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by Terrance M. Carney

Manned Spacecraft Center
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SUMMARY

A solution to the problem of finding the energy-optimal trajectory for terminal rendezvous with a satellite station in circular orbit by using the classical calculus of variations is described. A terminal-stage engine with constant thrust and fixed burning time is specified, and both the optimal initial conditions and thrust orientation time history are determined by appealing to the criterion that the energy gain be maximized. The problem is formulated in three dimensions, with the constraint that the terminal stage must turn through a prescribed initial angle between the target orbital plane and the rendezvous-vehicle velocity vector.

Numerical solutions to the two-point boundary-value problem were generated by using an adjoint iteration technique. Near-earth orbital rendezvous trajectories were calculated for orbital altitudes of 100, 200, and 500 nautical miles, initial thrust-to-mass ratios of 0.25g and 1g, and burning times of 100 and 200 seconds. The thrust angles for these solutions are found to lie very close to the local horizontal in elevation and to maintain a nearly constant bearing in azimuth with respect to the orbital plane of the station.

Quasi-optimal trajectories, for which the thrust vector is constrained to the local horizontal plane and to a constant azimuth angle with respect to the station plane, were computed. These trajectories compare very favorably with the optimal case. A degradation in effective characteristic velocity of only 0.063 percent is shown in the worst case (high thrust, long burning time, low altitude, and maximum plane-change angle). Gravity-turn trajectories, which were computed for the coplanar case only, proved to be slightly less efficient than the quasi-optimal trajectories in each case. The maximum coplanar effective characteristic velocity degradation is 0.040 percent for the quasi-optimal mode, as compared with a 0.046-percent degradation for the gravity-turn case.

INTRODUCTION

Studies of trajectories for satellite rendezvous and of the associated guidance problem have appeared frequently in the literature of the past few years. A summary of a large sector of this work is presented in reference 1, which is particularly valuable for its extensive bibliography. Typical studies directed toward satellite rendezvous trajectories are found in reference 2 where coasting trajectories for various incremental velocity changes at rendezvous are examined.

A logical extension of previous work is consideration of the finite thrust problem, and, in particular, the generation of optimal thrusting trajectories. Reference 3 uses bounded thrust and a coast to achieve a point-to-point transfer in a specified time for examining optimization of the coplanar rendezvous problem. This approach treats the linearized equations of motion in an attempt to develop a guidance scheme amenable to onboard computation.

The current paper contains the results of an investigation of terminal-stage guidance in three dimensions for rendezvous with a satellite station in circular orbit. The particular task was one of determining the initial conditions and steering for a constant-thrust vehicle with fixed burning time such that the total energy gained during the terminal stage is maximized, and a desired change of the orbital plane is achieved.

Classical techniques of the calculus of variations were applied to determine the optimal steering function, and an adjoint iteration technique was employed to yield numerical solutions. Rendezvous with a near-earth satellite was chosen as a significant problem to demonstrate the method. For this problem, parametric effects were investigated by varying the ratio of thrust to initial mass, the burning time, the station orbital altitude, and the required angular change of the orbital plane. The values of these parameters were: initial thrust-to-mass ratios of 0.25g and 1g; burning times of 100 and 200 seconds; station altitudes of 100, 200, and 500 nautical miles; and plane changes up to the limiting capability for each combination of thrust and burning time.

The use of analytical guidance schemes which provide reasonable approximations to the optimal solutions is generally more practical than attempting to mechanize the exact optimal steering scheme. The basis of such schemes is a simplified steering law to specify the thrust direction. Two simple laws were investigated. The first, called the quasi-optimal case, was derived from inspection of the optimal solutions. This scheme consisted of constraining the thrust vector to the local horizontal and holding a constant bearing angle with respect to the satellite orbital plane. The second scheme was the gravity turn, where the thrust vector is always alined with the velocity vector. These laws were evaluated by comparing their energy gain with that of the optimal solution.

The author wishes to acknowledge the assistance of Mrs. Beverly P. Latimer and Miss Harriette A. Seals of the Analysis and Computation Division, NASA Langley Research Center, who programed and checked the digital computer mechanization of this problem.

SYMBOLS

A	coefficient matrix for adjoint differential equation set
B	transformation matrix relating initial value perturbations to terminal errors
c^*	effective rocket exhaust velocity, ft/sec
E	total energy per unit mass, ft^2/sec^2
F	modified function to be optimized
f	arbitrary function
g	acceleration of gravity, ft/sec^2
h_s	satellite station altitude, nautical miles
I	integral
K	constant
m	mass of vehicle, slugs
m(0)	initial mass of vehicle, slugs
n	iteration cycle number
p	adjoint variable
r_e	radius of earth, ft
r_f	rendezvous vehicle radius from center of earth, ft
r_s	station radius from center of earth, ft
T	rocket thrust, lb
t	time, sec
u	state variable in expanded system

V total velocity of vehicle, ft/sec

V_c characteristic velocity, ft/sec

V_s station velocity, ft/sec

X,Y,Z coordinates of rotating space station centered axes (fig. 1), ft

α thrust elevation angle with respect to local horizontal, radians

β angle between rendezvous vehicle velocity vector and station velocity, in the station horizontal plane, deg

γ gravitational constant, gr_e^2, ft^3/sec^2

ΔE change in energy during rendezvous stage, ft^2/sec^2

ΔV_c effective characteristic velocity, including potential energy contribution, ft/sec

$\Delta\tilde{V}$ differential effective characteristic velocity, ft/sec

δu perturbation on state variable u

θ thrust elevation angle with respect to satellite reference axes, radians

λ Lagrange multiplier, ft

τ burning time, sec

φ constraint parameter

ψ thrust azimuth angle, radians

Ω_s rate of station rotation, radians/sec

Subscripts:

i,j,k state and adjoint variable indices

max maximum

opt optimal

quasi-opt quasi-optimal

1,2,3 Lagrange multiplier indices; constraint indices

Operators:

$(\dot{\ })$	differentiation with respect to time
$(\ddot{\ })$	second differentiation with respect to time
$(\bar{\ })$	boundary value specified by problem constraints
$\delta(\ )$	variational operator

ANALYSIS

Formulation of the Problem

The mathematical model employed for this investigation of satellite rendezvous was a mass-particle with three degrees of freedom moving in a rotating cartesian axis system oriented in a satellite station. The station was assumed to be in a circular orbit about a uniformly spherical earth. These axes and the associated notation are illustrated in figure 1.

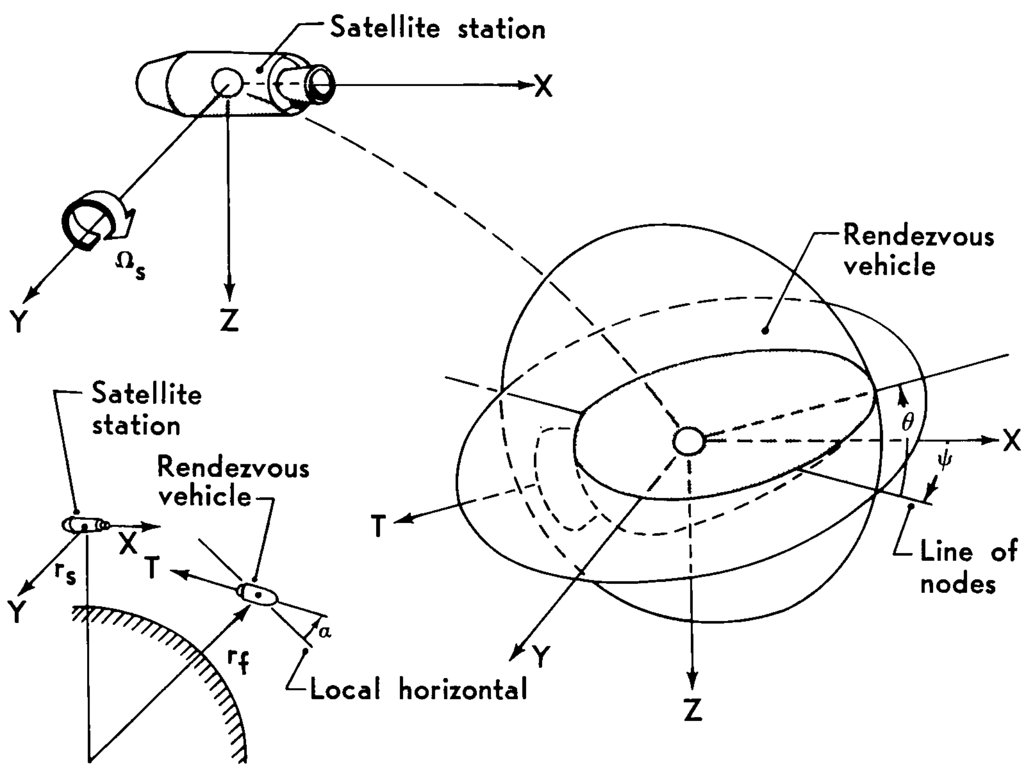

Figure 1.- Axis system and angle definitions.

The equations of motion for this model are

$$\left.\begin{aligned}
\ddot{X} &= \frac{T}{m}\cos\theta\cos\psi + 2\Omega_s\dot{Z} - X\left(\frac{\Upsilon}{r_f^3} - \Omega_s^2\right) \\
\ddot{Y} &= \frac{T}{m}\cos\theta\sin\psi - \frac{Y\Upsilon}{r_f^3} \\
\ddot{Z} &= -\frac{T}{m}\sin\theta - 2\Omega_s\dot{X} + \left(r_s - Z\right)\left(\frac{\Upsilon}{r_f^3} - \Omega_s^2\right)
\end{aligned}\right\} \tag{1}$$

where

$$\frac{T}{m} = \frac{T}{m(0)}\,\frac{1}{\left[1 - \frac{T}{m(0)}\left(\frac{t}{c^*}\right)\right]}$$

and

$$r_f = \left[X^2 + Y^2 + \left(r_s - Z\right)^2\right]^{\frac{1}{2}}$$

A rendezvous vehicle with constant thrust directed along one vehicle axis and a constant burning rate was assumed. The angles θ and ψ are the control variables and may be chosen arbitrarily. Reference 2 contains a derivation of these equations of motion.

The quantity selected for optimization was the change in total energy over the terminal stage. The total energy is expressed as

$$E = \frac{V^2}{2} - \frac{\Upsilon}{r_f} \tag{2}$$

This criterion was used rather than the familiar minimization of expended mass in order that the burning time of the fixed-thrust engine could be specified and all the initial conditions left free. The physical interpretation of the criterion is best illustrated by an example. Consider the problem of rendezvous from a parking orbit consisting of an initial boost, a coast period, and a terminal boost. The energy (eq. (2)) remains constant over any coasting orbit. Since the energy of the satellite station orbit is fixed, maximization of the change in energy over the terminal stage leads to the lowest possible energy for the intermediate coast and hence to the minimum energy cut-off condition for launch boost.

The geometrical boundary conditions for the problem were then left free at the initiation of terminal-stage burning except for the specification of the angle between the orbital plane of the station and the coasting ascent plane of the rendezvous vehicle. It was required that the origin of the station-centered coordinates be reached with no relative velocity remaining at the specified time of burnout.

To force satisfaction of the geometrical end constraints, the problem was inverted and run backward in time starting from the origin. The boundary condition on plane change can then be expressed as

$$\tan \overline{\beta(0)} = \left.\frac{\dot{Y}}{\dot{X} + \Omega_s\left(r_s - Z\right)}\right|_{t=0} \tag{3}$$

For the backward problem, the optimization task becomes one of minimizing the initial total energy. The integral corresponding to the energy, together with the equations of motion (eq. (1)) and the boundary condition (eq. (3)) adjoined as constraints, was varied to determine the Euler-Lagrange equations. Appendix A contains a detailed description of this process by which the equations for the control variables, the Lagrange multipliers, and their associated boundary values are determined.

Trajectories which satisfy the Euler-Lagrange equations and match the boundary conditions correspond to relative minimums for the optimization problem, neglecting questions of existence and uniqueness of the numerical solution. In order to determine the global minimum solution within the stated constraints, all possible relative minimums must be rigorously investigated. Similarly, to determine the validity of the numerical solution, the effect of perturbations on its stability must be determined. For the current problem, the axis system employed affords a well-conditioned set of equations of motion. Judicious search for alternate relative minimums, together with engineering judgment, was used to determine that the results represented global minimums.

Iteration by the Adjoint Method

Once the differential equations governing the variational problem have been written and the requisite number of boundary conditions has been specified, the problem is defined in the mathematical sense. Unfortunately, equations (1) are nonlinear and must be integrated numerically. To make this problem computable, the two-point boundary-value problem must be converted to an initial-value problem. A standard approach to this process is to estimate initial values for the unknown boundary conditions, integrate over the interval, and then observe the errors in the known end values and operate on them to improve the initial estimates. The logical process by which the end errors are converted to initial-value corrections should force convergence to the desired end values within a reasonable number of iterations.

For this study, an iteration process described in reference 4 was employed. This technique has also been employed recently in reference 5 for the solution of variational problems which are by nature two-point boundary-value problems. The iteration process, together with the resulting equations for application to this problem, is derived in appendix B.

RESULTS AND DISCUSSION

Range of Parameters

Numerical results were generated for the problem of rendezvous with a near-earth satellite in circular orbit. The rendezvous vehicle was assumed to have constant thrust directed along one vehicle axis and a specified thrust-time duration. A constant mass rate was used which corresponded to an effective exhaust velocity of 10,000 feet per second (specific impulse of 310 sec).

The ranges of parameters chosen for investigation were initial thrust-to-mass ratio, burning-time duration, orbital altitude, and plane-change angle. The values used in this investigation are tabulated as follows:

$T/m(0)$, g	τ, sec	$\overline{\beta(0)}$, deg
0.25	100	0
		.5
		1.0
		1.5
	200	0
		1.0
		2.0
		3.0
1.0	100	0
		2.0
		4.0
		6.0
	200	0
		4.0
		8.0
		10.0

All cases were run for orbital altitudes of 100, 200, and 500 nautical miles. To determine the range of the plane-change angle, an upper bound in $\overline{\beta}$ was calculated from the following expression:

$$\overline{\beta}_{max} = \sin^{-1}\left(\frac{V_c}{V_s}\right) \tag{4}$$

where

$$V_c = c^* \ln\left[\frac{m(0)}{m(\tau)}\right]$$

There is a different characteristic velocity V_c for each thrust and burning-time combination.

Adjoint Iteration Check

The combined variational problem and adjoint iteration scheme were tested and found to be rapidly convergent even for very poor guesses of the initial multiplier values. Table I illustrates the convergence process for two cases. The conditions of case 1 are: $T/m(0) = 0.25g$, burning time of 100 sec, station altitude of 100 nautical miles, and a plane-change angle of $1\frac{1}{2}^\circ$. The sum of the errors in end conditions $\sum_i \delta u_i^2(\tau)$ drops seven orders of magnitude in four passes. Case 2 conditions are $T/m(0) = 1g$, burning time of 200 sec, and a plane-change angle of 10°. This more severe case required 12 passes, but the error is seen to be uniformly convergent and no difficulties occurred during iteration. Case 2 displayed the slowest convergence of all the cases computed.

Optimal Trajectories

The profiles of the optimum trajectories, together with time histories of the thrust vector orientation, are presented in figures 2 to 5. The origins of the curves of the thrust elevation angle are displaced for clarity. The thrust elevation angle measured with respect to the local horizontal α exhibits variations with time and required angular change of plane. However, the maximum departure from the local horizontal encountered is less than 0.055 radian. The thrust azimuth angle is nearly constant for all the cases run.

The results from the optimal trajectories were used to evaluate the energy decrement resulting from use of finite burning times as compared with impulsive velocity changes. In order to make a direct comparison, effective characteristic velocities were calculated from the energy increment realized by using the following relation:

$$\Delta V_c = V_s\left[1 - \left(1 + \frac{2\Delta E}{V_s^2}\right)^{\frac{1}{2}}\right] \tag{5}$$

The variation of effective characteristic velocity and of percentage velocity differential compared with an ideal single impulse with plane change and altitude is shown in figure 6. As would be expected, the energy change realized decreases with increasing change of orbital plane. This decrease

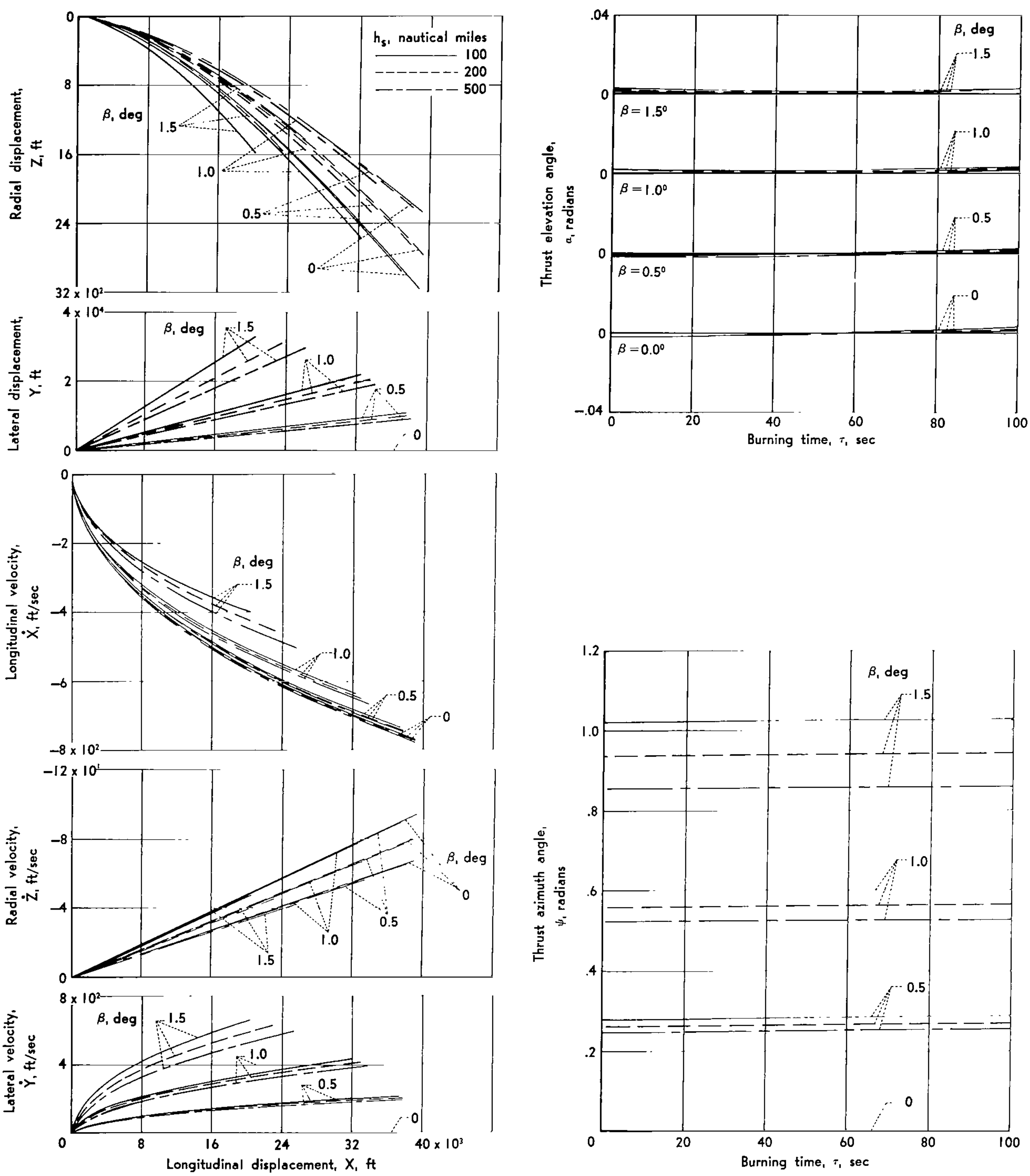

Figure 2.- Energy optimal trajectories. $\frac{T}{m(0)} = 0.25g$; $\tau = 100$ sec.

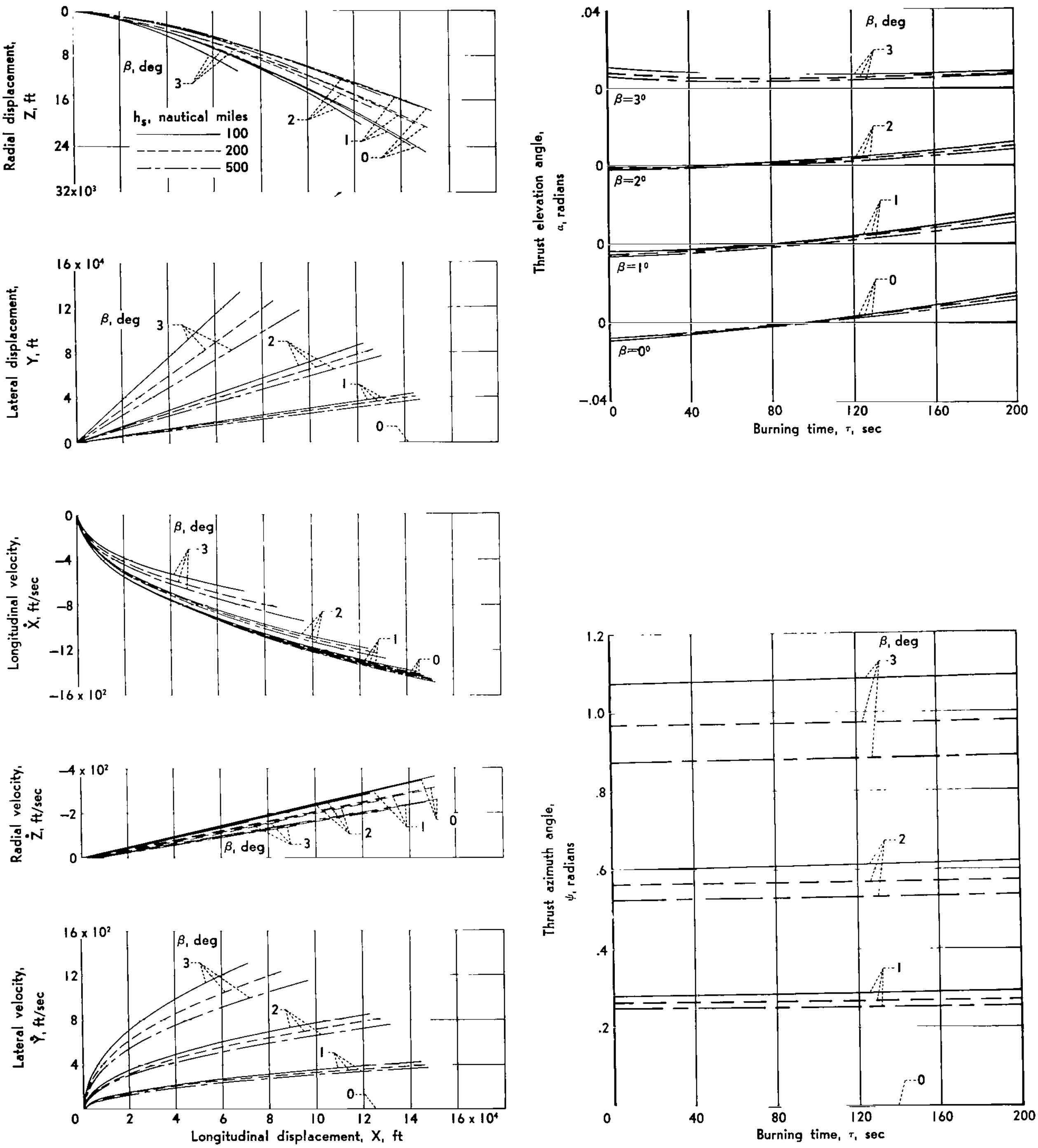

Figure 3.- Energy optimal trajectories. $\frac{T}{m(0)} = 0.25g$; $\tau = 200$ sec.

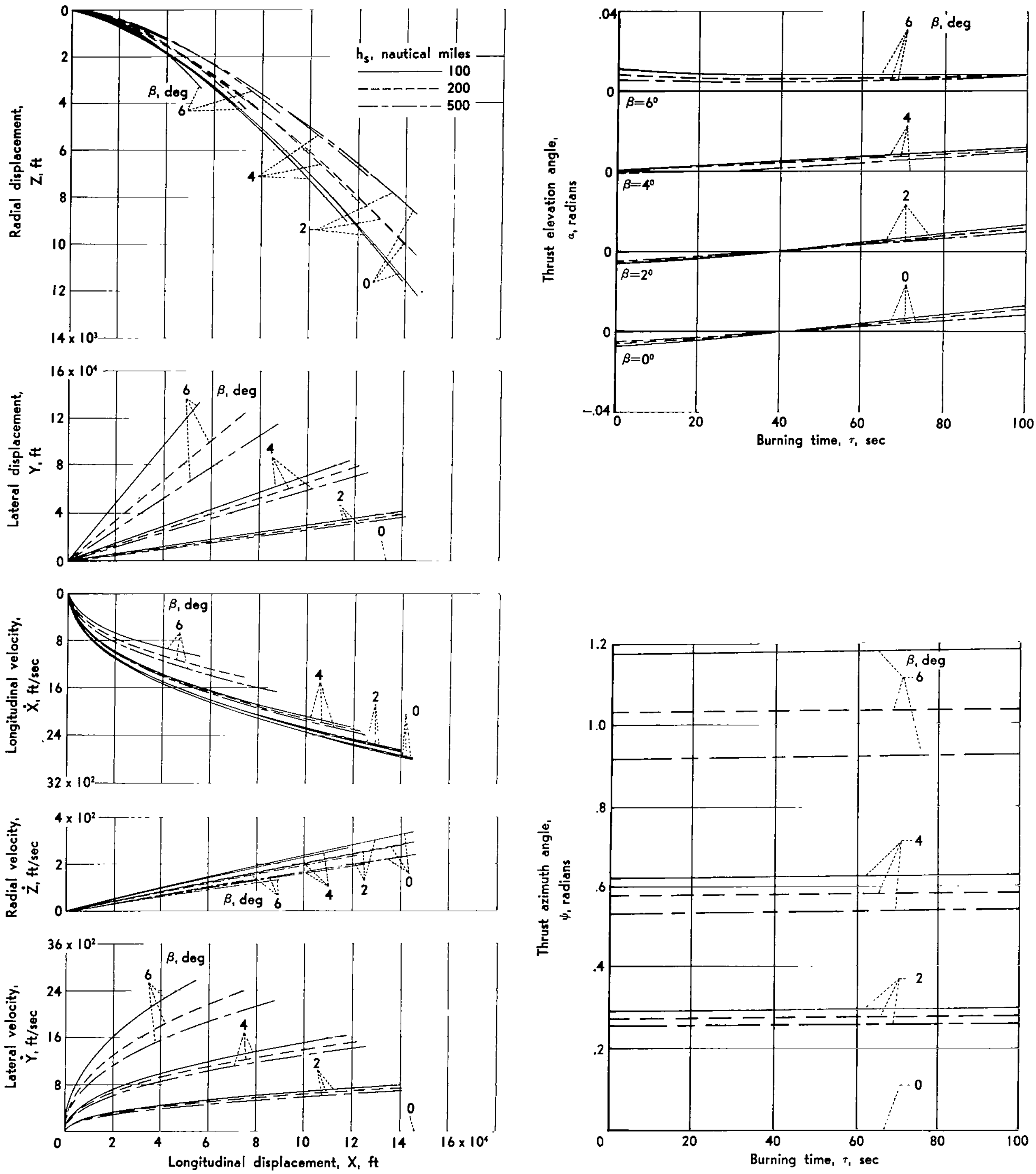

Figure 4.- Energy optimal trajectories. $\frac{T}{m(0)} = 1g$; $\tau = 100$ sec.

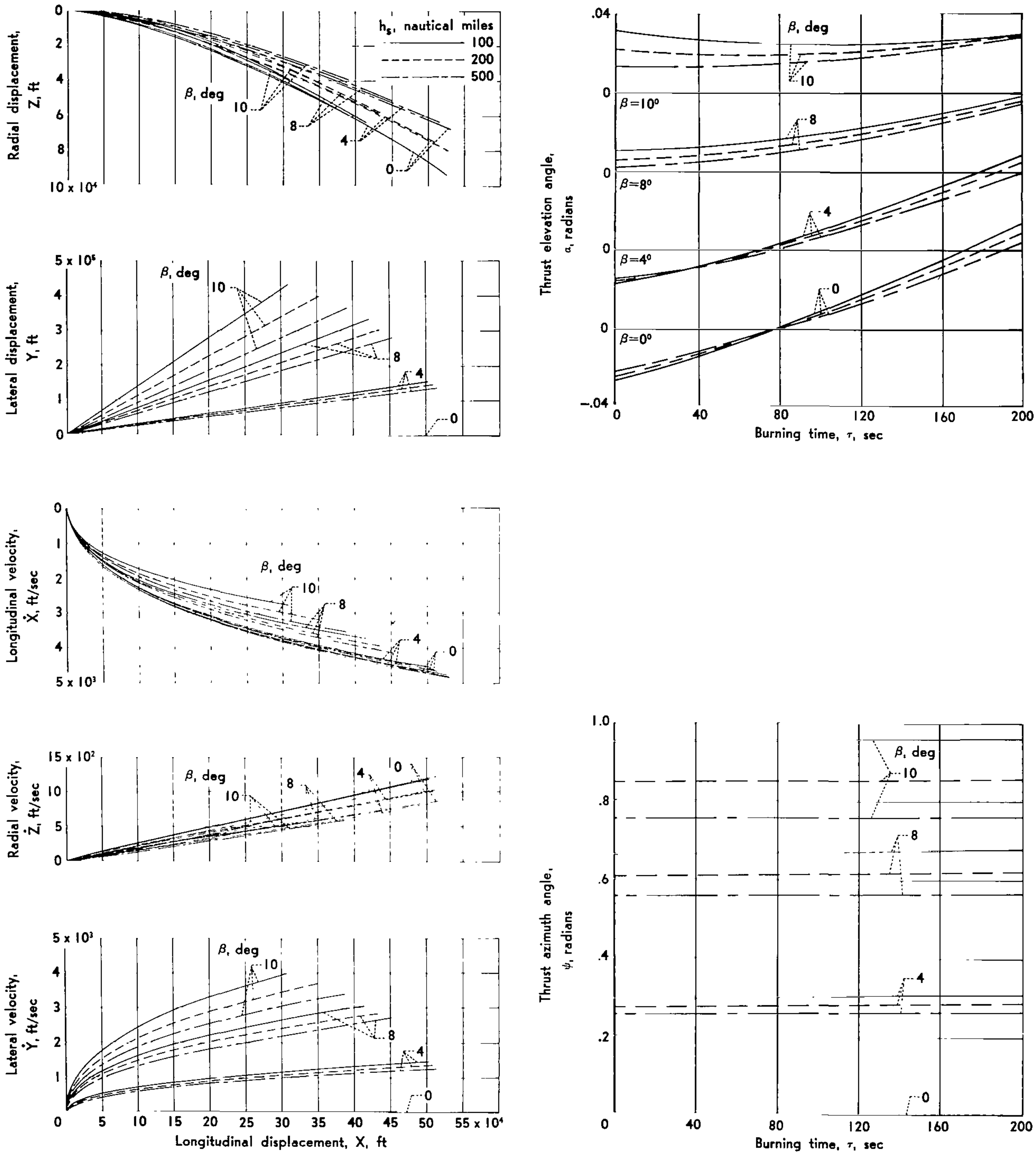

Figure 5.- Energy optimal trajectories. $\frac{T}{m(0)} = 1g$; $\tau = 200$ sec.

TABLE I. - ADJOINT ITERATION

Case	Cycle	$\lambda_1(\tau)$	$\dot{\lambda}_1(\tau)$	$\lambda_2(\tau)$	$\dot{\lambda}_2(\tau)$	$\lambda_3(\tau)$	$\dot{\lambda}_3(\tau)$	$\Delta\dot{\gamma}(0)$	$\Delta\lambda_1(0)$	$\Delta\dot{\lambda}_1(0)$	$\Delta\dot{\lambda}_2(0)$
1	1	0	0	0	0	0	0	-0.2165×10^{3}	0.2687×10^{2}	0.7932×10^{1}	0
	2	770.5	- 7.940	7,127	1.197	46.32	- .9176	$-.5436\times10^{1}$	$.6105\times10^{2}$	-.3094	$-.2038\times10^{1}$
	3	678.5	- 7.633	7,244	3.328	36.19	- .7944	$.9569\times10^{-2}$	$.4607\times10^{1}$	$.1020\times10^{-1}$	$.3459\times10^{-1}$
	4	675.0	- 7.645	7,246	3.292	35.84	- .7923	$-.9460\times10^{-3}$	$-.3771\times10^{-1}$	$-.3789\times10^{-3}$	$-.1348\times10^{-2}$
2	1	0	0	0	0	0	0	-0.3651×10^{4}	0.1328×10^{4}	0.2996×10^{2}	0
	2	4,846	-30.62	17,733	15.14	643.4	-5.687	$-.7745\times10^{3}$	$.2946\times10^{4}$	$-.7171\times10^{1}$	$-.1107\times10^{2}$
	3	429	-23.32	26,984	31.18	- 324.5	- .8022	$-.3099\times10^{3}$	$.2362\times10^{4}$	-.5812	-.7151
	4	-1,940	-23.76	34,818	34.56	- 882.3	1.881	$-.1125\times10^{3}$	$.1391\times10^{4}$	$-.1536\times10^{1}$	$-.1309\times10^{1}$
	5	-3,567	-22.97	39,397	37.31	-1,266	3.879	$-.3531\times10^{2}$	$.8611\times10^{3}$	-.6680	-.5354
	6	-4,516	-22.78	41,631	38.52	-1,488	4.998	$-.9836\times10^{1}$	$.4158\times10^{3}$	-.2489	-.1968
	7	-4,960	-22.75	42,528	38.97	-1,590	5.493	$-.2522\times10^{1}$	$.1665\times10^{3}$	$-.7707\times10^{-1}$	$-.6233\times10^{-1}$
	8	-5,134	-22.76	42,843	39.12	-1,630	5.677	- .6174	$.5842\times10^{2}$	$-.2166\times10^{-1}$	$-.1816\times10^{-1}$
	9	-5,194	-22.77	42,946	39.17	-1,643	5.738	- .1472	$.1877\times10^{2}$	$-.5823\times10^{-2}$	$-.5068\times10^{-2}$
	10	-5,213	-22.77	42,977	39.18	-1,648	5.757	$-.3442\times10^{-1}$	$.5686\times10^{1}$	$-.1531\times10^{-2}$	$-.1379\times10^{-2}$
	11	-5,219	-22.78	42,986	39.19	-1,649	5.762	$-.7843\times10^{-2}$	$.1653\times10^{1}$	$-.3964\times10^{-3}$	$-.3679\times10^{-3}$
	12	-5,220	-22.78	42,988	39.19	-1,649	5.764	$-.1831\times10^{-2}$	.4656	$-.1005\times10^{-3}$	$-.9680\times10^{-4}$

is less marked at higher altitudes, primarily because a smaller addition of velocity normal to the station plane is required to rotate the orbital velocity through the desired angle (orbital velocity decreases with altitude). The ratio of characteristic velocities similarly decreases with increasing orbital-plane change, which indicates that the cost of finite burning time increases with the turning angle, even when normalized by the characteristic velocity requirement.

For the cost criterion, finite burning times have a small advantage over impulsive velocity corrections in the coplanar case with the initial point unconstrained if only a terminal impulse is allowed. This is not totally unexpected, since, in general, two impulses would be required to achieve rendezvous from arbitrary initial conditions, and the sum of the two velocity increments would be less than the single increment for an equal change in energy. A heuristic explanation for this statement can be drawn from consideration of the rate of change of energy

$$\dot{E} = \vec{V} \cdot \dot{\vec{V}} + \frac{\gamma}{r_f^2} \dot{r}_f \tag{6}$$

$\Delta\lambda_3(0)$	$\Delta\dot{\lambda}_3(0)$	Error $\sum_i \delta u_i^2(\tau)$	$E_i(\tau)$	$\delta\lambda_1(0)$	$\delta\dot{\lambda}_1(0)$	$\delta\lambda_2(0)$	$\delta\dot{\lambda}_2(0)$	$\delta\lambda_3(0)$	$\delta\dot{\lambda}_3(0)$
-0.5000×10^{2}	-0.9913	0.50153×10^{5}	-0.3466×10^{9}	-0.7705×10^{3}	0.7940×10^{1}	-0.7127×10^{4}	-1.197	-46.32	0.9176
$.1529\times10^{1}$	$-.4955\times10^{-1}$	$.37632\times10^{5}$	$-.3459\times10^{9}$	$.9198\times10^{2}$	$-.3064$	$-.1166\times10^{3}$	-2.130	10.13	$-.1232$
$.1380\times10^{-1}$	$-.4592\times10^{-2}$	$.2123\times10^{2}$	$-.3458\times10^{9}$	$.3529\times10^{1}$	$.1127\times10^{-1}$	$-.2583\times10^{1}$	$.03538$	$.3502$	$-.002109$
$.6820\times10^{-2}$	$.1797\times10^{-2}$	$.1472\times10^{-2}$	$-.3458\times10^{9}$						
-0.9944×10^{3}	-0.9325×10^{1}	0.1608×10^{8}	-0.4420×10^{9}	-0.4846×10^{4}	0.3061×10^{2}	-0.17733×10^{5}	-0.1514×10^{2}	-0.6434×10^{3}	0.5687×10^{1}
$.2289\times10^{3}$	$-.2349\times10^{1}$	$.9333\times10^{7}$	$-.4142\times10^{9}$	$.4416\times10^{4}$	$-.7291\times10^{1}$	$-.9251\times10^{4}$	$-.1603\times10^{2}$	$.9679\times10^{3}$	$-.4885\times10^{1}$
$-.1572\times10^{2}$	$-.2926\times10^{1}$	$.5674\times10^{7}$	$-.4010\times10^{9}$	$.2369\times10^{4}$	$.4321$	$-.7833\times10^{4}$	$-.3386\times10^{1}$	$.5577\times10^{3}$	$-.2683\times10^{1}$
$.3366\times10^{2}$	$-.1435\times10^{1}$	$.1949\times10^{7}$	$-.3935\times10^{9}$	$.1627\times10^{4}$	$-.7897$	$-.4580\times10^{4}$	$-.2750\times10^{1}$	$.3833\times10^{3}$	$-.1998\times10^{1}$
$.2148\times10^{2}$	$-.8573$	$.7432\times10^{6}$	$-.3900\times10^{9}$	$.9492\times10^{3}$	$-.1911$	$-.2233\times10^{4}$	$-.1203\times10^{1}$	$.2224\times10^{3}$	$-.1119\times10^{1}$
$.1235\times10^{2}$	$-.3924$	$.1731\times10^{6}$	$-.3888\times10^{9}$	$.4441\times10^{3}$	$-.2388\times10^{-1}$	$-.8971\times10^{3}$	$-.4562$	$.1026\times10^{3}$	$-.4953$
$.5444\times10^{1}$	$-.1510$	$.2777\times10^{5}$	$-.3885\times10^{9}$	$.1736\times10^{3}$	$.1009\times10^{-1}$	$-.3160\times10^{3}$	$-.1517$	$.3947\times10^{2}$	$-.1838$
$.2027\times10^{1}$	$-.5148\times10^{-1}$	$.3417\times10^{4}$	$-.3884\times10^{9}$	$.5990\times10^{2}$	$.8058\times10^{-2}$	$-.1020\times10^{3}$	$-.4653\times10^{-1}$	$.1344\times10^{2}$	$-.6086\times10^{-1}$
$.6751$	$-.1621\times10^{-1}$	$.3528\times10^{3}$	$-.3884\times10^{9}$	$.1904\times10^{2}$	$.3532\times10^{-2}$	$-.3099\times10^{2}$	$-.1359\times10^{-1}$	$.4231\times10^{1}$	$-.1877\times10^{-1}$
$.2093$	$-.4844\times10^{-2}$	$.3238\times10^{2}$	$-.3884\times10^{9}$	$.5723\times10^{1}$	$.1262\times10^{-2}$	$-.9030\times10^{1}$	$-.3841\times10^{-2}$	$.1263\times10^{1}$	$-.5524\times10^{-2}$
$.6187\times10^{-1}$	$-.1393\times10^{-2}$	$.2736\times10^{1}$	$-.3884\times10^{9}$	$.1654\times10^{1}$	$.4071\times10^{-3}$	$-.2549\times10^{1}$	$-.1058\times10^{-2}$	$.3633$	$-.1571\times10^{-2}$
$.1765\times10^{-1}$	$-.3895\times10^{-3}$	$.2171$	$-.3884\times10^{9}$						

It is clear that the incremental change in energy is maximized if the vehicle is accelerated when the velocity is as high as possible (and with acceleration principally along the velocity vector). Hence for this problem, it is advantageous to thrust before the vehicle reaches apogee (minimum velocity) as long as no plane change is required. In the latter case, the advantage of turning a smaller velocity vector negates the benefits of accelerating at high velocity.

From inspection of characteristics of the optimal trajectories, a quasi-optimal flight was defined for which the thrust vector was constrained to the local horizontal plane and held at a constant bearing angle with respect to the satellite orbital plane. It was necessary to iterate to determine the bearing angle necessary to achieve a prescribed plane change.

Figure 7 is a typical plot of the losses in effective characteristic velocity in the quasi-optimal case. The parameter plotted is differential effective characteristic velocity, defined as

$$\Delta\tilde{V} = \Delta V_{c,quasi\text{-}opt} - \Delta V_{c,opt} \tag{7}$$

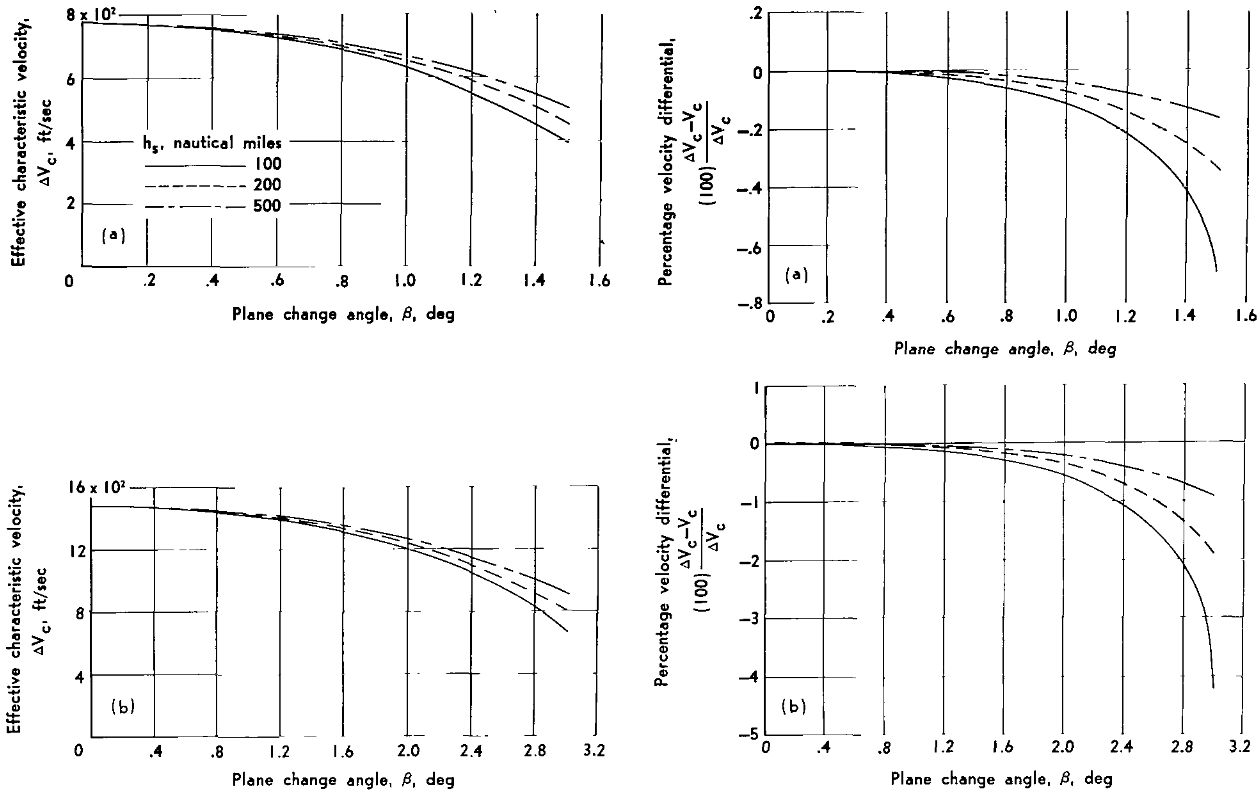

Figure 6.- Effective characteristic velocity variation with orbital altitude and inclination.

(a) $\frac{T}{m(0)} = 0.25g$; $\tau = 100$ sec. (b) $\frac{T}{m(0)} = 0.25g$; $\tau = 200$ sec.

A negative $\Delta\tilde{V}$ indicates that the energy gain over the quasi-optimal trajectory is less than that over the optimal trajectory. Note that the losses decrease and then increase with increases in orbital-plane change. This variation can be correlated with the time histories of optimal thrust elevation angle in figures 2 to 5 where it can be seen that this angle most closely approaches zero for medium orbital-plane changes. The comparison between optimal and quasi-optimal paths is very favorable, which indicates that guidance modes based on the quasi-optimal path should be highly efficient. The maximum penalty for using the quasi-optimal guidance mode occurs for the maximum plane change and is only 0.063 percent for the worst case encountered ($T/m(0) = 1g$, $\tau = 200$ sec, $h_s = 100$ nautical miles).

A second simplified flight path, the gravity-turn trajectory, was tested for the coplanar cases.

In the gravity turn, the thrust is always collinear with the velocity vector which lies close to the local horizontal for rendezvous with a

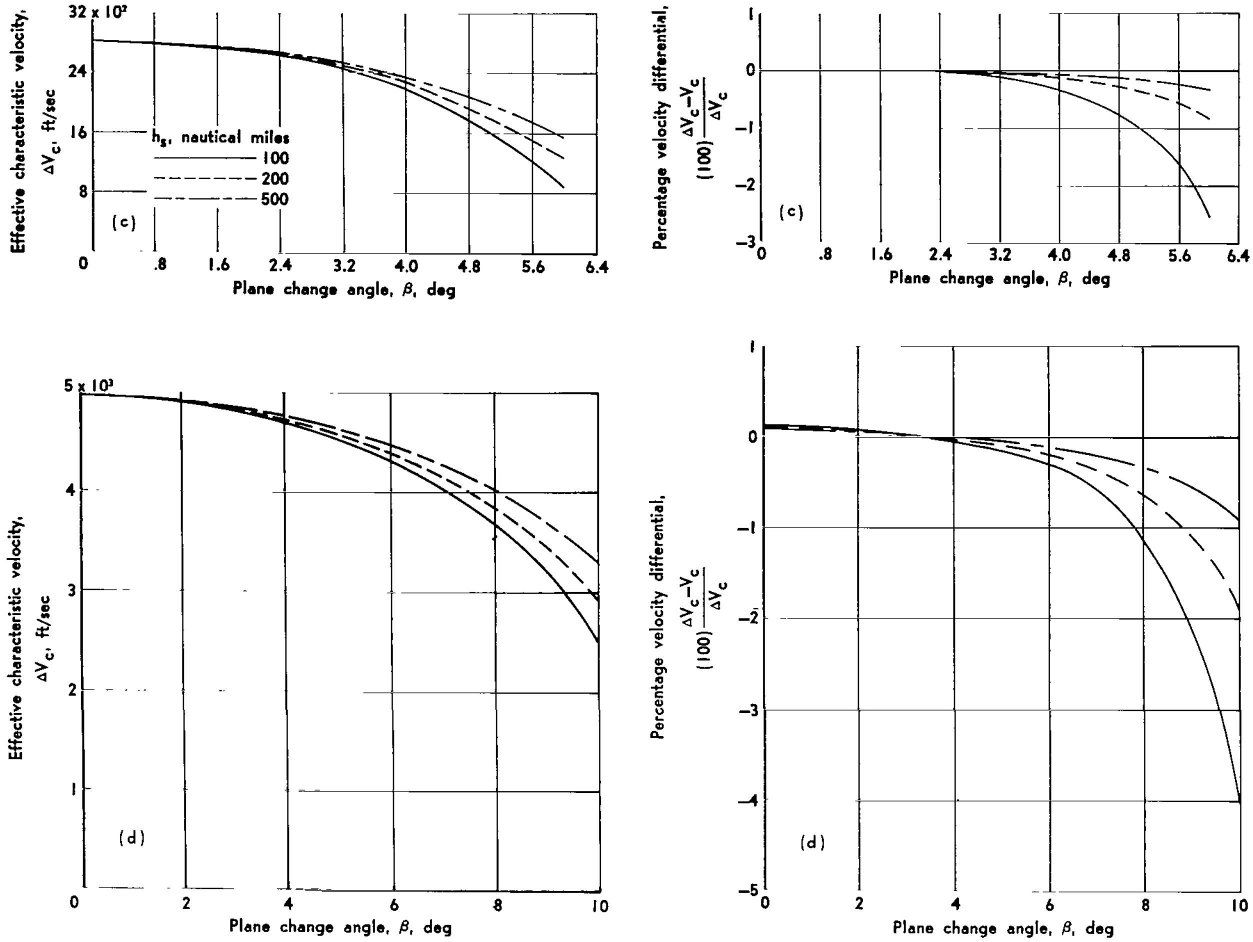

Figure 6.- Concluded.

(c) $\frac{T}{m(0)} = 1g$; $\tau = 100$ sec. (d) $\frac{T}{m(0)} = 1g$; $\tau = 200$ sec.

satellite in circular orbit when the velocity gains over the terminal stage are of the order considered here. The resultant effective characteristic velocities were calculated by using equation (5) and are listed in table II, together with comparable figures for the optimal and quasi-optimal cases.

The quasi-optimal mode proved to be slightly more efficient than the gravity-turn mode. Effective characteristic velocity showed a maximum degradation of 0.040 percent for the case where $T/m(0) = 1g$, $\tau = 200$ sec, and $h_s = 100$ nautical miles, as compared with a 0.046-percent degradation for the gravity turn.

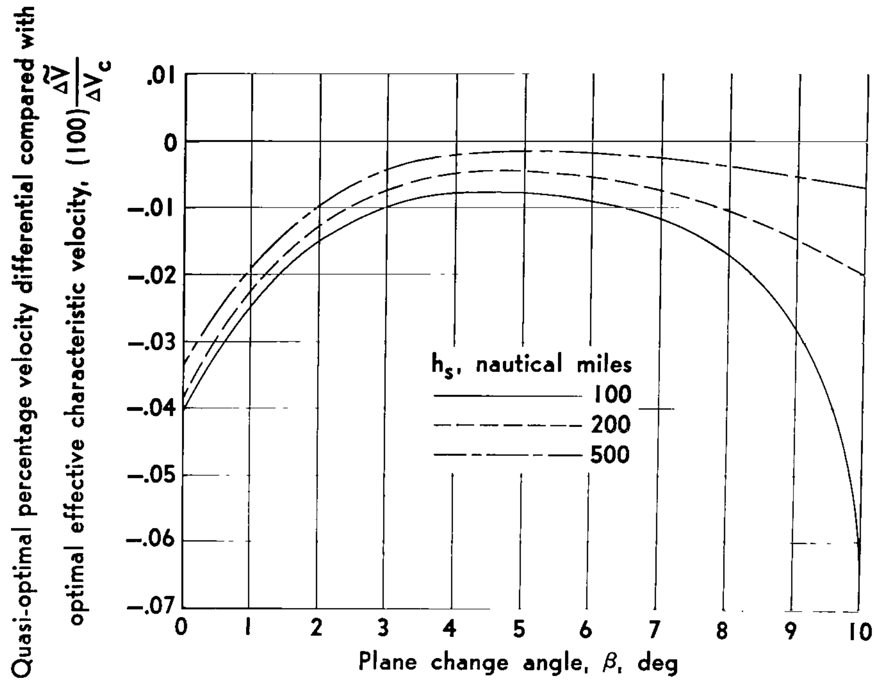

Figure 7.- Variation of effective characteristic velocity differential with orbital inclination angle and orbital altitude for quasi-optimal trajectories. $\frac{T}{m(0)} = 1g$; $\tau = 200$ sec.

TABLE II.- EFFECTIVE CHARACTERISTIC VELOCITY FOR COPLANAR CASE

h_s, nautical miles	$\frac{T}{m(0)}$, g	τ, sec	$\Delta V_{c,opt}$, ft/sec	$\tilde{\Delta V}$, ft/sec	
				Quasi-optimal	Gravity turn
100	0.25	100	774.26872	-0.00201	-0.00202
		200	1,493.2660	- .0591	- .0592
	1.0	100	2,791.8851	- .1049	- .1101
		200	4,977.6283	-1.9763	-2.2839
200	0.25	100	774.26198	-0.00162	-0.00163
		200	1,493.1625	- .0468	- .0486
	1.0	100	2,791.7875	- .0849	- .0898
		200	4,976.2501	-1.9188	-2.1217
500	0.25	100	774.25523	-0.00127	-0.00130
		200	1,493.0751	- .0375	- .0389
	1.0	100	2,791.7053	- .0682	- .0724
		200	4,975.0704	-1.6754	-1.8387

CONCLUDING REMARKS

Energy-optimal trajectories for terminal-stage rendezvous with a satellite station in circular orbit which used a constant-thrust rocket with a fixed burning time have been derived by using the calculus of variations. The problem is three dimensional, including the constraint that the terminal stage is to turn through a given initial plane-change angle. Numerical solutions to the resulting two-point boundary-value problem have been generated by using an adjoint iteration technique.

Near-earth orbital rendezvous was examined for orbital altitudes of 100, 200, and 500 nautical miles, initial thrust-to-mass ratios of 0.25g and 1g, and burning times of 100 and 200 seconds. The resulting prescribed thrust angles lie very close to the local horizontal in elevation, and are nearly constant with respect to the target orbital plane in azimuth.

Quasi-optimal trajectories were computed by holding the thrust vector in the local horizontal at a fixed azimuth angle. These trajectories, which show a maximum degradation in effective characteristic velocity of 0.063 percent, compare very favorably with the optimal trajectories. Gravity turns were also run for the coplanar case.

The quasi-optimal path proved to be the more efficient of the simplified trajectories. A degradation in effective characteristic velocity of 0.040 percent was shown for the worst coplanar case as compared with 0.046-percent degradation for the gravity turn.

Manned Spacecraft Center,
National Aeronautics and Space Administration,
Houston, Texas, March 26, 1963.

APPENDIX A

OPTIMIZATION OF ENERGY CHANGE FOR NONLINEAR EQUATIONS OF MOTION

The description of the problem of rendezvous of a mass-particle with a satellite in circular orbit is provided by the following equations of motion:

$$\ddot{X} = \frac{T}{m} \cos \theta \cos \psi + 2\Omega_s \dot{Z} - X\left(\frac{\Upsilon}{r_f^3} - \Omega_s^2\right) \tag{A1}$$

$$\ddot{Y} = \frac{T}{m} \cos \theta \sin \psi - Y \frac{\Upsilon}{r_f^3} \tag{A2}$$

$$\ddot{Z} = -\frac{T}{m} \sin \theta - 2\Omega_s \dot{X} + \left(r_s - Z\right)\left(\frac{\Upsilon}{r_f^3} - \Omega_s^2\right) \tag{A3}$$

The variables are defined with respect to the satellite in a rotating system. (See fig. 1.) It is required that the change in total energy over the burning period be maximized. The variational notation used in reference 6 is employed.

The energy is expressed as

$$E = \frac{V^2}{2} - \frac{\Upsilon}{r_f} \tag{A4}$$

where

$$V^2 = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 + 2\Omega_s\left[\dot{X}\left(r_s - Z\right) + \dot{Z}X\right] - \Omega_s^2\left[X^2 + \left(r_s - Z\right)^2\right]$$

and the integral to be maximized is

$$I = \int_0^{\tau} \dot{E}\, dt \tag{A5}$$

where

$$\dot{E} = \frac{T}{m}\left\{\left[\dot{X} + \Omega_s\left(r_s - Z\right)\right]\cos \theta \cos \psi + \dot{Y} \cos \theta \sin \psi - \left(\dot{Z} + \Omega_s X\right)\sin \theta\right\}$$

Three constraints, ϕ_1, ϕ_2, and ϕ_3, arise from requiring satisfaction of the dynamic equations (A1) to (A3). A fourth constraint is introduced when it is required that a specified change in the orbital plane of the vehicle be made. This constraint can be expressed as

$$\varphi_4 = \dot{Y} - \left[\dot{X} + \Omega_s\left(r_s - Z\right)\right] \tan \overline{\beta(0)}\Big|_{t=0} = 0 \tag{A6}$$

The first three constraints are adjoined to the function to be optimized with Lagrange multipliers

$$F = \dot{E} + \lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 \tag{A7}$$

The integral to be operated on is altered accordingly

$$I = \int_0^T \left(\dot{E} + \lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3\right)dt \tag{A8}$$

and the variation taken

$$\begin{aligned}
\delta I = \int_0^T \Bigg[&\left(\frac{\partial\dot{E}}{\partial X} + \lambda_1 \frac{\partial\varphi_1}{\partial X} + \lambda_2 \frac{\partial\varphi_2}{\partial X} + \lambda_3 \frac{\partial\varphi_3}{\partial X}\right)\delta X + \left(\frac{\partial\dot{E}}{\partial\dot{X}} + \lambda_3 \frac{\partial\varphi_3}{\partial\dot{X}}\right)\delta\dot{X} \\
&+ \lambda_1 \delta\ddot{X} + \left(\lambda_1 \frac{\partial\varphi_1}{\partial Y} + \lambda_2 \frac{\partial\varphi_2}{\partial Y} + \lambda_3 \frac{\partial\varphi_3}{\partial Y}\right)\delta Y + \left(\frac{\partial\dot{E}}{\partial\dot{Y}}\right)\delta\dot{Y} + \lambda_2 \delta\ddot{Y} \\
&+ \left(\frac{\partial\dot{E}}{\partial Z} + \lambda_1 \frac{\partial\varphi_1}{\partial Z} + \lambda_2 \frac{\partial\varphi_2}{\partial Z} + \lambda_3 \frac{\partial\varphi_3}{\partial Z}\right)\delta Z + \left(\frac{\partial\dot{E}}{\partial\dot{Z}} + \lambda_1 \frac{\partial\varphi_1}{\partial\dot{Z}}\right)\delta\dot{Z} \\
&+ \lambda_3 \delta\ddot{Z} + \left(\frac{\partial\dot{E}}{\partial\theta} + \lambda_1 \frac{\partial\varphi_1}{\partial\theta} + \lambda_2 \frac{\partial\varphi_2}{\partial\theta} + \lambda_3 \frac{\partial\varphi_3}{\partial\theta}\right)\delta\theta \\
&+ \left(\frac{\partial\dot{E}}{\partial\psi} + \lambda_1 \frac{\partial\varphi_1}{\partial\psi} + \lambda_2 \frac{\partial\varphi_2}{\partial\psi}\right)\delta\psi\Bigg]dt
\end{aligned} \tag{A9}$$

Integrating by parts in the usual manner

$$
\begin{aligned}
\delta I = &\left[\lambda_1\ \delta\dot{X} + \lambda_2\ \delta\dot{Y} + \lambda_3\ \delta\dot{Z} + \left(\frac{\partial\dot{E}}{\partial\dot{X}} + \lambda_3 \frac{\partial\varphi_3}{\partial\dot{X}} - \dot{\lambda}_1\right)\delta X + \left(\frac{\partial\dot{E}}{\partial\dot{Y}} - \dot{\lambda}_2\right)\delta Y \right.\\
&\left. + \left(\frac{\partial\dot{E}}{\partial\dot{Z}} + \lambda_1 \frac{\partial\varphi_1}{\partial\dot{Z}} - \dot{\lambda}_3\right)\delta Z\right]\Bigg|_{t=0}^{\tau} + \int_0^{\tau} \left\{\left[\frac{\partial\dot{E}}{\partial X} + \lambda_1 \frac{\partial\varphi_1}{\partial X} + \lambda_2 \frac{\partial\varphi_2}{\partial X}\right.\right.\\
&\left. + \lambda_3 \frac{\partial\varphi_3}{\partial X} - \frac{d}{dt}\left(\frac{\partial\dot{E}}{\partial\dot{X}} + \lambda_3 \frac{\partial\varphi_3}{\partial\dot{X}} - \dot{\lambda}_1\right)\right]\delta X + \left[\lambda_1 \frac{\partial\varphi_1}{\partial Y} + \lambda_2 \frac{\partial\varphi_2}{\partial Y} + \lambda_3 \frac{\partial\varphi_3}{\partial Y}\right.\\
&\left. - \frac{d}{dt}\left(\frac{\partial\dot{E}}{\partial\dot{Y}} - \dot{\lambda}_2\right)\right]\delta Y + \left[\frac{\partial\dot{E}}{\partial Z} + \lambda_1 \frac{\partial\varphi_1}{\partial Z} + \lambda_2 \frac{\partial\varphi_2}{\partial Z} + \lambda_3 \frac{\partial\varphi_3}{\partial Z}\right.\\
&\left. - \frac{d}{dt}\left(\frac{\partial\dot{E}}{\partial\dot{Z}} + \lambda_1 \frac{\partial\varphi_1}{\partial\dot{Z}} - \dot{\lambda}_3\right)\right]\delta Z + \left(\frac{\partial\dot{E}}{\partial\theta} + \lambda_1 \frac{\partial\varphi_1}{\partial\theta} + \lambda_2 \frac{\partial\varphi_2}{\partial\theta} + \lambda_3 \frac{\partial\varphi_3}{\partial\theta}\right)\delta\theta\\
&\left. + \left(\frac{\partial\dot{E}}{\partial\psi} + \lambda_1 \frac{\partial\varphi_1}{\partial\psi} + \lambda_2 \frac{\partial\varphi_2}{\partial\psi}\right)\delta\psi\right\} dt
\end{aligned}
\tag{A10}
$$

Considering the boundary conditions, all the physical variables are defined at $t = \tau$, since it is specified that the vehicle is to back off from a rendezvous. Therefore, all the Lagrange multipliers are unspecified at $t = \tau$. At $t = 0$, constraint equation (A6) is applied, but the end conditions are otherwise free. The variation of equation (A6) is written

$$
\delta\dot{Y} = \left(\delta\dot{X} - \Omega_s\ \delta Z\right)\tan \overline{\beta(0)}\Big|_{t=0}
\tag{A11}
$$

This relation is substituted to eliminate $\delta\dot{Y}$ from equation (A10), and the fundamental lemma of the calculus of variations is applied to the boundary-value bracket. This leads to the following end conditions on the multipliers:

$$
\left.
\begin{aligned}
\lambda_1(0) &= -\lambda_2 \tan \overline{\beta(0)}\Big|_{t=0}\\
\dot{\lambda}_1(0) &= \frac{T}{m}\cos\theta\cos\psi\Big|_{t=0}\\
\dot{\lambda}_2(0) &= \frac{T}{m}\cos\theta\sin\psi\Big|_{t=0}\\
\lambda_3(0) &= 0\\
\dot{\lambda}_3(0) &= \lambda_2\ \Omega_s \tan \overline{\beta(0)} - \frac{T}{m}\sin\theta\Big|_{t=0}
\end{aligned}
\right\}
\tag{A12}
$$

Note that $\lambda_2(0)$ is free, but $\dot{Y}(0)$ is defined by equation (A6).

Again, applying the fundamental lemma to the quantities under the integral and thus setting the coefficient of each variation equal to zero yields the Euler-Lagrange equations.

$$\left.\begin{aligned}
\ddot{\lambda}_1 &= 2\Omega_s \dot{\lambda}_3 - \left[\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \frac{3\gamma X^2}{r_f^5}\right]\lambda_1 + \frac{3\gamma XY}{r_f^5}\lambda_2 - \frac{3\gamma X(r_s - Z)}{r_f^5}\lambda_3 \\
&\quad + \frac{T}{m}\left[\left(\Omega_s - \dot{\theta}\cos\psi\right)\sin\theta - \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\cos\theta\right] \\
\ddot{\lambda}_2 &= \frac{3\gamma XY}{r_f^5}\lambda_1 - \left(\frac{\gamma}{r_f^3} - \frac{3\gamma Y^2}{r_f^5}\right)\lambda_2 - \frac{3\gamma Y(r_s - Z)}{r_f^5}\lambda_3 - \frac{T}{m}\left[\dot{\theta}\sin\psi\sin\theta\right. \\
&\quad \left. + \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\cos\theta\right] \\
\ddot{\lambda}_3 &= -2\Omega_s \dot{\lambda}_1 - \frac{3\gamma X(r_s - Z)}{r_f^5}\lambda_1 - \frac{3\gamma Y(r_s - Z)}{r_f^5}\lambda_2 - \left[\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right)\right. \\
&\quad \left. - \frac{3\gamma(r_s - Z)^2}{r_f^5}\right]\lambda_3 + \frac{T}{m}\left[\frac{\dot{m}}{m}\sin\theta + \left(\Omega_s\cos\psi - \dot{\theta}\right)\cos\theta\right] \\
\tan\theta &= -\frac{\left(\dot{Z} + X\Omega_s - \lambda_3\right)\cos\psi}{\dot{X} + \Omega_s(r_s - Z) - \lambda_1} \\
\tan\psi &= \frac{\dot{Y} - \lambda_2}{\dot{X} + \Omega_s(r_s - Z) - \lambda_1}
\end{aligned}\right\} \quad \text{(A13)}$$

These equations, together with the constraint equations (A1) to (A3), the six boundary values at $t = 0$ specified by starting at rendezvous (all relative velocities and displacements zero), and the six boundary values at $t = \tau$ from equations (A6) and (A11), completely specify the problem.

APPENDIX B

EQUATIONS FOR ADJOINT ITERATION

A general method for the satisfaction of dynamic two-point boundary-value problems is described in reference 4. This method is applied to the optimization problem of this paper. In this appendix, the iteration technique is first developed in general terms, and then the specific substitutions used in solving this problem are stated.

The set of three second-order differential equations (eqs. (1) to (3)), together with the associated Lagrange multipliers (eq. (A13)), is considered as the equivalent set of 12 first-order differential equations:

$$\dot{u}_i = f_i(u_j,t) \tag{B1}$$

where

$$u_i(0) = \overline{u_i(0)} \qquad (i = 1, 2, \ldots, 6)$$

$$u_i(\tau) = \overline{u_i(\tau)} \qquad (i = 6, 7, \ldots, 11)$$

The variable δu_i is defined as a first-order perturbation about u

$$\delta\dot{u}_i = \sum_j \frac{\partial f_i}{\partial u_j} \delta u_j \tag{B2}$$

The equations adjoint to the set of equations (B2) are described by

$$\dot{p}_i = -\sum_j \frac{\partial f_j}{\partial u_i} p_j \tag{B3}$$

If the set of equations (B2) is multiplied by p_i and the set of equations (B3) by δu_i and the resulting relations added and summed on i, then

$$\sum_i \left(p_i \delta\dot{u}_i + \dot{p}_i \delta u_i\right) = \sum_i \sum_j \frac{\partial f_i}{\partial u_j} \delta u_j p_i - \sum_i \sum_j \frac{\partial f_j}{\partial u_i} p_j \delta u_i \tag{B4}$$

It is seen that the right-hand side vanishes because of the symmetry of the indices. Finally, integration of equation (B4) from $t = 0$ to $t = \tau$ yields the desired adjoint relationship

$$\sum_i p_i(0)\delta u_i(0) = \sum_i p_i(\tau)\delta u_i(\tau) \tag{B5}$$

Now, if $\delta u_i(\tau)$ is defined as error observed in the end boundary values

$$\delta u_i(\tau) = \overline{u_i(\tau)} - u_i(\tau) \tag{B6}$$

It is seen that equation (B5) gives a means of reflecting these terminal errors back to determine initial value improvements.

To determine the six required values of $\delta u_i(0)$, the adjoint equations (B3) are integrated backward from $t = \tau$ to $t = 0$ for six distinct sets of end conditions. For convenience, these end conditions were chosen to be identically zero except for one value for each set which corresponds to a known end condition.

$$\left.\begin{aligned} p_{i,k}(\tau) &= 1 && (k = i + 5) \\ p_{i,k}(\tau) &= 0 && (k \neq i + 5;\ i = 1, 2, \ldots, 12;\ k = 1, 2, \ldots, 6) \end{aligned}\right\} \tag{B7}$$

Once this integration is performed, the resulting values are substituted into equation (B5)

$$\sum_i p_{i,k}(0)\ \delta u_i(0) = \delta u_{i+k}(\tau) \tag{B8}$$

and the six equations yielded are solved simultaneously for $\delta u_i(0)$. Finally, improved initial estimates are calculated

$$u_i(0) = u_i^*(0) + \delta u_i(0) \qquad (i = 7, 8, \ldots, 12) \tag{B9}$$

This iteration process is cycled until the end errors fall within a desired tolerance.

It must be noted that the $\delta u_i(0)$ corrections only satisfy conditions for small perturbations about the nominal solution. An additional test was provided in the actual mechanization of the problem to prevent the computation from diverging if the initial errors were too large. This test required satisfaction of the relation

$$\sum_i \left[\frac{\delta u_i(\tau)}{u_i(0)}\right]^2_{n+1} < \sum_i \left[\frac{\delta u_i(\tau)}{u_i(0)}\right]^2_n \tag{B10}$$

Now the substitutions required to state the explicit problem are defined as:

$$\left.\begin{aligned}
\dot{u}_1 &= \dot{X} = f_1 & \dot{u}_7 &= \dot{\lambda}_1 = f_7 \\
\dot{u}_2 &= \ddot{X} = f_2 & \dot{u}_8 &= \ddot{\lambda}_1 = f_8 \\
\dot{u}_3 &= \dot{Y} = f_3 & \dot{u}_9 &= \ddot{\lambda}_2 = f_9 \\
\dot{u}_4 &= \dot{Z} = f_4 & \dot{u}_{10} &= \dot{\lambda}_3 = f_{10} \\
\dot{u}_5 &= \ddot{Z} = f_5 & \dot{u}_{11} &= \ddot{\lambda}_3 = f_{11} \\
\dot{u}_6 &= \ddot{Y} = f_6 & \dot{u}_{12} &= \dot{\lambda}_2 = f_{12}
\end{aligned}\right\} \tag{B11}$$

The desired boundary values on δu_i (eq. (B2)) are

$$\left.\begin{aligned}
\delta u_i(\tau) &= 0;\ i = 1, 2, \ldots, 6 \\
\delta u_6(0) &= \left[\dot{X} + \Omega_s\left(r_s - Z\right)\right]\tan\overline{\beta(0)} - \dot{Y}\Big|_{t=0} \\
\delta u_7(0) &= -\lambda_2 \tan\overline{\beta(0)} - \lambda_1\Big|_{t=0} \\
\delta u_8(0) &= \frac{T}{m}\cos\theta\cos\psi - \dot{\lambda}_1\Big|_{t=0} \\
\delta u_9(0) &= \frac{T}{m}\cos\theta\sin\psi - \dot{\lambda}_2\Big|_{t=0} \\
\delta u_{10}(0) &= -\lambda_3\Big|_{t=0} \\
\delta u_{11}(0) &= \lambda_2\Omega_s \tan\overline{\beta(0)} - \frac{T}{m}\sin\theta - \dot{\lambda}_3\Big|_{t=0}
\end{aligned}\right\} \tag{B12}$$

Equation (B2) is not integrated since all of its boundary values cannot be specified. Equation (B3) is integrated, and since it is linear and will be used with six distinct sets of starting values on p_i, it is most easily treated as a vector-matrix equation.

$$\left[\dot{p}_{i,j}\right] = \left[A_{i,k}\right]\left[p_{k,j}\right] \tag{B13}$$

where the matrix $\left[p_{k,j}\right]$ is of dimension 12×6 and $\left[A_{i,k}\right]$ is a 12×12 square matrix. In the matrix $[p]$, the j columns are associated with the six sets of end conditions. The matrix $[p(\tau)]$ is defined as

$$\left.\begin{aligned} p_{i,j}(\tau) &= 0 && (i \neq j + 5) \\ p_{i,j}(\tau) &= 0 && (i = 1, \ldots, 5, 12) \\ p_{i,j}(\tau) &= 1 && (i = j + 5 = 6, 7, \ldots, 11) \end{aligned}\right\} \tag{B14}$$

Integrate $[\dot{p}]$ from $t = 0$ to $t = \tau$ together with the set $\dot{u}_i$. At $t = \tau$, the following relation is solved to find the required initial improvements:

$$\left\{\delta u_i(\tau)\right\} = [B]\left\{\delta u_i(0)\right\} \tag{B15}$$

where $\left\{\delta u_i\right\}$ is a six vector and $[B]$ is a 6×6 matrix defined from the elements of $[p]$ as

$$[B]^{-1} = \left[p_{j,i}\right] \tag{B16}$$

where

$$j = 1, 2, \ldots, 6$$

$$i = 7, 8, \ldots, 12$$

This process is equivalent to the solution of equation (B8).

The elements of $\left[A_{i,k}\right]$ are listed below, where missing elements are understood to be zero.

$$A_{1,2} = \frac{T}{m} \sin\theta \cos\psi \frac{\partial\theta}{\partial X} + \left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right) - \frac{3\gamma X^2}{r_f^{\,5}} \tag{B17}$$

$$A_{1,4} = \frac{T}{m} \sin\theta \sin\psi \frac{\partial\theta}{\partial X} - \frac{3\gamma XY}{r_f^{\,5}} \tag{B18}$$

$$A_{1,6} = \frac{T}{m} \cos\theta \frac{\partial\theta}{\partial X} + \frac{3\gamma X\left(r_s - Z\right)}{r_f^{\,5}} \tag{B19}$$

$$\begin{aligned} A_{1,8} = &- \frac{3\gamma}{r_f^{\,5}} \left\{ X\lambda_1\left(3 - \frac{5X^2}{r_f^{\,2}}\right) + \left[Y\lambda_2 - \left(r_s - Z\right)\lambda_3\right]\left(1 - \frac{5X^2}{r_f^{\,2}}\right)\right\} \\ &- \frac{T}{m}\left\{\left[\left(\Omega_s - \dot\theta \cos\psi\right)\cos\theta + \left(\frac{\dot m}{m}\cos\psi + \dot\psi \sin\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial X} \right. \\ &\left. - \frac{\partial\dot\theta}{\partial X}\cos\psi \sin\theta - \frac{\partial\dot\psi}{\partial X}\sin\psi\cos\theta\right\} \end{aligned} \tag{B20}$$

$$\begin{aligned} A_{1,10} = &- \frac{3\gamma}{r_f^{\,5}}\left[Y\lambda_1\left(1 - \frac{5X^2}{r_f^{\,2}}\right) - X\lambda_2\left(1 - \frac{5Y^2}{r_f^{\,2}}\right) - \frac{5XY\left(r_s - Z\right)}{r_f^{\,2}}\lambda_3\right] \\ &+ \frac{T}{m}\left\{\left[\dot\theta \sin\psi \cos\theta - \left(\frac{\dot m}{m}\sin\psi - \dot\psi\cos\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial X} \right. \\ &\left. + \frac{\partial\dot\theta}{\partial X}\sin\psi\sin\theta - \frac{\partial\dot\psi}{\partial X}\cos\psi\cos\theta\right\} \end{aligned} \tag{B21}$$

$$\begin{aligned} A_{1,12} = &\frac{3\gamma}{r_f^{\,5}}\left\{\left(r_s - Z\right)\lambda_1\left(1 - \frac{5X^2}{r_f^{\,2}}\right) - \frac{5XY\left(r_s - Z\right)}{r_f^{\,2}}\lambda_2 \right. \\ &\left. - X\lambda_3\left[1 - \frac{5\left(r_s - Z\right)^2}{r_f^{\,2}}\right]\right\} - \frac{T}{m}\left\{\left[\frac{\dot m}{m}\cos\theta - \left(\Omega_s \cos\psi - \dot\theta\right)\sin\theta\right]\frac{\partial\theta}{\partial X} \right. \\ &\left. - \frac{\partial\dot\theta}{\partial X}\cos\theta\right\} \end{aligned} \tag{B22}$$

$$A_{2,1} = -1 \tag{B23}$$

$$A_{2,2} = \frac{T}{m}\left(\sin\theta\cos\psi\frac{\partial\theta}{\partial\dot{X}} + \cos\theta\sin\psi\frac{\partial\psi}{\partial\dot{X}}\right) \tag{B24}$$

$$A_{2,4} = \frac{T}{m}\left(\sin\theta\sin\psi\frac{\partial\theta}{\partial\dot{X}} - \cos\theta\cos\psi\frac{\partial\psi}{\partial\dot{X}}\right) \tag{B25}$$

$$A_{2,6} = \frac{T}{m}\cos\theta\frac{\partial\theta}{\partial\dot{X}} - 2\Omega_s \tag{B26}$$

$$\begin{aligned} A_{2,8} = -\frac{T}{m}\Bigg\{&\left[\left(\Omega_s - \dot{\theta}\cos\psi\right)\cos\theta + \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\dot{X}} \\ &+ \left[\dot{\theta}\sin\theta\sin\psi + \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\cos\theta\right]\frac{\partial\psi}{\partial\dot{X}} \\ &- \frac{\partial\dot{\theta}}{\partial\dot{X}}\cos\psi\sin\theta - \frac{\partial\dot{\psi}}{\partial\dot{X}}\sin\psi\cos\theta\Bigg\} \end{aligned} \tag{B27}$$

$$\begin{aligned} A_{2,10} = \frac{T}{m}\Bigg\{&\left[\dot{\theta}\sin\psi\cos\theta - \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\dot{X}} \\ &+ \left[\dot{\theta}\cos\psi\sin\theta + \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\cos\theta\right]\frac{\partial\psi}{\partial\dot{X}} \\ &+ \frac{\partial\dot{\theta}}{\partial\dot{X}}\sin\psi\sin\theta - \frac{\partial\dot{\psi}}{\partial\dot{X}}\cos\psi\cos\theta\Bigg\} \end{aligned} \tag{B28}$$

$$\begin{aligned} A_{2,12} = -\frac{T}{m}\Bigg\{&\left[\frac{\dot{m}}{m}\cos\theta - \left(\Omega_s\cos\psi - \dot{\theta}\right)\sin\theta\right]\frac{\partial\theta}{\partial\dot{X}} \\ &- \Omega_s\sin\psi\cos\theta\frac{\partial\psi}{\partial\dot{X}} - \frac{\partial\dot{\theta}}{\partial\dot{X}}\cos\theta\Bigg\} \end{aligned} \tag{B29}$$

$$A_{3,2} = -\frac{3\gamma XY}{r_f^{\,5}} \tag{B30}$$

$$A_{3,4} = \frac{\gamma}{r_f^{\,3}}\left(1 - \frac{3Y^2}{r_f^{\,2}}\right) \tag{B31}$$

$$A_{3,6} = \frac{3\gamma Y\left(r_s - Z\right)}{r_f^{\,5}} \tag{B32}$$

$$A_{3,8} = -\frac{3\gamma}{r_f^{\,5}}\left[Y\lambda_1\left(1 - \frac{5X^2}{r_f^{\,2}}\right) + X\lambda_2\left(1 - \frac{5Y^2}{r_f^{\,2}}\right) - \frac{5XY\left(r_s - Z\right)}{r_f^{\,2}}\,\lambda_3\right]$$
$$+ \frac{T}{m}\left(\frac{\partial\dot{\theta}}{\partial Y}\cos\psi\,\sin\theta + \frac{\partial\dot{\psi}}{\partial Y}\sin\psi\,\cos\theta\right) \tag{B33}$$

$$A_{3,10} = -\frac{3\gamma}{r_f^{\,5}}\left\{\left[X\lambda_1 - \left(r_s - Z\right)\lambda_3\right]\left(1 - \frac{5Y^2}{r_f^{\,2}}\right) + Y\lambda_2\left(3 - \frac{5Y^2}{r_f^{\,2}}\right)\right\} \tag{B34}$$

$$A_{3,12} = -\frac{3\gamma}{r_f^{\,5}}\left\{\frac{5XY\left(r_s - Z\right)}{r_f^{\,2}}\,\lambda_1 - \left(r_s - Z\right)\lambda_2\left(1 - \frac{5Y^2}{r_f^{\,2}}\right)\right.$$
$$\left. + Y\lambda_3\left[1 - \frac{5\left(r_s - Z\right)^2}{r_f^{\,2}}\right]\right\} + \frac{T}{m}\,\frac{\partial\dot{\theta}}{\partial Y}\cos\theta \tag{B35}$$

$$A_{4,2} = \frac{T}{m}\left(\sin\theta\,\cos\theta\,\frac{\partial\theta}{\partial\dot{Y}} + \cos\theta\,\sin\psi\,\frac{\partial\psi}{\partial\dot{Y}}\right) \tag{B36}$$

$$A_{4,3} = -1 \tag{B37}$$

$$A_{4,4} = \frac{T}{m}\left(\sin\theta \sin\psi \frac{\partial\theta}{\partial\dot{Y}} - \cos\theta \cos\psi \frac{\partial\psi}{\partial\dot{Y}}\right) \tag{B38}$$

$$A_{4,6} = \frac{T}{m} \cos\theta \frac{\partial\theta}{\partial\dot{Y}} \tag{B39}$$

$$\begin{aligned} A_{4,8} = -\frac{T}{m}\Bigg\{&\left[\left(\Omega_s - \dot{\theta}\cos\psi\right)\cos\theta + \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\dot{Y}} \\ &+ \left[\dot{\theta}\sin\psi\sin\theta + \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\cos\theta\right]\frac{\partial\psi}{\partial\dot{Y}} \\ &- \frac{\partial\dot{\theta}}{\partial\dot{Y}}\cos\psi\sin\theta - \frac{\partial\dot{\psi}}{\partial\dot{Y}}\sin\psi\cos\theta\Bigg\} \end{aligned} \tag{B40}$$

$$\begin{aligned} A_{4,10} = \frac{T}{m}\Bigg\{&\left[\dot{\theta}\sin\psi\cos\theta - \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\dot{Y}} \\ &+ \left[\dot{\theta}\cos\psi\sin\theta + \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\cos\theta\right]\frac{\partial\psi}{\partial\dot{Y}} \\ &+ \frac{\partial\dot{\theta}}{\partial\dot{Y}}\sin\psi\sin\theta - \frac{\partial\dot{\psi}}{\partial\dot{Y}}\cos\psi\cos\theta\Bigg\} \end{aligned} \tag{B41}$$

$$\begin{aligned} A_{4,12} = -\frac{T}{m}\Bigg\{&\left[\frac{\dot{m}}{m}\cos\theta - \left(\Omega_s\cos\psi - \dot{\theta}\right)\sin\theta\right]\frac{\partial\theta}{\partial\dot{Y}} \\ &- \Omega_s\sin\psi\cos\theta\frac{\partial\psi}{\partial\dot{Y}} - \frac{\partial\dot{\theta}}{\partial\dot{Y}}\cos\theta\Bigg\} \end{aligned} \tag{B42}$$

$$A_{5,2} = \frac{T}{m}\left(\sin\theta\cos\psi\frac{\partial\theta}{\partial Z} + \cos\theta\sin\psi\frac{\partial\psi}{\partial Z}\right) + \frac{3\gamma X\left(r_s - Z\right)}{r_f^{\,5}} \tag{B43}$$

$$A_{5,4} = \frac{T}{m}\left(\sin\,\theta\,\sin\,\psi\,\frac{\partial\theta}{\partial Z} - \cos\,\theta\,\cos\,\psi\,\frac{\partial\psi}{\partial Z}\right) + \frac{3\gamma Y\left(r_s - Z\right)}{r_f^{\,5}} \tag{B44}$$

$$A_{5,6} = \frac{T}{m}\cos\,\theta\,\frac{\partial\theta}{\partial Z} + \left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right) - \frac{3\gamma\left(r_s - Z\right)^2}{r_f^{\,5}} \tag{B45}$$

$$\begin{aligned}A_{5,8} = \frac{3\gamma}{r_f^{\,5}}&\left\{\left(r_s - Z\right)\lambda_1\left(1 - \frac{5X^2}{r_f^{\,2}}\right) - \frac{5XY\left(r_s - Z\right)}{r_f^{\,2}}\,\lambda_2 - X\lambda_3\left[1 - \frac{5\left(r_s - Z\right)^2}{r_f^{\,2}}\right]\right\}\\ &- \frac{T}{m}\left\{\left[\left(\Omega_s - \dot{\theta}\cos\,\psi\right)\cos\,\theta + \left(\frac{\dot{m}}{m}\cos\,\psi + \dot{\psi}\sin\,\psi\right)\sin\,\theta\right]\frac{\partial\theta}{\partial Z}\right.\\ &+ \left[\dot{\theta}\sin\,\psi\,\sin\,\theta + \left(\frac{\dot{m}}{m}\sin\,\psi - \dot{\psi}\cos\,\psi\right)\cos\,\theta\right]\frac{\partial\psi}{\partial Z}\\ &\left.- \frac{\partial\dot{\theta}}{\partial Z}\cos\,\psi\,\sin\,\theta - \frac{\partial\dot{\psi}}{\partial Z}\sin\,\psi\,\cos\,\theta\right\}\end{aligned} \tag{B46}$$

$$\begin{aligned}A_{5,10} = -\frac{3\gamma}{r_f^{\,5}}&\left\{\frac{5XY\left(r_s - Z\right)}{r_f^{\,2}}\,\lambda_1 - \left(r_s - Z\right)\lambda_2\left(1 - \frac{5Y^2}{r_f^{\,2}}\right)\right.\\ &\left.+ Y\lambda_3\left[1 - \frac{5\left(r_s - Z\right)^2}{r_f^{\,2}}\right]\right\} + \frac{T}{m}\left\{\left[\dot{\theta}\sin\,\psi\,\cos\,\theta - \left(\frac{\dot{m}}{m}\sin\,\psi\right.\right.\right.\\ &\left.\left.- \dot{\psi}\cos\,\psi\right)\sin\,\theta\right]\frac{\partial\theta}{\partial Z} + \left[\dot{\theta}\cos\,\psi\,\sin\,\theta + \left(\frac{\dot{m}}{m}\cos\,\psi\right.\right.\\ &\left.\left.\left.+ \dot{\psi}\sin\,\psi\right)\cos\,\theta\right]\frac{\partial\psi}{\partial Z} + \frac{\partial\dot{\theta}}{\partial Z}\sin\,\psi\,\sin\,\theta - \frac{\partial\dot{\psi}}{\partial Z}\cos\,\psi\,\cos\,\theta\right\}\end{aligned} \tag{B47}$$

$$A_{5,12} = - \frac{3\gamma}{r_f^{\,5}} \left\{ \left(X\lambda_1 + Y\lambda_2\right)\left[1 - \frac{5\left(r_s - Z\right)^2}{r_f^{\,2}}\right] - \left(r_s - Z\right)\lambda_3\left[3 - \frac{5\left(r_s - Z\right)^2}{r_f^{\,2}}\right]\right\}$$

$$- \frac{T}{m}\left\{\left[\frac{\dot{m}}{m} \cos \theta - \left(\Omega_s \cos \psi - \dot{\theta}\right)\sin \theta\right]\frac{\partial\theta}{\partial Z}\right.$$

$$\left. - \Omega_s \sin \psi \cos \theta \frac{\partial\psi}{\partial Z} - \frac{\partial\dot{\theta}}{\partial Z} \cos \theta\right\} \tag{B48}$$

$$A_{6,2} = \frac{T}{m} \sin \theta \cos \psi \frac{\partial\theta}{\partial\dot{Z}} - 2\Omega_s \tag{B49}$$

$$A_{6,4} = \frac{T}{m} \sin \theta \sin \psi \frac{\partial\theta}{\partial\dot{Z}} \tag{B50}$$

$$A_{6,5} = - 1 \tag{B51}$$

$$A_{6,6} = \frac{T}{m} \cos \theta \frac{\partial\theta}{\partial\dot{Z}} \tag{B52}$$

$$A_{6,8} = - \frac{T}{m}\left\{\left[\left(\Omega_s - \dot{\theta} \cos \psi\right)\cos \theta + \left(\frac{\dot{m}}{m} \cos \psi + \dot{\psi} \sin \psi\right)\sin \theta\right]\frac{\partial\theta}{\partial\dot{Z}}\right.$$

$$\left. - \frac{\partial\dot{\theta}}{\partial\dot{Z}} \cos \psi \sin \theta - \frac{\partial\dot{\psi}}{\partial\dot{Z}} \sin \psi \cos \theta\right\} \tag{B53}$$

$$A_{6,10} = \frac{T}{m}\left\{\left[\dot{\theta} \sin \psi \cos \theta - \left(\frac{\dot{m}}{m} \sin \psi - \dot{\psi} \cos \psi\right)\sin \theta\right]\frac{\partial\theta}{\partial\dot{Z}}\right.$$

$$\left. + \frac{\partial\dot{\theta}}{\partial\dot{Z}} \sin \psi \sin \theta - \frac{\partial\dot{\psi}}{\partial\dot{Z}} \cos \psi \cos \theta\right\} \tag{B54}$$

$$A_{6,12} = - \frac{T}{m}\left\{\left[\frac{\dot{m}}{m} \cos \theta - \left(\Omega_s \cos \psi - \dot{\theta}\right)\sin \theta\right]\frac{\partial\theta}{\partial\dot{Z}} - \frac{\partial\dot{\theta}}{\partial\dot{Z}} \cos \theta\right\} \tag{B55}$$

$$A_{7,2} = \frac{T}{m}\left(\sin\theta\cos\psi\frac{\partial\theta}{\partial\lambda_1} + \cos\theta\sin\psi\frac{\partial\psi}{\partial\lambda_1}\right) \tag{B56}$$

$$A_{7,4} = \frac{T}{m}\left(\sin\theta\sin\psi\frac{\partial\theta}{\partial\lambda_1} - \cos\theta\cos\psi\frac{\partial\psi}{\partial\lambda_1}\right) \tag{B57}$$

$$A_{7,6} = \frac{T}{m}\cos\theta\frac{\partial\theta}{\partial\lambda_1} \tag{B58}$$

$$\begin{aligned} A_{7,8} = {} & \left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right) - \frac{3\gamma X^2}{r_f^{\,5}} - \frac{T}{m}\left\{\left[\left(\Omega_s - \dot{\theta}\cos\psi\right)\cos\theta\right.\right. \\ & \left. + \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\lambda_1} + \left[\dot{\theta}\sin\psi\sin\theta\right. \\ & \left. + \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\cos\theta\right]\frac{\partial\psi}{\partial\lambda_1} - \frac{\partial\dot{\theta}}{\partial\lambda_1}\cos\psi\sin\theta \\ & \left. - \frac{\partial\dot{\psi}}{\partial\lambda_1}\sin\psi\cos\theta\right\} \end{aligned} \tag{B59}$$

$$\begin{aligned} A_{7,10} = {} & -\frac{3\gamma XY}{r_f^{\,5}} + \frac{T}{m}\left\{\left[\dot{\theta}\sin\psi\cos\theta - \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\lambda_1}\right. \\ & + \left[\dot{\theta}\cos\psi\sin\theta + \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\cos\theta\right]\frac{\partial\psi}{\partial\lambda_1} \\ & \left. + \frac{\partial\dot{\theta}}{\partial\lambda_1}\sin\psi\sin\theta - \frac{\partial\dot{\psi}}{\partial\lambda_1}\cos\psi\cos\theta\right\} \end{aligned} \tag{B60}$$

$$\begin{aligned} A_{7,12} = {} & \frac{3\gamma X\left(r_s - Z\right)}{r_f^{\,5}} - \frac{T}{m}\left\{\left[\frac{\dot{m}}{m}\cos\theta - \left(\Omega_s\cos\psi - \dot{\theta}\right)\sin\theta\right]\frac{\partial\theta}{\partial\lambda_1}\right. \\ & \left. - \Omega_s\sin\psi\cos\theta\frac{\partial\psi}{\partial\lambda_1} - \frac{\partial\dot{\theta}}{\partial\lambda_1}\cos\theta\right\} \end{aligned} \tag{B61}$$

$$A_{8,7} = -1 \tag{B62}$$

$$A_{8,8} = \frac{T}{m}\left(\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_1} \cos\psi \sin\theta + \frac{\partial\dot{\psi}}{\partial\dot{\lambda}_1} \sin\psi \cos\theta\right) \tag{B63}$$

$$A_{8,10} = \frac{T}{m}\left(\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_1} \sin\psi \sin\theta - \frac{\partial\dot{\psi}}{\partial\dot{\lambda}_1} \cos\psi \cos\theta\right) \tag{B64}$$

$$A_{8,12} = 2\Omega_s + \frac{T}{m}\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_1} \cos\theta \tag{B65}$$

$$A_{9,2} = \frac{T}{m}\left(\sin\theta \cos\psi \frac{\partial\theta}{\partial\lambda_2} + \cos\theta \sin\psi \frac{\partial\psi}{\partial\lambda_2}\right) \tag{B66}$$

$$A_{9,4} = \frac{T}{m}\left(\sin\theta \sin\psi \frac{\partial\theta}{\partial\lambda_2} - \cos\theta \cos\psi \frac{\partial\psi}{\partial\lambda_2}\right) \tag{B67}$$

$$A_{9,6} = \frac{T}{m} \cos\theta \frac{\partial\theta}{\partial\lambda_2} \tag{B68}$$

$$\begin{aligned} A_{9,8} = &- \frac{3\gamma XY}{r_f^{\,5}} - \frac{T}{m}\left\{\left[\left(\Omega_s - \dot{\theta}\cos\psi\right)\cos\theta + \left(\frac{\dot{m}}{m}\cos\psi + \dot{\psi}\sin\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\lambda_2}\right. \\ &+ \left[\dot{\theta}\sin\psi\sin\theta + \left(\frac{\dot{m}}{m}\sin\psi - \dot{\psi}\cos\psi\right)\cos\theta\right]\frac{\partial\psi}{\partial\lambda_2} \\ &\left. - \frac{\partial\dot{\theta}}{\partial\lambda_2}\cos\psi\sin\theta - \frac{\partial\dot{\psi}}{\partial\lambda_2}\sin\psi\cos\theta\right\} \end{aligned} \tag{B69}$$

$$A_{9,10} = \frac{\gamma}{r_f^{\,3}}\left(1 - \frac{3Y^2}{r_f^{\,2}}\right) + \frac{T}{m}\left\{\left[\dot{\theta} \sin \psi \cos \theta - \left(\frac{\dot{m}}{m} \sin \psi - \dot{\psi} \cos \psi\right)\sin \theta\right]\frac{\partial\theta}{\partial\lambda_2}\right.$$

$$+ \left[\dot{\theta} \cos \psi \sin \theta + \left(\frac{\dot{m}}{m} \cos \psi + \dot{\psi} \sin \psi\right)\cos \theta\right]\frac{\partial\psi}{\partial\lambda_2}$$

$$\left. + \frac{\partial\dot{\theta}}{\partial\lambda_2} \sin \psi \sin \theta - \frac{\partial\dot{\psi}}{\partial\lambda_2} \cos \psi \cos \theta \right\} \tag{B70}$$

$$A_{9,12} = \frac{3\gamma Y\left(r_s - Z\right)}{r_f^{\,5}} - \frac{T}{m}\left\{\left[\frac{\dot{m}}{m} \cos \theta - \left(\Omega_s \cos \psi - \dot{\theta}\right)\sin \theta\right]\frac{\partial\theta}{\partial\lambda_2}\right.$$

$$\left. - \Omega_s \sin \psi \cos \theta \frac{\partial\psi}{\partial\lambda_2} - \frac{\partial\dot{\theta}}{\partial\lambda_2} \cos \theta \right\} \tag{B71}$$

$$A_{10,8} = \frac{T}{m}\left(\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_2} \cos \psi \sin \theta + \frac{\partial\dot{\psi}}{\partial\dot{\lambda}_2} \sin \psi \cos \theta\right) \tag{B72}$$

$$A_{10,9} = -1 \tag{B73}$$

$$A_{10,10} = \frac{T}{m}\left(\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_2} \sin \psi \sin \theta - \frac{\partial\dot{\psi}}{\partial\dot{\lambda}_2} \cos \psi \cos \theta\right) \tag{B74}$$

$$A_{10,12} = \frac{T}{m} \frac{\partial\dot{\theta}}{\partial\dot{\lambda}_2} \cos \theta \tag{B75}$$

$$A_{11,2} = \frac{T}{m} \sin \theta \cos \psi \frac{\partial\theta}{\partial\lambda_3} \tag{B76}$$

$$A_{11,4} = \frac{T}{m} \sin\theta \sin\psi \frac{\partial\theta}{\partial\lambda_3} \tag{B77}$$

$$A_{11,6} = \frac{T}{m} \cos\theta \frac{\partial\theta}{\partial\lambda_3} \tag{B78}$$

$$A_{11,8} = \frac{3\gamma X\left(r_s - Z\right)}{r_f^{\;5}} - \frac{T}{m}\left\{\left[\left(\Omega_s - \dot\theta\cos\psi\right)\cos\theta + \left(\frac{\dot m}{m}\cos\psi + \dot\psi\sin\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\lambda_3} - \frac{\partial\dot\theta}{\partial\lambda_3}\cos\psi\sin\theta\right\} \tag{B79}$$

$$A_{11,10} = \frac{3\gamma Y\left(r_s - Z\right)}{r_f^{\;5}} + \frac{T}{m}\left\{\left[\dot\theta\sin\psi\cos\theta - \left(\frac{\dot m}{m}\sin\psi - \dot\psi\cos\psi\right)\sin\theta\right]\frac{\partial\theta}{\partial\lambda_3} + \frac{\partial\dot\theta}{\partial\lambda_3}\sin\psi\sin\theta\right\} \tag{B80}$$

$$A_{11,12} = \left(\frac{\gamma}{r_f^{\;3}} - \Omega_s^{\;2}\right) - \frac{3\gamma\left(r_s - Z\right)^2}{r_f^{\;5}} - \frac{T}{m}\left\{\left[\frac{\dot m}{m}\cos\theta - \left(\Omega_s\cos\psi - \dot\theta\right)\sin\theta\right]\frac{\partial\theta}{\partial\lambda_3} - \frac{\partial\dot\theta}{\partial\lambda_3}\cos\theta\right\} \tag{B81}$$

$$A_{12,8} = -2\Omega_s + \frac{T}{m}\frac{\partial\dot\theta}{\partial\lambda_3}\cos\psi\sin\theta \tag{B82}$$

$$A_{12,10} = \frac{T}{m}\frac{\partial\dot\theta}{\partial\lambda_3}\sin\psi\sin\theta \tag{B83}$$

$$A_{12,11} = -1 \tag{B84}$$

$$A_{12,12} = \frac{T}{m}\frac{\partial\dot{\theta}}{\partial\lambda_3}\cos\theta \tag{B85}$$

where

$$\frac{\partial\psi}{\partial\dot{X}} = -\frac{\sin\psi\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} = -\frac{\partial\psi}{\partial\lambda_1} = -\frac{\partial\dot{\psi}}{\partial\dot{\lambda}_1}$$

$$\frac{\partial\psi}{\partial\dot{Y}} = \frac{\cos^2\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} = -\frac{\partial\psi}{\partial\lambda_2} = -\frac{\partial\dot{\psi}}{\partial\dot{\lambda}_2}$$

$$\frac{\partial\psi}{\partial Z} = \frac{\Omega_s\sin\psi\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} = -\frac{\partial\dot{\psi}}{\partial\dot{Z}}$$

$$\frac{\partial\theta}{\partial X} = -\frac{\Omega_s\cos\psi\cos^2\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$\frac{\partial\theta}{\partial\dot{X}} = -\frac{\cos^2\psi\sin\theta\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} = -\frac{\partial\theta}{\partial\lambda_1} = -\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_1}$$

$$\frac{\partial\theta}{\partial\dot{Y}} = -\frac{\sin\psi\cos\psi\sin\theta\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} = -\frac{\partial\theta}{\partial\lambda_2} = -\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_2}$$

$$\frac{\partial\theta}{\partial Z} = \frac{\Omega_s\cos^2\psi\sin\theta\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$\frac{\partial\theta}{\partial\dot{Z}} = -\frac{\cos\psi\cos^2\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} = -\frac{\partial\theta}{\partial\lambda_3} = -\frac{\partial\dot{\theta}}{\partial\dot{\lambda}_3}$$

$$\dot{\psi} = -\left\{\left(\frac{Y\gamma}{r_f^3} + \dot{\lambda}_2\right)\cos\psi + \left[\Omega_s\dot{Z} - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \dot{\lambda}_1\right]\sin\psi\right\}\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$\dot{\theta} = \left(\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) + \dot{\lambda}_3\right]\cos\theta + \left\{\left(\frac{Y\gamma}{r_f^3} + \dot{\lambda}_2\right)\sin\psi\right.\right.$$

$$\left.\left. - \left[\Omega_s\dot{Z} - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \dot{\lambda}_1\right]\cos\psi\right\}\sin\theta\right)\frac{\cos\theta\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$\frac{\partial\dot{\psi}}{\partial X} = \left\{\frac{3\gamma XY}{r_f^5}\cos\psi + \left[\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \frac{3\gamma X^2}{r_f^5}\right]\sin\psi\right\}\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$\frac{\partial\dot{\psi}}{\partial\dot{X}} = \left\{\left(\frac{Y\gamma}{r_f^3} - \dot{\lambda}_2\right)\cos 2\psi + \left[\dot{Z}\Omega_s - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \dot{\lambda}_1\right]\sin 2\psi\right\}\left[\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

$$\frac{\partial\dot{\psi}}{\partial Y} = -\left[\frac{\gamma}{r_f^3}\left(1 - \frac{3Y^2}{r_f^2}\right)\cos\psi + \frac{3\gamma XY}{r_f^5}\sin\psi\right]\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$\frac{\partial\dot{\psi}}{\partial\dot{Y}} = \left\{\left(\frac{Y\gamma}{r_f^3} - \dot{\lambda}_2\right)\sin 2\psi - \left[\dot{Z}\Omega_s - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \dot{\lambda}_1\right]\cos 2\psi\right\}\left[\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

$$\frac{\partial\dot{\psi}}{\partial Z} = -\frac{3\gamma\left(r_s - Z\right)}{r_f^5}\left(Y\cos\psi - X\sin\psi\right)\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} - \left\{\left(\frac{Y\gamma}{r_f^3} - \dot{\lambda}_2\right)\cos 2\psi\right.$$

$$\left. + \left[\dot{Z}\Omega_s - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \dot{\lambda}_1\right]\sin 2\psi\right\}\Omega_s\left[\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

$$\frac{\partial\dot{\psi}}{\partial\lambda_1} = -\left\{\left(\frac{Y\gamma}{r_f^{\,3}} - \dot{\lambda}_2\right)\cos 2\psi + \left[\dot{Z}\Omega_s - X\left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right)\right.\right.$$

$$\left.\left. - \dot{\lambda}_1\right]\sin 2\psi\right\}\left[\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

$$\frac{\partial\dot{\psi}}{\partial\lambda_2} = -\left\{\left(\frac{Y\gamma}{r_f^{\,3}} - \dot{\lambda}_2\right)\sin 2\psi - \left[\dot{Z}\Omega_s - X\left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right)\right.\right.$$

$$\left.\left. - \dot{\lambda}_1\right]\cos 2\psi\right\}\left[\frac{\cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

$$\frac{\partial\dot{\theta}}{\partial X} = \left\{\frac{3\gamma X\left(r_s - Z\right)\cos\theta}{r_f^{\,5}} - \left[\frac{3\gamma XY\sin\psi}{r_f^{\,5}} - \left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right.\right.\right.$$

$$\left.\left.\left. - \frac{3\gamma X^2}{r_f^{\,5}}\right)\cos\psi\right]\sin\theta\right\}\frac{\cos\psi\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$+ \left(\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right) + \dot{\lambda}_3\right]\sin 2\theta - \left\{\left(\frac{Y\gamma}{r_f^{\,3}} + \dot{\lambda}_2\right)\sin\psi\right.\right.$$

$$\left.\left. - \left[\Omega_s\dot{Z} - X\left(\frac{\gamma}{r_f^{\,3}} - \Omega_s^{\,2}\right) - \dot{\lambda}_1\right]\cos\psi\right\}\cos 2\theta\right)\Omega_s\left[\frac{\cos\psi\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

$$\frac{\partial\dot{\theta}}{\partial\dot{X}} = \frac{\Omega_s \cos^2\theta \cos\psi}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} - \left(\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\Upsilon}{r_f^{\,3}} - \Omega_s^{\,2}\right) + \dot{\lambda}_3\right]\cos 2\psi \cos\psi \cos\theta\right.$$

$$+ \left\{3\left(\frac{Y\Upsilon}{r_f^{\,3}} - \dot{\lambda}_2\right)\cos\psi \sin\psi + \left[\Omega_s\dot{Z} - X\left(\frac{\Upsilon}{r_f^{\,3}} - \Omega_s^{\,2}\right)\right.\right.$$

$$\left.\left.\left. - \dot{\lambda}_1\right]\left[\cos^2\psi\left(2 + \cos 2\theta\right) - 1\right]\right\}\sin\theta\right)\frac{\cos\theta \cos^2\psi}{\left[\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1\right]^2}$$

$$\frac{\partial\dot{\theta}}{\partial Y} = \left\{\frac{3\Upsilon Y\left(r_s - Z\right)\cos\theta}{r_f^{\,5}} + \left[\frac{\Upsilon}{r_f^{\,3}}\left(1 - \frac{3Y^2}{r_f^{\,2}}\right)\sin\psi\right.\right.$$

$$\left.\left. - \frac{3\Upsilon XY}{r_f^{\,5}}\cos\psi\right]\sin\theta\right\}\frac{\cos\psi \cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}$$

$$\frac{\partial\dot{\theta}}{\partial\dot{Y}} = \left\{-\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\Upsilon}{r_f^{\,3}} - \Omega_s^{\,2}\right) + \dot{\lambda}_3\right]\cos 2\theta \cos\theta \sin\psi\right.$$

$$+ \left(\frac{Y\Upsilon}{r_f^{\,3}} + \dot{\lambda}_2\right)\left[1 - \sin^2\psi\left(2 + \cos 2\theta\right)\right]\sin\theta - \left[\Omega_s\dot{Z}\right.$$

$$\left. - X\left(\frac{\Upsilon}{r_f^{\,3}} - \Omega_s^{\,2}\right) - \dot{\lambda}_1\right]\left(\sin 2\theta \sin\psi\right.$$

$$\left.\left. - \cos 2\theta \cos\psi\right)\sin\psi \sin\theta\right\}\frac{\cos^2\psi \cos\theta}{\left[\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1\right]^2}$$

$$\frac{\partial\dot{\theta}}{\partial Z} = \left\{\left[\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \frac{3\gamma\left(r_s - Z\right)^2}{r_f^5}\right]\cos\theta + \frac{3\gamma\left(r_s - Z\right)}{r_f^5}\left(X\cos\psi\right.\right.$$

$$\left.\left. + Y\sin\psi\right)\sin\theta\right\}\frac{\cos\psi\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} + \left\{\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right)\right.\right.$$

$$\left. + \dot{\lambda}_3\right]\cos 2\theta\cos\psi\cos\theta + \left(\frac{Y\gamma}{r_f^3} + \dot{\lambda}_2\right)\left(2 + \cos 2\theta\right)\cos\psi\sin\psi\sin\theta$$

$$+ \left[\Omega_s\dot{Z} - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) - \dot{\lambda}_1\right]\left[1 - \cos^2\psi\left(2\right.\right.$$

$$\left.\left.\left. + \cos 2\theta\right)\right]\sin\theta\right\}\frac{\Omega_s\cos^2\psi\cos\theta}{\left[\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1\right]^2}$$

$$\frac{\partial\dot{\theta}}{\partial\dot{Z}} = -\frac{\Omega_s\cos^2\psi\sin\theta\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1} + \left(\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) + \dot{\lambda}_3\right]\sin 2\theta\right.$$

$$+ \left\{\left(\frac{Y\gamma}{r_f^3} + \dot{\lambda}_2\right)\sin\psi + \left[\Omega_s\dot{Z} - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right)\right.\right.$$

$$\left.\left.\left. - \dot{\lambda}_1\right]\cos\psi\right\}\cos 2\theta\right)\left[\frac{\cos\psi\cos\theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

$$\frac{\partial\dot{\theta}}{\partial\lambda_1} = \left\{\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right) + \dot{\lambda}_3\right]\cos 2\theta\cos\theta\cos\psi\right.$$

$$+ \left(\frac{Y\gamma}{r_f^3} + \dot{\lambda}_2\right)\left(2 + \cos 2\theta\right)\cos\psi\sin\psi\sin\theta + \left[\Omega_s\dot{Z} - X\left(\frac{\gamma}{r_f^3} - \Omega_s^2\right)\right.$$

$$\left.\left. - \dot{\lambda}_1\right]\left[1 - \cos^2\psi\left(2 + \cos 2\theta\right)\right]\sin\theta\right\}\frac{\cos^2\psi\cos\theta}{\left[\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1\right]^2}$$

$$\frac{\partial\dot{\theta}}{\partial\lambda_2} = \Bigg\{\left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\Upsilon}{{r_f}^3} - {\Omega_s}^2\right) + \dot{\lambda}_3\right]\cos 2\theta \cos \theta \sin \psi$$

$$- \left(\frac{Y\Upsilon}{{r_f}^3} + \dot{\lambda}_2\right)\left[1 - \sin^2\psi\left(2 + \cos 2\theta\right)\right]\sin \theta - \left[\Omega_s\dot{Z} - X\left(\frac{\Upsilon}{{r_f}^3} - {\Omega_s}^2\right)\right.$$

$$\left. - \dot{\lambda}_1\right]\left(2 + \cos 2\theta\right)\sin \psi \cos \psi \sin \theta\Bigg\}\frac{\cos^2\psi \cos \theta}{\left[\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1\right]^2}$$

$$\frac{\partial\dot{\theta}}{\partial\lambda_3} = \Bigg(- \left[\Omega_s\dot{X} - \left(r_s - Z\right)\left(\frac{\Upsilon}{{r_f}^3} - {\Omega_s}^2\right) + \dot{\lambda}_3\right]\sin 2\theta + \Bigg\{\left(\frac{Y\Upsilon}{{r_f}^3} + \dot{\lambda}_2\right)\sin \psi$$

$$- \left[\Omega_s\dot{Z} - X\left(\frac{\Upsilon}{{r_f}^3} - {\Omega_s}^2\right) - \dot{\lambda}_1\right]\cos \psi\Bigg\} \cos 2\theta\Bigg)\left[\frac{\cos \psi \cos \theta}{\dot{X} + \Omega_s\left(r_s - Z\right) - \lambda_1}\right]^2$$

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