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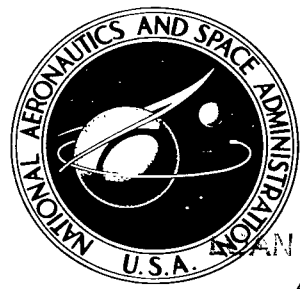

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TRANSPORT EQUATIONS FOR A PARTIALLY IONIZED GAS IN AN ELECTRIC FIELD

by Peter M. Sockol

Lewis Research Center
Cleveland, Ohio

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SUMMARY

Transport equations for a partially ionized gas in an electric field are derived from the Boltzmann equation by the Grad 13-moment method. Charged particle interactions are described by Coulomb forces, while all other particle interactions are described by a rigid elastic sphere model. First approximations to currents, stresses, and heat fluxes are obtained. The resulting transport relations are suitable for use in situations where the electrons have an elevated temperature.

INTRODUCTION

There are two methods currently in use in the description of transport phenomena in multicomponent gas mixtures. The Chapman-Enskog method (refs. 1 and 2) produces a set of approximations to currents, stresses, and heat flows. The theory is restricted, however, to small departures from local equilibrium; in particular there is no provision for assigning separate temperatures to the components of the mixture. The Grad method (refs. 3 to 8) results in a set of partial differential equations for the coefficients in the expansions of the species distribution functions about local Maxwellians. The first few expansion coefficients are simply related to currents, stresses, and heat fluxes. This theory permits greater departures from equilibrium than does the Chapman-Enskog solution and, moreover, permits the assignment of different temperatures to each component. For small departures from the local Maxwellians, a series of approximations can be generated for currents, stresses, and heat flows from the differential equations.

In the presence of an electric field, the electrons in an ionized gas are usually characterized by a higher temperature than the heavier particles. Previous treatments have been limited to either such low degrees of ionization that only electron-neutral and ion-neutral interactions need be considered (refs. 9 and 10) or to such high degrees of ionization that the gas may be treated as fully ionized (refs. 8 and 11). In the present work, the description of the general n-component mixture obtained by the Grad method (refs. 4 to 7) is worked out in detail for a three-component mixture of electrons, ions, and neutral atoms. A first approximation to currents, stresses, and heat flows is obtained through an iterative scheme.

There is some question as to the applicability of the standard collision integral (see eq. (2)) in the case of Coulomb interactions because of the long range nature of these forces. Comparisons with the usually preferred Fokker-Planck treatment, however, show good agreement. The second approximation to the electrical conductivity as obtained by the Chapman-Enskog method differs by only 2.1 percent from the Fokker-Planck value (refs. 11 and 12), while the value obtained by the 13-moment method (ref. 8) differs by 2.6 percent from that obtained with the Fokker-Planck equation. The problem of collective phenomena, such as the interaction of a high-speed electron with its wake of plasma oscillation (ref. 13), is not included in the previous treatments and has been neglected herein.

Cartesian tensor notation is used throughout this report. Greek subscripts are used for tensor indices with summation on double subscripts. English superscripts refer to species.

THEORY

The theory has been treated in detail elsewhere (refs. 3 to 7). A brief resumé is given here. The distribution function f^j for each species is assumed to satisfy the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \xi_\alpha \frac{\partial}{\partial x_\alpha} + \frac{e^j}{m^j} E_\alpha \frac{\partial}{\partial \xi_\alpha}\right) f^j = \sum_k I^{jk} \tag{1}$$

where ξ_α is the particle velocity, x_α the position, and E_α the electric field. The collision integrals I^{jk} are given by

$$I^{jk} = \int \left[f^j(\vec{\xi}\,')f^k(\vec{\xi}_1') - f^j(\vec{\xi})f^k(\vec{\xi}_1)\right] gb \, db \, d\epsilon \, d\vec{\xi}_1 \tag{2}$$

where $g = |\vec{\xi}_1 - \vec{\xi}|$, b is the impact parameter, ϵ the azimuthal angle, and $\vec{\xi}\,', \vec{\xi}_1'$ are the final values of the velocities $\vec{\xi}, \vec{\xi}_1$ after an encounter. (All symbols are defined in appendix A.)

The f^j are expanded in Hermite polynomials (ref. 14) about local Maxwellian distributions $f^{j(0)}$. In the 13-moment approximation, f^j takes the form

$$f^j = f^{j(0)}\left[1 + a_\alpha^{j(1)} v_\alpha + \frac{1}{2} a_{\alpha\beta}^{j(2)} v_\alpha v_\beta + \frac{1}{10} a_\alpha^{j(3)} v_\alpha (v^2 - 5)\right] \tag{3}$$

where

$$v_\alpha \equiv (\alpha^j)^{1/2} c_\alpha \qquad \alpha^j \equiv \frac{m^j}{kT^j} \qquad c_\alpha \equiv \xi_\alpha - w_\alpha \tag{4}$$

$$kT^j \equiv \frac{p^j}{n^j} \tag{5}$$

$$f^j(0) = n^j\left(\frac{\alpha^j}{2\pi}\right)^{3/2} \exp\left(-\frac{1}{2}\, v^2\right) \tag{6}$$

$$a_\alpha^{j(1)} = (\alpha^j)^{1/2}\, u_\alpha^j \tag{7}$$

$$a_{\alpha\beta}^{j(2)} = \frac{1}{p^j}\left(P_{\alpha\beta}^j - p^j\delta_{\alpha\beta}\right) \equiv \frac{1}{p^j}\, p_{\alpha\beta}^j \tag{8}$$

$$a_\alpha^{j(3)} = \frac{2}{p^j}\,(\alpha^j)^{1/2}\left(q_\alpha^j - \frac{5}{2}\, p^j u_\alpha^j\right) \equiv \frac{2}{p^j}\,(\alpha^j)^{1/2}\, R_\alpha^j \tag{9}$$

In these equations, u_α^j is the drift velocity of species j with respect to the mean mass velocity w_α, $p_{\alpha\beta}^j$ is the nonhydrostatic stress, and R_α^j is a reduced heat flux.

In order for the approximation of equation (3) to be valid, the expansion coefficients must be small compared to unity. In particular this requires that

$$u_\alpha^j \ll (kT^j/m^j)^{1/2} \quad \text{and} \quad p_{\alpha\beta}^j \ll p^j$$

Equations for n^j, p^j, u_α^j, $p_{\alpha\beta}^j$, and R_α^j are generated by taking moments of the Boltzmann equation. The higher moments and the collision integrals occurring in these equations are evaluated in terms of preceding quantities by means of the approximation to f^j in equation (3). The result for an m-component mixture is 13m coupled partial differential equations in the 13m dependent variables n^j, p^j, u_α^j, $p_{\alpha\beta}^j$, and R_α^j.

The collision integrals have been considered in detail in the literature (refs. 3 to 7), and the results for elastic collisions are listed in convenient form in reference 5. For the Coulomb interaction between charged particles, the integration over the impact parameter b has been cut off at the Debye length (ref. 15):

$$h = \left(4\pi \sum_j \frac{n^j e^{j2}}{kT^j}\right)^{-1/2} \qquad \text{(cgs units)} \tag{10}$$

The integration over velocity is taken from reference 5 with a small correction (see appendix B). The rigid elastic sphere model has been used for all other interactions. The collision terms, in general, are exceedingly complex. The following restrictions, however, result in a considerable reduction in complexity for a plasma:

$$m^e \ll m^a \qquad m^i = m^a \qquad T^i = T^a \tag{11}$$

Furthermore, as in references 5, 7, and 8, all terms quadratic in u_α^j, $p_{\alpha\beta}^j$, R_α^j have been neglected.

MOMENT EQUATIONS

When the moment equations for m^j, $m^j c_\alpha$, and $1/2\ m^j c^2$ are summed over the species, the conservation equations for the mixture are obtained:

Continuity:

$$\frac{D\rho}{Dt} + \rho \frac{\partial w_\alpha}{\partial x_\alpha} = 0 \tag{12}$$

Momentum:

$$\rho \frac{Dw_\gamma}{Dt} = \sigma E_\gamma - \frac{\partial P_{\alpha\gamma}}{\partial x_a} \tag{13}$$

Energy:

$$\frac{D}{Dt}\left(\frac{3}{2}\ p\right) + \frac{3}{2}\ p \frac{\partial w_\alpha}{\partial x_\alpha} + \frac{\partial q_\alpha}{\partial x_\alpha} + P_{\alpha\beta} \frac{\partial w_\beta}{\partial x_\alpha} = E_\alpha J_\alpha \tag{14}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + w_\alpha \frac{\partial}{\partial x_\alpha} \tag{15}$$

$$\sigma \equiv \sum_j e^j n^j \qquad J_\alpha \equiv \sum_j e^j \Gamma_\alpha^j \tag{16}$$

$$p \equiv \sum_j p^j \qquad P_{\alpha\beta} \equiv \sum_j P_{\alpha\beta}^j \qquad \text{etc.} \tag{17}$$

$$\Gamma_\gamma^j \equiv n^j u_\gamma^j \tag{18}$$

The 13 moment equations for each species are generated from m^j, $m^j c_\gamma$, $\frac{1}{2}\ m^j c^2$, and $m^j\left(c_\gamma c_\lambda - \frac{1}{3}\ c^2 \delta_{\gamma\lambda}\right)$ $\frac{1}{2}\ m^j c_\gamma \left(c^2 - \frac{5}{\alpha^j}\right)$

Continuity:

$$\frac{Dn^j}{Dt} + n^j \frac{\partial w_\alpha}{\partial x_\alpha} + \frac{\partial \Gamma^j_\alpha}{\partial x_\alpha} = 0 \tag{19}$$

Momentum:

$$m^j\left(\frac{D\Gamma^j_\gamma}{Dt} + \Gamma^j_\gamma \frac{\partial w_\alpha}{\partial x_\alpha} + \Gamma^j_\alpha \frac{\partial w_\gamma}{\partial x_\alpha}\right) + \frac{\partial P^j_{\alpha\gamma}}{\partial x_\alpha} - n^j F^j_\gamma$$

$$= - \sum_k \eta^{jk}\left(n^k \Gamma^j_\gamma - n^j \Gamma^k_\gamma\right) + \sum_k \alpha^{jk}\left(n^k R^j_\gamma - \frac{m^j}{m^k} n^j R^k_\gamma\right) \tag{20}$$

where equation (13) is used to define a new force

$$F^j_\gamma \equiv e^j E_\gamma - m^j \frac{Dw_\gamma}{Dt} = \left(e^j - \frac{m^j \sigma}{\rho}\right) E_\gamma + \frac{m^j}{\rho} \frac{\partial P_{\alpha\gamma}}{\partial x_\alpha} \tag{21}$$

Energy:

$$\frac{D}{Dt}\left(\frac{3}{2} p^j\right) + \frac{3}{2} p^j \frac{\partial w_\alpha}{\partial x_\alpha} + \frac{\partial q^j_\alpha}{\partial x_\alpha} - F^j_\alpha \Gamma^j_\alpha + P^j_{\alpha\beta} \frac{\partial w_\beta}{\partial x_\alpha} = - \sum_k \beta^{jk} n^j n^k (T^j - T^k) \tag{22}$$

Stress:

$$\frac{Dp^j_{\gamma\lambda}}{Dt} + p^j_{\gamma\lambda} \frac{\partial w_\alpha}{\partial x_\alpha} + \frac{4}{5}\left\{\frac{\partial q^j_\gamma}{\partial x_\lambda}\right\} - 2\left\{F^j_\gamma \Gamma^j_\lambda\right\} + 2\left\{P^j_{\alpha\gamma} \frac{\partial w_\lambda}{\partial x_\alpha}\right\} = - \sum_k \omega^{jk} p^k_{\gamma\lambda} \tag{23}$$

where for any second-rank tensor $A_{\alpha\beta}$

$$\{A_{\alpha\beta}\} \equiv \frac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha} - \frac{2}{3} A_{\gamma\gamma}\delta_{\alpha\beta}\right) \tag{24}$$

Heat flux:

$$\frac{DR^j_\gamma}{Dt} + R^j_\gamma \frac{\partial w_\alpha}{\partial x_\alpha} + R^j_\alpha \frac{\partial w_\gamma}{\partial x_\alpha} + \frac{k}{m^j}\left(\frac{7}{2} p^j_{\gamma\alpha} \frac{\partial T^j}{\partial x_\alpha} + \frac{5}{2} p^j \frac{\partial T^j}{\partial x_\gamma} + T^j \frac{\partial p^j_{\alpha\gamma}}{\partial x_\alpha}\right) + \frac{5}{2} k\Gamma^j_\gamma \frac{DT^j}{Dt}$$

$$- \frac{1}{m^j} F^j_\alpha p^j_{\gamma\alpha} + \frac{2}{5}\left(q^j_\gamma \frac{\partial w_\alpha}{\partial x_\alpha} + q^j_\alpha \frac{\partial w_\gamma}{\partial x_\alpha} + q^j_\alpha \frac{\partial w_\alpha}{\partial x_\gamma}\right) = - \sum_k \nu^{jk} R^k_\gamma + \sum_k \gamma^{jk}\left(n^k \Gamma^j_\gamma - n^j \Gamma^k_\gamma\right) \tag{25}$$

Collision coefficients for the system of electrons, ions, and neutral atoms are listed in appendix B.

In writing the collision terms as they appear on the right sides of equations (23) and (24) and in appendix B, an order-of-magnitude analysis has been used to discard several terms. The complete analysis is too complicated to be reproduced here, but a sample portion is given in appendix C. The end result is that the collision terms appear in their given forms, and the electron stress and heat flux equations uncouple from those for the ions and atoms provided that certain inequalities are satisfied. The most stringent restriction is given by

$$Z \frac{Q^{ea}}{Q^{ia}} \left(\frac{T^e}{T^a}\right)^{1/2} \left(\frac{m^e}{m^a}\right)^{1/2} \ll 1 \tag{26}$$

The collision coefficients corresponding to the neglected terms are set equal to zero in appendix B.

TRANSPORT APPROXIMATIONS

It has been proposed that approximations to currents, stresses, and heat fluxes be generated from the moment equations by an iterative scheme (refs. 1, 6, 7, and 8). Let T and L be a macroscopic time and length characterizing a flow. It is assumed that T^{-1} and wL^{-1} are small compared with the relaxation frequencies appearing as coefficients on the right sides of the moment equations. In addition, it is assumed that gradients of currents, stresses, and heat flows are small. Then, as a first approximation, a set of algebraic equations is obtained for Γ_γ^j, $p_{\gamma\lambda}^j$, and R_γ^j as functions of n^j, T^j, E_γ, and w_γ and their gradients. Higher approximations are obtained by substituting the value derived in the previous approximation into those terms originally neglected on the left side of the equations.

The first approximations are now considered in detail. Under the preceding assumptions equations (20), (23), and (25) become

$$-\frac{\partial p^j}{\partial x_\gamma} + n^j F_\gamma^j = \sum_k \eta^{jk}\left(n^k \Gamma_\gamma^j - n^j \Gamma_\gamma^k\right) - \sum_k \alpha^{jk}\left(n^k R_\gamma^j - \frac{m^j}{m^k} n^j R_\gamma^k\right) \tag{27}$$

$$-2p^j\left\{\frac{\partial w_\lambda}{\partial x_\gamma}\right\} + 2\left\{e^j E_\gamma \Gamma_\lambda^j\right\} = \sum_k \omega^{jk}\, p_{\gamma\lambda}^k \tag{28}$$

$$-\frac{5}{2} p^j \frac{k}{m^j} \frac{\partial T^j}{\partial x_\gamma} + \frac{e^j}{m^j} E_\alpha p_{\gamma\alpha}^j = \sum_k \nu^{jk} R_\gamma^k - \sum_k \gamma^{jk}\left(n^k \Gamma_\gamma^j - n^j \Gamma_\gamma^k\right) \tag{29}$$

with

$$F_\gamma^j = \left(e^j - \frac{m^j \sigma}{\rho}\right) E_\gamma + \frac{m^j}{\rho} \frac{\partial p}{\partial x_\gamma} \tag{30}$$

The vector equations (27) are not independent. As the Γ_γ^j are referred to the mass velocity w_γ

$$\sum_k m^k \Gamma_\gamma^k = 0 \tag{31}$$

Inverting equations (28) gives

$$p_{\gamma\lambda}^e = -2\mu^e \left\{\frac{\partial w_\lambda}{\partial x_\gamma}\right\} - 2\pi^e \left\{eE_\gamma \Gamma_\lambda^e\right\} \tag{32}$$

$$p_{\gamma\lambda}^i = -2\mu^i \left\{\frac{\partial w_\lambda}{\partial x_\gamma}\right\} + 2\pi^i \left\{ZeE_\gamma \Gamma_\lambda^i\right\} \tag{33}$$

$$p_{\gamma\lambda}^a = -2\mu^a \left\{\frac{\partial w_\lambda}{\partial x_\gamma}\right\} - 2\pi^a \left\{ZeE_\gamma \Gamma_\lambda^i\right\} \tag{34}$$

where the viscosities μ are given by

$$\left.\begin{aligned} \mu^e &= \frac{p^e}{\omega^{ee}} \\ \mu^i &= \frac{\omega^{aa} p^i - \omega^{ia} p^a}{\omega^{ii}\omega^{aa} - \omega^{ia}\omega^{ai}} \\ \mu^a &= \frac{\omega^{ii} p^a - \omega^{ai} p^i}{\omega^{ii}\omega^{aa} - \omega^{ia}\omega^{ai}} \end{aligned}\right\} \tag{35}$$

and the coefficients π^j by

$$\left.\begin{aligned} \pi^e &= (\omega^{ee})^{-1} \\ \pi^i &= \omega^{aa}(\omega^{ii}\omega^{aa} - \omega^{ia}\omega^{ai})^{-1} \\ \pi^a &= \frac{\omega^{ai}}{\omega^{aa}} \pi^i \end{aligned}\right\} \tag{36}$$

Inverting equations (29) gives

$$R_\gamma^e = -\lambda^e \frac{\partial T^e}{\partial x_\gamma} - \frac{e}{m^e} \tau^e E_\alpha p_{\gamma\alpha}^e + D^{ei}\left(n^i \Gamma_\gamma^e - n^e \Gamma_\gamma^i\right) + D^{ea}\left(n^a \Gamma_\gamma^e - n^e \Gamma_\gamma^a\right) \tag{37}$$

$$R_\gamma^i = -\lambda^i \frac{\partial T^a}{\partial x_\gamma} + \frac{Ze}{m^a} \tau^i E_\alpha p_{\gamma\alpha}^i + D^i\left(n^a \Gamma_\gamma^i - n^i \Gamma_\gamma^a\right) \tag{38}$$

$$R_\gamma^a = -\lambda^a \frac{\partial T^a}{\partial x_\gamma} - \frac{Ze}{m^a} \tau^a E_\alpha p_{\gamma a}^i + D^a\left(n^i \Gamma_\gamma^a - n^a \Gamma_\gamma^i\right) \tag{39}$$

where the thermal conductivities λ^j are given by

$$\left.\begin{aligned} \lambda^e &= \frac{5}{2} \frac{k}{m^e} \frac{p^e}{\nu^{ee}} \\ \lambda^i &= \frac{5}{2} \frac{k}{m^a} \frac{\nu^{aa} p^i - \nu^{ia} p^a}{\nu^{ii}\nu^{aa} - \nu^{ia}\nu^{ai}} \\ \lambda^a &= \frac{5}{2} \frac{k}{m^a} \frac{\nu^{ii} p^a - \nu^{ai} p^i}{\nu^{ii}\nu^{aa} - \nu^{ia}\nu^{ai}} \end{aligned}\right\} \tag{40}$$

the coefficients τ^j by

$$\left.\begin{aligned} \tau^e &= (\nu^{ee})^{-1} \\ \tau^i &= \nu^{aa}(\nu^{ii}\nu^{aa} - \nu^{ia}\nu^{ai})^{-1} \\ \tau^a &= \frac{\nu^{ai}}{\nu^{aa}} \tau^i \end{aligned}\right\} \tag{41}$$

and the coefficients D by

$$\left.\begin{aligned} D^{ei} &= \frac{\Upsilon^{ei}}{\nu^{ee}} \\ D^{ea} &= \frac{\Upsilon^{ea}}{\nu^{ee}} \\ D^i &= \frac{\nu^{aa} + \nu^{ia}}{\nu^{ii}\nu^{aa} - \nu^{ia}\nu^{ai}} \Upsilon^{ia} \\ D^a &= \frac{\nu^{ii} + \nu^{ai}}{\nu^{ii}\nu^{aa} - \nu^{ia}\nu^{ai}} \Upsilon^{ia} \end{aligned}\right\} \tag{42}$$

The final step is the inversion of equations (27) with equation (31) used in place of the ion equation. Equations (37) to (39) are used to eliminate the reduced heat fluxes R_γ^j. The following diffusion equations result:

$$\Gamma_\gamma^e = -\frac{1}{\eta^e} d_\gamma^e + \frac{\theta^e}{\eta^i} d_\gamma^a \tag{43}$$

$$\Gamma_\gamma^a = -\frac{\theta^i}{\eta^i}\frac{n^a}{n^i + n^a} d_\gamma^e - \frac{1}{\eta^i} d_\gamma^a \tag{44}$$

$$\Gamma_\gamma^i = -\Gamma_\gamma^a - \frac{m^e}{m^a}\Gamma_\gamma^e \tag{45}$$

where the forces d_γ^j are defined by

$$\left.\begin{aligned} n^e d_\gamma^e &\equiv \frac{\partial p^e}{\partial x_\gamma} - n^e F_\gamma^e + (\alpha^{ei} n^i + \alpha^{ea} n^a)\left(\lambda^e \frac{\partial T^e}{\partial x_\gamma} + \frac{e}{m^e}\tau^e E_\alpha p_{\gamma\alpha}^e\right) \\ (n^i + n^a) d_\gamma^a &\equiv \frac{\partial p^a}{\partial x_\gamma} - n^a F_\gamma^a - \alpha^{ea} n^a\left(\lambda^e \frac{\partial T^e}{\partial x_\gamma} + \frac{e}{m^e}\tau^e E_\alpha p_{\gamma\alpha}^e\right) \\ &\quad + \alpha^{ia}\left[\left(n^i\lambda^a - n^a\lambda^i\right)\frac{\partial T^a}{\partial x_\gamma} + \frac{Ze}{m^a}(n^i\tau^a + n^a\tau^i) E_\alpha p_{\gamma\alpha}^i\right] \end{aligned}\right\} \tag{46}$$

and the transport coefficients are given by

$$\left.\begin{aligned} \eta^e &= \left[\eta^{ei} - (\alpha^{ei} n^i + \alpha^{ea} n^a) D^{ei}\right]\frac{n^i}{n^e} + \left[\eta^{ea} - (\alpha^{ei} n^i + \alpha^{ea} n^a) D^{ea}\right]\frac{n^a}{n^e} \\ \eta^i &= \eta^{ia} - \alpha^{ia}(n^i D^a + n^a D^i) \\ \theta^e &= \frac{1}{\eta^e}\left[\eta^{ei} - \eta^{ea} - (\alpha^{ei} n^i + \alpha^{ea} n^a)(D^{ei} - D^{ea})\right] \\ \theta^i &= \frac{1}{\eta^e}\left[\eta^{ea} - \alpha^{ea}(n^i D^{ei} + n^a D^{ea})\right] \end{aligned}\right\} \tag{47}$$

It has been assumed that

$$\frac{m^e}{m^a} n^e \ll n^i \qquad \frac{m^e}{m^a}\eta^{ia} \ll \eta^{ea} \qquad n^e \eta^{ea} \ll n^i \eta^{ia} \tag{48}$$

In general the term in d_γ^a in equation (43) is negligible.

SUMMARY OF RESULTS

The 13 moment transport equations have been obtained for a partially ionized gas in an electric field. All particle interactions have been described in terms of binary collisions; the shielded Coulomb model has been used for those between charged particles and the rigid elastic sphere model for all interactions involving neutral atoms. According to Grad these equa-

tions should be valid even in cases where there is an appreciable variation in properties over the distance of a mean free path.

First approximations to currents, stresses, and heat fluxes have been obtained from the 13 moment equations. These approximations include expressions for diffusion coefficients, viscosities, and thermal conductivities and exhibit cross-coupling between currents and heat fluxes. The resultant transport relations are suitable for use in situations where the electrons have an elevated temperature.

National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio, February 19, 1964

APPENDIX A

SYMBOLS

$a_\alpha^{j(n)}$	expansion coefficient in distribution function, eq. (3), where n = 0, 1, 2, 3
b	impact parameter
c_α	peculiar velocity, $\xi_\alpha - w_\alpha$
D^j	coefficient, eqs. (42)
d_α^j	diffusion force, eqs. (46)
E_α	electric field
e	electronic charge
e^j	charge on particle of species j
F_α^j	force, eq. (21)
f^j	distribution function of species j
g	magnitude of relative velocity
h	Debye length, eq. (10)
I^{jk}	collision integral, eq. (2)
J_α	electric current
k	Boltzmann constant
m^j	mass of particle of species j
n^j	number density
$P_{\alpha\beta}^j$	stress tensor
p^j	pressure, $1/3\ P_{\alpha\alpha}^j$
$p_{\alpha\beta}^j$	nonhydrostatic stress tensor, $P_{\alpha\beta}^j - p^j\delta_{\alpha\beta}$
Q^{jk}	rigid sphere collision cross section
q_α^j	heat flux
R_α^j	reduced heat flux, $q_\alpha^j - 5/2\ p^j u_\alpha^j$

T^j kinetic temperature of species j

t time

u_α^j diffusion velocity

v_α dimensionless peculiar velocity

w_α mean mass velocity

x_α position vector

Z ion charge in units of e

α^{jk} coefficient, eq. (20)

β^{jk} coefficient, eq. (22)

Γ_α^j diffusion current, $n^j u_\alpha^j$

γ^{jk} coefficient, eq. (25)

$\delta_{\alpha\beta}$ Kronecker delta

ϵ azimuthal angle

η^j coefficient, eqs. (47)

η^{jk} coefficient, eq. (20)

θ^j coefficient, eqs. (47)

κ^{jk} collision parameter, eqs. (B22) to (B25)

Λ parameter, eqs. (B26) to (B28)

λ^j thermal conductivity, eqs. (40)

μ^j viscosity, eqs. (35)

ν^{jk} coefficient, eq. (25)

ξ_α particle velocity

π^j coefficient, eqs. (36)

ρ mass density of mixture

σ charge density of mixture

τ^j coefficient, eqs. (41)

ω^{jk} coefficient, eq. (23)

Subscripts:

$\alpha,\beta,\gamma,\lambda$ tensor indices

Superscripts:

a atom

e electron

i ion

j,k any species

(') velocities after collision

APPENDIX B

COLLISION COEFFICIENTS

The collision coefficients appearing in the species moment equations for the electron, ion, neutral-atom system are as follows:

$$\eta^{ei} = \eta^{ie} = \frac{4}{3}\sqrt{2}\, Z^2 m^e \kappa^{ee}(\ln \Lambda - \gamma) \tag{B1}$$

$$\eta^{ea} = \eta^{ae} = \frac{8}{3}\, m^e \kappa^{ea} \tag{B2}$$

$$\eta^{ia} = \eta^{ai} = \frac{8}{3}\, m^a \kappa^{ia} \tag{B3}$$

$$\alpha^{ei} = \frac{m^a}{m^e}\,\alpha^{ie} = \frac{4}{5}\sqrt{2}\, Z^2 \frac{m^e}{kT^e}\,\kappa^{ee}\left(\ln \Lambda - \gamma - \frac{2}{3}\right) \tag{B4}$$

$$\alpha^{ea} = \frac{m^a}{m^e}\,\alpha^{ae} = -\frac{8}{15}\frac{m^e}{kT^e}\,\kappa^{ea} \tag{B5}$$

$$\alpha^{ia} = \alpha^{ai} = -\frac{4}{15}\frac{m^a}{kT^a}\,\kappa^{ia} \tag{B6}$$

$$\beta^{ei} = \beta^{ie} = 4\sqrt{2}\, Z^2 \frac{m^e}{m^a}\, k\kappa^{ee}(\ln \Lambda - \gamma) = 3\frac{k}{m^a}\,\eta^{ei} \tag{B7}$$

$$\beta^{ea} = \beta^{ae} = 8\frac{m^e}{m^a}\, k\kappa^{ea} = 3\frac{k}{m^a}\,\eta^{ea} \tag{B8}$$

$$\omega^{ee} = \frac{8}{5}\,\kappa^{ee}\left[\left(\ln \Lambda^{ee} - \gamma + \frac{1}{2}\right)n^e + \sqrt{2}\, Z^2\left(\ln \Lambda - \gamma + \frac{1}{2}\right)n^i\right] + \frac{16}{5}\,\kappa^{ea} n^a \tag{B9}$$

$$\omega^{ii} = \frac{8}{5}\, Z^4\left(\frac{m^e}{m^a}\right)^{1/2}\left(\frac{T^e}{T^a}\right)^{3/2}\kappa^{ee}\left(\ln \Lambda^{ii} - \gamma + \frac{1}{2}\right)n^i + \frac{64}{15}\,\kappa^{ia} n^a \tag{B10}$$

$$\omega^{ia} = -\frac{16}{15}\,\kappa^{ia} n^i \tag{B11}$$

$$\omega^{ai} = -\frac{16}{15}\,\kappa^{ia} n^a \tag{B12}$$

$$\omega^{aa} = \frac{64}{15}\,\kappa^{ia} n^i + \frac{16}{5}\,\kappa^{aa} n^a \tag{B13}$$

$$(\omega^{ei} \approx \omega^{ie} \approx \omega^{ea} \approx \omega^{ae} \approx 0)$$

$$\nu^{ee} = \kappa^{ee}\left[\frac{16}{15}\left(\ln \Lambda^{ee} - \gamma + \frac{1}{2}\right)n^e + \frac{26}{15}\sqrt{2}\, Z^2\left(\ln \Lambda - \gamma - \frac{8}{13}\right)n^i\right] + \frac{52}{15}\kappa^{ea}n^a \tag{B14}$$

$$\nu^{ii} = \frac{16}{15} Z^4\left(\frac{m^e}{m^a}\right)^{1/2}\left(\frac{T^e}{T^a}\right)^{3/2} \kappa^{ee}\left(\ln \Lambda^{ii} - \gamma + \frac{1}{2}\right)n^i + \frac{59}{15}\kappa^{ia}n^a \tag{B15}$$

$$\nu^{ia} = -\frac{9}{5}\kappa^{ia}n^i \tag{B16}$$

$$\nu^{ai} = -\frac{9}{5}\kappa^{ia}n^a \tag{B17}$$

$$\nu^{aa} = \frac{59}{15}\kappa^{ia}n^i + \frac{32}{15}\kappa^{aa}n^a \tag{B18}$$

$$(\nu^{ei} \approx \nu^{ie} \approx \nu^{ea} \approx \nu^{ae} \approx 0)$$

$$\gamma^{ei} = 2\sqrt{2}\, Z^2 kT^e\kappa^{ee}\left(\ln \Lambda - \gamma - \frac{2}{3}\right) \tag{B19}$$

$$\gamma^{ea} = -\frac{4}{3} kT^e\kappa^{ea} \tag{B20}$$

$$\gamma^{ia} = \gamma^{ai} = -\frac{2}{3} kT^a\kappa^{ia} \tag{B21}$$

$$(\gamma^{ie} \approx \gamma^{ae} \approx 0)$$

where

$$\kappa^{ee} = \sqrt{\pi}\, e^4 m^{e-1/2}(kT^e)^{-3/2} \tag{B22}$$

$$\kappa^{ea} = \sqrt{\frac{2}{\pi}}\, Q^{ea}\left(\frac{kT^e}{m^e}\right)^{1/2} \tag{B23}$$

$$\kappa^{ia} = \sqrt{\frac{1}{\pi}}\, Q^{ia}\left(\frac{kT^a}{m^a}\right)^{1/2} \tag{B24}$$

$$\kappa^{aa} = \sqrt{\frac{1}{\pi}}\, Q^{aa}\left(\frac{kT^a}{m^a}\right)^{1/2} \tag{B25}$$

$$\Lambda = \frac{2hkT^e}{Ze^2} \tag{B26}$$

$$\Lambda^{ee} = Z\Lambda \tag{B27}$$

$$\Lambda^{ii} = \frac{T^a}{ZT^e}\,\Lambda \tag{B28}$$

and γ is Euler's constant, 0.577.

It should be noted that equation (A6) of reference 5 should be corrected to read

$$\int_0^\infty y^{k-1}e^{-y}\log\left[1+\left(\frac{y}{z}\right)^2\right]dy = 2(k-1)!\,\mathrm{Re}\left[(C+iS)\sum_{p=0}^{k-1}\frac{(iz)^p}{p!} + \sum_{\substack{q=0\\k\geq 2}}^{k-2}(iz)^q\sum_{p=q}^{k-2}\frac{(p-q)!}{(p+1)!}\right] \tag{B29}$$

With this correction the last two equations in section (5-3) of reference 5 become

$$a_{rs}^k = \frac{2^{k+1}(k-1)!}{1.3\ldots(2k+1)}\left(\ln\frac{1}{z} - \gamma + \sum_{\substack{p=1\\k\geq 2}}^{k-1}\frac{1}{p}\right)$$

$$b_{rs}^k = \frac{2^{k+1}(k-1)!}{1.3\ldots(2k+1)}\left(\ln\frac{1}{z} - \gamma - \frac{1}{2} + \sum_{\substack{p=1\\k\geq 2}}^{k-1}\frac{1}{p}\right)$$

APPENDIX C

SIMPLIFICATION OF HEAT-FLUX EQUATIONS

In the evaluation of the collision integrals in the ion heat-flux equation these two terms appear:

$$\frac{20}{3}\sqrt{2}\,Z^2\frac{m^e}{m^a}kT^e\left(1-\frac{T^a}{T^e}\right)\kappa^{ee}(\ln\Lambda-\gamma)n^e n^i u_\gamma^i$$

and

$$\frac{16}{15}Z^4\left(\frac{m^e}{m^a}\right)^{1/2}\left(\frac{T^e}{T^a}\right)^{3/2}\kappa^{ee}\left(\ln\Lambda^{ii}-\gamma+\frac{1}{2}\right)n^i R_\gamma^i \tag{C1}$$

The first term will be negligible compared with the second if the following inequality is satisfied:

$$\frac{1}{Z^2}\left(\frac{m^e}{m^a}\right)^{1/2}\left(\frac{T^a}{T^e}\right)^{3/2}\left|\frac{p^e u_\gamma^i}{R_\gamma^i}\right| \ll 1 \tag{C2}$$

where $p^e = n^e kT^e$ and $\ln\Lambda^{ii}$ has been assumed to be of the same order as $\ln\Lambda$.

For the fully ionized gas ($n^i \gg n^a$), the ion current is very small, and the inequality is assumed to hold (ref. 5). For $n^i \approx n^a$ and $n^i \ll n^a$, equations (38), (42), (45), (B15) to (B18), (B21), (B22), and (B24) give

$$\left|\frac{p^e u_\gamma^i}{R_\gamma^i}\right| = -\frac{p^e}{p^i}\frac{\nu^{ii}\nu^{aa}-\nu^{ia}\nu^{ai}}{\nu^{aa}+\nu^{ia}}\frac{kT^a}{\gamma^{ia}(n^a+n^i)}$$

$$\approx -\frac{p^e}{p^i}\frac{\nu^{ii}kT^a}{\gamma^{ia}(n^a+n^i)}$$

$$\approx \frac{p^e}{p^i}\frac{Z^4\left(\frac{m^e}{m^a}\right)^{1/2}\left(\frac{T^e}{T^a}\right)^{3/2}\kappa^{ee}\ln\Lambda^{ii}n^i+\kappa^{ia}n^a}{\kappa^{ia}(n^a+n^i)}$$

$$\approx \frac{p^e}{p^i}\left[\frac{Z^4}{Q^{ia}}\left(\frac{e^2}{kT^a}\right)^2\ln\Lambda^{ii}\frac{n^i}{n^i+n^a}+\frac{n^a}{n^i+n^a}\right] \tag{C3}$$

Under conditions in which the moment equations are assumed to hold $n^e \approx Zn^i$. Equations (C2) and (C3) lead to these two inequalities in the two density ranges:

$$Z^3\left(\frac{m^e}{m^a}\right)^{1/2}\left(\frac{T^a}{T^e}\right)^{1/2}\left(\frac{e^2}{kT^a}\right)^2\frac{\ln \Lambda^{ii}}{Q^{ia}} \ll 1 \qquad \text{(C4)}$$

$$\frac{1}{Z}\left(\frac{m^e}{m^a}\right)^{1/2}\left(\frac{T^a}{T^e}\right)^{1/2} \ll 1 \qquad \text{(C5)}$$

These relations have been presumed to hold.

Analyses similar to this one have been used to simplify the other moment equations.

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