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*The importance of mathematics in the Space Age**

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The distance and immensity of outer space, as well as the enormity of wasted expense and danger of a "miss," clearly place unusually high desirability on mathematical technique as the best approach to many problems in this area.

As a former full-time worker at the challenging and never-ending task of leading young minds into the ways of mathematical thinking, I welcome the privilege of reviewing before this important group of mathematics teachers some new developments in the significance of their profession.

What is the meaning of "The Space Age," a phrase which was born and has now come into common use (by nonscientists as well as by scientists), all within the past five or six years? Everyone *feels,* without knowing, the meaning of this phrase, just as they did for all other common words of the English language, long before Samuel Johnson and Noah Webster composed their dictionaries. I believe that this required meaning, which everyone feels he knows without putting it into words, is essentially and fundamentally a mathematical concept: "space" means three degrees of freedom for positional movement—the greatest number that is practically possible.

Before the Space Age, all life we know of was confined to movement at the surface of the earth; it had only two degrees of freedom, as viewed from a large-scale standpoint. As a small-scale model of this habitat, a fairly accurate suggestion has been to consider the earth as like an ordinary billiard ball, of which 71 percent of the surface is wet. The earth's deviations from a true mathematical sphere, including its general flattening, the roughness of the highest mountains, and water of the deepest seas would be approximately represented to scale by a wet ball sufficiently round and smooth as to be quite acceptable for official billiard playing! On a two-inch billiard ball a general flattening of only three thousandths of an inch with local high points and water films only one thousandth of an inch from a perfect spherical surface would be approximately as smooth and wet as any true scale model of the earth. Thus all life before the Space Age has been confined to within 6 miles of a perfectly spheroidal surface extending many thousands of miles. The deviations of human beings from strictly two-dimensional activity was thus less than 1 part in a thousand, and by the principles of the calculus of finite differences, the error committed by considering human activity to be strictly superficial would be less than 1 part in a million. All history before the Space Age could be properly designated the "Surface Age."

Although man's large-scale activities before the Space Age were essentially

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two-dimensional, his small-scale works were, as we all know, fully three-dimensional. But it required many centuries of abstract mathematical thinking, summarized by the ancient Greek mathematician Euclid in his *Solid Geometry*, to show that the same principles used for describing and measuring small-scale objects shaped as spheres, cubes, pyramids, and cones could be extended to indefinite distances in all directions from the observer. These concepts and principles involving mathematical space were applied to and derived from progressively larger scale problems involving the physical world: in surveying, in navigation, and in astronomy. Thus it might be said that the ancient Greek mathematicians and astronomers were mentally prepared for the Space Age some two thousand years before it became an actuality, at least so far as the required geometrical concepts were concerned. However, much in the way of spreading and developing such knowledge, as well as in advancing the technical arts, remained to be done before man could think practically of physical escape from his essentially two-dimensional range of activity.

A most important practical step in progress toward the Space Age was in navigation, namely, the revolution from thinking in terms of a flat surface as representing the earth to considering the surface of the earth as spherical. Thus we might speak with reference to the geography of ancient times as the "Flat Surface Age." In the fifteenth century of the Christian era this was revolutionized by the activities of da Gama, Columbus, and Magellan to become the "Spheroidal Surface Age" which likewise, in turn, became the "Space Age," because of astronautical achievements beginning in A.D. 1957.

Although the ancient astronomers had cited clear proofs of the spheroidal form of the earth, such as the circularity of its shadow during lunar eclipses, and Eratosthenes had actually succeeded in making a good estimate of the earth's diameter by measuring the change in celestial direction of the vertical with north-south change of position, there had been little or no practical application of this new knowledge to travel and navigation. Most sailing was within short ranges of latitude and not far from landmarks, especially within the Mediterranean Sea and, to a lesser degree, around other coasts of Europe and Asia. Long cross-country travel, such as the famous journey from Italy to China by Marco Polo, could be accomplished by landmarks alone. Even da Gama's epoch-making cruise around South Africa to India depended on following a continuous coastline. On all these journeys the earth's surface could, for all practical navigational purposes, be considered flat.

The continuing need of European commercial interests to find more economically feasible routes to the Far East inspired the application by Columbus of mathematical reasoning and astronomical knowledge to practical navigation. On a flat earth, one could naturally think of reaching a point to the east of oneself only by sailing eastward; whereas consideration of a spheroidal surface clearly indicated also an alternative westward route. That the monarchs of Spain were most easily convinced by this mathematical reasoning of Columbus may be no mere accident in terms of the history of mathematics. Less than a century before Ferdinand and Isabella, Spain was dominated by the Moslems, whose culture then led the world in mathematics and astronomy. At any rate, Columbus successfully inaugurated the "Spheroidal Surface Age" by sailing westward from Europe for India and thus making a landfall on islands which he therefore called "The Indies." The native peoples there he naturally called "Indians," as, indeed, they remain officially designated even to this day, although they and their country are nearly 10,000 miles from India.

From the time of Columbus until the Space Age, all progress in practical navigation and means of travel was directed toward facilitation of movement only around the spheroidal surface of the earth. Even the invention of air travel for movement within a few miles above the solid or liquid surface of the earth made little change from a mathematical standpoint; the air available for such travel was still within the same few miles of a spheroidal earth surface, being merely an upper and lighter layer of the earth's fluid surface. Nevertheless, the "Air Age," which developed approximately with the present century, could be cons dered as a transition between strictly surface travel and the full three-dimensional travel possible in space. The mobility of the aviator in the third dimension, while relatively small, is real, and so forms a logical introduction to some end-point piloting problems of space travel. Perhaps even more important was the more recent development of jet propulsion in aircraft, which formed a basis for progress in the similar rocket propulsion so necessary for acceleration and launching of spacecraft.

Columbus did not have to make new discoveries in mathematics or astronomy; he had only to apply knowledge already hundreds of years old to the travel problems of his day. Similarly with the abstract problems which had to be met for successful inauguration of the Space Age: most of the necessary basic mathematics, physics, and astronomy had existed for a long time and needed only to be properly adapted and applied. But such adaptation and application of abstract principles to real circumstances, or even recognition of the possible appropriateness of such application, is often the part of the solution to such problems which takes the longest to discover and which is most difficult to bring into practical effect.

The age of flat-earth travel ("Plane Sailing," in modern navigational language) led to the development of much of what we now call "Elementary Mathematics." The most primitive of its aspects, land surveying (necessitated by floods of the Nile, Tigris, and Euphrates rivers), is said to have been the inspiration for much of plane geometry and plane trigonometry. Then when navigation required estimation of distances to landmarks for plotting and using charts of sailing waters, these mathematical arts were ready for adaptation to the problem. Euclidean geometry perfected the general abstract considerations of physical forms and their properties. Its natural extension by the ancient Greeks to the third dimension, which we nowadays call solid geometry, included special studies of the properties of spheres and spheroids, thus laying the foundation for advance to the "Spheroidal Surface Age" to be initiated by Columbus many centuries later. On the other hand, trigonometry is more useful than abstract geometry for problems of practical measurement, since by using available measurements of segments and angles, inaccessible segments and angles may be computed. Original development of trigonometry and the general arts of computation took place in the Middle East, by the Egyptians, by the Phoenicians, in India, and, more recently, in the Moslem Arabian culture. Our decimal arithmetic with its so-called Arabic numerals probably originated in India, while the generalization and abstraction of arithmetic principles covered by the Arabian term "Algebra" had similar origins. Lack of these particular arts of computation among the ancient Greeks and Romans severely handicapped their practice of navigation, and may partly account for the delay of advancement beyond "Plane Sailing" concepts until the time of Columbus. Combination of trigonometry with the geometry of the sphere to form the science of spherical trigonometry made possible the measurements on the surface of a sphere so necessary to successful development of the resultant "Spheroidal Surface Age," and are still today essential for air and surface navigation. Also, only when Greek geometry and Arabian arithmetic and algebra were finally combined in Europe during the sixteenth and seventeenth centuries to form the single subject called mathematics was the stage of abstract thought set for most of the developments of modern science and technology, including the Space Age.

Almost all the developments which have produced the Space Age depend more or less directly on mathematical and physical principles invented or, at least, collected and synthesized by Sir Isaac Newton just before 1700. Newton not only enunciated the various laws of mechanical forces and motions and the combinations of their effects which describe and predict all motions, including those in outer space, but he worked out the geometric and algebraic approaches for mathematical application of these principles in space. Of these geometric approaches, the coordinate or analytic geometry invented by Descartes just before Newton's time is perhaps the most important. This solid analytic geometry, as we now call it, which presents scalar directional measurements of points in space with reference to a zero-point or origin which is the intersection of three planes of reference, thus furnishes an approach for expressing directional relations between point positions on geometric figures so that the forms and theorems of Euclidean geometry may be studied by means of algebra. Furthermore, forms and relationships never dreamt of in Greek geometry may be invented and analyzed directly by algebraic concepts and manipulations. Many such algebraic relationships describing motions in space of masses such as planets of the solar system had also been developed just before Newton's time by the astronomer John Kepler. Using Kepler's kinematical astronomy, Descartes' mathematics, and his own hypothesis of universal gravitational forces between masses, Newton was able to lay the foundation of the science of mechanics which is now used in the solution of all mathematical problems of the Space Age, from rocket design and propulsion to deduction of the path followed by a rocket flying through the earth's atmosphere, a spacecraft orbiting about the earth or moon, or a probe approaching some more distant planet.

The basic approach for calculating all such motions is Newton's equation of motion, $a=\frac{F}{m}$. This states that acceleration a or rate of change of velocity of any freely moving body occurs only by action of a force on the body, and that this acceleration is always in the direction of the impressed force and is measured by the magnitude of the force F divided by the mass m of the body. For freely moving bodies in outer space, Newton suggested that the main impressed force could only be what we on the earth call gravitational attraction, and that such attraction between any two bodies in the universe is proportional both to their masses and to the inverse square of the distance r separating them; that is,

$$F=\frac{mM}{r^2}\cdot$$

But acceleration would always cause change of distance, which thereby would cause change of the gravitational force, which in turn would cause a change in the acceleration—a never-ending chain of changes which flows evenly along with passage of time. How could one deal mathematically with such continuously flowing variables? Algebra had no ready means for such an operation, although some researches of the ancient Greek geometers into certain problems such as those related to squaring the circle had used some such manner of thinking. Newton solved this key difficulty by inventing and developing "fluxions" (literally, *flow-functions*), which today we call differentials or time-derivatives, and teach in a subject called calculus. Thus, the acceleration of a mass was considered to be merely the time-derivative of its velocity, which in turn was merely the time-derivative of its position in space. Newton's equation of motion stating that attractive force equals mass times acceleration could, using his "fluxions," be explicitly written for the effect of, say, the earth on a space-probe of unit mass. The equation would be written in terms of the position of the probe with respect to the earth and the time-derivatives (or differentials) of this position as what we would call nowadays a "differential equation,"

$$\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{\text{mass of earth}}{r^2},$$

where t is the time coordinate. Solving such an equation to find the position of the probe at any time requires not only algebra, but an operation inverse to that of differentiation (i.e., forming the time-derivative), an operation which we now call integration.

If the space probe were considered to be affected by the gravitation of the moon as well as that of the earth, there would be another similar equation for the moon which must be solved simultaneously with the equation for the earth. This sounds easy, but it has never yet been completely and explicitly accomplished; it is the famous "Problem of Three Bodies." We have thus reached the most advanced frontier of the present mathematical art by considering such a simple question as the motion of a spacecraft from the earth to the moon. However, such problems as this one, which cannot be solved with closed explicitness and complete accuracy, can usually be solved in numerical form by approximation methods to any degree of accuracy which available computational facilities will permit. Practical means for the very extensive computations needed in such cases have been vastly extended in recent years with technical developments such as electronic digital computing machines. The mathematical aspect and basis of such computing operations is called numerical analysis. Thus we see how Newton's original mathematical theories of bodies falling under gravitational forces (which are said by tradition to have been applied first to the homely case of a falling apple) are used nowadays to predict and determine rocket trajectories, manned space capsule orbits, and the schedule of deep space probes to the planets Venus and Mars, as well as to the moon. Such is the universal adaptability and applicability of sound mathematical principles. Indeed, other basic Newtonian principles are used, also, not only to get such a manned capsule off the ground, but to get it back with comfort and safety to the astronaut. Newton's Law that "for every action there is an equal and opposite reaction" is the basis for the possibility of rockets' attaining the 5 miles per second necessary to attain a free orbit around the earth. In launching plans the air resistance to acceleration must be taken into account. The suggestion that such air resistance is usually proportional to the air density, to the square of the velocity, and to the area of the cross section of the moving object was first stated by Newton. Reentry of the astronaut to land on the earth at a safe velocity is performed by greatly increasing his cross-sectional area and, hence, air resistance, by a parachute. Newton's general mathematical statement,

$$\text{air resistance} = \rho A v^2$$

where ρ is the air density, A the effective cross section, and v the velocity of the astronaut's capsule and appendages, is thus applicable to both the launching and reentry operations.

The fact that any precise description of the orbit of an astronaut's capsule requires plenty of Descartes' solid analytic geometry is indicated by Figure 1. Here the Z-axis is parallel to the axis of the earth with positive direction northward. Note that the positive direction of the X-axis is toward the point of the vernal equinox in the sky, and to refer the satellite to longitudes on the earth requires much of the spherical trigonometry which is the basis of all mathematical astronomy, as does effective tracking of any artificial satellite in its orbit. To solve the more special problems of space science in which it is sought to develop subtle conclusions as to the shape of the earth from irregularities of a satellite's orbit, or to explain such experimentally suggested results as the shape and extent of a ring of ionized particles circulating in the magnetic field of the earth, the classical mathematical subjects algebra, geometry, trigonometry, and differential and integral calculus are all of essential and key importance.

I shall illustrate this last point by a few remarks about one or two of these problems on which I have worked in recent years. One problem is: how would the spin of a space vehicle be affected by a natural magnetic field such as that of the earth, the force field which directs our compass needles? The principles of electrophysics tell us that a closed electrically conducting loop, spinning in a magnetic field, is slowed down by a torque proportional to the conductivity and enclosed area of the loop, to its angular velocity, and to the square of the effective magnetic field perpendicular to its axis of spin. For material objects such as actual satellite parts, this torque must be formulated by application of the integral calculus to the special analytic geometry of the part in question. Thus, for a thin cylindrical shell having a height equal to its diameter we find by integration over the shell for this braking torque C an equation such as

$$C = \left(18\pi - \frac{160}{3}\right)\sigma\mu^2 H^2 \omega r^4 \Delta r,$$

where σ is the electric conductivity, μH the effective magnetic field inside the metal due to the earth's field, ω the spin rate in

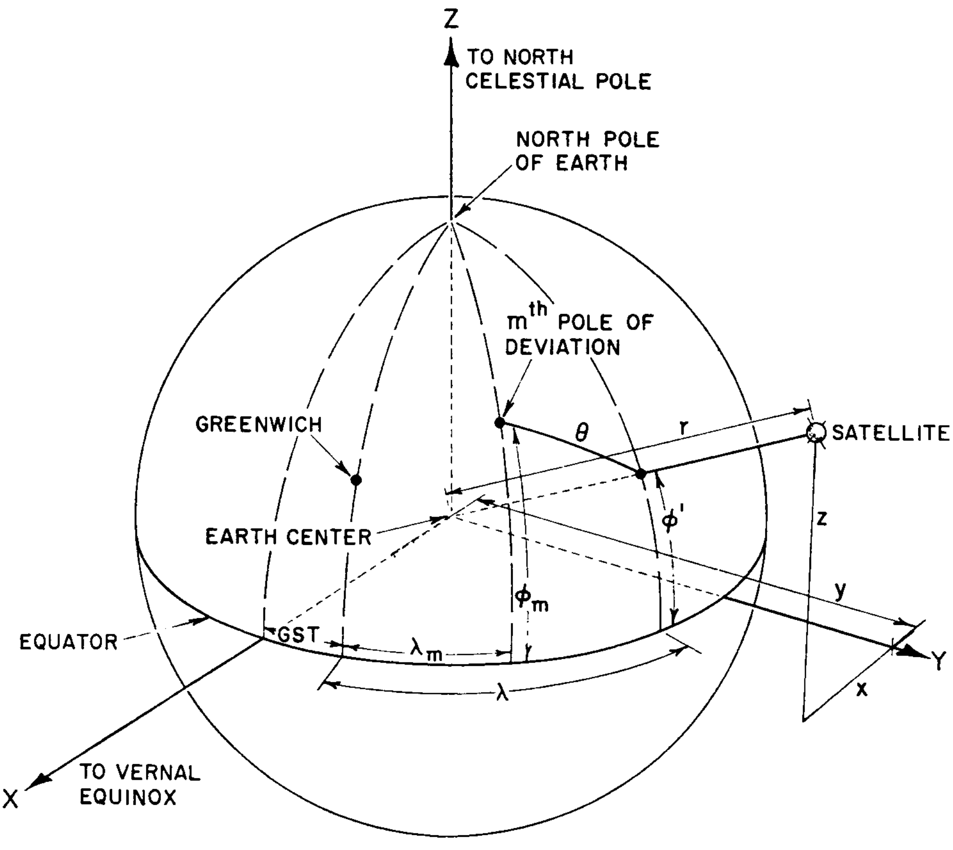

Figure 1

Geometrical relations of the inertial and geographical coordinate systems as to positions of a satellite and any pole (center) of deviation of the geoidal potential function from exact spherical form.

rotations per second, r the radius of the cylindrical shell, and Δr the thickness of that shell. After the torques for all parts of any satellite have been computed, one may apply Newton's equation of motion stated above, adapted to relating such torque to angular acceleration, to predict the angular spin rate of the satellite at any future time.

Satellite spin rates are often followed as part of the operations of tracking them, in order that the mathematical theory of spin damping may be compared with the observed facts. One good example of this was the so-called Solar Radiation Satellite, which was launched in 1960. Figure 2 shows a graph in which this satellite's observed spin rates are plotted as open or dark circles. The curve through this set of observations represents predicted spin rates according to the mathematical theory of magnetic damping described above, assuming the known magnetic field of the earth. The representation seems satisfactory. Conversely, if such a satellite were spinning in the unknown magnetic field about some other planet, this field could be calculated from the spin-damping theory, provided the spin rate could be observed. Also, for many purposes the spin rate of satellites needs to be planned ahead, and this mathematical theory would form a practical basis for such planning.

In order to measure the spin rate or rotation of an artificial space vehicle, it is necessary, as for any natural celestial body, to observe periodic variations in the radiation intensity received from that

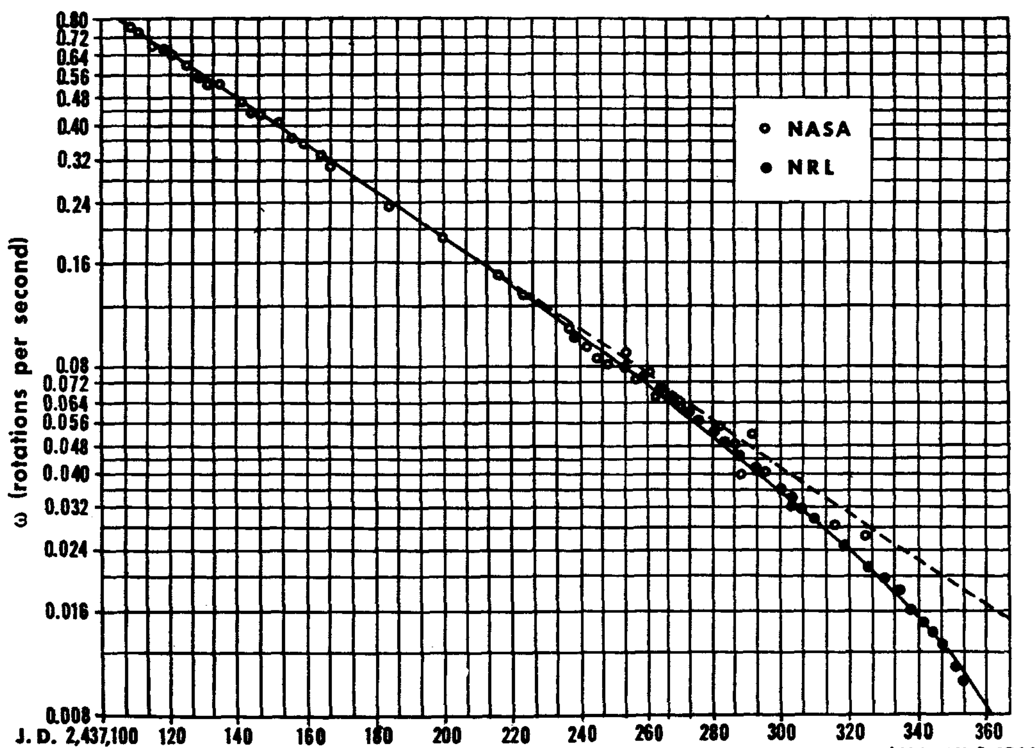

Figure 2

Observed spin rate vs. time for Satellite 1960 *Eta* 2 (*Solar Radiation I*). *Straight line represents exponential decay with relaxation time of* 66 *days. Curve includes additional linear decay of* 3.3×10^{-5} *rotations per second per day. Extrapolating the curve shows rotation to have stopped on J.D.* (*Julian date*) 2,437,397 (*April* 8, 1961). (*Julian date is defined as the number of days elapsed since Greenwich Mean Noon, January* 1, 4713 B.C.)

body. Such measurements of rotation could be made very easily and accurately by optical observations on an artificial body, provided its surface for reflecting sunlight were at least partially fabricated as polyhedral, with many highly reflecting facets. Solar reflection from the facets would produce a set of discontinuous flashes for which the flash rate per second is given by the equation

$$n = \frac{\omega N(\cos L)}{273},$$

where ω is the spin rate per second, N the total number of such facets which would completely cover a sphere, and L is the "latitude" on such a sphere of the currently reflecting facets. Such a scheme is presently being used by Bell Telephone Laboratories on their Telstar satellite, in order to determine its ω and L, quantities which are essential to know for the successful relay of TV and other signals between America and other continents. Another incidental advantage of such reflecting facets on a space vehicle is that the optical brightness of each flash would be greatly increased over that from a smooth, nonfaceted surface. Such a device, called a "heliotrope," for transmitting signals over great distances by solar reflection from a flat mirror, was first invented and its advantages proved mathematically by Carl Friedrich Gauss, who also applied it to surveying, in the course of his field operations for determining the precise size and shape of the earth.

Thus mathematics is of special practical importance when applied to scientific and technical problems outside the range of feasible hit-and-miss experimentation. A new emphasis in mathematical thinking for the Space Age is demanded by the fact that "space" implies activity in three equally important dimensions, so that the predominately two-dimensional thinking appropriate to our former confinement within a few miles of the wide surface of the earth becomes grossly inadequate. For mathematical education this development suggests renewed attention, at as early an age as possible, to studies involving three-dimensional (solid) geometry and to spherical trigonometry. Also in all mathematical subjects the "word problem" provides most important practice in bridging the gap between the principles of mathematics and the conditions of physical reality; that is, practice in applied mathematics.

The preceding examples illustrate the general application of mathematics in the Space Age, as in all technological science, to problems for which experimentation for direct measurement of all special cases involved might be impossible or hopelessly expensive. A mathematical approach using rigorous thinking may yield general formulation having widespread applicability. Numerical analysis can render the most complicated mathematical formulations susceptible to approximate computation of particular cases for practical use. In all such work on Space Age problems, the basic mathematical subjects algebra, geometry (both plane and solid), and trigonometry (both plane and spherical) are most often used as the international language of rigorous thinking. Study of them is a key to powerful results in fundamental research in many fields, and teaching of them constitutes a most important contribution to our progress in the Space Age.

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