NASA TECHNICAL REPORT

NASA TR R-172

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GENERALIZED TREATMENT OF PLANE ELECTROMAGNETIC WAVES PASSING THROUGH AN ISOTROPIC INHOMOGENEOUS PLASMA SLAB AT ARBITRARY ANGLES OF INCIDENCE

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1963

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For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $1.25

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SUMMARY

The differential equations describing the passage of plane electromagnetic waves through a plane-parallel inhomogeneous plasma slab at an arbitrary angle of incidence are solved by numerical integration between appropriate boundary conditions. The methods used are described and a typical solution is discussed.

INTRODUCTION

Analytical solutions to the wave equation in an inhomogeneous plasma can be derived for simple electron density distributions (refs. 1, 2, and 3), but such solutions are often inadequate for application to a given practical problem. One approach (ref. 4) considers the plasma as a multilayered series of homogeneous slabs. This technique has been derived for convenient application to waves normal to the plasma slab; however, extension to include oblique incidence seems questionable because derivatives of the plasma properties which generally appear in the propagation equations for inhomogeneous media do not explicitly appear in the multilayered-slab approach. The W.K.B. approximation (refs. 5 and 6) gives good solutions for arbitrary electron density distributions so long as the conditions for its validity are met. These conditions are not met if plasma properties vary appreciably within a wavelength; and since such plasmas are frequently of interest, another method for obtaining solutions is needed.

The purpose of the work reported herein is to reduce the propagation equations to a form suitable for numerical integration on an electronic computing machine. The physical model postulates arbitrarily polarized, plane, monochromatic electromagnetic waves impinging at an arbitrary angle of incidence on a slab of plasma bounded by infinite parallel planes, which separate the plasma region from vacuum. The plasma properties are assumed to vary only in the direction of the normal to the planes. In general, discontinuities in the plasma properties and their first derivatives are permitted only at the boundaries; however, at normal incidence, the continuity requirement on the first derivative can be removed.

SYMBOLS

In all derivations mks units are used, but in tables and graphs, linear dimensions are shown in centimeter units.

c	speed of light, 3×10^8 meters per sec
$\vec{E}$	electric field intensity, volts/meter
F_y	z-dependent part of y-component of electric field intensity in plasma, volts/meter
f	frequency, sec^{-1}
G_y	z-dependent part of y-component of magnetic field intensity in plasma, ampere-turns/meter
$\vec{H}$	magnetic field intensity, ampere-turns/meter
k_0	propagation constant, ω/c, meters^{-1}
$\vec{k}$	propagation vector, $k_x\vec{u}_x + k_z\vec{u}_z = k_0 \sin \theta\vec{u}_x + k_0 \cos \theta\vec{u}_z$, meters^{-1}
M,N	arbitrary constants determined by solution of wave equation, volts/meter
N_e	number of electrons per cubic meter
n	index of refraction, $\sqrt{\frac{\epsilon}{\epsilon_0}}$
R_1,R_2	voltage reflection coefficients of perpendicular and parallel components of wave, respectively
$\vec{r}$	position vector, $x\vec{u}_x + y\vec{u}_y + z\vec{u}_z$, meters
T_1,T_2	voltage transmission coefficients of perpendicular and parallel components of wave, respectively
$\vec{u}_x$	unit vector in x-direction
$\vec{u}_y$	unit vector in y-direction
$\vec{u}_z$	unit vector in z-direction

$$V = 1 - \frac{1}{\left(\frac{\omega}{\omega_p}\right)^2 + \left(\frac{\nu}{\omega_p}\right)^2}$$

$$V^* = V - \sin^2\theta$$

$$W = \frac{\nu/\omega_p}{\omega/\omega_p} \frac{1}{\left(\frac{\omega}{\omega_p}\right)^2 + \left(\frac{\nu}{\omega_p}\right)^2}$$

x,y,z	Cartesian coordinates
X	x-dependent part of solution of wave equation
Z	z-dependent part of solution of wave equation
z_0	thickness of plasma slab, meters
z_1	ramp depth for trapezoidal electron density, meters
γ	amplitude factor defined by equation (110), radians
δ_r	phase angle of reflected wave, radians
δ_t	phase angle of transmitted wave, radians
ϵ	permittivity, farads/meter
ϵ_0	permittivity of free space, 8.854×10^{-12} farads/meter
θ	angle of incidence, deg
θ_t	angle of transmission, deg
λ	wavelength, meters
μ_0	permeability of free space = $4\pi \times 10^{-7}$ henry/meter
ν	mean collision frequency of electrons with neutral particles, collisions per second
ξ	phase difference between parallel and perpendicular components of incident wave, which accounts for elliptical polarization, radians
ϕ	angle between incident electric vector and y-axis, deg

ψ phase factor defined by equation (111), radians

ω frequency of propagating wave (radians per second)

ω_p plasma frequency (radians per second), $2\pi \times 8.97 \times \sqrt{N_e}$

$|\ |$ magnitude of a complex number

$'$ differentiation with respect to z

Subscripts:

E components of X and Z which are solutions to the electric field intensity

fs wave in free space (includes incident and reflected components)

H components of X and Z which are solutions to the magnetic field intensity

i incident wave

p wave transmitted into plasma

r reflected wave

t wave which has transmitted through plasma slab

x x-component of wave

y y-component of wave

z z-component of wave

τfs tangential component of wave in free space (including incident and reflected waves)

τp tangential component of wave which is transmitted into plasma

0 amplitude of wave or free-space parameter, as appropriate

00 amplitude of arbitrarily oriented incident wave

1 component of the wave which is perpendicular to plane of incidence

2 component of the wave which is parallel to plane of incidence

Note that the subscripts are used in combination. For instance, E_{r0y} is the amplitude of the y-component of the reflected electric vector.

THEORY

Electromagnetic Waves in an Inhomogeneous Plasma

The wave equations for $\vec{E}$ and $\vec{H}$ in inhomogeneous media are (refs. 1 and 7):

$$\nabla^2\vec{E} + k_0^2 n^2 \vec{E} = -\vec{\nabla}\left(\vec{E} \cdot \frac{\vec{\nabla} n^2}{n^2}\right) \tag{1}$$

$$\nabla^2\vec{H} + k_0^2 n^2 \vec{H} = -\frac{\vec{\nabla} n^2}{n^2} \times \left(\vec{\nabla} \times \vec{H}\right) \tag{2}$$

where the index of refraction n is a function of position.

Let an electromagnetic wave be incident upon an inhomogeneous plasma slab as shown in figure 1. The plane of incidence of the wave is defined as the plane which contains $\vec{u}_z$ and $\vec{k}_0$. The electric vector is, by definition, orthogonal to $\vec{k}_0$; however, there is no imposed restriction on the orientation of $\vec{E}$ with respect to the plane of incidence. Although a ray path is shown in the diagram for visual purposes, the method presented does not depend upon explicit knowledge of the ray path. It is of interest to note (see appendix) that the angle of the emerging ray is identical to that of the incident wave, even for the inhomogeneous case discussed in this analysis. If the variation of the index of refraction is restricted to one dimension so that

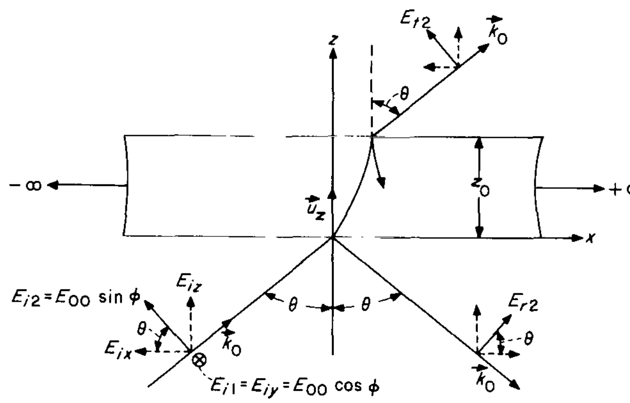

Figure 1.- Geometry of plane-wave interaction with an inhomogeneous plasma slab. $N_e = N_e(z)$; $\nu = \nu(z)$.

$$n = n(z) \tag{3}$$

and if the wave in free space is independent of y, then the solutions to equations (1) and (2) are independent of y. (See appendix.) If the condition postulated by equation (3) is used and all derivatives with respect to y are ignored, equations (1) and (2) become

$$\frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial z^2} + k_0^2 n^2(z)\vec{E} = -\frac{\partial}{\partial x}\left[E_z \frac{\partial n^2(z)}{\partial z}\right]\vec{u}_x - \frac{\partial}{\partial z}\left[E_z \frac{\partial n^2(z)}{\partial z}\right]\vec{u}_z \tag{4}$$

$$\frac{\partial^2 \vec{H}}{\partial x^2} + \frac{\partial^2 \vec{H}}{\partial z^2} + k_0^2 n^2(z)\vec{H} = \frac{1}{n^2(z)} \frac{\partial n^2(z)}{\partial z}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{u}_x + \frac{1}{n^2(z)} \frac{\partial n^2(z)}{\partial z} \frac{\partial H_y}{\partial z} \vec{u}_y \tag{5}$$

Solutions to equation (4) for arbitrary orientation of $\vec{E}$ can be obtained by solving three differential equations in which E_x, E_y, and E_z are dependent variables. The solutions of equation (5) can be found from the solutions of equation (4) by means of Maxwell's equations. However, Stratton (ref. 8, p. 492) has shown that the wave in the plasma can be uniquely defined by two rather than three differential equations in which the dependent variables are $E_1 = E_y$ and $E_2 = \sqrt{E_x^2 + E_z^2}$ and, since E_2 is related to H_y through Maxwell's equations, the two differential equations for which solutions are required are:

For the E_1 case:

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + k_0^2 n^2(z) E_y = 0 \tag{6}$$

For the E_2 case:

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} + k_0^2 n^2(z) H_y = \frac{1}{n^2(z)} \frac{\partial n^2(z)}{\partial z} \frac{\partial H_y}{\partial z} \tag{7}$$

If a product solution to equations (6) and (7) is assumed, it follows that

$$E_y(x,z) = Z_E(z) X_E(x) \tag{8}$$

and

$$H_y(x,z) = Z_H(z) X_H(x) \tag{9}$$

The substitution of equations (8) and (9) into equations (6) and (7) results in a solution for X_E and X_H. Solving for the X's yields for E_y and H_y

$$E_y(x,z) = F_y(z) e^{\pm i k_x x} \tag{10}$$

$$H_y(x,z) = G_y(z) e^{\pm i k_x x} \tag{11}$$

But from the arguments of the appendix and from figure 1, $k_x = k_0 \sin\theta$. If the condition is made that the wave propagates in the positive x-direction, the negative sign on the exponent can be dropped. The positive sign on $k_x x$ implies also a time factor of $e^{-i\omega t}$. Equations (10) and (11) now become

$$E_y(x,z) = F_y(z)e^{ik_0 x\sin\theta} \tag{12}$$

$$H_y(x,z) = G_y(z)e^{ik_0 x\sin\theta} \tag{13}$$

Substitution of equations (12) and (13) into equations (6) and (7) further reduces the wave equations to

$$\frac{d^2F_y(z)}{dz^2} + k_0{}^2\left[n^2(z) - \sin^2\theta\right]F_y(z) = 0 \tag{14}$$

$$\frac{d^2G_y(z)}{dz^2} - \frac{dn^2(z)}{n^2(z)dz}\frac{dG_y(z)}{dz} + k_0{}^2\left[n^2(z) - \sin^2\theta\right]G_y(z) = 0 \tag{15}$$

The solution to the wave equation in free space is obtained by setting $n^2 = 1$ and the right-hand side of equations (1) and (2) equal to zero. If the propagation vector $\vec{k}_0$ lies in the xz-plane, then the incident waves will be of the form

$$\left.\begin{aligned} E_{iy} &= E_{i0y}e^{i\vec{k}_{0i}\cdot\vec{r}} \\ H_{iy} &= H_{i0y}e^{i\vec{k}_{0i}\cdot\vec{r}} \end{aligned}\right\} \tag{16}$$

From inspection of figure 1, it can readily be determined that

$$\begin{aligned} \vec{k}_{0i} \cdot \vec{r} &= k_0\left(\sin\theta\vec{u}_x + \cos\theta\vec{u}_z\right) \cdot \left(x\vec{u}_x + y\vec{u}_y + z\vec{u}_z\right) \\ &= k_0 x\sin\theta + k_0 z\cos\theta \end{aligned} \tag{17}$$

Likewise, the reflected waves have the form

$$\left.\begin{aligned} E_{ry} &= E_{r0y}e^{i\vec{k}_{0r}\cdot\vec{r}} \\ H_{ry} &= H_{r0y}e^{i\vec{k}_{0r}\cdot\vec{r}} \end{aligned}\right\} \tag{18}$$

with

$$\vec{k}_{0r} \cdot \vec{r} = k_0 x \sin\theta - k_0 z \cos\theta \tag{19}$$

and the transmitted waves will be

$$\left.\begin{aligned} E_{ty} &= E_{t0y}e^{i\vec{k}_{0t}\cdot\vec{r}} \\ H_{ty} &= H_{t0y}e^{i\vec{k}_{0t}\cdot\vec{r}} \end{aligned}\right\} \tag{20}$$

with

$$\vec{k}_{0t} \cdot \vec{r} = k_0 x \sin\theta + k_0 z \cos\theta \tag{21}$$

Boundary Conditions and Reduction of Wave Equations to Computational Form

Case 1: E perpendicular to the plane of incidence.- When E is perpendicular to the plane of incidence, the boundary conditions at the surfaces $z = 0$ and $z = z_0$ require that the tangential components of $\vec{E}$ and $\vec{H}$ be continuous across each surface. At the boundary $z = 0$,

$$E_{\tau fs} = E_{\tau p} \tag{22}$$

$$H_{\tau fs} = H_{\tau p} \tag{23}$$

Equation (22) requires that

$$\vec{u}_z \times \left(E_{i0y}\vec{u}_y + E_{r0y}\vec{u}_y\right)e^{ik_0 x\sin\theta} = \vec{u}_z \times \vec{u}_y F_y(0)e^{ik_0 x\sin\theta} \tag{24}$$

or

$$E_{i0y} + E_{r0y} = F_y(0) \tag{25}$$

Now, since

$$\vec{H} = \frac{1}{i\omega\mu_0} \vec{\nabla} \times \vec{E} \tag{26}$$

equation (23) becomes

$$\vec{u}_y\left(\frac{\partial}{\partial z} E_{iy} + \frac{\partial}{\partial z} E_{ry}\right) = \vec{u}_y e^{ik_0 x \sin\theta} F_y{}'(0) \tag{27}$$

or

$$ik_0 \cos\theta\left(E_{i0y} - E_{r0y}\right) = F_y{}'(0) \tag{28}$$

Similarly, the boundary conditions at $z = z_0$ lead to the equations

$$F_y(z_0) = E_{t0y} e^{ik_0 z_0 \cos\theta} \tag{29}$$

and

$$F_y{}'(z_0) = ik_0 \cos\theta E_{t0y} e^{ik_0 z_0 \cos\theta} \tag{30}$$

Thus, the problem is reduced to solving the differential equation (14):

$$\frac{d^2F_y(z)}{dz^2} + k_0{}^2\left[n^2(z) - \sin^2\theta\right] F_y(z) = 0$$

subject to the four boundary conditions (eqs. (25), (28), (29), and (30)). The wave equation and the boundary conditions are composed of complex quantities. It is therefore necessary to separate real and imaginary parts prior to machine programing. If the boundary conditions are divided by E_{i0y} and if the following definitions are expressed:

$$\frac{E_{r0y}}{E_{i0y}} = |R_1| e^{i\delta_{r1}} \tag{31}$$

$$\frac{E_{t0y}}{E_{i0y}} = |T_1| e^{i\delta_{t1}} \tag{32}$$

then the boundary conditions (eqs. (25), (28), (29), and (30)) reduce to

$$1 + |R_1| e^{i\delta_{r1}} = \frac{F_y(0)}{E_{i0y}} \tag{33}$$

$$ik_0 \cos\theta \left(1 - |R_1| e^{i\delta_{r1}}\right) = \frac{F_y'(0)}{E_{i0y}} \tag{34}$$

$$\frac{F_y(z_0)}{E_{i0y}} = |T_1| e^{i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)} \tag{35}$$

$$\frac{F_y'(z_0)}{E_{i0y}} = ik_0 \cos\theta |T_1| e^{i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)} \tag{36}$$

If the boundary conditions are divided by

$$|T_1| e^{i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)}$$

and if further notations are introduced, such that

$$I_1 = \frac{e^{-i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)}}{|T_1|} \tag{37}$$

and

$$\bar{R}_1 = \frac{|R_1|}{|T_1|} e^{-i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)} \tag{38}$$

the boundary conditions (eqs. (33) to (36)) now become

$$I_1 + \bar{R}_1 e^{i\delta r l} = \frac{F_y(0)}{E_{i0y}} I_1 \tag{39}$$

$$ik_0 \cos\theta \left(I_1 - \bar{R}_1 e^{i\delta r l}\right) = \frac{F_y'(0) I_1}{E_{i0y}} \tag{40}$$

$$\frac{F_y(z_0) I_1}{E_{i0y}} = 1 \tag{41}$$

$$\frac{F_y'(z_0) I_1}{E_{i0y}} = ik_0 \cos\theta \tag{42}$$

If the differential equation (14) is multiplied by I_1 / E_{i0y}, the normalization of the equations is complete. If the following definitions are expressed:

$$\frac{F_y(z) I_1}{E_{i0y}} = r(z) + is(z) \tag{43}$$

$$n^2(z) = V(z) + iW(z) \tag{44}$$

$$V^*(z) = V(z) - \sin^2\theta \tag{45}$$

the differential equation (14) becomes

$$\frac{d^2(r + is)}{dz^2} + k_0^2(V^* + iW)(r + is) = 0 \tag{46}$$

Equation (46) can be separated into real and imaginary parts, and they must independently vanish; therefore,

$$\left.\begin{aligned} \frac{d^2r}{dz^2} + k_0{}^2(V^*r - Ws) &= 0 \\ \frac{d^2s}{dz^2} + k_0{}^2(Wr + V^*s) &= 0 \end{aligned}\right\} \quad (47)$$

Also, $n^2(z)$ is interpreted as the square of the complex index of refraction with

$$V = 1 - \frac{1}{\left(\frac{\omega}{\omega_p}\right)^2 + \left(\frac{\nu}{\omega_p}\right)^2}$$

$$W = \frac{\nu/\omega_p}{\omega/\omega_p} \frac{1}{\left(\frac{\omega}{\omega_p}\right)^2 + \left(\frac{\nu}{\omega_p}\right)^2}$$

From the definition (43), the boundary conditions at z_0 (eqs. (41) and (42)) become

$$\left.\begin{aligned} r(z_0) &= 1 \\ s(z_0) &= 0 \end{aligned}\right\} \quad (48)$$

$$\left.\begin{aligned} r'(z_0) &= 0 \\ s'(z_0) &= k_0 \cos\theta \end{aligned}\right\} \quad (49)$$

The parameters I_1 and $\overline{R}_1$ can be solved from the boundary conditions at $z = 0$,

$$I_1 + \overline{R}_1 e^{i\delta_{r1}} = r(0) + is(0) \quad (50)$$

$$ik_0 \cos\theta\left(I_1 - \overline{R}_1 e^{i\delta_{r1}}\right) = r'(0) + is'(0) \quad (51)$$

The solutions to these quantities are:

$$I_1 = \frac{1}{2}\left[r(0) + \frac{s'(0)}{k_0 \cos\theta}\right] + \frac{1}{2}\, i\left[s(0) - \frac{r'(0)}{k_0 \cos\theta}\right]$$

$$\equiv \frac{1}{2}(A_1 + iB_1) \tag{52}$$

and

$$\bar{R}_1 = \left\{\frac{1}{2}\left[r(0) - \frac{s'(0)}{k_0 \cos\theta}\right] + \frac{1}{2}\, i\left[s(0) + \frac{r'(0)}{k_0 \cos\theta}\right]\right\} e^{-i\delta_{r1}}$$

$$\equiv \frac{1}{2}(C_1 + iD_1)e^{-i\delta_{r1}} \tag{53}$$

The values $r(0)$, $r'(0)$, $s(0)$, and $s'(0)$ are found from the solution of the wave equation (47).

From equation (37), the magnitude of I_1 is

$$|I_1| = \frac{1}{|T_1|} \tag{54}$$

Therefore,

$$|T_1| = \frac{2}{\sqrt{A_1^2 + B_1^2}} \tag{55}$$

And from equation (38), the magnitude of $\bar{R}_1$ is

$$\left|\bar{R}_1\right| = \frac{|R_1|}{|T_1|} \tag{56}$$

Therefore, the reflection coefficient is

$$|R_1| = \sqrt{\frac{C_1^2 + D_1^2}{A_1^2 + B_1^2}} \tag{57}$$

The quantity δ_{r1} is the phase of the reflected $\vec{E}$ vector and δ_{t1} is the phase of the transmitted $\vec{E}$ vector. The phase factors are computed by first making equation (53) equal to the definition of equation (38). Thus,

$$\frac{1}{2}\left(C_1 + iD_1\right)e^{-i\delta_{r1}} = \frac{|R_1|}{|T_1|}\, e^{-i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)} \tag{58}$$

But, from equation (56)

$$|\bar{R}_1| = \frac{|R_1|}{|T_1|} = \frac{1}{2}\sqrt{C_1^2 + D_1^2}$$

Therefore,

$$\frac{C_1 + iD_1}{\sqrt{C_1^2 + D_1^2}} = e^{-i\left(k_0 z_0 \cos\theta + \delta_{t1} - \delta_{r1}\right)} \tag{59}$$

and

$$\tan\left(k_0 z_0 \cos\theta + \delta_{t1} - \delta_{r1}\right) = -\frac{D_1}{C_1} \tag{60}$$

or

$$k_0 z_0 \cos\theta + \delta_{t1} - \delta_{r1} = \arc\tan\left(-\frac{D_1}{C_1}\right) \tag{61}$$

Similarly, if (52) and (37) are equated,

$$I_1 = \frac{1}{2}\left(A_1 + iB_1\right) = \frac{e^{-i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)}}{|T_1|} \tag{62}$$

But, from equations (54) and (55)

$$\frac{1}{|T_1|} = |I_1| = \frac{1}{2}\sqrt{A_1^2 + B_1^2}$$

Therefore,

$$\frac{A_1 + iB_1}{\sqrt{A_1^2 + B_1^2}} = e^{-i\left(k_0 z_0 \cos\theta + \delta_{t1}\right)} \tag{63}$$

and

$$k_0 z_0 \cos\theta + \delta_{t1} = \text{arc tan}\left(-\frac{B_1}{A_1}\right) \tag{64}$$

Equations (61) and (64) are equations with which to solve for the two unknowns δ_{r1} and δ_{t1}. The solutions for the phase factors are:

$$\delta_{t1} = \text{arc tan}\left(-\frac{B_1}{A_1}\right) - k_0 z_0 \cos\theta \tag{65}$$

$$\delta_{r1} = \text{arc tan}\left(-\frac{B_1}{A_1}\right) - \text{arc tan}\left(-\frac{D_1}{C_1}\right) \tag{66}$$

Case 2: E parallel to the plane of incidence.- As in case 1, the boundary conditions at the surfaces $z = 0$ and $z = z_0$ require continuity of the tangential components of $\vec{E}$ and $\vec{H}$.

At $z = 0$, equation (23) establishes the requirement that

$$\vec{u}_z \times \left(H_{i0y}\vec{u}_y + H_{r0y}\vec{u}_y\right) = \vec{u}_z \times G_y(0)\vec{u}_y \tag{67}$$

or

$$H_{i0y} + H_{r0y} = G_y(0) \tag{67a}$$

And, since

$$\vec{E} = -\frac{1}{i\omega\epsilon}\vec{\nabla} \times \vec{H} \tag{68}$$

equation (22) requires that

$$\vec{u}_z \times \left(-\frac{1}{i\omega\epsilon_0}\vec{\nabla} \times \vec{H}_i - \frac{1}{i\omega\epsilon_0}\vec{\nabla} \times \vec{H}_r\right) = \frac{-\vec{u}_z \times \vec{\nabla} \times G_y(0)\vec{u}_y}{i\omega\epsilon(0)} e^{ik_0 x \sin\theta} \tag{69}$$

or

$$ik_0 \cos\theta\left(H_{i0y} - H_{r0y}\right) = \frac{1}{n^2(0)} G_y'(0) \tag{69a}$$

Similarly, the boundary conditions at $z = z_0$ require that

$$G_y(z_0) = H_{t0y}e^{ik_0 z_0 \cos\theta} \tag{70}$$

and

$$G_y'(z_0) = n^2(z_0)ik_0 \cos\theta H_{t0y}e^{ik_0 z_0 \cos\theta} \tag{71}$$

As in case 1, the problem is reduced to solving equation (15):

$$\frac{d^2G_y(z)}{dz^2} - \frac{1}{n^2(z)}\frac{dn^2(z)}{dz}\frac{dG_y(z)}{dz} + k_0^2\left[n^2(z) - \sin^2\theta\right]G_y(z) = 0$$

subject to the boundary conditions (67a), (69a), (70), and (71). If the boundary conditions are normalized as in case 1, they reduce to

$$I_2 + \overline{R}_2 e^{i\delta_{r2}} = \frac{G_y(0)}{H_{i0y}} I_2 = t(0) + iu(0) \tag{72}$$

$$ik_0 n^2(0)\cos\theta\left(I_2 - \overline{R}_2 e^{i\delta_{r2}}\right) = \frac{G_y'(0)}{H_{i0y}} I_2 = t'(0) + iu'(0) \tag{73}$$

$$\frac{G_y(z_0)}{H_{i0y}} I_2 = t(z_0) + iu(z_0) = 1 \tag{74}$$

$$\begin{aligned}\frac{G_y{}'(z_0)}{H_{i0y}} I_2 &= t'(z_0) + iu'(z_0) \\ &= ik_0 n^2(z_0)\cos\theta \\ &= ik_0 V(z_0)\cos\theta - k_0 W(z_0)\cos\theta\end{aligned} \tag{75}$$

where

$$n^2(0) = V(0) + iW(0) \tag{76}$$

$$n^2(z_0) = V(z_0) + iW(z_0) \tag{77}$$

$$\frac{H_{r0y}}{H_{i0y}} = |R_2| e^{i\delta_{r2}} \tag{78}$$

$$\frac{H_{t0y}}{H_{i0y}} = |T_2| e^{i\delta_{t2}} \tag{79}$$

$$I_2 = \frac{e^{-i\left(k_0 z_0 \cos\theta + \delta_{t2}\right)}}{|T_2|} \tag{80}$$

$$\overline{R}_2 = \frac{|R_2|}{|T_2|} e^{-i\left(k_0 z_0 \cos\theta + \delta_{t2}\right)} \tag{81}$$

$$|I_2| = \frac{1}{|T_2|} \tag{82}$$

$$|\overline{R}_2| = \frac{|R_2|}{|T_2|} \tag{83}$$

$$\frac{G_y(z)}{H_{i0y}} I_2 = t(z) + iu(z) \tag{84}$$

If the differential equation (15) is multiplied by I_2/H_{i0y}, the following equation results:

$$\frac{d^2}{dz^2}\left[t(z) + iu(z)\right] - \frac{\left[V(z) - iW(z)\right]}{\left[V^2(z) + W^2(z)\right]}\left\{\left[\frac{dV(z)}{dz}\frac{dt(z)}{dz} - \frac{dW(z)}{dz}\frac{du(z)}{dz}\right] + i\left[\frac{dW(z)}{dz}\frac{dt(z)}{dz} + \frac{dV(z)}{dz}\frac{du(z)}{dz}\right]\right\} + k_0^2\left\{\left[V^*(z)t(z) - W(z)u(z)\right] + i\left[W(z)t(z) + V^*(z)u(z)\right]\right\} = 0 \tag{85}$$

As before, the real and imaginary parts must independently vanish, whereby two simultaneous differential equations result:

$$\frac{d^2t(z)}{dz^2} - \frac{V(z)}{\left[V^2(z) + W^2(z)\right]}\left[\frac{dV(z)}{dz}\frac{dt(z)}{dz} - \frac{dW(z)}{dz}\frac{du(z)}{dz}\right] - \frac{W(z)}{\left[V^2(z) + W^2(z)\right]}\left[\frac{dW(z)}{dz}\frac{dt(z)}{dz} + \frac{dV(z)}{dz}\frac{du(z)}{dz}\right] + k_0^2\left[V^*(z)t(z) - W(z)u(z)\right] = 0 \tag{86a}$$

$$\frac{d^2u(z)}{dz^2} - \frac{V(z)}{\left[V^2(z) + W^2(z)\right]}\left[\frac{dW(z)}{dz}\frac{dt(z)}{dz} + \frac{dV(z)}{dz}\frac{du(z)}{dz}\right] + \frac{W(z)}{\left[V^2(z) + W^2(z)\right]}\left[\frac{dV(z)}{dz}\frac{dt(z)}{dz} - \frac{dW(z)}{dz}\frac{du(z)}{dz}\right] + k_0^2\left[W(z)t(z) + V^*(z)u(z)\right] = 0 \tag{86b}$$

By separating real and imaginary parts in equations (74) and (75) the boundary conditions for t and u at z_0 are:

$$\left.\begin{aligned} t(z_0) &= 1 \\ u(z_0) &= 0 \end{aligned}\right\} \tag{87}$$

$$\left.\begin{aligned} t'(z_0) &= -k_0 W(z_0)\cos\theta \\ u'(z_0) &= k_0 V(z_0)\cos\theta \end{aligned}\right\} \tag{88}$$

The boundary conditions, and the solutions of u and t at $z = 0$, determine the reflection and transmission terms.

From equations (72) and (73)

$$\begin{aligned} I_2 &= \frac{1}{2}\left(\left\{t(0) + \frac{-W(0)t'(0) + V(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}\right. \\ &\quad \left. + i\left\{u(0) + \frac{-V(0)t'(0) - W(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}\right) \\ &= \frac{1}{2}\left(A_2 + iB_2\right) \end{aligned} \tag{89}$$

and

$$\begin{aligned} R_2 e^{i\delta_{r2}} &= \frac{1}{2}\left(\left\{t(0) + \frac{W(0)t'(0) - V(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}\right. \\ &\quad \left. + i\left\{u(0) + \frac{V(0)t'(0) + W(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}\right) \\ &= \frac{1}{2}\left(C_2 + iD_2\right) \end{aligned} \tag{90}$$

The transmission and reflection coefficients are:

$$|T_2| = \frac{1}{|I_2|} = \frac{2}{\sqrt{A_2^2 + B_2^2}} \tag{91}$$

$$|R_2| = \frac{|\bar{R}_2|}{|I_2|} = \frac{\sqrt{C_2^2 + D_2^2}}{\sqrt{A_2^2 + B_2^2}} \tag{92}$$

Equations (89) and (90) (with definitions (80) and (81)) are equivalent in form to the corresponding equations for E perpendicular to the plane of incidence; therefore, the phase terms of the parallel component will be

$$\delta_{t2} = \text{arc tan}\left(-\frac{B_2}{A_2}\right) - k_0 z_0 \cos\theta \tag{93}$$

$$\delta_{r2} = \text{arc tan}\left(-\frac{B_2}{A_2}\right) - \text{arc tan}\left(-\frac{D_2}{C_2}\right) \tag{94}$$

Arbitrary Orientation of Electric Vector With Respect to the Plane of Incidence

If the electric vector is oriented at an arbitrary angle with respect to the plane of incidence, it may be resolved into vector components parallel to and perpendicular to the plane of incidence. The purpose of this section is to see how the components recombine after the wave interacts with the plasma.

Case 1: Transmission coefficient for E at arbitrary orientation to plane of incidence.- From figure 1, let ϕ be the angle between the electric vector and the y-axis. If the amplitude of the electric vector is E_{00}, then

$$\left.\begin{aligned} E_{i0y} &= E_{00} \cos\phi \\ E_{i0x} &= -E_{00} \sin\phi \cos\theta \\ E_{i0z} &= E_{00} \sin\phi \sin\theta \end{aligned}\right\} \tag{95}$$

and since (eq. 26)

$$\vec{H} = \frac{1}{i\omega\mu_0} \vec{\nabla} \times \vec{E}$$

$$H_{i0y} = \frac{1}{i\omega\mu_0}\left(\frac{\partial E_{ix}}{\partial z} - \frac{\partial E_{iz}}{\partial x}\right)e^{-i(k_0 x\sin\theta + k_0 z\cos\theta)}$$

$$= -\sqrt{\frac{\epsilon_0}{\mu_0}}\, E_{00} \sin \phi \tag{96}$$

the transmitted waves are:

$$E_{ty} = |T_1| E_{00} \cos \phi e^{i\left(k_0 x\sin\theta + k_0 z\cos\theta + \delta_{t1}\right)}$$

$$= E_{t1} \tag{97}$$

$$H_{ty} = -\sqrt{\frac{\epsilon_0}{\mu_0}} |T_2| E_{00} \sin \phi e^{i\left(k_0 x\sin\theta + k_0 z\cos\theta + \delta_{t2}\right)} \tag{98}$$

But

$$E_{t2} = \sqrt{E_{tx}{}^2 + E_{tz}{}^2} \tag{99}$$

where E_{tx} and E_{tz} can be determined from the relationship (eq. (68))

$$\vec{E} = -\frac{1}{i\omega\epsilon_0} \vec{\nabla} \times \vec{H}$$

Performing this operation yields

$$\vec{E}_{t2} = \left(-\cos \theta\vec{u}_x + \sin \theta\vec{u}_z\right) | T_2 | E_{00} \sin \phi e^{i\left(k_0 x\sin\theta + k_0 z\cos\theta + \delta_{t2}\right)} \tag{100}$$

and, from equation (99)

$$E_{t2} = |T_2| E_{00} \sin \phi e^{i\left(k_0 x\sin\theta + k_0 z\cos\theta + \delta_{t2}\right)}$$

$$= -H_{ty}\sqrt{\frac{\mu_0}{\epsilon_0}} \tag{101}$$

The transmitted electric vector is the vector sum of its components, or

$$E_t = \sqrt{E_{t1}^2 + E_{t2}^2} \tag{102}$$

Substituting equation (97) and equation (101) into equation (102) and dividing by E_{00} gives

$$\begin{aligned}\frac{E_t}{E_{00}} &= \left(T_1^2\cos^2\phi e^{2i\delta_{t1}} + T_2^2\sin^2\phi e^{2i\delta_{t2}}\right)^{1/2} e^{i\left(k_0x\sin\theta+k_0z\cos\theta\right)} \\ &= \left[T_1^2\cos^2\phi + T_2^2\sin^2\phi e^{2i\left(\delta_{t2}-\delta_{t1}\right)}\right]^{1/2} e^{i\left(k_0x\sin\theta+k_0z\cos\theta+\delta_{t1}\right)}\end{aligned} \tag{103}$$

Let

$$\begin{aligned}&\left[T_1^2\cos^2\phi + T_2^2\sin^2\phi e^{2i\left(\delta_{t2}-\delta_{t1}\right)}\right]^{1/2} \\ &= \left\{\left[T_1^2\cos^2\phi + T_2^2\sin^2\phi \cos 2\left(\delta_{t2} - \delta_{t1}\right)\right] + i\left[T_2^2\sin^2\phi \sin 2\left(\delta_{t2} - \delta_{t1}\right)\right]\right\}^{1/2} \\ &= \gamma e^{i\psi}\end{aligned} \tag{104}$$

and define

$$A_3 = T_1^2\cos^2\phi + T_2^2\sin^2\phi \cos 2\left(\delta_{t2} - \delta_{t1}\right) \tag{105}$$

$$B_3 = T_2^2\sin^2\phi \sin 2\left(\delta_{t2} - \delta_{t1}\right) \tag{106}$$

Then

$$A_3 + iB_3 = \gamma^2 e^{2i\psi} \tag{107}$$

$$A_3 = \gamma^2\cos 2\psi \tag{108}$$

$$B_3 = \gamma^2\sin 2\psi \tag{109}$$

$$A_3^2 + B_3^2 = \gamma^4 \tag{110}$$

$$\psi = \frac{1}{2} \text{ arc } \tan\left(\frac{B_3}{A_3}\right) \tag{111}$$

Therefore,

$$\frac{E_t}{E_{00}} = \left(A_3^2 + B_3^2\right)^{1/4} e^{i\left[k_0 x \sin\theta + k_0 z \cos\theta + \delta_{t1} + \frac{1}{2} \arctan\left(\frac{B_3}{A_3}\right)\right]} \tag{112}$$

$$|T| = \left|\frac{E_t}{E_{00}}\right| = \left(A_3^2 + B_3^2\right)^{1/4} \tag{113}$$

It may be noted, in general, that a polarization shift is implied by equation (112).

Case 2: Reflection coefficient for E at arbitrary orientation to plane of incidence.- As in case 1 (eqs. (95) and (96))

$$E_{i0y} = E_{00} \cos \phi$$

$$H_{i0y} = -\sqrt{\frac{\epsilon_0}{\mu_0}} E_{00} \sin \phi$$

But from equations (18), (19), (31), (78), (95), and (96), the reflected components are:

$$E_{ry} = |R_1| E_{00} \cos \phi e^{i\left(k_0 x \sin\theta - k_0 z \cos\theta + \delta_{r1}\right)} \tag{114}$$

$$H_{ry} = -\sqrt{\frac{\epsilon_0}{\mu_0}} |R_2| E_{00} \sin \phi e^{i\left(k_0 x \sin\theta - k_0 z \cos\theta + \delta_{r2}\right)} \tag{115}$$

From the relationship (eq. (68)),

$$\vec{E} = -\frac{1}{i\omega\epsilon_0} \vec{\nabla} \times \vec{H}$$

equation (115) determines the parallel component of the electric vector. When equation (68) is used,

$$\vec{E}_{r2} = \left(\cos\,\theta\vec{u}_x + \sin\,\theta\vec{u}_z\right)|R_2|E_{00}\,\sin\,\phi e^{i\left(k_0 x\sin\theta - k_0 z\cos\theta + \delta_{r2}\right)} \tag{116}$$

and

$$E_{r2} = \sqrt{E_{rx}{}^2 + E_{rz}{}^2} = |R_2|E_{00}\,\sin\,\phi e^{i\left(k_0 x\sin\theta - k_0 z\cos\theta + \delta_{r2}\right)} \tag{117}$$

Combining equations (114) and (117), with the definition

$$E_r = \sqrt{E_{r1}{}^2 + E_{r2}{}^2} \tag{118}$$

and dividing by E_{00} yields the reflected component

$$\begin{aligned} \frac{E_r}{E_{00}} &= \left(R_1{}^2\cos^2\phi e^{2i\delta_{r1}} + R_2{}^2\sin^2\phi e^{2i\delta_{r2}}\right)^{1/2} e^{i\left(k_0 x\sin\theta - k_0 z\cos\theta\right)} \\ &= \left[R_1{}^2\cos^2\phi + R_2{}^2\sin^2\phi e^{2i\left(\delta_{r2} - \delta_{r1}\right)}\right]^{1/2} e^{i\left(k_0 x\sin\theta - k_0 z\cos\theta + \delta_{r1}\right)} \end{aligned} \tag{119}$$

The form of equation (119) is similar to that of equation (103). Therefore,

$$\frac{E_r}{E_{00}} = \left(A_4{}^2 + B_4{}^2\right)^{1/4} e^{i\left[k_0 x\sin\theta - k_0 z\cos\theta + \delta_{r1} + \frac{1}{2}\arctan\left(\frac{B_4}{A_4}\right)\right]} \tag{120}$$

$$|R| = \left|\frac{E_r}{E_{00}}\right| = \left(A_4{}^2 + B_4{}^2\right)^{1/4} \tag{121}$$

where

$$A_4 = R_1{}^2\cos^2\phi + R_2{}^2\sin^2\phi\,\cos\,2\left(\delta_{r2} - \delta_{r1}\right) \tag{122}$$

$$B_4 = R_2{}^2\sin^2\phi\,\sin\,2\left(\delta_{r2} - \delta_{r1}\right) \tag{123}$$

Arbitrary Polarization of Incident Wave

The electric vector of an elliptically polarized wave can be resolved into two orthogonal linearly polarized components. Suppose the incident wave of figure 1 is elliptically polarized so that

$$\vec{E}_{iy} = E_{i0y}\vec{u}_y e^{i\vec{k}_0\cdot\vec{r}+i\xi}$$

where ξ is the phase difference between the perpendicular and parallel components. If E_{i0y} is replaced by $E_{i0y}e^{i\xi}$, the definitions of the reflection and transmission coefficients are altered so that equations (31) and (32) become:

$$\frac{E_{r0y}}{E_{i0y}} = |R_1|e^{i\left(\delta_{r1}+\xi\right)} \qquad (124)$$

$$\frac{E_{t0y}}{E_{i0y}} = |T_1|e^{i\left(\delta_{t1}+\xi\right)} \qquad (125)$$

The boundary conditions (33) to (36) are also modified so that they become:

$$I_1 + \overline{R}_1 e^{i\delta_{r1}} = \frac{F_y(0)}{E_{i0y}} I_1 e^{-i\xi} \qquad (126)$$

$$ik_0 \cos\theta\left(I_1 - \overline{R}_1 e^{i\delta_{r1}}\right) = \frac{F_y'(0)}{E_{i0y}} I_1 e^{-i\xi} \qquad (127)$$

$$\frac{F_y(z_0)}{E_{i0y}} I_1 e^{-i\xi} = 1 \qquad (128)$$

$$\frac{F_y'(z_0)}{E_{i0y}} I_1 e^{-i\xi} = ik_0 \cos\theta \qquad (129)$$

However, the additional factor ξ can be absorbed by redefining r and s (eq. (43))

$$r(z) + is(z) = \frac{F_y(z)}{E_{i0y}} I_1 e^{-i\xi} \tag{130}$$

If the wave equation is multiplied by $\frac{I_1}{E_{i0y}} e^{-i\xi}$, then the differential equations which involve $r(z)$, $s(z)$, and the boundary conditions remain unaltered. Thus, the appropriate parameters for an elliptically polarized wave are obtained by adding the phase factor ξ to the solutions for δ_{r1} and δ_{t1}.

RESULTS

Test Cases for Machine Program

The numerical integration of the propagation equations was performed on an IBM 7090 electronic data processing system by the Runge-Kutta method with an accuracy of 10^{-7} per integration.

As a first check of the computer program, analytical and machine calculations were compared for the homogeneous collision-free plasma $\left(\nu/\omega_p = 0\right)$. The value of $\frac{\omega}{\omega_p} = \frac{3}{\sqrt{5}}$ (hence an index of refraction of 1/1.5) was chosen because Rossi (ref. 9, pp. 137 and 377) has computed convenient parameters for $n = 1/1.5$, which aided hand calculations. The plasma slab was arbitrarily chosen to be 6 free-space wavelengths deep. Figure 2 shows that the agreement was good.

Machine computations were also carried out for cases with collisions in order to observe the behavior of the transmission coefficient when losses are introduced. The results of these calculations are in figure 2. For the lossy plasmas, hand-calculated transmission coefficients at $\theta = 0$ (normal incidence) were in excellent agreement with the machine results. The interference effects, which appear so strongly for $\nu/\omega_p = 0$ are caused by multiple internal reflections. As the plasma losses become more predominant the waves which undergo multiple reflections are damped out, as figure 2 shows. Figure 2 also shows that the calculated transmission coefficient drops sharply as θ exceeds the critical angle, which is defined as the angle of incidence for which the refraction angle is 90°. For angles of incidence greater than the critical angle ($\theta \approx 42°$) the transmitted wave is exponentially damped (ref. 8, p. 498).

Figure 3 is a plot of phase difference between the reflected y-components of $\vec{E}$ and $\vec{H}$ as a function of the angle of incidence. Note that figure 3 represents

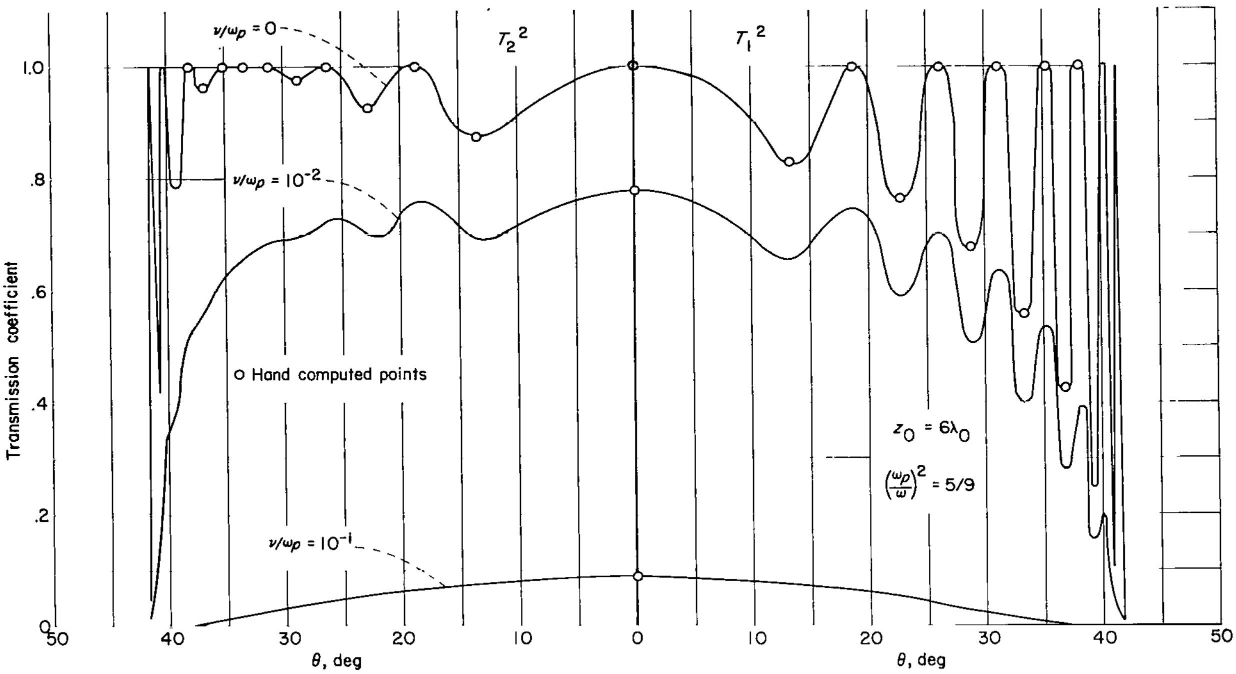

Figure 2.- Transmission coefficients for a homogeneous plasma slab.

the phase difference between $E_1\left(=E_y\right)$ and $E_2\left(=\sqrt{\frac{\mu_0}{\epsilon_0}}\,H_y e^{i\pi}\right)$ provided 180° is added to every point on the ordinate. An interesting feature of this plot is the abrupt change in phase difference at $\theta \approx 34^{\circ}$, which is the Brewster, or polarizing, angle for this medium (ref. 8, p. 497).

Table I lists all of the computed results obtained for the homogeneous plasma with $\frac{\omega}{\omega_p} = \frac{3}{\sqrt{5}}$.

Plots of reflection and transmission coefficients at normal incidence for a plasma slab with constant collision frequency and a trapezoidal electron density distribution like that shown in figure 4 were given in reference 2. A comparison of the machine-computed coefficients with those of reference 2 is shown in figure 5 as a function of the trapezoid base width. The agreement is excellent, but the comparison is unfortunately limited to normal incidence.

TABLE I.- COMPUTER RESULTS FOR THE HOMOGENEOUS PLASMA SLAB

θ, deg	T_1	T_2	R_1	R_2	δ_{r1}, radians	δ_{r2}, radians	δ_{t1}, radians	δ_{t2}, radians	$\delta_{r2} - \delta_{r1}$, radians	$\delta_{t2} - \delta_{t1}$, radians
(a) For $\frac{\nu}{\omega_p} = 0$; $z_0 = 6\lambda_0 = 18$ cm; $\frac{\omega}{\omega_p} = \frac{3}{\sqrt{5}}$										
0	1.0000	------	0	------	1.558	-------	-37.712	-------	-------	-------
13.415	.9094	0.9356	.4160	0.3531	-.0119	-3.154	-38.253	-38.254	-3.142	-0.0003
18.823	1.000	1.0000	.0059	.0039	1.558	4.700	-32.554	-32.554	3.142	.0008
22.867	.8735	.9621	.4868	.2729	-.0119	3.128	-33.177	-33.179	3.140	-.0012
26.161	1.0000	1.0000	.0087	.0032	1.554	-1.585	-33.854	-33.852	-3.139	.0021
28.954	.8216	.9867	.5700	.1628	-.0140	-3.158	-34.572	-34.575	-3.144	-.0028
31.359	.9999	1.0000	.0145	.0017	1.547	4.694	-29.074	-29.069	3.147	-.0049
33.449	.7471	.9999	.6647	.0106	-.0160	3.120	-29.900	-29.906	3.136	-.0054
35.261	.9998	1.0000	.0216	.0018	1.541	1.550	-30.813	-30.804	.0090	.0090
36.834	.6422	.9803	.7665	.1975	-.0188	-.0287	-31.763	-31.773	-.0099	-.0099
38.170	.9991	.9999	.0433	.0105	1.518	1.538	-26.550	-26.529	.0207	.0207
39.287	.5005	.8851	.8658	.4654	-.0153	-.0270	-27.623	-27.635	-.0117	-.0117
40.203	.9943	.9993	.1068	.0381	1.453	1.509	-28.911	-28.855	.0556	.0556
40.905	.3187	.6487	.9479	.7610	-.0185	-.0376	-30.082	-30.101	-.0191	-.0191
41.407	.9277	.9858	.3733	.1677	1.179	1.381	-25.526	-25.324	.2022	.2022
41.713	.0974	.2173	.9952	.9761	-.0225	-.0503	-26.594	-26.621	-.0277	-.0277
(b) For $\frac{\nu}{\omega_p} = 10^{-2}$; $z_0 = 6\lambda_0 = 18$ cm; $\frac{\omega}{\omega_p} = \frac{3}{\sqrt{5}}$										
0	0.7773	------	0.0019	------	0.0728	-------	-37.711	-------	-------	-------
13.415	.8106	0.8315	.3737	0.3163	-.0203	-3.162	-38.250	-38.251	-3.141	-0.0010
18.823	.8624	.8690	.0586	.0390	.0529	3.198	-32.552	-32.552	3.145	.0003
22.867	.7683	.8367	.4326	.2398	-.0207	3.124	-33.173	-33.177	3.144	-.0033
26.161	.8354	.8523	.0815	.0306	.0500	-3.080	-33.850	-33.850	-3.130	.0007
28.954	.7112	.8335	.5005	.1396	-.0229	-3.151	-34.566	-34.573	-3.128	-.0068
31.359	.7952	.8283	.1179	.0147	.0495	3.240	-29.068	-29.067	3.191	.0014
33.449	.6340	.8117	.5761	.0101	-.0251	3.592	-29.891	-29.903	3.617	-.0117
35.261	.7316	.7902	.1777	.0161	.0301	-.0522	-30.801	-30.800	-.0824	.0011
36.834	.5308	.7527	.6561	.1571	-.0282	-.0918	-31.747	-31.766	-.0636	-.0188
38.170	.6240	.7214	.2823	.0790	.0136	-.0200	-26.518	-26.519	-.0337	-.0005
39.287	.3968	.6277	.7338	.3528	-.0272	-.0805	-27.586	-27.612	-.0533	-.0256
40.203	.4371	.5779	.4677	.2196	-.0229	-.0604	-28.796	-28.809	-.0375	-.0128
40.905	.2287	.4017	.8042	.5571	-.0358	-.0985	-29.966	-30.006	-.0627	-.0399
41.407	.1679	.2823	.7469	.5307	-.0891	-.1836	-31.143	-31.215	-.0945	-.0727
41.713	.0682	.1305	.8704	.7323	-.0898	-.2093	-25.628	-25.726	-.1195	-.0983
(c) For $\frac{\nu}{\omega_p} = 10^{-1}$; $z_0 = 6\lambda_0 = 18$ cm; $\frac{\omega}{\omega_p} = \frac{3}{\sqrt{5}}$										
0	0.0914	------	0.0326	------	-0.1332	-------	-37.589	-------	-------	-------
13.415	.2758	0.2800	.2332	0.1957	-.0982	-3.231	-38.118	-38.122	-3.132	-0.0041
18.823	.2526	.2596	.2205	.1508	-.1393	-3.258	-38.683	-38.692	-3.118	-.0088
22.867	.2243	.2357	.2708	.1466	-.1107	3.076	-33.005	-33.020	3.187	-.0145
26.161	.1986	.2116	.2706	.1086	-.1479	3.078	-33.650	-33.672	3.226	-.0214
28.954	.1693	.1855	.3182	.0874	-.1287	-3.107	-34.339	-34.369	-2.978	-.0294
31.359	.1420	.1586	.3330	.0521	-.1617	-2.937	-35.068	-35.108	-2.775	-.0401
33.449	.1135	.1310	.3784	.0288	-.1546	4.160	-29.557	-29.609	4.315	-.0517
35.261	.0875	.1036	.4088	.0482	-.1845	-1.008	-30.359	-30.428	-.8236	-.0686
36.834	.0630	.0774	.4542	.0964	-.1930	-.7193	-31.190	-31.278	-.5263	-.0872
38.170	.0426	.0540	.4972	.1526	-.2232	-.6653	-32.013	-32.127	-.4420	-.1133
39.287	.0266	.0350	.5430	.2149	-.2494	5.615	-26.512	-26.655	5.865	-.1430
40.203	.0157	.0212	.5880	.2793	-.2850	-.7094	-27.202	-27.381	-.4244	-.1794
40.905	.0092	.0128	.6261	.3370	-.3212	-.7643	-27.733	-27.949	-.4431	-.2162
41.407	.0059	.0082	.6548	.3823	-.3533	-.8176	-28.090	-28.338	-.4643	-.2482
41.713	.0043	.0061	.6724	.4108	-.3756	-.8557	-28.289	-28.559	-.4801	-.2701

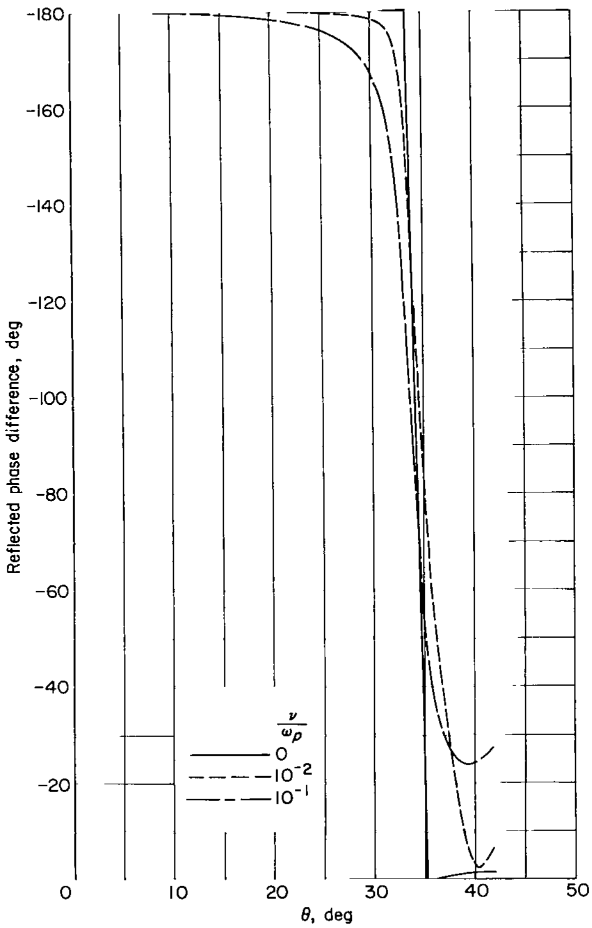

Figure 3.- Phase difference of reflected components (E_y and H_y) for the homogeneous plasma slab. $z_0 = 6\lambda_0$; $(\omega_p/\omega)^2 = 5/9$.

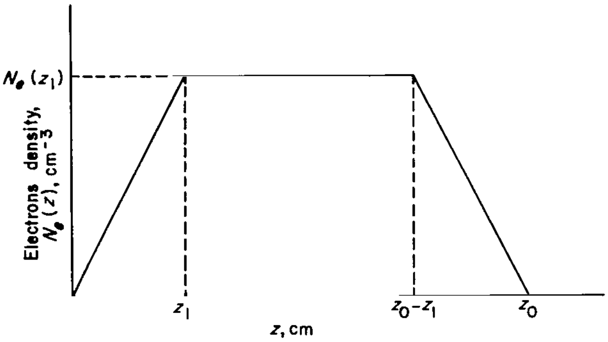

Figure 4.- Trapézoidal distribution of electron density (from ref. 2).

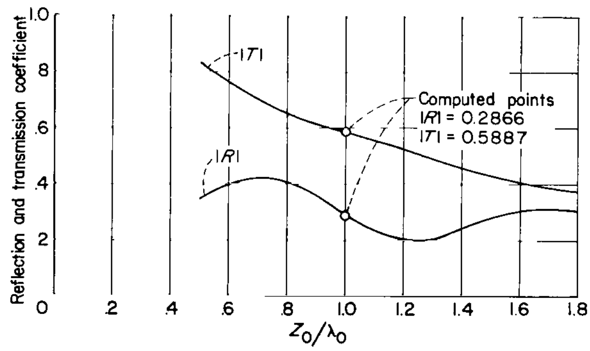

Figure 5.- Reflection and transmission coefficients as a function of trapezoid base width (from ref. 2). $f = 10^9$ cps; $\nu = 8.241 \times 10^8$ sec^{-1}; $N_e(z_1) = 9.5273 \times 10^9$ cm^{-3}; $z_1 = 0.25\lambda_0$.

Practical Application

The plasma surrounding a reentering body provides an interesting application of the program. The dimensions of many shock layers of current interest are comparable to or smaller than telemetry wavelengths; hence, a W.K.B. solution of the wave equation is not valid.

Typical distributions (ref. 10) of electron density and collision frequency are given in figure 6 along a line perpendicular to the body. The N_e curve which continually increases from the shock to the body corresponds to a completely inviscid shock layer. The N_e curve which has a maximum between the shock and the body is a flow profile which includes a viscous boundary layer.

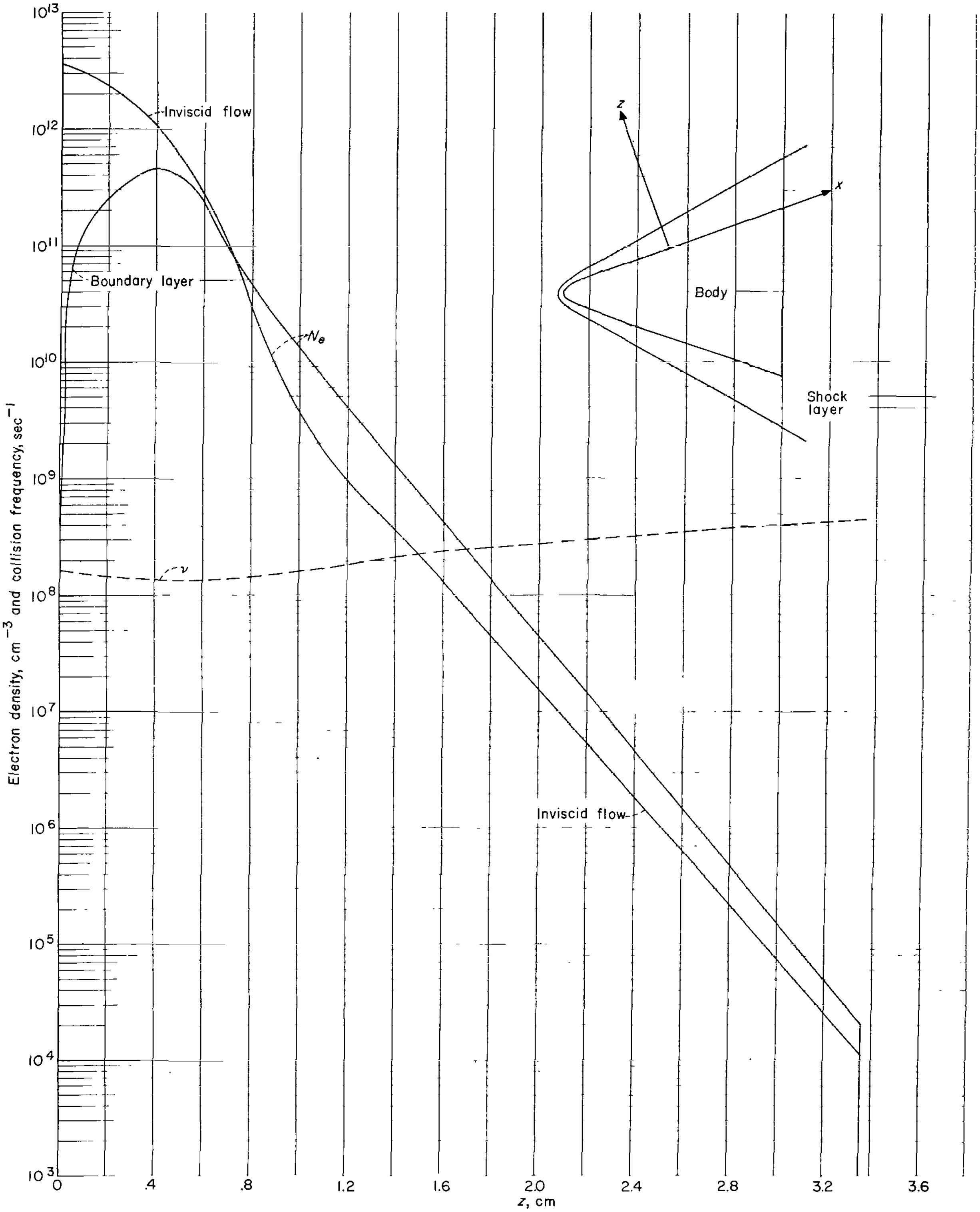

Figure 6.- Electron-density and collision-frequency distribution in a shock layer.

The $y = 0$ plane defines the H-plane (E_1) and the $x = 0$ plane corresponds to the E-plane (E_2) of a slot antenna whose major dimension is along the x-axis.

The plane-wave transmission coefficients are plotted in figures 7 and 8 for the two flow-field approximations. The shapes of the two curves are similar. The transmitted perpendicular component of $\vec{E}$ is maximum at normal incidence and transmission gradually decreases with increasing angle of incidence. On the other hand, the transmitted parallel components of $\vec{E}$ gradually increase with increasing angle of incidence. At near grazing incidence the parallel transmission coefficient drops rapidly to zero. It must be emphasized that figures 7 and 8 are the result of calculations based on an idealized plasma model; that is, transmission and reflection coefficients have been calculated for plane monochromatic waves incident at various angles on a plane parallel plasma slab in which the plasma properties vary only in the direction normal to the bounding planes. In lieu of a better method, these calculated transmission coefficients can be applied to the free-space radiation pattern of an antenna. Whether the resultant radiation pattern is a good approximation of the actual pattern depends on many factors. Consequently, some caution should be exercised in using this approach.

The computed results for several angles of incidence for the typical plasma layers with and without a boundary layer are given in table II.

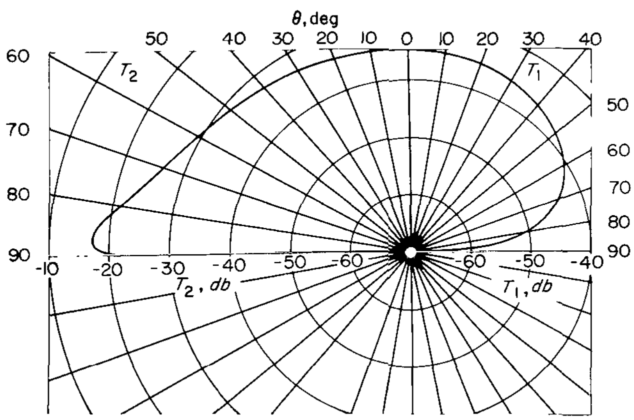

Figure 7.- Transmission coefficients in decibels as function of angle of incidence for the inviscid flow field.

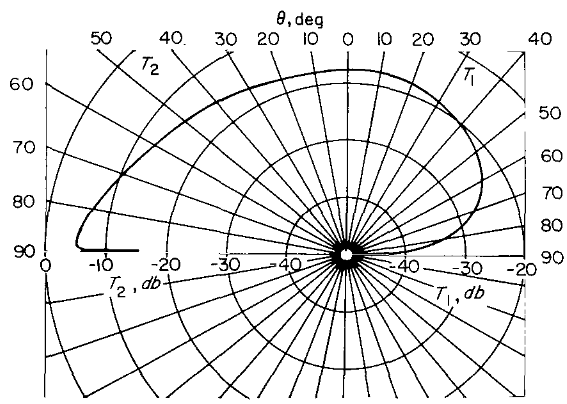

Figure 8.- Transmission coefficients in decibels as function of angle of incidence for the flow field which includes a boundary layer.

TABLE II.- COMPUTER RESULTS FOR THE INHOMOGENEOUS PLASMA SLAB

θ, deg	T_1, db	T_2, db	$1 - R_1^2$, db	$1 - R_2^2$, db	δ_{r1}, radians	δ_{r2}, radians	δ_{t1}, radians	δ_{t2}, radians	$\delta_{r2} - \delta_{r1}$, radians	$\delta_{t2} - \delta_{t1}$, radians
(a) Invicid flow results; $f = 244.3 \times 10^6$ cps; $z_0 = 3.36$ cm										
0	-34.8	-----	-22.1	-----	-3.106	------	-1.409	-------	-----	-------
10	-35.0	-34.9	-22.1	-22.0	-3.106	0.0367	-1.410	-1.417	3.143	-0.0072
20	-35.4	-34.5	-22.3	-21.8	-3.108	.0384	-1.410	-1.411	3.146	-.0004
30	-36.1	-33.8	-22.7	-21.4	-3.110	.0417	-1.411	-1.400	3.152	.0114
45	-37.8	-32.2	-23.6	-20.5	-3.116	.0510	-1.413	-1.372	3.167	.0411
60	-40.8	-29.4	-25.2	-18.9	-3.124	.0720	-1.416	-1.320	3.196	.0962
70	-44.1	-26.5	-26.9	-17.1	-3.129	.1051	-1.418	-1.250	3.234	.1682
85	-56.0	-17.6	-32.9	-10.8	-3.138	.4001	-1.422	-.7979	3.539	.6239
89.5	-76.0	-18.9	-42.9	-7.9	-3.141	2.199	-1.423	.5358	5.340	1.959
(b) Boundary-layer results; $f = 244.3 \times 10^6$ cps; $z_0 = 3.36$ cm										
0	-18.1	-----	-13.7	-----	-2.977	------	-1.338	-------	-----	-------
10	-18.2	-18.0	-13.8	-13.8	-2.979	0.1592	-1.340	-1.342	3.138	-0.0026
20	-18.6	-17.6	-14.1	-13.6	-2.986	.1666	-1.345	-1.330	3.153	.0152
30	-19.3	-17.0	-14.5	-13.1	-2.998	.1804	-1.354	-1.308	3.179	.0411
45	-21.1	-15.4	-15.6	-12.0	-3.024	.2196	-1.373	-1.249	3.244	.1239
60	-24.0	-12.9	-17.5	-10.1	-3.058	.3052	-1.399	-1.133	3.364	.2652
70	-27.3	-10.3	-19.4	-8.0	-3.085	.4323	-1.418	-.9766	3.517	.4413
85	-39.1	-5.4	-25.9	-3.5	-3.127	1.249	-1.450	-.1782	4.376	1.271
89.5	-59.1	-15.6	-36.1	-8.7	-3.140	2.807	-1.459	.9538	5.947	2.413

SUMMARY EQUATIONS FOR MACHINE CALCULATIONS

Accurate solutions of the wave propagation equations for a one-dimensional plane-parallel plasma slab can be obtained by numerical integration of the following differential equations:

E_1 case:

$$\frac{d^2r}{dz^2} + k_0^2(V^*r - Ws) = 0$$

$$\frac{d^2s}{dz^2} + k_0^2(Wr + V^*s) = 0$$

E_2 case:

$$\frac{d^2t}{dz^2} - \frac{V}{V^2 + W^2}\left(\frac{dV}{dz}\frac{dt}{dz} - \frac{dW}{dz}\frac{du}{dz}\right) - \frac{W}{V^2 + W^2}\left(\frac{dW}{dz}\frac{dt}{dz} + \frac{dV}{dz}\frac{du}{dz}\right) + k_0^2\left(V^*t - Wu\right) = 0$$

$$\frac{d^2u}{dz^2} - \frac{V}{V^2 + W^2}\left(\frac{dW}{dz}\frac{dt}{dz} + \frac{dV}{dz}\frac{du}{dz}\right) + \frac{W}{V^2 + W^2}\left(\frac{dV}{dz}\frac{dt}{dz} - \frac{dW}{dz}\frac{du}{dz}\right) + k_0^2\left(V^*u + Wt\right) = 0$$

$$V^* = V - \sin^2\theta$$

$$V = 1 - \frac{1}{\left(\frac{\omega}{\omega_p}\right)^2 + \left(\frac{\nu}{\omega_p}\right)^2}$$

$$W = \frac{\nu/\omega_p}{\omega/\omega_p}\,\frac{1}{\left(\frac{\omega}{\omega_p}\right)^2 + \left(\frac{\nu}{\omega_p}\right)^2}$$

At normal incidence only the first set of equations need be solved. The starting point for the solution to the equations is z_0, and the following boundary conditions must be met:

$$r(z_0) = 1$$

$$s(z_0) = 0$$

$$r'(z_0) = 0$$

$$s'(z_0) = k_0 \cos\theta$$

$$t(z_0) = 1$$

$$u(z_0) = 0$$

$$t'(z_0) = -k_0 W(z_0) \cos\theta$$

$$u'(z_0) = k_0 V(z_0) \cos\theta$$

The boundary conditions at the surface $z = 0$ determine reflection coefficients, transmission coefficients, and phase factors such that:

$$|T_1| = \frac{2}{\sqrt{A_1^2 + B_1^2}}$$

$$|T_2| = \frac{2}{\sqrt{A_2^2 + B_2^2}}$$

$$|R_1| = \sqrt{\frac{C_1^2 + D_1^2}{A_1^2 + B_1^2}}$$

$$|R_2| = \sqrt{\frac{C_2^2 + D_2^2}{A_2^2 + B_2^2}}$$

$$\delta_{r1} = \text{Reflected phase} = \text{arc tan}\left[\frac{(-B_1)}{A_1}\right] - \text{arc tan}\left[\frac{(-D_1)}{C_1}\right]$$

$$\delta_{t1} = \text{Transmitted phase} = \text{arc tan}\left[\frac{(-B_1)}{A_1}\right] - k_0 z_0 \cos\theta$$

$$\delta_{r2} = \text{Reflected phase} = \text{arc tan}\left[\frac{(-B_2)}{A_2}\right] - \text{arc tan}\left[\frac{(-D_2)}{C_2}\right]$$

$$\delta_{t2} = \text{Transmitted phase} = \text{arc tan}\left[\frac{(-B_2)}{A_2}\right] - k_0 z_0 \cos\theta$$

$$A_1 = r(0) + \frac{s'(0)}{k_0 \cos\theta}$$

$$B_1 = s(0) - \frac{r'(0)}{k_0 \cos\theta}$$

$$C_1 = r(0) - \frac{s'(0)}{k_0 \cos\theta}$$

$$D_1 = s(0) + \frac{r'(0)}{k_0 \cos\theta}$$

$$A_2 = t(0) + \left\{\frac{-W(0)t'(0) + V(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}$$

$$B_2 = u(0) - \left\{\frac{V(0)t'(0) + W(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}$$

$$C_2 = t(0) + \left\{\frac{W(0)t'(0) - V(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}$$

$$D_2 = u(0) + \left\{\frac{V(0)t'(0) + W(0)u'(0)}{k_0 \cos\theta\left[W^2(0) + V^2(0)\right]}\right\}$$

For arbitrary orientation of the electric vector with respect to the plane of incidence,

$$|T| = \left(A_3{}^2 + B_3{}^2\right)^{1/4}$$

$$|R| = \left(A_4{}^2 + B_4{}^2\right)^{1/4}$$

$$A_3 = T_1{}^2\cos^2\phi + T_2{}^2\sin^2\phi \cos 2\left(\delta_{t2} - \delta_{t1}\right)$$

$$B_3 = T_2{}^2\sin^2\phi \sin 2\left(\delta_{t2} - \delta_{t1}\right)$$

$$A_4 = R_1{}^2\cos^2\phi + R_2{}^2\sin^2\phi \cos 2\left(\delta_{r2} - \delta_{r1}\right)$$

$$B_4 = R_2{}^2\sin^2\phi \sin 2\left(\delta_{r2} - \delta_{r1}\right)$$

If the incident wave is elliptically polarized, the polarization phase factor ξ is added to δ_{r1} and δ_{t1}. The solutions to the wave equation remain unchanged.

CONCLUSIONS

The equations and the associated boundary conditions which describe the interaction of electromagnetic waves with a one-dimensional inhomogeneous plasma slab were developed in a form suitable for machine programing. The program solves for the reflection coefficients, transmission coefficients, and phase factors for the components of the electric vector perpendicular to and parallel to the plane of incidence. From these results, a wave for which the electric vector is of arbitrary orientation to the plane of incidence and of arbitrary polarization can be completely defined.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., April 23, 1963.

APPENDIX

IMPLICATIONS OF BOUNDARY CONDITIONS

The purpose of this appendix is to show:

(1) That the behavior of the wave in the plasma as a function of x is of the form $e^{ik_0 x \sin\theta}$

(2) That the exit angle of the wave is equal to the angle of incidence

(3) That the solutions of the wave equations are independent of the y-coordinate if the incident waves are independent of y.

The general form of equation (6) implies solutions which are functionally dependent upon all the coordinates x, y, and z; that is,

$$E_y = E_y(x,y,z) \tag{A1}$$

If by arbitrary choice of coordinates, the incident and reflected waves in free space are independent of y, then the solution of the wave equation is

$$E_y(x,0,z) = e^{ik_{0x}x}\left(Me^{ik_{0z}z} + Ne^{-ik_{0z}z}\right) \tag{A2}$$

The $e^{-ik_{0x}x}$ term is dropped, so that the waves are required to propagate in the positive x-direction. The coefficients M and N are interpreted as incident and reflected wave amplitudes, of such character that:

$$\begin{aligned} E_{iy}(x,0,z) &= Me^{ik_{0x}x+ik_{0z}z} \\ &= E_{i0y}e^{ik_0 x\sin\theta+ik_0 z\cos\theta} \end{aligned} \tag{A3}$$

$$\begin{aligned} E_{ry}(x,0,z) &= Ne^{ik_{0x}x-ik_{0z}z} \\ &= E_{r0y}e^{ik_0 x\sin\theta-ik_0 z\cos\theta} \end{aligned} \tag{A4}$$

where, from figure 1, $k_{0x} = k_0 \sin\theta$, $k_{0z} = k_0 \cos\theta$, and θ is the angle of incidence.

The boundary conditions require that the tangential components of $\vec{E}$ be continuous across the surface $z = 0$. The boundary condition for the perpendicular component is:

$$\vec{u}_z \times E_y\vec{u}_y \Big|_{fs} = \vec{u}_z \times E_y\vec{u}_y \Big|_{p} \tag{A5}$$

After the vector operations are performed, the boundary condition becomes

$$E_{yfs} = E_{yp} \tag{A6}$$

The value of E_{yfs} is the sum of equations (A3) and (A4). Therefore, at $z = 0$,

$$E_{i0y}e^{ik_0x\sin\theta} + E_{r0y}e^{ik_0x\sin\theta} = E_y(x,y,0) \tag{A7}$$

At $x = y = 0$, equation (A7) becomes

$$E_{i0y} + E_{r0y} = E_y(0,0,0) \tag{A7a}$$

The plasma properties at a point on the surface $z = 0$ are indistinguishable from those at any other point. Hence, equation (A7a) must be satisfied at any point on the surface $z = 0$. At another point $(0,y_2)$, equation (A7) is

$$E_{i0y} + E_{r0y} = E_y(0,y_2,0) \tag{A7b}$$

Equations (A7a) and (A7b) are identical. Therefore, the electric vector in the plasma is independent of the coordinate y, and wave propagation must occur in the xz-plane.

At another point $(x_2,0)$ equation (A7) becomes

$$\left(E_{i0y} + E_{r0y}\right)e^{ik_0x_2\sin\theta} = E_y\left(x_2,0,0\right) \tag{A7c}$$

The presence of the sinusoidal factor $e^{ik_0 x_2 \sin\theta}$ implies that the energy transmitted into the plasma varies as a function of x. This implication is physically unreasonable for incident plane waves, and the right-hand side of equation (A7c) must be

$$E_y\left(x_2,0,0\right) = E_y(0,0)e^{ik_0 x_2 \sin\theta} \tag{A8}$$

Thus, the right-hand side of equation (A7) has the form

$$\begin{aligned} E_y &= E_y(z)e^{ik_0 x \sin\theta} \\ &\equiv F_y(z)e^{ik_0 x \sin\theta} \end{aligned} \tag{A9}$$

The solution to the wave equation in free space for $z \geqq z_0$ is

$$E_{ty} = E_{t0y}e^{ik_{t0x}x+ik_{t0z}z} \tag{A10}$$

An $e^{-ik_{t0z}z}$ term does not appear because the wave is outgoing. If θ_t is the angle between the z-axis and the direction of propagation, then

$$E_{ty} = E_{t0y}e^{ik_0 x \sin\theta_t + ik_0 z \cos\theta_t} \tag{A11}$$

One of the necessary boundary conditions requires that equations (A9) and (A11) must be equal at $z = z_0$. Therefore,

$$F_y\left(z_0\right)e^{ik_0 x \sin\theta} = E_{t0y}e^{ik_0 x \sin\theta_t + ik_0 z_0 \cos\theta_t} \tag{A12}$$

At $x = 0$, equation (A12) becomes

$$F_y\left(z_0\right) = E_{t0y}e^{ik_0 z_0 \cos\theta_t} \tag{A12a}$$

Since the plasma properties are indistinguishable over the surface, equation (A12a) must be identically satisfied for every point (x,y) at $z = z_0$.

Therefore, equation (A12a) completely specifies $F_y(z_0)$. Substituting equation (A12a) into equation (A12) leads to the identity

$$e^{ik_0 x \sin\theta} \equiv e^{ik_0 x \sin\theta_t} \tag{A13}$$

Hence

$$\theta = \theta_t$$

The continuity requirements for the component H_y, lead to the following equation:

$$\left(H_{i0y} + H_{r0y}\right) e^{ik_0 x \sin\theta} = H_y(x,y,0) \tag{A14}$$

This equation is of the same form as equation (A7), and arguments similar to those applied in the analysis of E_y show that

$$H_y(x,z) = G_y(z) e^{ik_0 x \sin\theta} \tag{A15}$$

It is of interest to note that H_y is also independent of y; hence all the solutions, E_x, E_y, E_z, H_x, H_y, and H_z are independent of y.

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