescribed changes in the orbital elements. The direct determination of the velocity increments required to make prescribed changes in the elements of planetary orbits proceeds from a knowledge of the time rates of change of the orbital elements in the presence of perturbing forces. The simplification inherent in the present method involves the replacement of a set of differential equations relating the time rates of change of the orbital elements to the components of the perturbing force by a set of ordinary equations relating the orbital element increments to the components of the impulse vector. These equations are then solved by conventional methods to obtain the impulse vector required to make prescribed changes in the orbital elements. This procedure necessitates the omission of second and higher order quantities. However, a 7090 digital computer program has been used to compute the errors involved, and the results obtained indicate that the errors are well within acceptable limits for the type of maneuvers contemplated. The equations have been mechanized on the 7090 digital computer and used to compute the impulsive velocity increments required to make independent changes in the orbital elements of a variety of earth and lunar orbits. In addition, the velocity increments required to produce orbital element changes of the type required to execute lunar reconnaissance maneuvers have been computed and the results presented in graphical and tabular form. Specifically, the velocity increments required to modify a lunar orbit with a semimajor axis of 1,400 statute miles and an eccentricity of 0.2 have been computed. Such an orbit would be of value if the object of a reconnaissance mission were to observe the lunar surface in the vicinity of perilune. Because the use of high eccentricity orbits can greatly reduce the velocity increments required to produce prescribed changes in orbit plane orientation, a lunar orbit with a semimajor axis of 2,200 statute miles and an eccentricity of 0.5 has been used to further illustrate the method. It is assumed that such orbits would be established by successive applications of the method described in this report.

INTRODUCTION

Considerable research by Ames Research Center and other research groups has been directed towards reducing the complexity of guidance and control systems for manned space missions. References 1 and 2, for example, present results which describe some relatively simple navigation and guidance systems that have application to the midcourse return phase and the earth reentry phase of a manned lunar mission. As part of a continuing effort in this direction, an analytical study was undertaken to determine the possibility of devising simple methods of computing the impulsive-thrust-vector requirements for producing prescribed changes in the elements of an orbiting space vehicle. In order to determine the impulses required to change the size and shape of established orbits by the methods described in this report, it is necessary to know the semimajor axis, the eccentricity, and the true anomaly in the established orbit. In the case of earth orbits, the necessary orbital element data can be obtained from ground tracking stations. However, in the case of extraterrestrial missions, other back-up methods may be desirable. One method by which the semimajor axis and the eccentricity can be obtained is from a series of range determinations by the use of the parallax method. The determination of range by this method requires a knowledge of the radius of the target body and the angle subtended at the vehicle by this radius. Before true anomaly can be determined it is necessary to establish the position and direction of the line of apsides. However, discussion of the various methods of determining the orbit from an orbiting space vehicle is considered outside the scope of the present report, where it is assumed that the required orbital element data are available. This study was motivated, in part, by the potential application of simplified computational procedures to space missions where fairly precise control of the orbital elements is desired, for example, lunar or planetary reconnaissance missions, the shaping of parking orbits prior to injection into lunar or interplanetary trajectories, and the removal of the secular changes produced by the perturbations of the planet's gravitational field (ref. 3). The present report has two main objectives:

(1) To describe a simple computational procedure for determining the impulsive velocity increments required to produce prescribed changes in the orbital elements of a space vehicle

(2) To illustrate the potential application of this procedure to a lunar reconnaissance mission where prescribed orbital element changes may be made with relatively simple on-board computations.

SYMBOLS

a semimajor axis, statute miles

a_{ij} matrix element in the ith row and the jth column

C_{ij} matrix element in the ith row and the jth column

e orbit eccentricity

F_h normal component of force

F_r radial component of force

F_θ transverse component of force

G universal constant of gravitation

$\bar{h}$ angular momentum per unit mass

$\hat{h}$ unit vector in the positive direction of the angular momentum vector, $\frac{\bar{h}}{h}$

i orbit plane inclination

$\hat{i},\hat{j},\hat{k}$ a triad of inertially fixed, mutually orthogonal unit vectors (fig. 20)

I_1,I_2,I_3 components of an impulse vector

I_a impulse required to make a prescribed change in the semimajor axis

I_{ae} impulse required to make prescribed changes in the semimajor axis and the eccentricity

I_e impulse required to make a prescribed change in the eccentricity

I_h normal component of the impulse vector

I_i normal impulse required to make a prescribed change in the orbit plane inclination

I_r radial component of the impulse vector

I_θ transverse component of the impulse vector

I_ω radial impulse required to make prescribed changes in the argument of perifocus

I_{ω_a} radial impulse required to null the changes induced in the argument of perifocus by I_a

$I_{\omega_{ae}}$ radial impulse required to null the changes induced in the argument of perifocus by I_{ae}

I_{ω_e} radial impulse required to null the changes induced in the argument of perifocus by I_e

I_Ω normal impulse required to make a prescribed change in the longitude of the ascending node

l semilatus rectum, $a(1 - e^2)$

m	mass of space vehicle
M	mass of the central body
$\overline{P}$	perturbing force vector
$\overline{r}$	position vector of vehicle in its orbit
$\hat{r}$	unit vector in the positive direction of the position vector, $\frac{\overline{r}}{r}$
$\overline{T}$	thrust vector
v	true anomaly
β	thrust direction measured from the local tangent to the flight path
β_a	direction of I_a as measured from the local tangent to the flight path
β_{ae}	direction of I_{ae} as measured from the local tangent to the flight path
β_e	direction of I_e as measured from the local tangent to the flight path
γ_a	direction of I_a as measured from the local horizontal
γ_{ae}	direction of I_{ae} as measured from the local horizontal
γ_e	direction of I_e as measured from the local horizontal
δ	increment of the associated variable
$\delta\omega_a$	changes in the argument of perifocus associated with prescribed changes in the semimajor axis
$\delta\omega_{ae}$	changes in the argument of perifocus associated with prescribed changes in the semimajor axis and the eccentricity
$\delta\omega_e$	changes in the argument of perifocus associated with prescribed changes in the orbit eccentricity
Δ	2×2 determinant
θ	argument of latitude, $\omega + v$
$\hat{\theta}$	unit vector in the direction of increasing anomaly
μ	dynamical constant of gravitation, $G(M + m)$
φ_{et}	extraterrestrial gravitational potential function
φ_{ob}	gravitational potential function for an oblate planet

ω	argument of perifocus as measured from the ascending node line
$\tilde{\omega}$	argument of perifocus, $\omega + \Omega$
Ω	longitude of ascending node line
$\hat{\Omega}$	unit vector in the direction of the ascending node line
$\nabla\varphi$	gradient of the gravitational potential function
$^\circ$	degrees of arc
$(\ )^{-1}$	inverse of a matrix
$(\ )^{T}$	transpose of a matrix

Subscripts

1,2,3	components of subscripted impulse
a,e,ae	orbital element changes resulting in changes in the subscripted variable
a,e,ae, i,Ω,ω	orbital element to be changed by the subscripted impulse
et	represents the effects of extraterrestrial gravitation of subscripted potential function
i,j	row and column, respectively, to which subscripted matrix element belongs
ob	represents the effects of oblateness of subscripted potential function
r,θ,h	radial, transverse, and normal components, respectively, of the subscripted variable
ω_a	orbital element to be changed by the impulse subscripted by a
ω_{ae}	orbital element to be changed by the impulse subscripted by ae
ω_e	orbital element to be changed by the impulse subscripted by e

ANALYSIS

The influence of thrust and gravitational forces on the elements that determine the size, shape, and orientation of the orbit of a space vehicle may be determined by the use of Lagrange's planetary equations. The five elements that determine the size, shape, and orientation of a satellite orbit have to be controlled by a force vector having three components. It follows that the application of a force vector at an arbitrary point on an orbit will, in general, produce changes in all the orbital elements. For example, the application of a force vector which has a component normal to the orbit plane has the effect of producing a gyroscopic precession of the orbit plane; and changes in the orbit plane orientation give rise to changes in the argument of perifocus. Moreover, a force which lies wholly in the orbit plane has no influence on the orbit plane orientation, but does produce changes in the semimajor axis, the orbit eccentricity, and the argument of perifocus. If the differential equations for the time rates of change of the orbital elements of a space vehicle in a non-Keplerian force field are considered, it is possible to make a direct determination of the impulses required to produce prescribed changes in the elements if second-order terms are neglected. If the second-order quantities are omitted the differential equations relating the time rates of change of the orbital elements to the components of the thrust vector can be replaced by a set of ordinary equations relating the orbital element increments to the components of the impulse vector. In addition to the influence of thrust which may be treated as a controlled perturbation, the equations for the time rates of change of the orbital elements can be used to study the effect of the earth's oblateness and extraterrestrial gravitational perturbations. If required, the terms representing the effect of the earth's oblateness can be used to determine the changes produced in the elements of a parking orbit by the earth's gravitational perturbations. These changes, which depend on the length of time spent in orbit and on the orbit inclination, consist of periodic and secular variations. In considering the effect of the gravitational forces produced by an oblate planet having a potential function given by an infinite series of zonal harmonics, it is found that the greatest influence is exerted by the second harmonic of the planet's gravitational field. Although the influence of higher gravitational harmonics may be relatively smaller than the second, the cumulative effect of these harmonics is significant if the time spent in orbit is long enough to permit the secular changes to accumulate.

Differential Equations

A set of differential equations which may be used to study the influence of a perturbing force on the orbital elements of an orbiting space vehicle is derived in reference 4. See also references 5 and 6. For an alternative vectorial derivation, see appendix B. These equations give the time rates of change of the orbital elements as functions of the radial, transverse, and the normal components of the perturbing force vector. Since the present study is concerned only with those elements that determine the size, shape, and orientation of an orbit, the influence of perturbing forces on the time of perifocal passage is omitted. The appropriate equations as derived in reference 4 are reproduced in appendix A. To conform to the most widely accepted space-age usage, changes of notation are introduced where necessary. The rates of change of the orbital elements are all expressed as functions of the true anomaly with the exception of the equation for the rate of change of the eccentricity. This equation is seen to depend on the true anomaly and the eccentric anomaly. For present applications, it is more convenient to have all the orbital element rates expressed as functions of the true anomaly only. For this reason, the eccentric anomaly is removed from the equation by expressing it as a function of the true anomaly and the eccentricity. The differential equations for the rates of change of the appropriate elements may be written in abbreviated form as follows:

$$\dot{\epsilon}_i = \frac{\partial \dot{\epsilon}_i}{\partial F_j} F_j = a_{ij} F_j \tag{1}$$

where

$$i = 1, 2 \ldots 5$$

$$j = 1, 2, 3$$

$$a_{ij} = a_{ij}(\epsilon_i)$$

The repeated subscript on the right-hand side of equation (1) denotes a summation. When equation (1) is written in full, the coefficients a_{ij} become the elements of a 5×3 matrix; ϵ_i are the five orbital elements that determine the size, shape, and orientation of the orbit and they are defined as follows:

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix} = \begin{pmatrix} a \\ e \\ \tilde{\omega} \\ i \\ \Omega \end{pmatrix} \tag{2}$$

The three mutually orthogonal components of the force vector are F_j. If subscripts r, θ, and h denote the radial, transverse, and normal components, respectively, the components F_j are defined by the following column vector:

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} F_r \\ F_\theta \\ F_h \end{pmatrix} \tag{3}$$

Equations For Impulsive Velocity Increments

Assuming that second-order quantities may be neglected, the relationship between the orbital element increments and the components of impulse is given by

$$\delta\epsilon_i = \frac{d\epsilon}{dt}\,\delta t = a_{ij}(F_j\delta t) = a_{ij}I_j \tag{4}$$

In terms of radial, transverse, and normal components, the impulse vector is defined as follows:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} I_r \\ I_\theta \\ I_h \end{pmatrix} \tag{5}$$

In matrix notation, equation (4) assumes the following form:

$$\begin{pmatrix} \delta a \\ \delta e \\ \delta\tilde{\omega} \\ \delta i \\ \delta\Omega \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & a_{43} \\ 0 & 0 & a_{53} \end{pmatrix} \begin{pmatrix} I_r \\ I_\theta \\ I_h \end{pmatrix} \tag{6}$$

It is clear from equation (6) that the problem of determining impulse consists in finding three unknown impulse components when there are five equations. Since it has five rows and three columns, the rank of the matrix which operates on the column vector of impulse components in equation (6) cannot exceed three. Hence, there cannot be more than three independent rows. It follows that any combination of these equations taken three at a time can be used to obtain an impulse vector having the magnitude and direction required to produce prescribed changes in the three corresponding elements. The influence of this impulse on the remaining two orbital elements is obtained from the condition of compatibility or consistency. The arrangement of the zero elements in the matrix operator gives an indication of the way in which the orbital elements are influenced by the components of impulse. For example, the fact that $a_{13} = a_{23} = 0$ denotes that changes in the semimajor axis and the eccentricity require in-plane forces. These two orbital elements are not influenced by forces normal to the orbit plane. Furthermore, the orbit plane inclination and the longitude of the ascending node line can only be changed by the application of forces normal to the orbit plane. This is evident from the fact that

$$a_{41} = a_{42} = a_{51} = a_{52} = 0$$

Since $a_{33} \neq 0$, it follows that the argument of perifocus undergoes changes when either in-plane or out-of-plane forces are applied. If the argument of perifocus is defined as the sum of the nodal longitude and the argument of the latitude of perifocus, then changes in orbit plane orientation give rise to changes in the argument of perifocus. The matrix element a_{33} is a measure of the change in the argument of perifocus associated with changes in the orbit plane orientation. The matrix elements a_{ij} are functions of the true anomaly and the argument of the latitude; hence, some of these elements vanish at certain orbital locations. The orbital locations where a given matrix element vanishes are points of ineffectiveness for the corresponding impulse component. However, such locations can be used to advantage if it is required to manipulate one orbital element without changing the others. To show the dependence of the matrix elements on the true anomaly and the argument of the latitude, equation (6) is written in full.

$$\begin{bmatrix} \delta a \\ \delta e \\ \delta\tilde{\omega} \\ \delta i \\ \delta\Omega \end{bmatrix} = \begin{bmatrix} \dfrac{2a^2 e \sin v}{\sqrt{\mu a(1-e^2)}} & \dfrac{2a^2(1+e\cos v)}{\sqrt{\mu a(1-e^2)}} & 0 \\[2ex] \sqrt{\dfrac{a(1-e^2)}{\mu}}\sin v & \sqrt{\dfrac{a(1-e^2)}{\mu}}\,\dfrac{\cos v(2+e\cos v)+e}{1+e\cos v} & 0 \\[2ex] -\dfrac{1}{e}\sqrt{\dfrac{a(1-e^2)}{\mu}}\cos v & \dfrac{1}{e}\sqrt{\dfrac{a(1-e^2)}{\mu}}\,\dfrac{\sin v(2+e\cos v)}{(1+e\cos v)} & \sqrt{\dfrac{a(1-e^2)}{\mu}}\,\dfrac{\sin(v+\omega)\tan[(1/2)]i}{(1+e\cos v)} \\[2ex] 0 & 0 & \sqrt{\dfrac{a(1-e^2)}{\mu}}\,\dfrac{\cos(v+\omega)}{(1+e\cos v)} \\[2ex] 0 & 0 & \sqrt{\dfrac{a(1-e^2)}{\mu}}\,\dfrac{\sin(v+\omega)}{\sin i(1+e\cos v)} \end{bmatrix} \begin{bmatrix} I_r \\ I_\theta \\ I_h \end{bmatrix} \tag{7}$$

The matrix element a_{43} which controls the orbit plane inclination is seen to vanish when the argument of the latitude is 90° or 270°, whereas a_{53}, which controls the longitude of the ascending node line, vanishes at the ascending and descending nodes. Hence, a normal impulse applied at a nodal position will alter the orbit plane inclination without changing the longitude of the ascending node line; whereas, a normal impulse applied when the argument of the latitude is 90° or 270° will alter the longitude of the ascending node line without changing the orbit plane inclination. The components of the impulse vector required to make prescribed changes in the semimajor axis, the eccentricity, and the argument of perifocus can be determined from the following matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} I_r \\ I_\theta \\ I_h \end{pmatrix} = \begin{pmatrix} \delta a \\ \delta e \\ \delta\tilde{\omega} \end{pmatrix} \tag{8}$$

Therefore

$$\begin{pmatrix} I_r \\ I_\theta \\ I_h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} \delta a \\ \delta e \\ \delta\tilde{\omega} \end{pmatrix} \tag{9}$$

The impulse vector obtained from equation (9) would produce the prescribed changes in the orbital elements; however, such an impulse would produce unwanted changes in the orbit plane orientation. Subsequent corrections to the orbit plane orientation would then re-introduce errors in the argument of perifocus. In view of this it is considered more expedient to effect the prescribed changes by the use of an impulse vector which lies wholly in the orbit plane. The influence of such an impulse on the semimajor axis, the eccentricity, and the argument of perifocus can be determined from the following matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} I_r \\ I_\theta \end{pmatrix} = \begin{pmatrix} \delta a \\ \delta e \\ \delta\omega \end{pmatrix} \tag{10}$$

The rank of the matrix which operates on the column vector of impulse components cannot exceed two, and therefore the number of linearly independent rows cannot exceed two. Hence, any combination of these equations taken two at a time can be used to obtain an impulse having the magnitude and direction required to produce prescribed changes in the corresponding elements. The impulse vector required to produce prescribed changes in the semimajor axis and the eccentricity is obtained from equation (10) in the following form:

$$\begin{pmatrix} I_r \\ I_\theta \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} \delta a \\ \delta e \end{pmatrix} \tag{11}$$

Compatibility or consistency of equation (10) requires that

$$\delta\omega = (a_{31}\ a_{32}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} \delta a \\ \delta e \end{pmatrix} \tag{12}$$

Therefore,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} I_r \\ I_\theta \end{pmatrix} = \begin{pmatrix} \delta a \\ \delta e \\ (a_{31}\ a_{32}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} \delta a \\ \delta e \end{pmatrix} \end{pmatrix} \tag{13}$$

Physically, this means that in producing the prescribed changes δa and δe by the application of the impulse vector $(I_r\ I_\theta)^T$, the change given by equation (12) has been induced in the argument of perifocus. From equation (7) it is seen that these induced errors can be removed without changing the remaining elements if a radial impulse is applied at perifocus or apofocus. In the event that the impulse vector should have an out-of-plane component I_h, equation (8) shows that the total change induced in the argument of perifocus is given by

$$\delta\tilde{\omega} = \left[\begin{pmatrix} a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} \delta a \\ \delta e \end{pmatrix} + a_{33}\, I_h \right] \tag{14}$$

The solution of equation (11) is given by

$$\begin{pmatrix} I_r \\ I_\theta \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} \delta a \\ \delta e \end{pmatrix} \tag{15}$$

$$\Delta = (a_{11} \quad a_{22} - a_{12} \quad a_{21}) \tag{16}$$

On substitution from equation (7), it is found that the impulse vector required to produce prescribed changes δa and δe in the semimajor axis and the eccentricity, respectively, is given by

$$\begin{pmatrix} I_r \\ \\ I_\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{\mu}{a^3(1-e^2)}}\,\frac{\cos v(2 + e\cos v) + e}{\sin v} & \sqrt{\frac{\mu}{a(1-e^2)^3}}\,\frac{(1 + e\cos v)^2}{\sin v} \\ \\ \frac{1}{2}\sqrt{\frac{\mu}{a^3(1-e^2)}}\,(1 + e\cos v) & -e\sqrt{\frac{\mu}{a(1-e^2)^3}}\,(1 + e\cos v) \end{pmatrix} \begin{pmatrix} \delta a \\ \\ \delta e \end{pmatrix} \tag{17}$$

<u>Radial impulse required to change the argument of perifocus</u>.- If the radial impulse required to produce an incremental change $\delta\omega$ in the argument of perifocus be denoted by I_ω, then from equation (7)

$$I_\omega = -\left[e\sqrt{\frac{\mu}{a(1-e^2)}}\,\frac{1}{\cos v}\right]\delta\omega \tag{18}$$

If this impulse were applied at perifocus or apofocus, it would have no effect on the remaining orbital elements. Hence, the radial impulse required to change the argument of perifocus by $\delta\omega$ without changing the remaining orbital elements is given by

$$\frac{\partial I\omega}{\partial\omega}\,\delta\omega \tag{18a}$$

where

$$\frac{\partial I\omega}{\partial\omega} = -\left[e\sqrt{\frac{\mu}{a(1-e^2)}}\,\frac{1}{\cos v}\right]_{v=0,180^\circ} \tag{18b}$$

Normal impulses required to change the orientation of the orbital plane.- If the impulse required to produce an incremental change δi in the orbit plane inclination be denoted by I_i, then from equation (7)

$$I_i = \left[\sqrt{\frac{\mu}{a(1 - e^2)}} \frac{(1 + e \cos v)}{\cos(v + \omega)}\right] \delta i \tag{19}$$

Likewise, if the impulse required to produce an incremental change $\delta\Omega$ in the longitude of the ascending node be denoted by I_Ω, then from equation (7)

$$I_\Omega = \left[\sqrt{\frac{\mu}{a(1 - e^2)}} \frac{(1 + e \cos v)}{\sin(v + \omega)} \sin i\right] \delta\Omega \tag{20}$$

From equations (19) and (20), it is seen that if a normal impulse were applied when the argument of the latitude $(v + \omega)$ was 0° or 180°, the orbit plane inclination would be changed without altering the longitude of the ascending node. Alternatively, if it were required to alter the longitude of the ascending node without changing the orbit plane inclination, the normal impulse would be applied when the argument of the latitude was 90° or 270°.

Impulsive Velocity Increments Required to Produce Prescribed Changes in the Orbital Elements of Noncircular Orbits

Simultaneous changes in the semimajor axis and the eccentricity.- To facilitate the computation of the velocity increments required to produce prescribed changes in the semimajor axis and the eccentricity of the orbit of a space vehicle, equation (15) may be rearranged as follows:

$$\begin{pmatrix} I_r \\ I_\theta \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} aa_{22} & -ea_{12} \\ -aa_{21} & ea_{11} \end{pmatrix} \begin{pmatrix} \frac{\delta a}{a} \\ \frac{\delta e}{e} \end{pmatrix} \tag{21}$$

The elements of this matrix may be obtained from equation (17). By letting $\delta a/a = m/100$ and $\delta e/e = n/100$, we can use equation (21) to obtain the impulse vector required to produce an m percentage change in the semimajor axis, and an n percentage change in the orbit eccentricity. The required impulse vector has the following components:

$$\begin{pmatrix} I_r \\ I_\theta \end{pmatrix} = \begin{pmatrix} C_{11} + C_{12} \\ C_{21} + C_{22} \end{pmatrix} \tag{22}$$

where the matrix elements C_{ij} have the following values,

$$C_{11} = -\left\{\frac{m}{200}\sqrt{\frac{\mu}{a(1 - e^2)}}\left[\frac{\cos v(2 + e \cos v) + e}{\sin v}\right]\right\} \tag{23}$$

$$C_{12} = \frac{ne}{100}\sqrt{\frac{\mu}{a(1 - e^2)^3}}\left[\frac{(1 + e \cos v)^2}{\sin v}\right] \tag{24}$$

$$C_{21} = \frac{m}{200}\sqrt{\frac{\mu}{a(1 - e^2)}}(1 + e \cos v) \tag{25}$$

$$C_{22} = -\left[\frac{ne^2}{100}\sqrt{\frac{\mu}{a(1 - e^2)^3}}(1 + e \cos v)\right] \tag{26}$$

The magnitude of the impulse vector is given by

$$I_{ae} = \sqrt{(C_{11} + C_{12})^2 + (C_{21} + C_{22})^2}$$

To achieve the desired result, the impulse vector must be inclined at an angle γ_{ae} to the local horizontal, where

$$\gamma_{ae} = \tan^{-1}\left(\frac{C_{11} + C_{12}}{C_{21} + C_{22}}\right) \tag{27}$$

The angle between the impulse vector and the local tangent to the flight path at the point of application of the impulse is given by β_{ae}, where

$$\beta_{ae} = \tan^{-1}\left(\frac{C_{11} + C_{12}}{C_{21} + C_{22}}\right) - \tan^{-1}\left(\frac{e \sin v}{1 + e \cos v}\right) \tag{28}$$

Changes in the argument of perifocus resulting from changes in the semimajor axis and the orbit eccentricity.- If the change in the argument of perifocus induced by the application of the impulse I_{ae} be denoted by $\delta\omega_{ae}$, then

$$\delta\omega_{ae} = (a_{31}\ a_{32}) \begin{pmatrix} C_{11} + C_{12} \\ C_{21} + C_{22} \end{pmatrix} \tag{29}$$

where

$$a_{31} = -\frac{1}{e}\sqrt{\frac{a(1 - e^2)}{\mu}} \cos v \tag{30}$$

and

$$a_{32} = \frac{1}{e}\sqrt{\frac{a(1 - e^2)}{\mu}}\ \frac{\sin v(2 + e \cos v)}{(1 + e \cos v)} \tag{31}$$

A radial impulse is required to null $\delta_{\omega_{ae}}$ without changing the remaining orbital elements, and it must be applied at perifocus or apofocus. If this impulse be denoted by $I_{\omega_{ae}}$, then from equation (18a)

$$I_{\omega_{ae}} = \frac{\partial I\omega}{\partial\omega}\, \delta\omega_{ae} \tag{32}$$

Prescribed changes in the semimajor axis.- To produce an m percentage change in the semimajor axis of a noncircular orbit, without changing the orbit eccentricity, requires the application of an impulse I_a, where

$$I_a = \sqrt{C_{11}^2 + C_{21}^2} \tag{33}$$

For the desired effect, the impulse vector must be applied at an angle γ_a to the local horizontal where

$$\gamma_a = \tan^{-1}\left(\frac{C_{11}}{C_{21}}\right) \tag{34}$$

The angle between the impulse vector and the local tangent to the flight path at the point of application of the impulse is given by β_a, where

$$\beta_a = \tan^{-1}\left(\frac{C_{11}}{C_{21}}\right) - \tan^{-1}\left(\frac{e \sin v}{1 + e \cos v}\right) \tag{35}$$

<u>Changes in the argument of perifocus resulting from changes in the semimajor axis</u>.- If the change in the argument of perifocus induced by the impulse I_a be denoted by $\delta\omega_a$, then

$$\delta\omega_a = (a_{31} \ a_{32}) \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} \tag{36}$$

where a_{31} and a_{32} have the values given in equations (30) and (31). As indicated previously, a radial impulse is required to null $\delta\omega_a$ without changing the remaining orbital elements and it must be applied at perifocus or apofocus. If the required impulse be denoted by I_{ω_a}, then from equation (18a),

$$I_{\omega_a} = \frac{\partial I\omega}{\partial \omega} \delta\omega_a \tag{37}$$

<u>Prescribed changes in the orbit eccentricity</u>.- The impulse required to produce an n percentage change in the eccentricity, without changing the semimajor axis, is given by

$$I_e = \sqrt{C_{12}^2 + C_{22}^2} \tag{38}$$

The impulse vector must be applied at an angle γ_e to the local horizontal where

$$\gamma_e = \tan^{-1}\left(\frac{C_{12}}{C_{22}}\right) \tag{39}$$

The corresponding angle between the impulse vector and the local tangent to the flight path is given by

$$\beta_e = \tan^{-1}\left(\frac{C_{12}}{C_{22}}\right) - \tan^{-1}\left(\frac{e \sin v}{1 + e \cos v}\right) \tag{40}$$

From equations (24), (26), and (40) it is seen that

$$\frac{C_{12}}{C_{22}}\left(\frac{e \sin v}{1 + e \cos v}\right) = -1 \tag{41}$$

hence, the angle β_e is always $\pm 90^\circ$. It follows that to change the orbit eccentricity without changing the semimajor axis, the impulse must be directed along the normal to the flight path.

Changes in the argument of perifocus resulting from changes in the orbit eccentricity.- The change in the argument of perifocus produced by the impulse I_e is given by

$$\delta\omega_e = (a_{31}\ a_{32})\begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix} \tag{42}$$

The radial impulse required to null $\delta\omega_e$ is given by

$$I_{\omega_e} = \frac{\partial I\omega}{\partial\omega}\delta\omega_e \tag{43}$$

Application to Lunar Reconnaissance Maneuvers

Since only a limited area of the lunar surface can be observed from one orbit, the orbit plane orientation must be changed to permit observation of other areas. Orbital plane rotation about any position vector can be accomplished by the application of an impulse normal to the orbital plane at any desired orbital location, as indicated in previous sections.

The orientation of an orbital plane is by convention defined in terms of the orbit plane inclination and the longitude of the ascending node line. This way of resolving the orbit plane orientation into components is convenient for establishing the orientation of the plane in space. However, for lunar or planetary reconnaissance work, there is no need to determine the orbital plane orientation in terms of the angular displacement of the node line from some fixed line, and the angular displacement of the orbital plane from some fixed plane, measured as a rotation about the node line. For the applications contemplated here, it is more convenient to consider the orbital plane rotation vector rather than its components. It is assumed that from some initial orbit which is unsatisfactory from the point of view of observation and photographic survey, an orbit with a satisfactory size and shape has been established by the methods previously indicated. When the surface of the target body has been observed and photographed from this orbit, the orbital plane can then be rotated about the position vector from the center of the target body to the vehicle by the application of an impulse normal to the orbital plane. This is evident from the following considerations: the precessional torque required to change the direction of the angular momentum vector $\bar{h}$ without changing its magnitude is given by

$$\bar{M} = \bar{\omega} \times \bar{h} \tag{44}$$

where $\bar{M}$ is the precessional torque vector and $\bar{\omega}$ is the precessional rate vector. This symbol is not to be confused with the symbol for the argument of perifocus. Solving this equation for $\bar{\omega}$ gives

$$\bar{\omega} = \frac{\bar{h} \times \bar{M}}{h^2} \tag{45}$$

however,

$$\bar{M} = \bar{r} \times \bar{T} \tag{46}$$

where $\bar{r}$ is the position vector and $\bar{T}$ is the thrust vector. On substituting from equation (46) in equation (45), the angular velocity $\bar{\omega}$ is obtained in the following form:

$$\bar{\omega} = \frac{\bar{h} \times (\bar{r} \times \bar{T})}{h^2} = \frac{F_h r}{h} \hat{r} \tag{47}$$

where F_h is the component of thrust normal to the orbit plane. Equation (47) shows that the orbital plane rotates about the position vector in the presence of a force which has a component in the direction of the normal to the orbital plane. It is seen that for a given force F_h, the angular velocity $\bar{\omega}$ is proportional to the radial distance r; and, hence, the impulse required to produce a prescribed change in orientation decreases with radial distance.

The normal impulse I_h required to produce a prescribed change δo in the orbit plane orientation can be obtained from equation (47) as follows:

$$\bar{\omega} = \frac{do}{dt}\hat{r} = \left(\frac{F_h r}{h}\right)\hat{r}$$

therefore

$$\delta o = \frac{do}{dt}\delta t = \frac{r}{h}\left(F_h \delta t\right) = \frac{r}{h} I_h$$

therefore

$$I_h = \frac{h}{r}\delta o = \left[\sqrt{\frac{\mu}{a(1 - e^2)}}\,(1 + e \cos v)\right]\delta o \tag{48}$$

RESULTS AND DISCUSSION

The results presented here have been divided into two parts: one part deals with the velocity increments required to make prescribed changes in the elements of earth orbits, and the other deals with the application of the method to lunar reconnaissance maneuvers.

Earth Orbital Element Changes

The results plotted in figures 1 and 2 show the velocity increments required to produce a range of prescribed changes in the semimajor axis of an earth orbit, subject to the constraint that the eccentricity retains its initial value. The results plotted in figures 1 and 2 are for an orbit with a semimajor axis of 8,120 statute miles. However, the curves in figure 1 refer to an orbit with an eccentricity of 0.2; whereas those of figure 2 refer to an orbit with an eccentricity of 0.5. Each figure shows the velocity increments required to produce five percentage increases in the semimajor axis. The percentage increases selected for computation were: 0.1, 0.3, 0.5, 0.7, and 0.9. For convenience, however, the corresponding increase in miles is noted on each curve. The curves superimposed on these plots show the directions which the velocity increments must have in order to produce the desired change. It is interesting to note that there are two points on each orbit where a tangential impulse can be used to produce a prescribed change in the semimajor axis without changing the eccentricity of the orbit. It is easy to show that this condition arises when the length of the position vector to the vehicle equals the length of the semimajor axis. The condition $r = a$ is satisfied when the true anomaly satisfies the following equation:

$$v = \cos^{-1}(-e)$$

Changes in the argument of perigee resulting from prescribed changes in the semimajor axis.- The results plotted in figures 3 and 4 show the changes in the argument of perigee resulting from the range of percentage increases in the semimajor axis of the earth orbit discussed in the preceding section. Each curve of figure 3 shows the changes in the argument of perigee resulting from the corresponding curve of velocity increments plotted in figure 1. Likewise, each curve of figure 4 shows the changes in the argument of perigee resulting from the corresponding curve of velocity increments plotted in figure 2.

Eccentricity changes.- The impulsive velocity increments required to produce prescribed changes in the orbit eccentricity subject to the constraint that the semimajor axis retains its initial value are plotted in figures 5 and 6. The results shown are for an earth orbit with a semimajor axis of 8,120 miles. However, the curves shown in figure 5 refer to an orbit with an eccentricity of 0.2; whereas those of figure 6 refer to an orbit with an eccentricity of 0.5. Each figure shows the velocity increments required to produce five percentage increases in the orbit eccentricity. The percentage increases selected for computation were: 0.1, 0.3, 0.5, 0.7, and 0.9. The curves superimposed on these plots show the directions the velocity increments must have in order to produce the required change in the eccentricity. To change the eccentricity without changing the semimajor axis, it has been shown in equation (41) that it is necessary to apply the impulse vector in a direction perpendicular to the local flight path, and in the plane of the orbit. See the curve of β in figures 5 and 6.

Changes in the argument of perigee resulting from prescribed changes in the orbit eccentricity.- The curves plotted in figures 7 and 8 show the incremental changes induced in the argument of perigee by the velocity increments used to produce the percentage changes in eccentricity discussed in the previous section. Each curve of figure 7 shows the changes in the argument of perigee induced by the corresponding curve of velocity increments plotted in figure 5. Likewise, each curve of figure 8 shows the changes in the argument of perigee induced by the velocity increments plotted in figure 6. It is of interest to note that there are two points on an orbit where a normal impulse can be applied to produce a prescribed change in the eccentricity without inducing a change in the argument of perigee. It is easy to show that these two points correspond to values of the true anomaly given by the following equation:

$$v = 180^{\circ} \pm \tan^{-1}\left(\frac{1 - e^2}{2e}\right)$$

Orbital plane inclination.- To produce a prescribed change in the orbital plane inclination, a velocity increment must be applied in the direction of the normal to the orbit plane. Equation (19) has been used to compute the magnitude of the impulsive velocity increments required to produce a 1° change in the orbit plane inclination of an earth orbit. The results are plotted in figure 9. The initial values of the orbital elements are as indicated on the figure.

Nodal longitude.- As in the case of orbit plane inclination, changes in the nodal longitude require that the corrective impulses be applied in the direction of the normal to the orbital plane. Equation (20) has been used to compute the impulsive velocity increments required to produce a 1° change in the longitude of the ascending node line of an earth orbit with an inclination of 45°. The results are plotted in figure 10. The initial values of the orbital elements are as indicated on the figure. It is to be noted that changes in the orbit plane inclination are coupled to changes in the longitude of the ascending node line, and vice versa, unless the impulses are applied when the argument of the latitude $(v + \omega)$ is 0°, 90°, 180°, or 270°.

Application to Lunar Reconnaissance

Advantage of using high eccentricity orbits for certain types of reconnaissance maneuvers.- It is evident from equation (48) that fuel requirements could be reduced for certain types of reconnaissance missions if the lunar surface area of interest were in the vicinity of perilune and subsequently the orbit plane were rotated about the line of apsides. The impulse required to produce a 1° change in orbit plane orientation is seen to be a minimum if the orbit plane is made to rotate about the line of apsides by the application of a normal impulse at apolune. The impulse required is a maximum if the plane change maneuver is executed at perilune. The ratio of the impulse required at apolune to that required at perilune is given by $(1 - e)/(1 + e)$. Equation (48) has been used to compute the velocity increments required to rotate various orbit planes through a 1° angle about the line of apsides. The results in table I give a good indication of the advantage of executing plane change maneuvers when the vehicle is at apolune of high eccentricity orbits.

Prescribed changes in the semimajor axis of a lunar orbit.- Equations (33) through (35) have been used to compute the impulse vector required to make prescribed changes in the semimajor axis of a lunar orbit, subject to the constraint that $\Delta e = 0$. The results for a lunar orbit with a semimajor axis of 1,600 miles are plotted in figures 11 and 12. The orbit eccentricities have the values indicated on the figures. The incremental changes in the argument of perilune resulting from changes in the semimajor axis are plotted in figures 13 and 14.

Changes in perilune height.- A lunar orbit with a semimajor axis of 1,400 miles and an eccentricity of 0.2 has a perilune height of 40 miles. The velocity increments required to reduce the perilune height by 5.6 miles were computed and plotted as a function of the true anomaly in figure 15. Curve 1 shows the velocity increments required to produce the required change in height by reducing the semimajor axis without changing the orbit eccentricity. Curve 1 of figure 16 shows the changes in the argument of perilune induced by changes in the semimajor axis. When the perilune height is changed without altering the orbital period (i.e., without changing the semimajor axis), curve 2 of figure 15 results. The corresponding plot of induced changes in the argument of perilune is given by curve 2 of figure 16. Because the use of high eccentricity orbits can greatly reduce the velocity increments required to produce prescribed changes in orbit plane orientation, the velocity increments required to modify a lunar orbit with a semimajor axis of 2,200 miles and an eccentricity of 0.5 have been computed. It is assumed that this orbit which has a perilune height of 20 miles would be established by successive applications of the method described in this report. The results plotted in figure 17 show the velocity increments required to reduce the height at perilune by 5.5 miles. Curve 1 of figure 17 shows the velocity increments required to reduce perilune height by reducing the semimajor axis without changing the orbit eccentricity. Curve 2 of figure 17 shows the velocity increments required to modify the height at perilune by the prescribed amount, by changing the orbit eccentricity without altering the semimajor axis. When equal percentage changes are made in the semimajor axis and the orbit eccentricity, curve 3 of figure 17 results. The corresponding curves of figure 18 show the changes induced in the argument of perilune by the velocity increments plotted in figure 17. The curves of figure 19 give the directions which the velocity increments plotted in figure 17 must have in order to produce the prescribed change in perilune height.

CONCLUDING REMARKS

Replacing a set of differential equations relating the time rates of change of the orbital elements to the components of a perturbing force with a set of ordinary equations relating the orbital element increments to the components of an impulse vector makes it possible to compute the impulsive velocity increments required for prescribed changes in the elements of lunar or planetary orbits. The necessity of omitting second and higher order quantities, which the method entails, gives rise to errors in the computed results. However, numerical analysis indicates that these errors are well within acceptable limits for the types of maneuvers contemplated. As an example, a lunar orbit with a semimajor axis of 1,600 miles and an eccentricity of 0.2 may be considered. In making a 5-mile change in the semimajor axis, it was found that the impulse computed on the basis of the simplifying assumptions made gave rise to a change in the semimajor axis which was approximately 100 feet in error. As a second example, an earth orbit with a semimajor axis of 8,000 miles and an eccentricity of 0.1 may be cited. In making an 8-mile change in the semimajor axis, it was found that the computed impulse gave rise of an incremental change in the semimajor axis which was approximately 50 feet in error.

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Moffett Field, Calif., Jan. 9, 1964

APPENDIX A

DIFFERENTIAL EQUATIONS FOR THE TIME RATES OF CHANGE OF THE ORBITAL ELEMENTS IN THE PRESENCE OF PERTURBING FORCES

It is shown in reference 4 that the variation of the orbital elements can be expressed in terms of the components of the disturbing force. The formulation is such that the rates of change of the orbital elements are expressed as functions of the radial, transverse, and normal components of the perturbing force vector. In terms of the notation of reference 4, the equations for the rates of change of the orbital elements assume the following form:

$$\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}}\left[S \tan \varphi \sin w + T \sec \varphi(1 + e \cos w)\right] \tag{A1}$$

$$\frac{de}{dt} = \sqrt{\frac{a}{\mu}} \cos \varphi \left[S \sin w + T(\cos w + \cos E)\right] \tag{A2}$$

$$\frac{d\tilde{\omega}}{dt} = \frac{1}{\sqrt{\mu a}} \left\{ \frac{-aS \cos^2\varphi \cos w + rT \sin w(2 + e \cos w) + rW \sin\varphi \tan[(1/2)i]\sin u}{\sin \varphi \cos \varphi} \right\} \tag{A3}$$

$$\frac{di}{dt} = \frac{rW \cos u}{\sqrt{\mu a} \cos \varphi} \tag{A4}$$

$$\frac{d\Omega}{dt} = \frac{rW \sin u}{\sqrt{\mu a} \cos \varphi \sin i} \tag{A5}$$

where

w true anomaly

$\sin \varphi$ eccentricity

E eccentric anomaly

u argument of the latitude (i.e., the sum of the argument of perifocus, measured from the ascending node line and the true anomaly)

S,T,W radial, transverse, and normal components, respectively, of the perturbing force vector

In the notation of the present report (see fig. 20)

$$S = \hat{r} \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \overline{T}) \tag{A6}$$

$$T = \hat{\theta} \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \overline{T}) \tag{A7}$$

$$W = \hat{h} \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \overline{T}) \tag{A8}$$

$$u = \omega + v \tag{A9}$$

$$w = v \tag{A10}$$

$$\sin \varphi = e \tag{A11}$$

The rates of change of the orbital elements are all expressed as functions of the true anomaly, with the exception of the equation for the rate of change of eccentricity. This equation is seen to depend on the true anomaly and the eccentric anomaly. For present applications it is more convenient to have all the orbital element rates expressed as functions of the true anomaly only. For this reason the eccentric anomaly is removed from the equation by expressing it as a function of the true anomaly and the eccentricity as follows:

$$r = \frac{a(1 - e^2)}{1 + e \cos v}$$

$$\frac{r}{a} = 1 - e \cos E$$

therefore

$$\cos E = \frac{\cos v + e}{1 + e \cos v}$$

therefore

$$\cos v + \cos E = \frac{\cos v(2 + e \cos v) + e}{1 + e \cos v} \tag{A12}$$

Substituting from equations (A6) through (A12) in equations (A1) through (A5) yields the following form of the equations for the rates of change of the orbital elements:

$$\frac{da}{dt} = \frac{2a^2}{\sqrt{\mu a(1 - e^2)}} [(e \sin v)\hat{r} + (1 + e \cos v)\hat{\theta}] \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T}) \tag{A13}$$

$$\frac{de}{dt} = \sqrt{\frac{a(1 - e^2)}{\mu}} \left\{(\sin v)\hat{r} + \left[\frac{\cos v(2 + e \cos v) + e}{1 + e \cos v}\right]\hat{\theta}\right\} \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T}) \tag{A14}$$

$$\frac{d\tilde{\omega}}{dt} = \sqrt{\frac{a(1 - e^2)}{\mu}} \left(\left(- \frac{1}{e} \cos v\right)\hat{r} + \left[\frac{1}{e} \frac{\sin v(2 + e \cos v)}{1 + e \cos v}\right]\hat{\theta}\right.$$

$$\left. + \left\{\frac{\sin(v + \omega)\tan[(1/2)i]}{1 + e \cos v}\right\}\hat{h}\right) \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T}) \tag{A15}$$

$$\frac{di}{dt} = \left[\sqrt{\frac{a(1 - e^2)}{\mu}} \frac{\cos(v + \omega)}{1 + e \cos v}\hat{h}\right] \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T}) \tag{A16}$$

$$\frac{d\Omega}{dt} = \left[\sqrt{\frac{a(1 - e^2)}{\mu}} \frac{\sin(v + \omega)}{\sin i(1 + e \cos v)}\hat{h}\right] \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T}) \tag{A17}$$

where

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\varphi}{\partial u}\hat{\theta} + \frac{1}{r \sin u}\frac{\partial\varphi}{\partial i}\hat{h} \tag{A18}$$

APPENDIX B

VECTOR DERIVATION OF THE EQUATIONS FOR ORBITAL ELEMENT PERTURBATIONS

The equation of motion of a particle of unit mass moving in an inverse-square-law central force field is

$$\frac{d^2\bar{r}}{dt^2} = -\frac{\mu}{r^2}\hat{r} \tag{B1}$$

Vector multiplication of each side of equation (B1) by the angular momentum vector $\bar{h}$ gives

$$\frac{d^2\bar{r}}{dt^2} \times \bar{h} = -\frac{\mu}{r^2}(\hat{r} \times \bar{h}) \tag{B2}$$

$$\bar{h} = \bar{r} \times \bar{v} = (r^2\dot{\theta})\hat{h} \tag{B3}$$

On substitution from equation (B3) in equation (B2) it is seen that

$$\frac{d^2\bar{r}}{dt^2} \times \bar{h} = (\mu\dot{\theta})\hat{\theta}$$

therefore

$$\frac{d}{dt}\frac{\bar{v} \times \bar{h}}{\mu} = \dot{\theta}\hat{\theta} = \frac{d}{dt}\hat{r} \tag{B4}$$

The integral of equation (B4) is given by

$$\frac{\bar{v} \times \bar{h}}{\mu} = \hat{r} + \bar{e} \tag{B5}$$

where $\bar{e}$ is a constant vector of integration. The vector $\bar{e}$ may be expressed in terms of its scalar magnitude and a vector of unit length as follows:

$$\bar{e} = e\hat{a} \tag{B6}$$

where $\hat{a}$ is a unit vector (fig. 20). Substituting equation (B6) in equation (B5) gives

$$\frac{\bar{v} \times \bar{h}}{\mu} = \hat{r} + e\hat{a} \tag{B7}$$

Equation (B7) may be solved to obtain the position vector $\bar{r}$. Scalar multiplication of each side by $\bar{r}$ gives the following equation for $\bar{r}$:

$$\frac{\bar{r} \cdot (\bar{v} \times \bar{h})}{\mu} = r + e(\bar{r} \cdot \hat{a})$$

therefore

$$\frac{(\bar{r} \times \bar{v}) \cdot \bar{h}}{\mu} = r(1 + e \cos v)$$

therefore

$$r = \frac{h^2/\mu}{1 + e \cos v} \tag{B8}$$

Vector multiplication of each side of equation (B7) by $\bar{h}$ gives

$$\bar{h} \times (\bar{v} \times \bar{h}) = \mu\bar{h} \times (e\hat{a} + \hat{r})$$

therefore

$$\bar{v} = \frac{\mu}{h} [\hat{\theta} + e(\hat{h} \times \hat{a})] \tag{B9}$$

With the notation of figure 20, the unit vector $\hat{a}$ may be expressed in the following form:

$$\hat{a} = (\cos v)\hat{r} - (\sin v)\hat{\theta} \tag{B10}$$

If the assumption of an inverse-square-law central force field is not satisfied, the equation of motion must be modified accordingly. In the presence of a perturbing force $\bar{P}$, the equation of motion becomes

$$\frac{d\bar{v}}{dt} = \bar{P} - \frac{\mu\bar{r}}{r^3} \tag{B11}$$

Furthermore, in the presence of the perturbing force vector $\bar{P}$, the assumption of constancy no longer applies to the vector $\bar{e}$. Hence,

$$\mu \frac{d\bar{e}}{dt} = \left\{\frac{d\bar{v}}{dt} \times \bar{h} + \bar{v} \times (\bar{r} \times \bar{P}) - \frac{\mu}{r^3} [(\bar{r} \times \bar{v}) \times \bar{r}]\right\} \tag{B12}$$

therefore

$$\mu \frac{d\bar{e}}{dt} = \left(\frac{dv}{dt} + \frac{\mu\bar{r}}{r^3}\right) \times \bar{h} + \bar{v} \times (\bar{r} \times \bar{P})$$

therefore

$$\mu \frac{d\bar{e}}{dt} = \bar{P} \times \bar{h} + \bar{v} \times (\bar{r} \times \bar{P}) \tag{B13}$$

The first term on the right side of equation (B13) may be written in the following alternative form:

$$\bar{P} \times \bar{h} = h(\hat{r}\hat{\theta} - \hat{\theta}\hat{r}) \cdot \bar{P} \tag{B14}$$

likewise,

$$\bar{r} \times \bar{P} = r(\hat{h}\hat{\theta} - \hat{\theta}\hat{h}) \cdot \bar{P} \tag{B15}$$

When substitutions are made from equations (B9) and (B15), the second term on the right side of equation (B13) is given by

$$\bar{v} \times (\bar{r} \times \bar{P}) = \frac{\mu r}{h} \left\{\hat{r}\hat{\theta} + e\left[\hat{a}\hat{\theta} + (\hat{\theta} \cdot \hat{a})\hat{h}\hat{h}\right]\right\} \cdot \bar{P} \tag{B16}$$

Substitution for $\hat{a}$ from equation (B10), equation (B16) becomes

$$\bar{v} \times (\bar{r} \times \bar{P}) = \frac{\mu r}{h} [(1 + e \cos v)\hat{r}\hat{\theta} - e \sin v(\hat{\theta}\hat{\theta} + \hat{h}\hat{h})] \cdot \bar{P} \tag{B17}$$

From equations (B14) and (B17) it follows that

$$\bar{P} \times \bar{h} + \bar{v} \times (\bar{r} \times \bar{P}) = \left\{\left[\frac{\mu r}{h} (1 + e \cos v) + h\right] \hat{r}\hat{\theta} - \left(\frac{\mu re}{h} \sin v\, \hat{\theta}\hat{\theta} + h\hat{\theta}\hat{r}\right) - \frac{\mu re}{h} \sin v\, \hat{h}\hat{h}\right\} \cdot \bar{P} \tag{B18}$$

The components of the term on the left side of equation (B13) may be obtained as follows:

$$\mu \frac{d\bar{e}}{dt} = \mu \left(\frac{\partial \bar{e}}{\partial t} + \bar{\omega} \times \bar{e} \right) \tag{B19}$$

where $\bar{\omega}$ is the angular velocity of the vector $\bar{e}$. As the vector $\bar{e}$ moves in the orbit plane, it rotates about the unit vector $\hat{h}$ which is normal to the plane. In addition, the orbit plane rotates about the instantaneous position vector. Hence, the vector $\bar{\omega}$ is given by

$$\bar{\omega} = \left[\dot{\omega}\hat{h} + \frac{\bar{h} \times (\bar{r} \times \bar{P})}{h^2} \right]$$

therefore

$$\bar{\omega} = \left[\dot{\omega}\hat{h} + \frac{r}{h} (\hat{r}\hat{h}) \cdot \bar{P} \right] \tag{B20}$$

By substitution from equation (B20) in equation (B19) it is found that

$$\mu \frac{d\bar{e}}{dt} = \left[\mu \left(\dot{e} \cos v + \dot{\omega} e \sin v \right) \hat{r} + \mu \left(\dot{\omega} e \cos v - \dot{e} \sin v \right) \hat{\theta} \right.$$

$$\left. - \frac{\mu r e}{h} \sin v (\hat{h}\hat{h}) \cdot \bar{P} \right] \tag{B21}$$

Equating coefficients of like vectors in equations (B18) and (B21) yields the following results:

$$\mu \left(\dot{e} \cos v + \dot{\omega} e \sin v \right) = 2h(\hat{\theta} \cdot \bar{P})$$

and

$$\mu \left(\dot{\omega} e \cos v - \dot{e} \sin v \right) = - \left(\frac{\mu r e}{h} \sin v \hat{\theta} + h\hat{r} \right) \cdot \bar{P}$$

On solving for $\dot{e}$ and $\dot{\omega}$ it is found that

$$\dot{e} = \left\{ \frac{r}{h} \left[\cos v(2 + e \cos v) + e \right] \hat{\theta} + \left(\frac{h}{\mu} \sin v \right) \hat{r} \right\} \cdot \bar{P} \tag{B22}$$

$$\dot{\omega} = \left[\frac{r}{eh} \sin v(2 + e \cos v) \hat{\theta} - \left(\frac{h}{\mu e} \cos v \right) \hat{r} \right] \cdot \bar{P} \tag{B23}$$

It is seen that for circular orbits there is a singularity in the equation for $\dot{\omega}$ because of the presence of e in the denominator. The physical significance of this is that for low eccentricity orbits the argument of perifocus is not well defined and does not exist for circular orbits.

Orbital Plane Orientation

Since the orientation of a plane in space is uniquely determined by the normal to its surface, the orientation of an orbit plane is determined by the angular momentum vector. This fact may be used to advantage in finding the time rates of change of the elements defining the orientation of an orbit plane in space. The longitude of the ascending node line and the inclination of the orbit plane are related to the unit vectors $\hat{k}$ and $\hat{h}$ as follows:

$$\hat{k} \times \hat{h} = (\sin i)\hat{\Omega}$$

therefore

$$\hat{k} \times \frac{d\hat{h}}{dt} = \left(\cos i \frac{di}{dt}\right) \hat{\Omega} + \left(\sin i \frac{d\Omega}{dt}\right) (\hat{k} \times \hat{\Omega}) \tag{B24}$$

and

$$\hat{k} \times \frac{d\hat{h}}{dt} = - \frac{r}{h} (\hat{k} \times \hat{\theta}) (\hat{h} \cdot \bar{P}) \tag{B25}$$

The unit vector $\hat{\theta}$ has the following components along and perpendicular to the ascending node line

$$\hat{\theta} = -(\sin \theta)\hat{\Omega} + \cos \theta(\hat{h} \times \hat{\Omega}) \tag{B26}$$

On substitution from equation (B26) in equation (B25), the following result is obtained:

$$\hat{k} \times \frac{d\hat{h}}{dt} = \frac{r}{h} \left[\sin \theta(\hat{k} \times \hat{\Omega}) + \cos \theta \cos i\hat{\Omega}\right] (\hat{h} \cdot \bar{P}) \tag{B27}$$

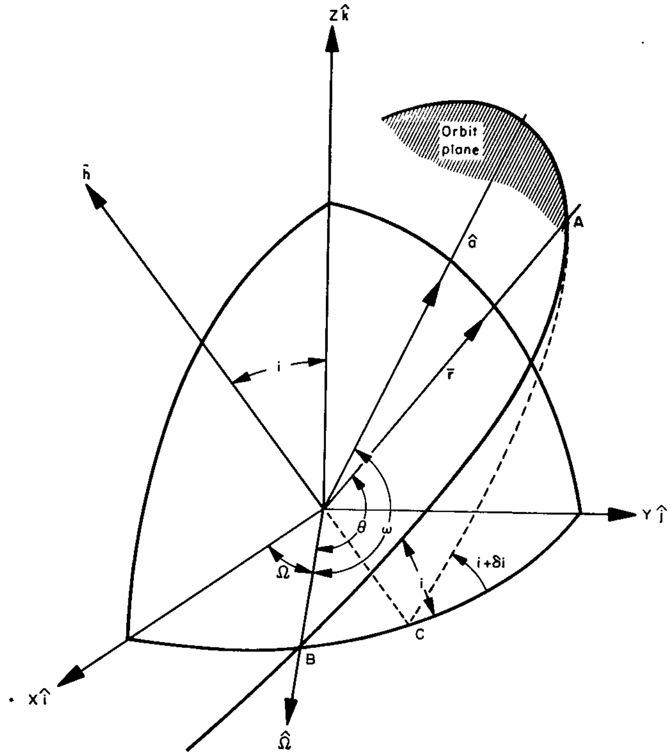
From equations (B24) and (B27) it follows that

$$\frac{di}{dt} = \frac{r \cos \theta}{h} (\hat{h} \cdot \bar{P}) \tag{B28}$$

$$\frac{d\Omega}{dt} = \frac{r \sin \theta}{h \sin i} (\hat{h} \cdot \bar{P}) \tag{B29}$$

Orbital Plane Orientation and the Argument of Perifocus

Since the orbital plane rotates about the instantaneous position vector $\bar{r}$, the argument of perifocus as measured from the line of nodes will vary in the manner indicated in sketch (a).



Sketch (a)

When the sine rule is applied to the spherical triangle ABC, it is seen that

$$\frac{\sin \theta}{\sin(i + \delta i)} = \frac{\sin(\theta + \delta\theta)}{\sin i}$$

therefore

$$\delta\theta = -(\tan \theta \cot i)\delta i$$

Since the position vector is assumed constant during the orbital plane rotation, it follows that $\delta\theta = \delta\omega$. Hence,

$$\frac{d\omega}{dt} = -(\tan \theta \cot i) \frac{di}{dt} \tag{B30}$$

Substituting for di/dt from equation (B28), the rate of change of the argument of perifocus produced by the rotation of the orbit plane is given by

$$\frac{d\omega}{dt} = - \frac{r \cot i \sin \theta}{h} (\hat{h} \cdot \bar{P}) \tag{B31}$$

If the argument of perifocus is measured from the inertially fixed x axis, then the rate of change associated with orientation changes is given by the following sum:

$$\frac{d\Omega}{dt} + \frac{d\omega}{dt} \tag{B32}$$

Substituting for $d\Omega/dt$ from equation (B29) in equation (B32), the required rate is obtained in the following form:

$$\frac{d\Omega}{dt} + \frac{d\omega}{dt} = \frac{r \sin \theta \tan[(1/2)i]}{h} (\hat{h} \cdot \bar{P}) \tag{B32a}$$

If equations (B32a) and (B23) are combined, the total rate of change of the argument of perifocus is given by

$$\frac{d\tilde{\omega}}{dt} = \left\{- \left(\frac{h}{\mu e} \cos v\right) \hat{r} + \frac{r}{eh} \sin v(2 + e \cos v)\hat{\theta} + \frac{r \sin (v + \omega)\tan[(1/2)i]}{h} \hat{h}\right\} \cdot \bar{P} \tag{B33}$$

The semimajor axis is defined as follows:

Let

$$a = \frac{(\bar{h} \cdot \bar{h})/\mu}{1 - \bar{e} \cdot \bar{e}} \tag{B34}$$

It is known from orbit theory that for closed orbits, the quantity $2a$ is the distance from perifocus to apofocus. The quantity a is the semimajor axis. In an ideal inverse-square-law central force field it remains constant. However, the force field assumed in the present study is noncentral so that the quantity a must be treated as a variable. From equation (B9), the square of velocity is given by

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right) \tag{B35}$$

therefore

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu} \tag{B36}$$

Differentiating equation (B36) with respect to time gives the following rate of change of a:

$$-\frac{1}{a^2}\frac{da}{dt} = -\frac{2}{r^2}\frac{dr}{dt} - \frac{2v}{\mu}\frac{dv}{dt}$$

therefore

$$\frac{1}{a^2}\frac{da}{dt} = \frac{2}{r^2}\left(\hat{r} \cdot \frac{d\bar{r}}{dt}\right) + \frac{2}{\mu}\left(\bar{v} \cdot \frac{d\bar{v}}{dt}\right)$$

therefore

$$\frac{1}{a^2}\frac{da}{dt} = \frac{2}{r^2}\left(\hat{r} \cdot \frac{d\bar{r}}{dt}\right) + \frac{2}{\mu}\bar{v} \cdot \left(\bar{P} - \frac{\mu\hat{r}}{r^2}\right)$$

therefore

$$\frac{da}{dt} = \frac{2a^2}{\mu}(\bar{v} \cdot \bar{P}) \tag{B37}$$

Substituting for $\bar{v}$ from equation (B9) yields

$$\frac{da}{dt} = \frac{2a^2}{h}[(e \sin v)\hat{r} + (1 + e \cos v)\hat{\theta}] \cdot \bar{P} \tag{B38}$$

Perturbing Forces

Perturbing forces are assumed to be generated by the earth's gravity field and extraterrestrial gravity fields. Both of these forces are assumed to be derivable from known gravitational potential functions. It is further assumed that thrust forces are available to introduce controlled perturbations. In terms of the gravity gradients and the thrust forces, the perturbing force vector $\bar{P}$ may be expressed as follows:

$$\bar{P} = \nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T} \tag{B39}$$

For convenience of reference the results are summarized below:

$$\frac{da}{dt} = \frac{2a^2}{h}[(e \sin v)\hat{r} + (1 + e \cos v)\hat{\theta}] \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T}) \tag{B40}$$

$$\frac{de}{dt} = \left\{\left(\frac{h}{\mu} \sin v\right)\hat{r} + \frac{r}{h}\left[\cos v(2 + e \cos v) + e\right]\hat{\theta}\right\} \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \bar{T}) \tag{B41}$$

$$\frac{d\tilde{\omega}}{dt} = \left(- \left(\frac{h}{\mu e} \cos v \right) \hat{r} + \frac{r}{eh} \left[\sin v(2 + e \cos v) \right] \hat{\theta} \right.$$

$$\left. + \left\{ \frac{r \sin(v + \omega) \tan[(1/2) i]}{h} \right\} \hat{h} \right) \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \overline{T}) \tag{B42}$$

$$\frac{di}{dt} = \left[\frac{r \cos(v + \omega)}{h} \hat{h} \right] \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \overline{T}) \tag{B43}$$

$$\frac{d\Omega}{dt} = \left[\frac{r \sin(v + \omega)}{h \sin i} \hat{h} \right] \cdot (\nabla\varphi_{ob} + \nabla\varphi_{et} + \overline{T}) \tag{B44}$$

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TABLE I.- VELOCITY INCREMENTS REQUIRED TO ROTATE THE LUNAR-ORBIT PLANE THROUGH 1° ABOUT THE LINE OF APSIDES

a, miles	e	Impulse, ft/sec	
		Perifocus	Apofocus
1600	0.1	87.4	71.5
1600	.2	96.8	64.6
1600	.3	107.7	58.0
2200	.1	74.5	61.0
2200	.3	91.9	49.5
2200	.5	116.8	39.0

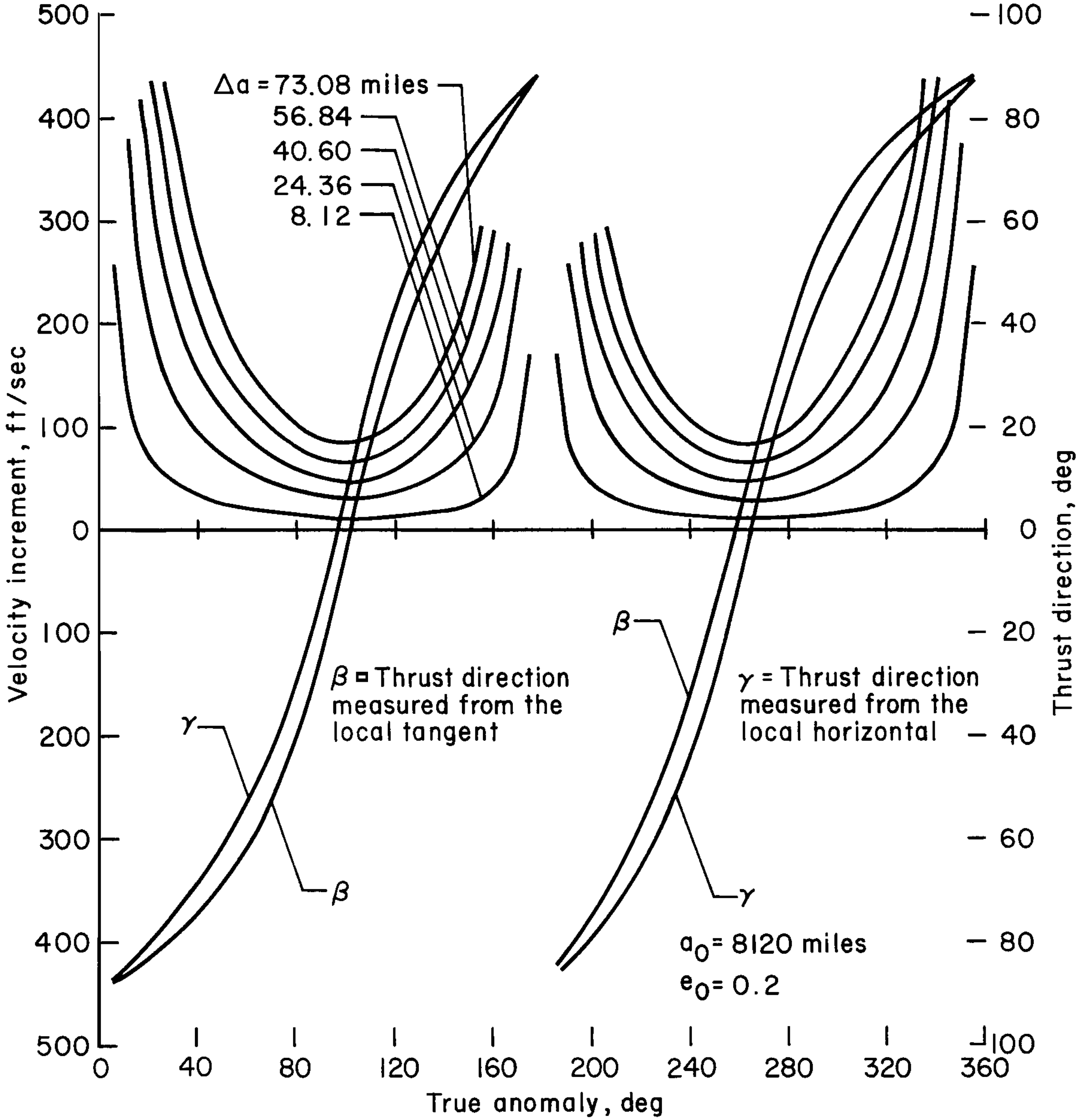


Figure 1.- Velocity increments required to make prescribed changes in the semimajor axis of an earth satellite subject to the constraint that $\Delta e = 0$.

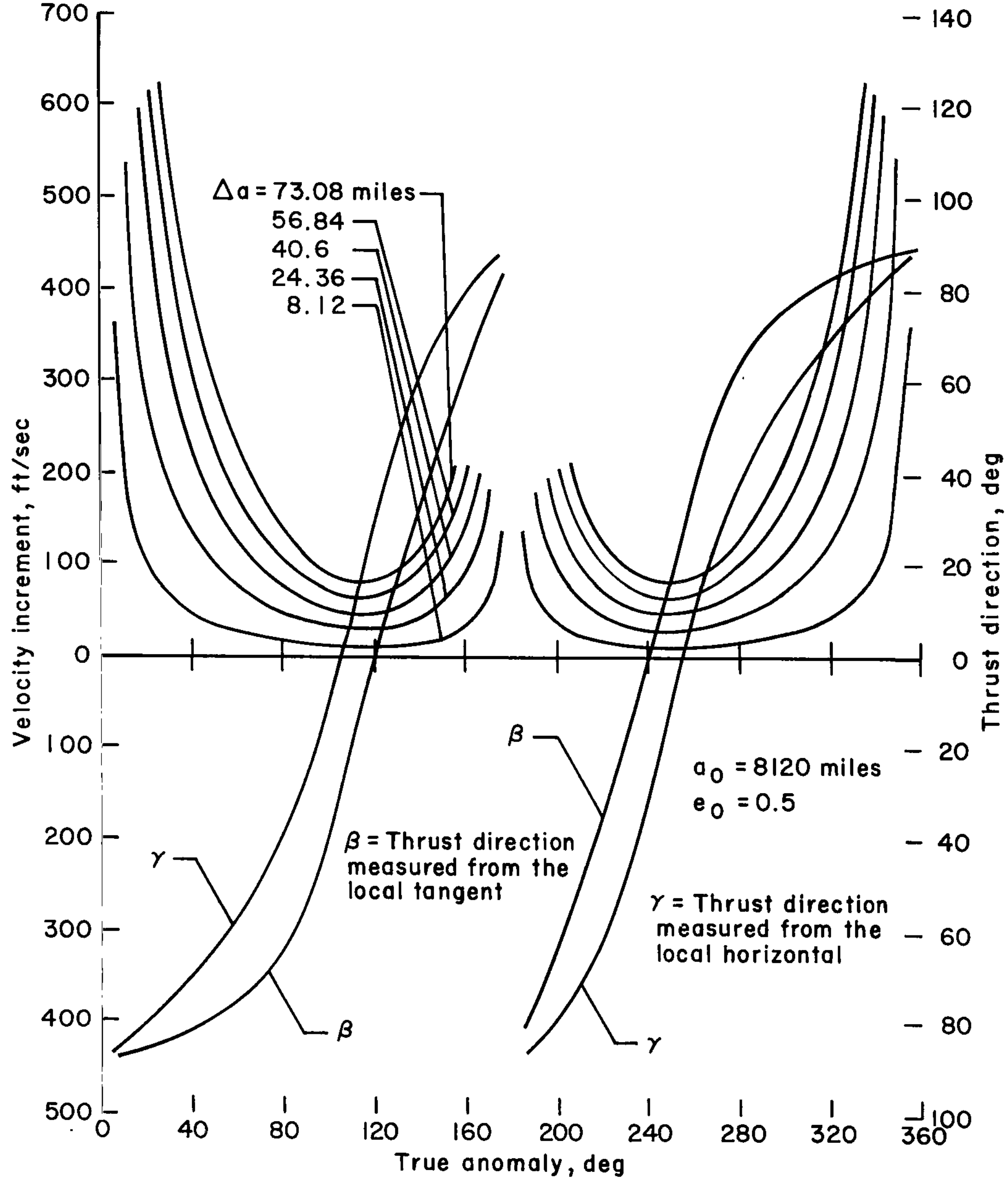


Figure 2.- Velocity increments required to make prescribed changes in the semimajor axis of an earth satellite subject to the constraint that $\Delta e = 0$.

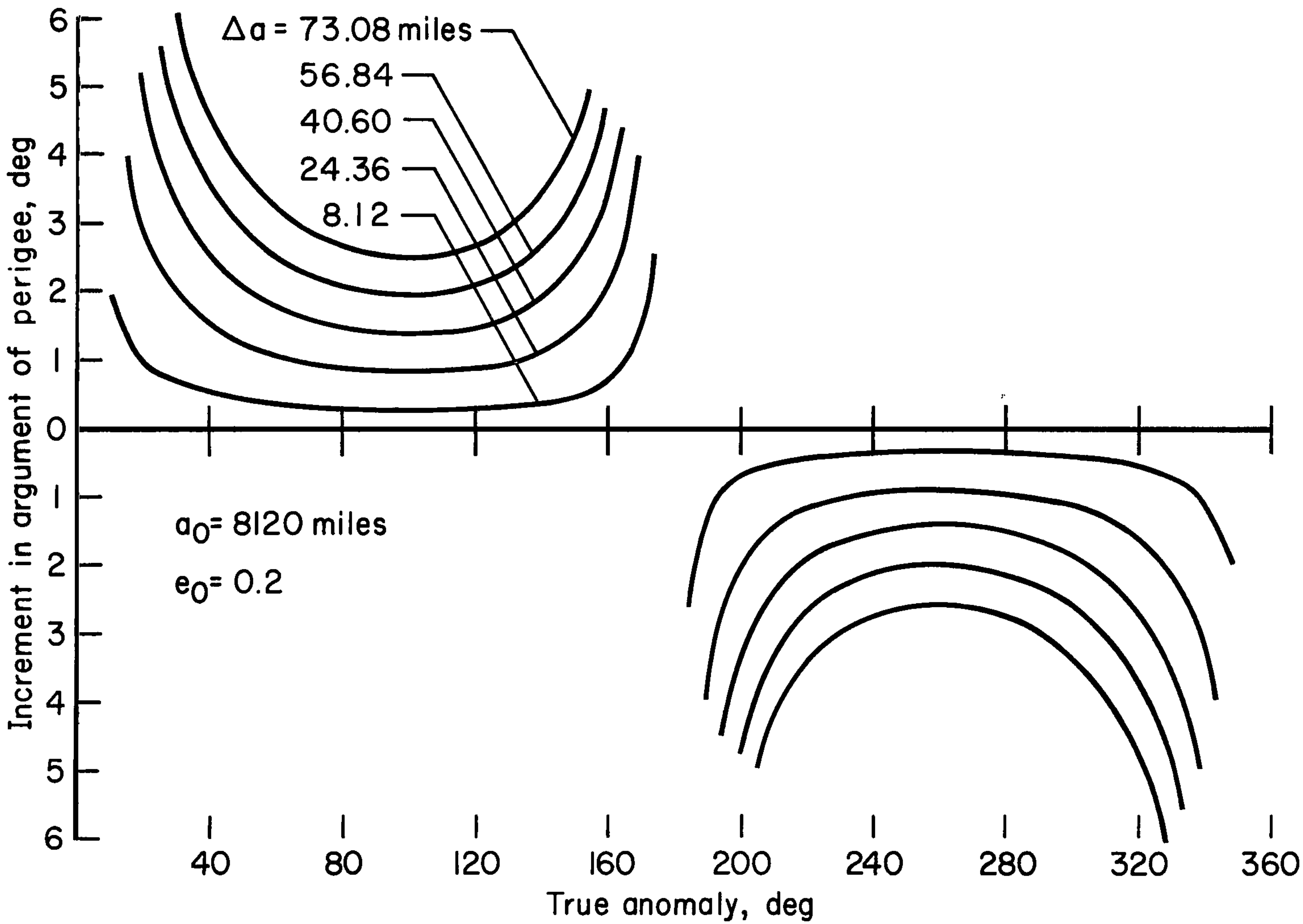


Figure 3.- Changes in the argument of perigee resulting from prescribed changes in the semimajor axis.

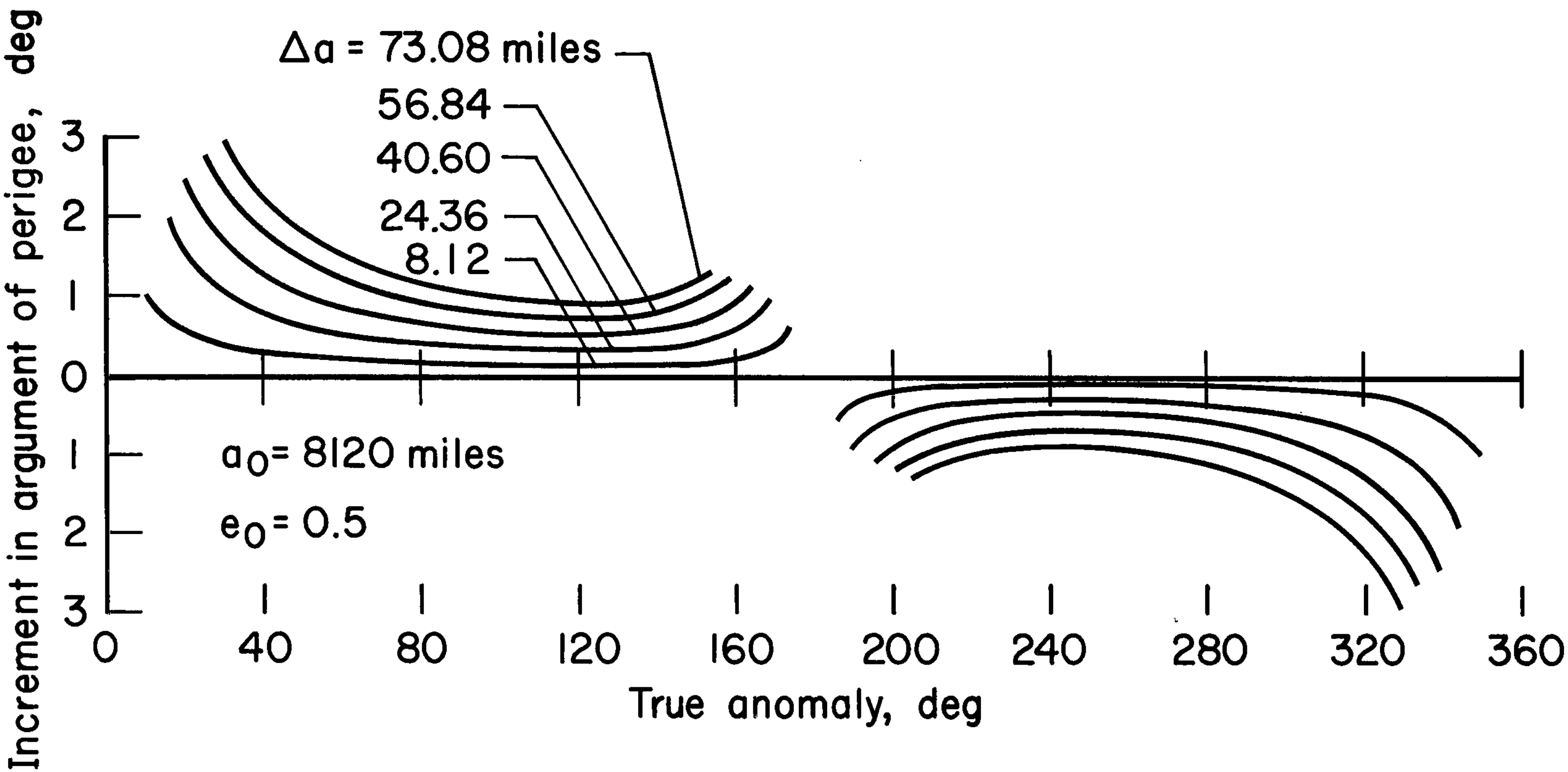


Figure 4.- Changes in the argument of perigee resulting from prescribed changes in the semimajor axis.

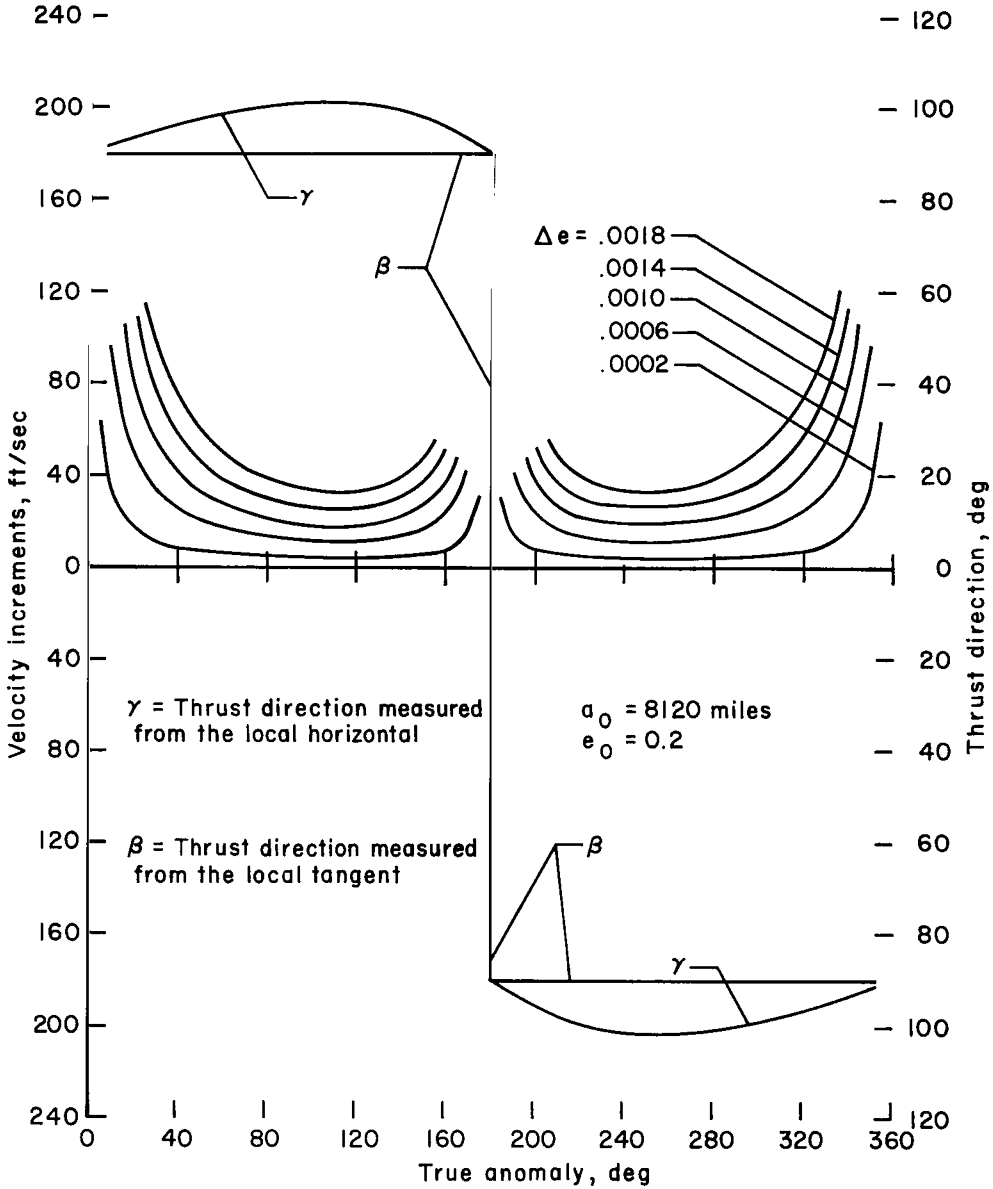


Figure 5.- Velocity increments required to make prescribed changes in the eccentricity of an earth satellite subject to the constraint that $\Delta a = 0$.

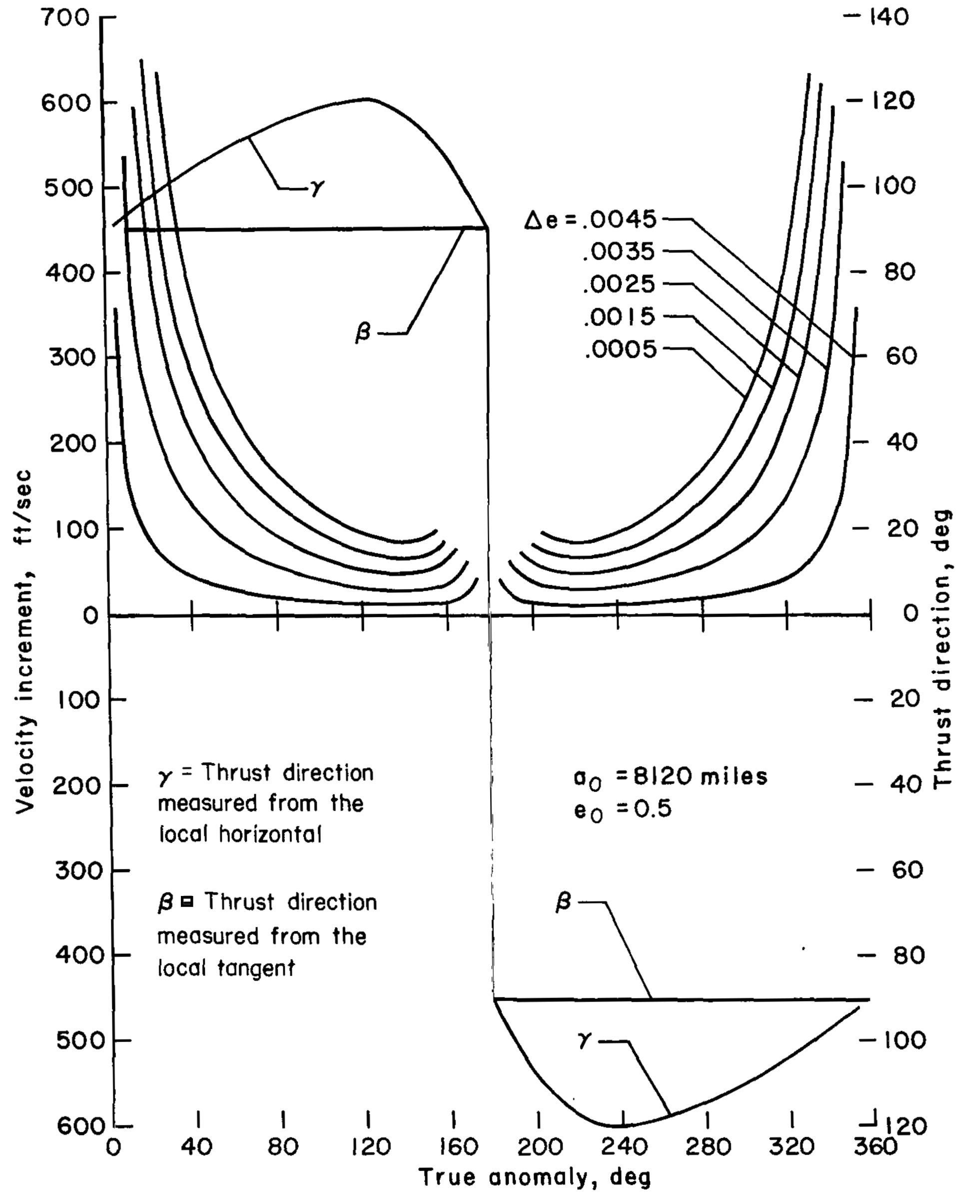


Figure 6.- Velocity increments required to make prescribed changes in the eccentricity of an earth satellite subject to the constraint that $\Delta a = 0$.

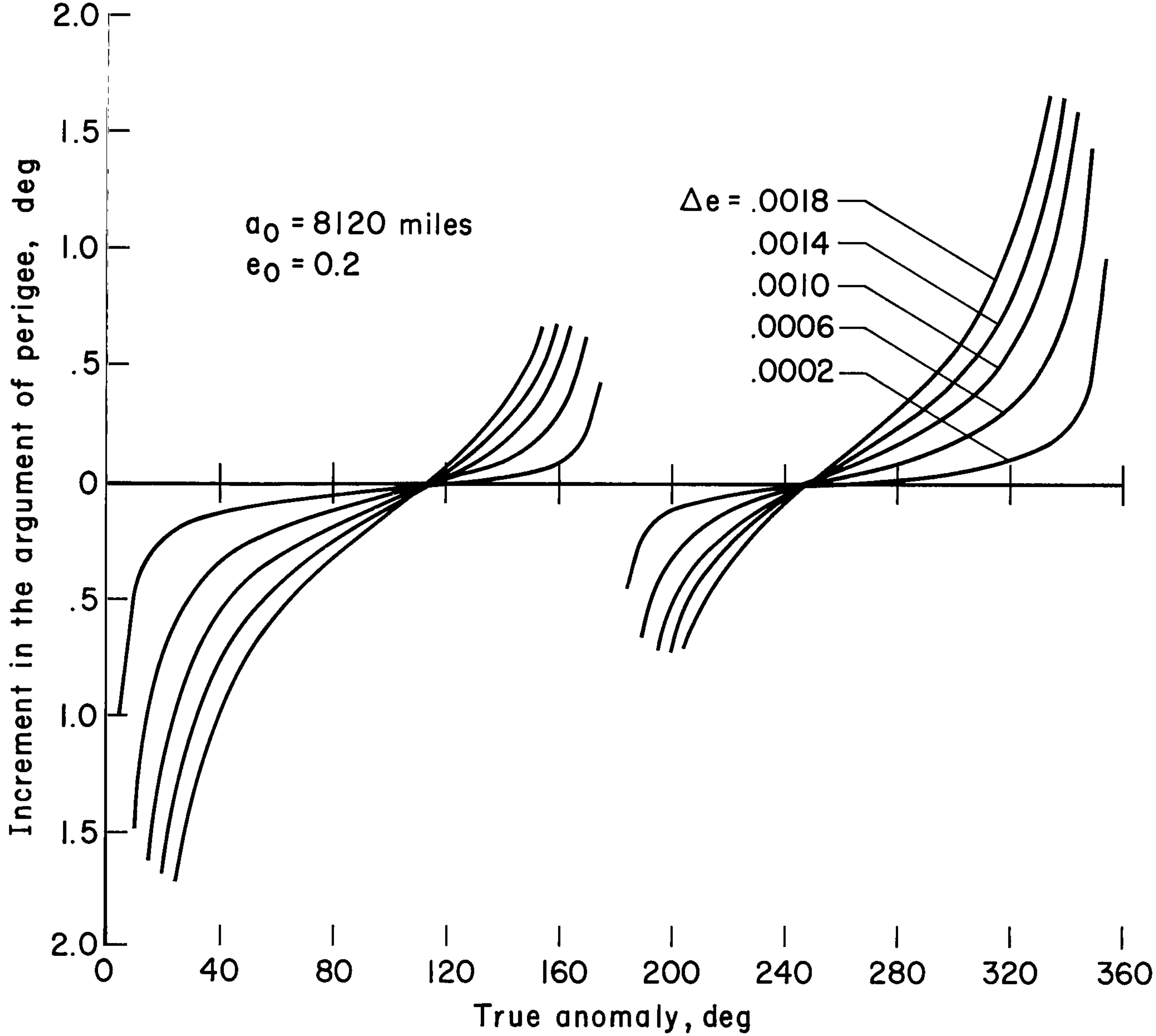


Figure 7.- Changes in the argument of perigee resulting from prescribed changes in the orbit eccentricity.

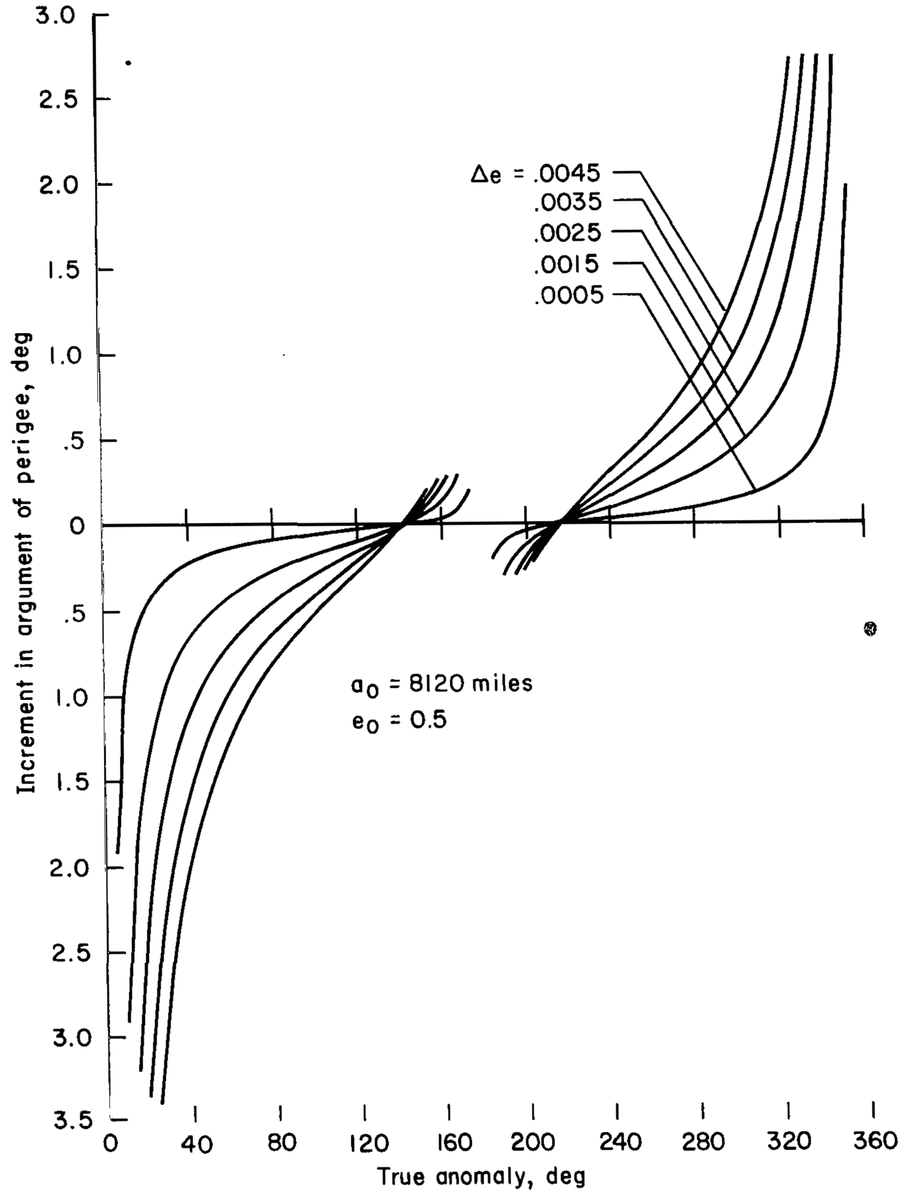


Figure 8.- Changes in the argument of perigee resulting from prescribed changed in the orbit eccentricity.

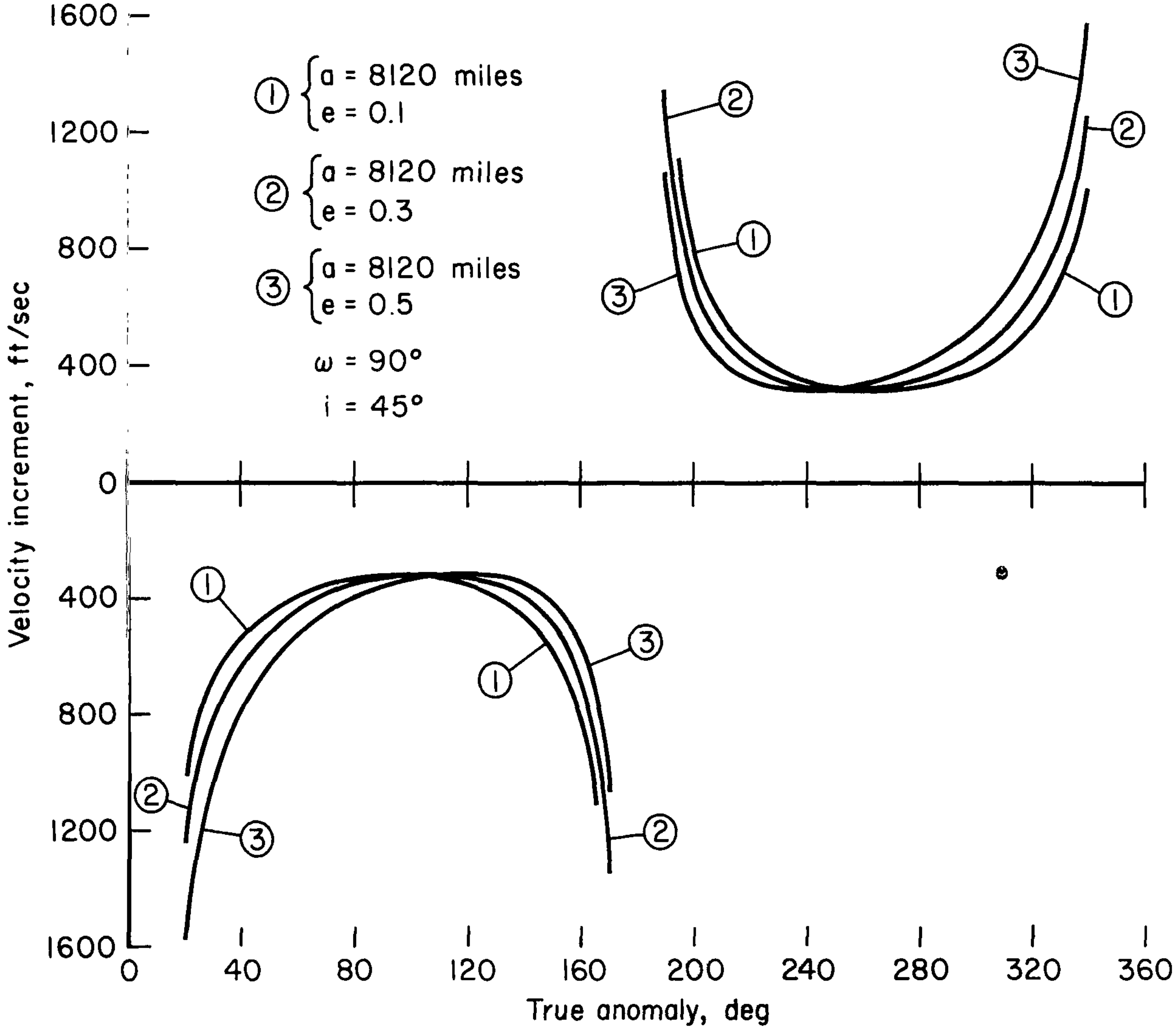


Figure 9.- Velocity increments required to produce a 1° change in the orbit plane inclination of an earth orbit.

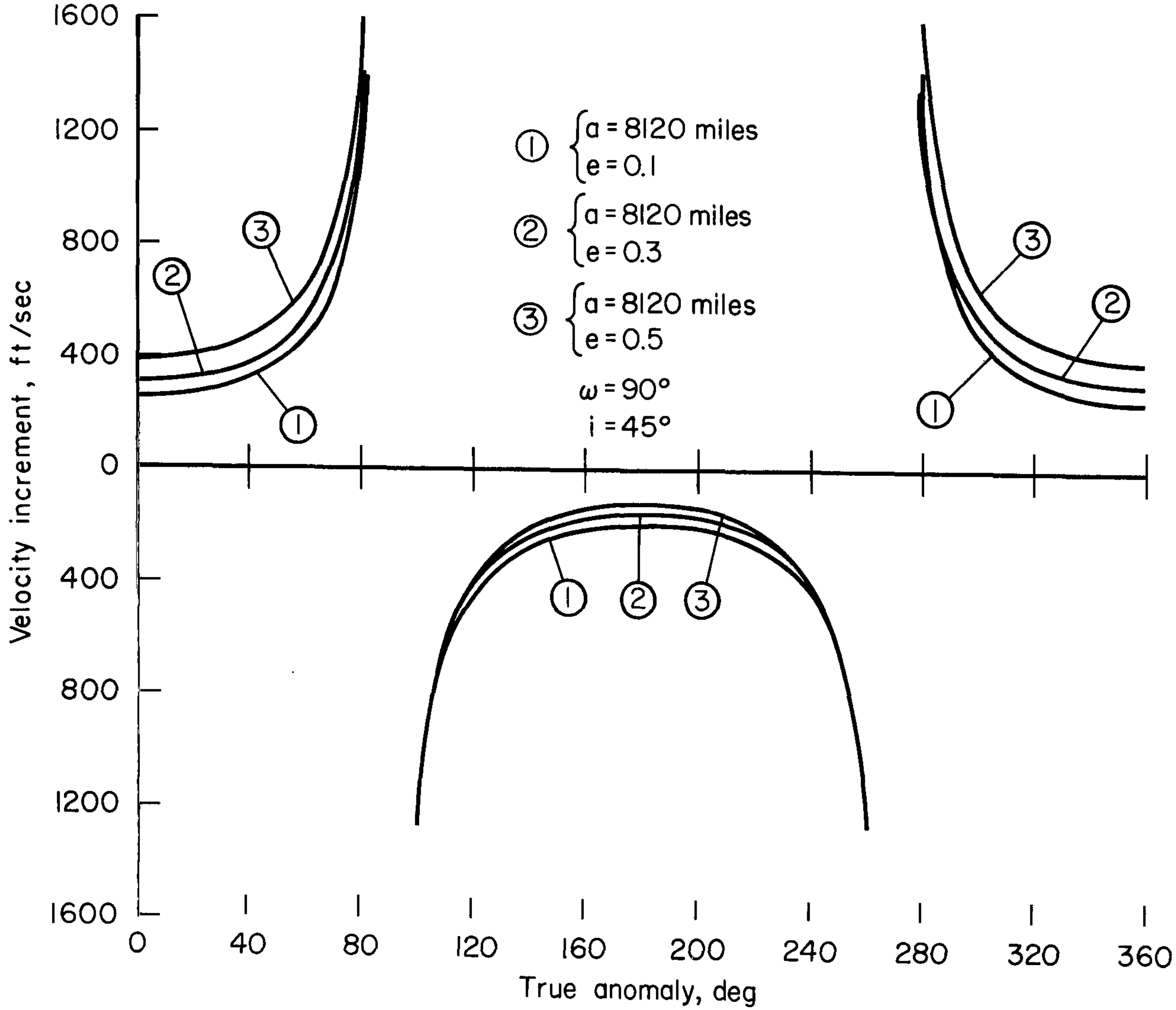


Figure 10.- Velocity increments required to produce a 1° change in the longitude of ascending node line.

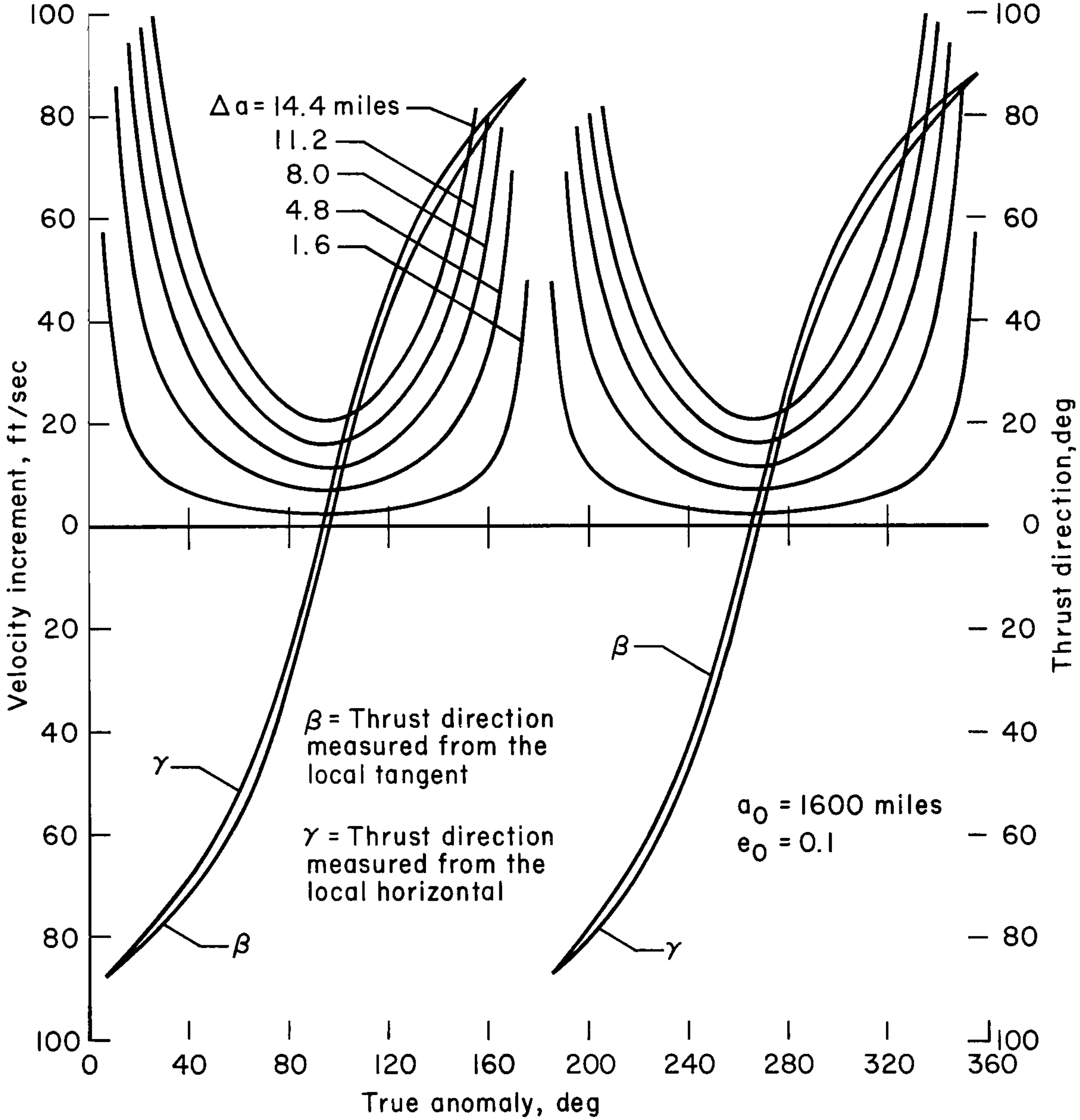


Figure 11.- Velocity increments required to make prescribed changes in the semimajor axis of a lunar orbit subject to the constraint that $\Delta e = 0$.

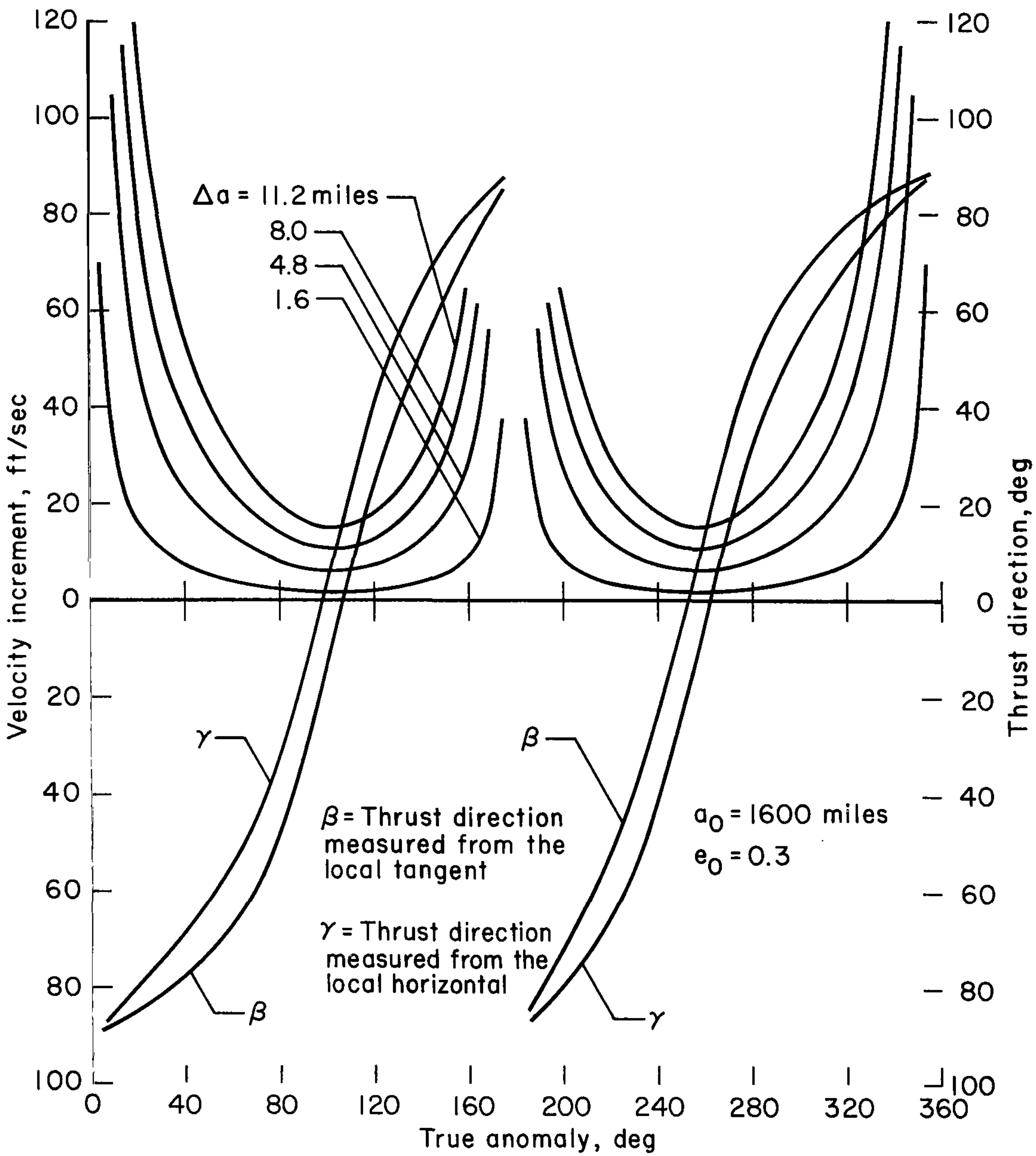
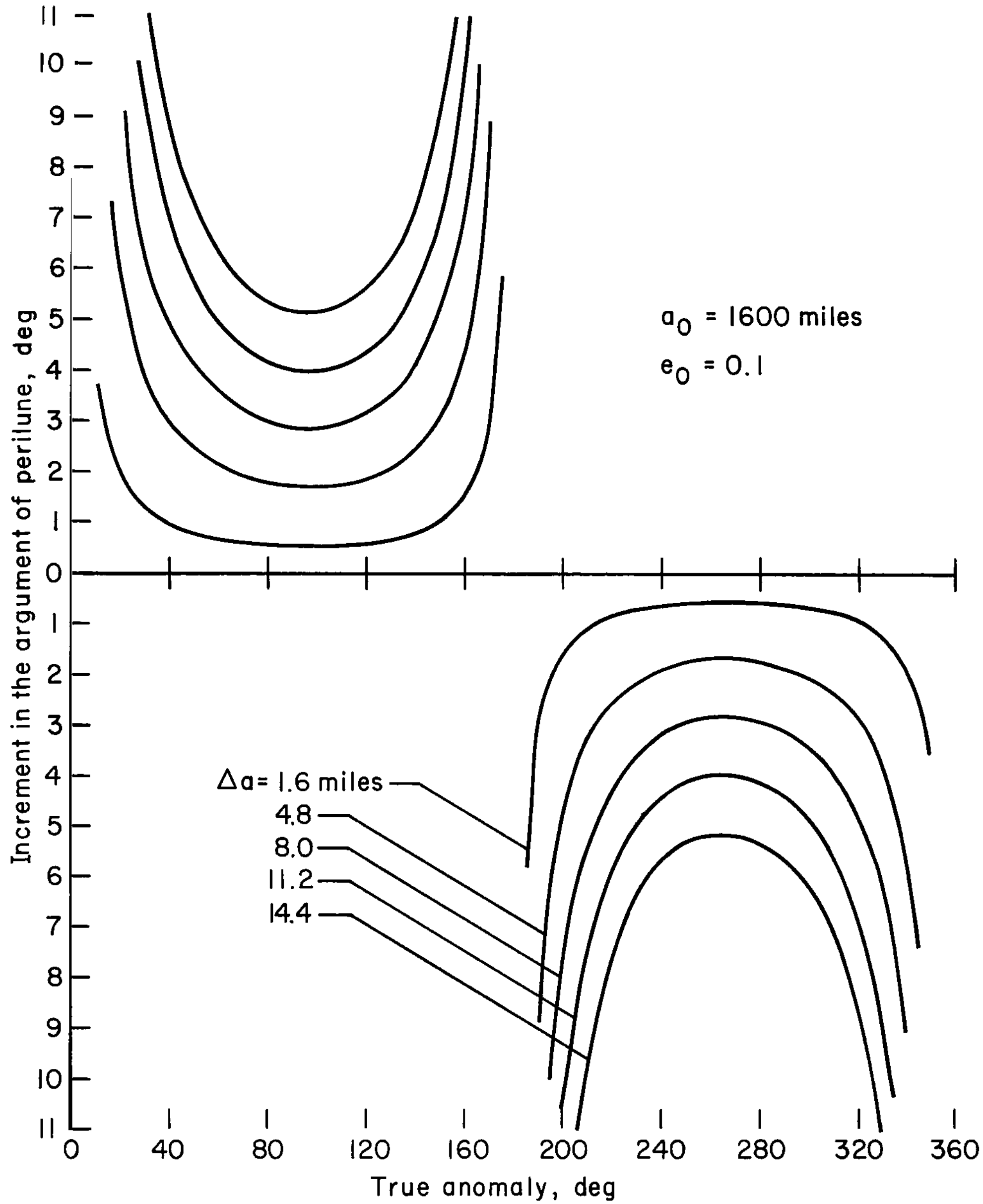


Figure 12.- Velocity increments required to make prescribed changes in the semimajor axis of a lunar orbit subject to the constraint that $\Delta e = 0$.



49 Figure 13.- Changes in the argument of perilune resulting from prescribed changes in the semimajor axis.

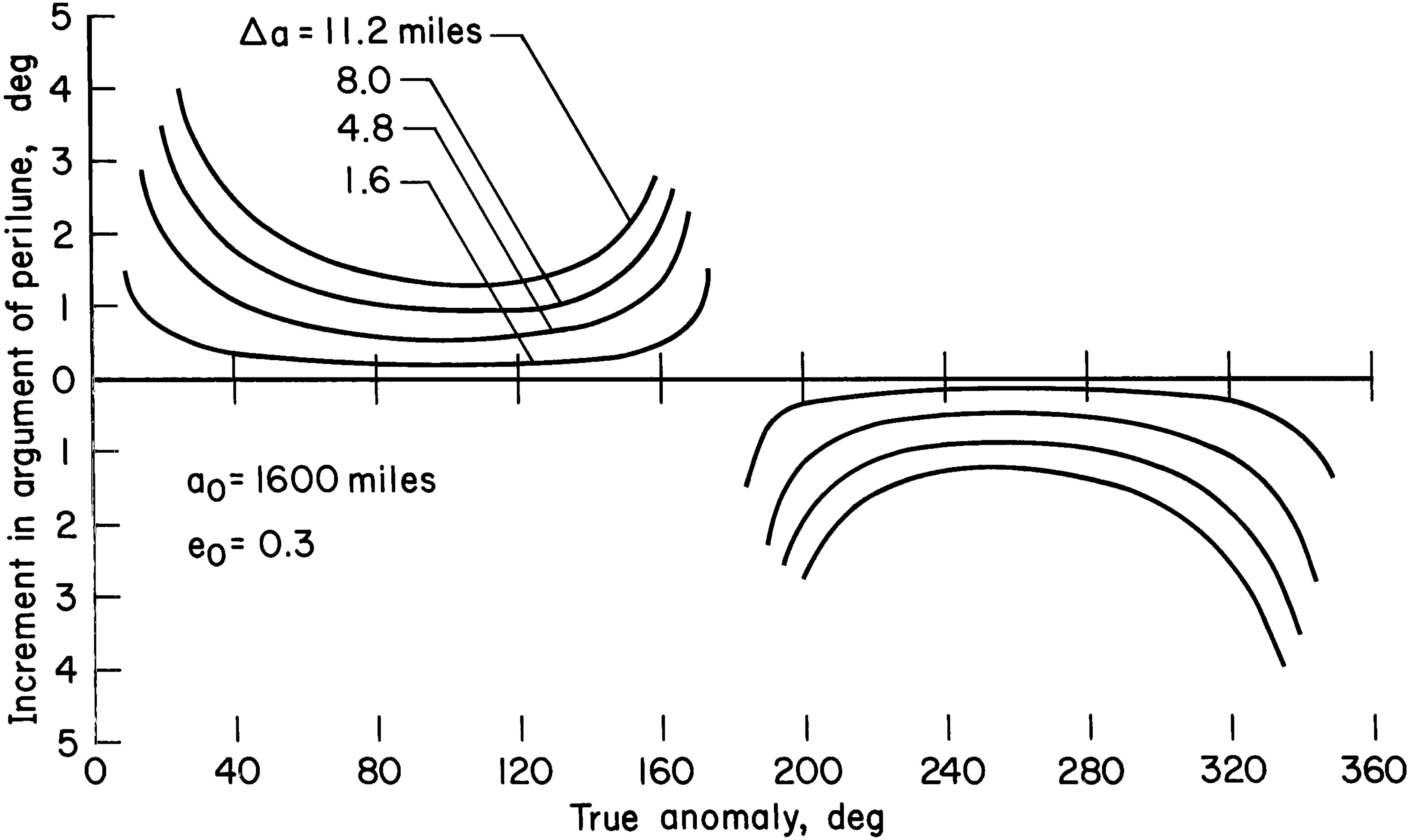


Figure 14.- Changes in the argument of perilune resulting from prescribed changes in the semimajor axis.

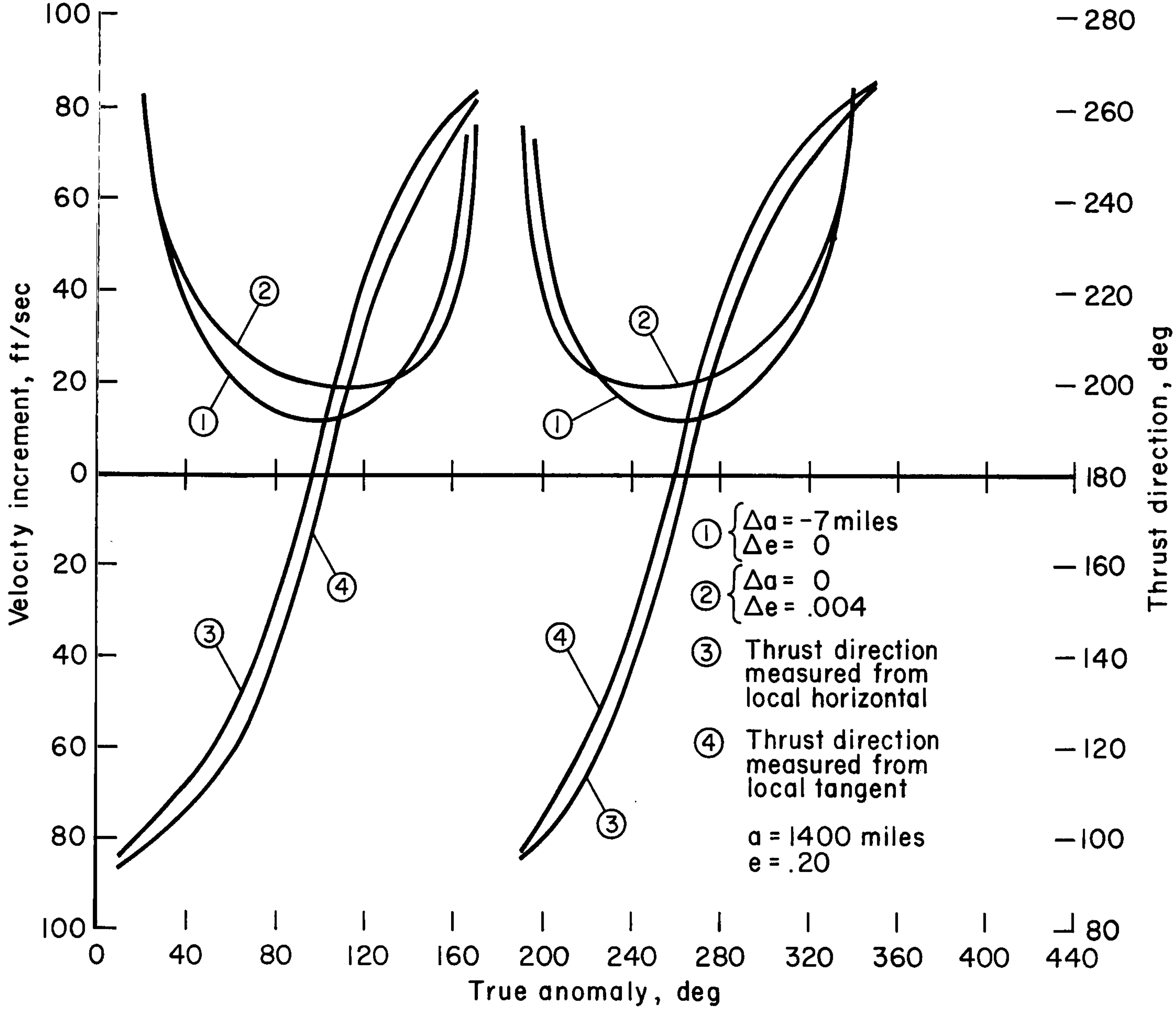


Figure 15.- Impulse vector required to reduce perilune height by 5.6 miles.

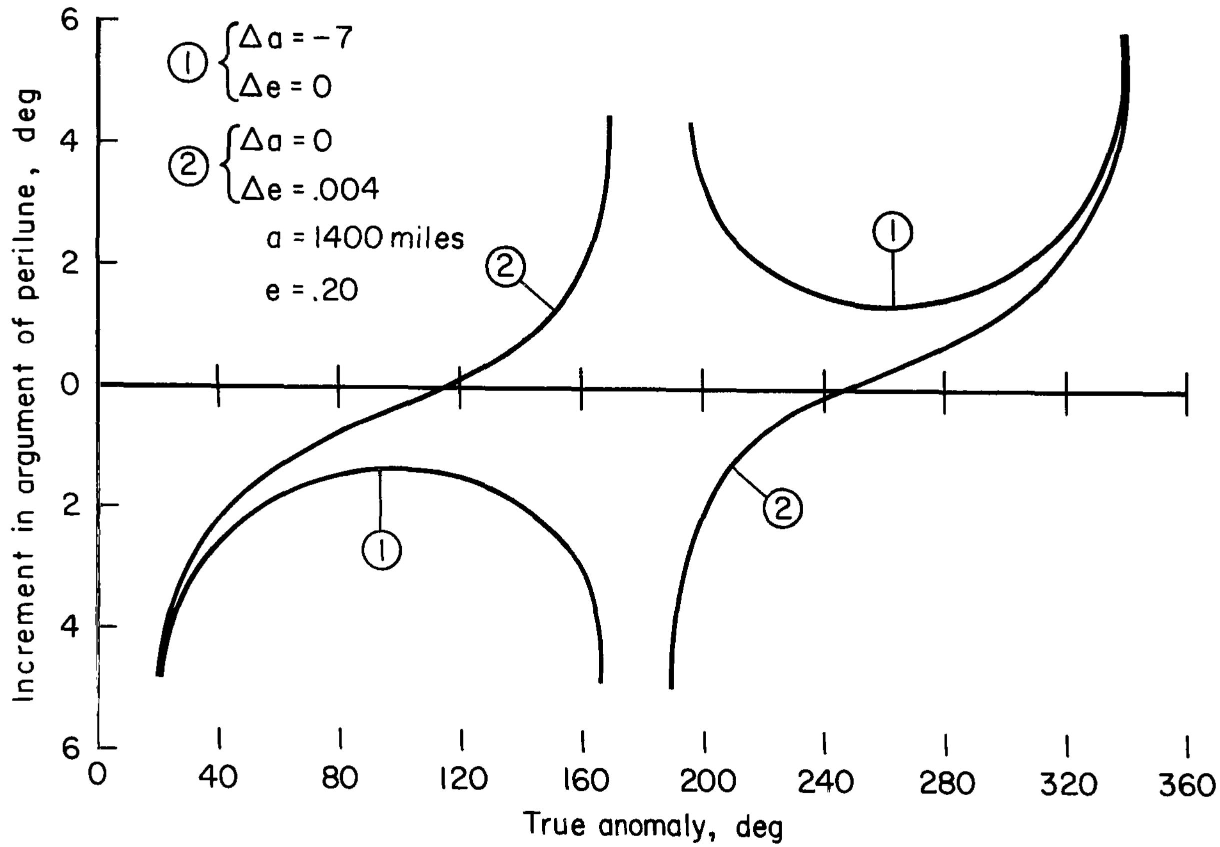


Figure 16.- Changes induced in the argument of perilune by the impulse vectors plotted in figure 15.

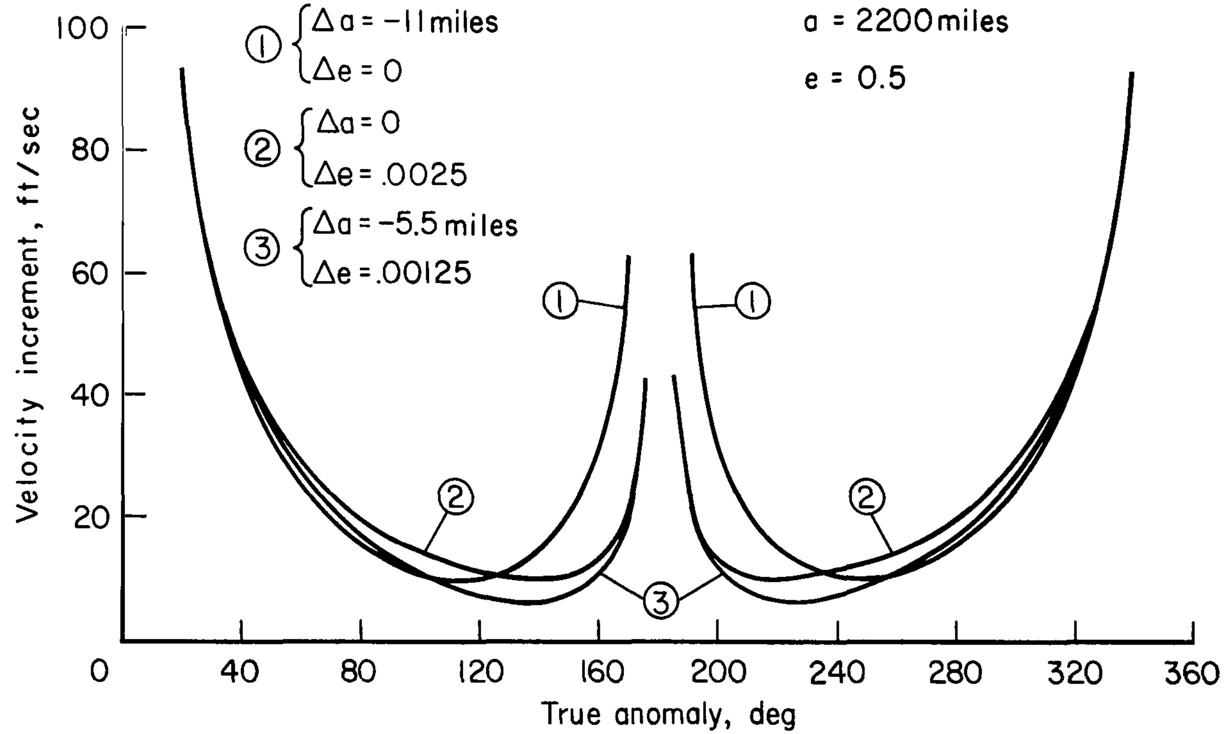


Figure 17.- Velocity increments required to reduce perilune height by 5.5 miles.

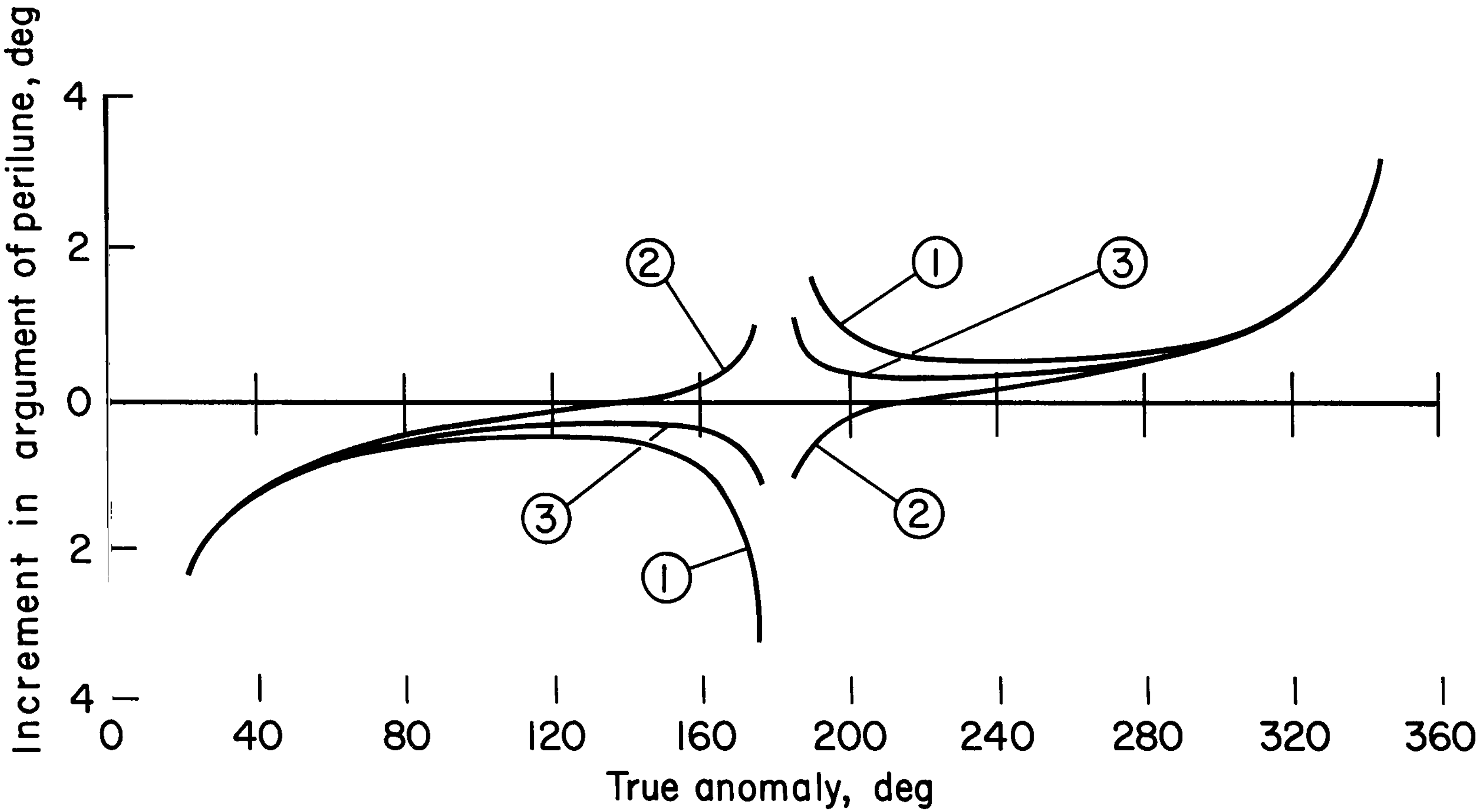


Figure 18.- Changes induced in the argument of perilune by the velocity increments plotted in figure 17.

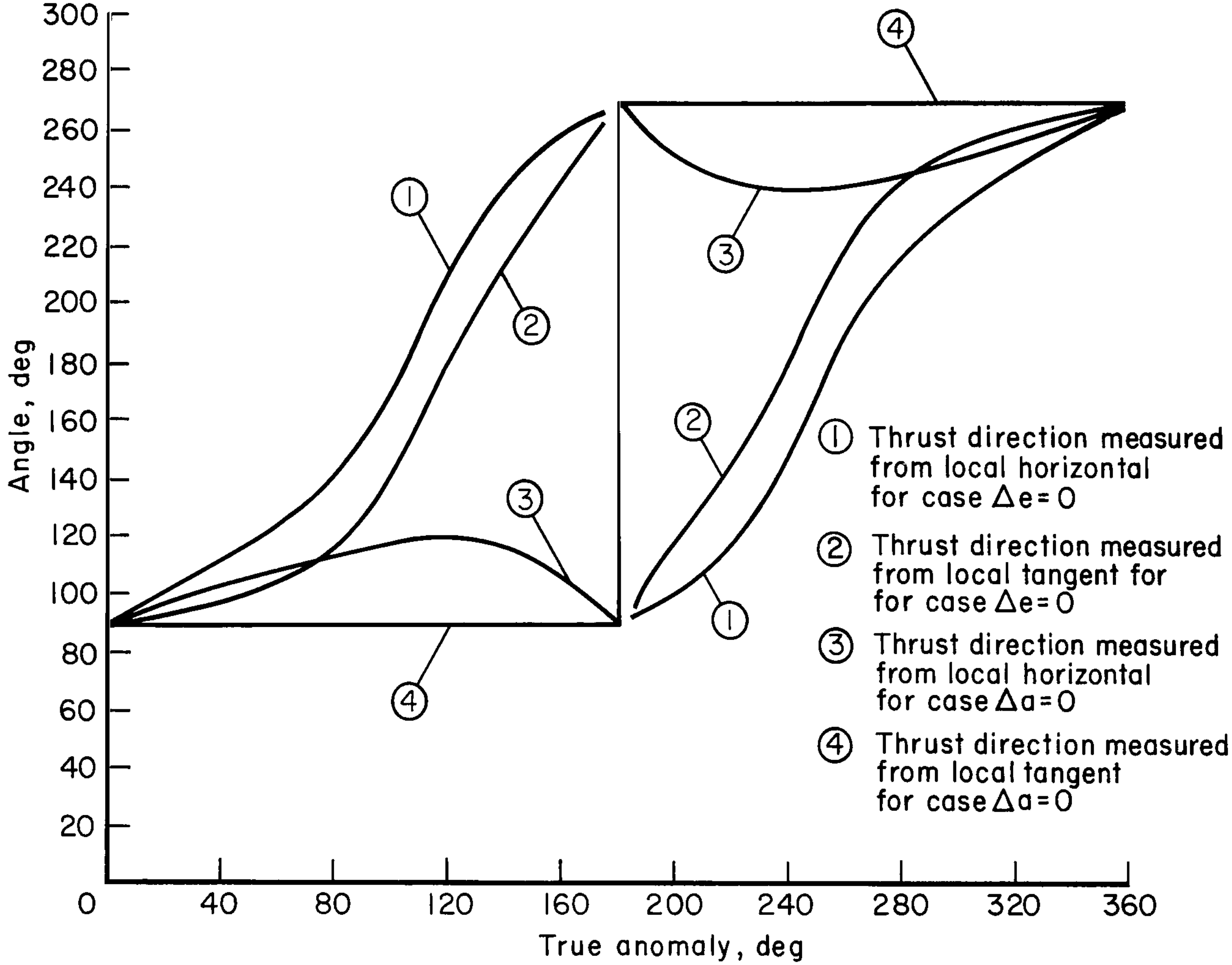


Figure 19.- Thrust direction for velocity increments plotted in figure 17.

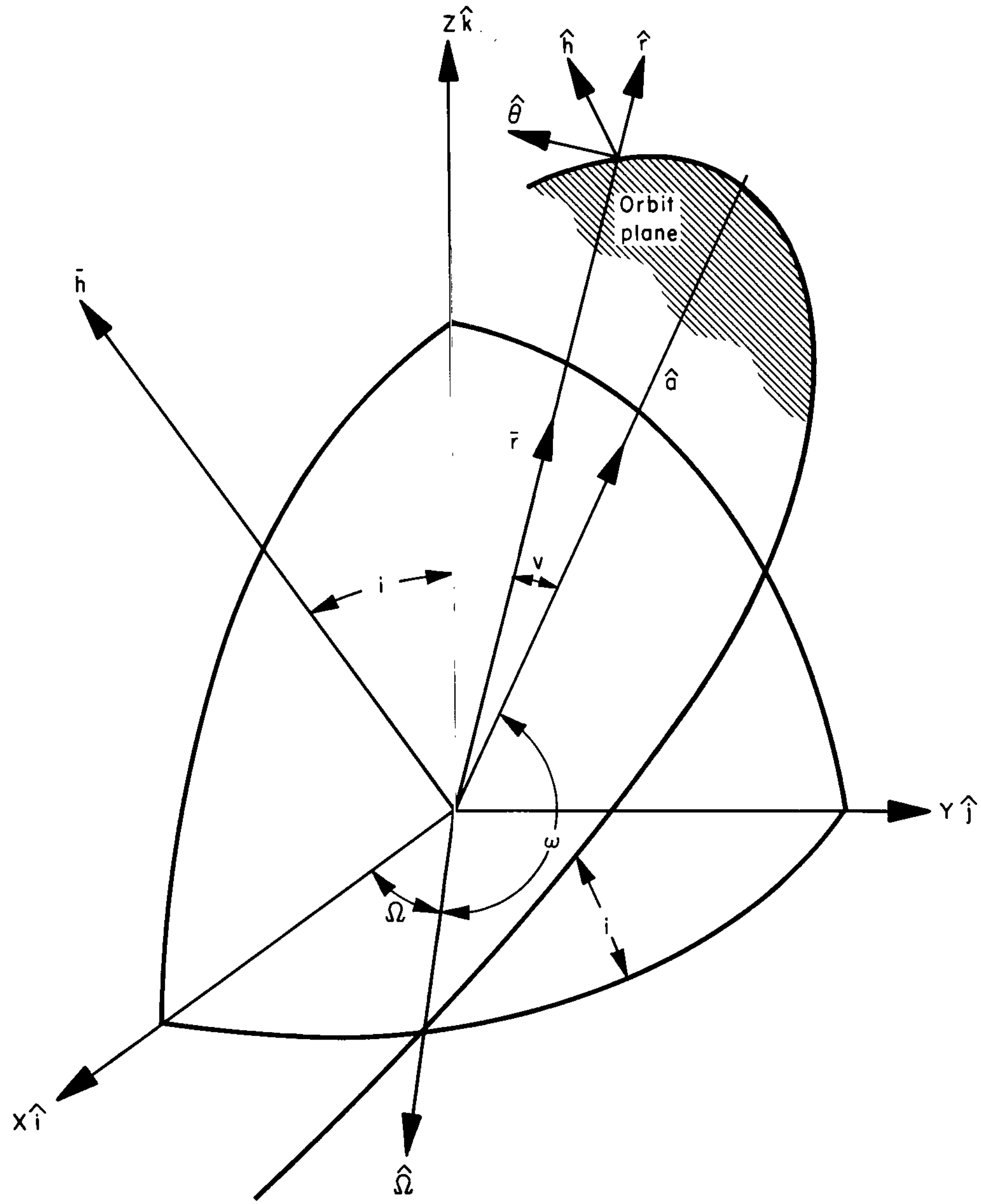


Figure 20.- Notation used in appendix B.

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