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EFFECT OF INITIAL VELOCITY ON ONE-DIMENSIONAL, AMBIPOLAR, SPACE-CHARGE CURRENTS

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Cleveland, Ohio



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SUMMARY

Oppositely directed flows of positive and negative charges constitute an ambipolar current. A general case is analyzed in which the charges traverse the evacuated space between plane-parallel boundaries and all particles of a given charge species are monoenergetic. It is assumed that the charges may possess nonvanishing initial kinetic energy. This represents an extension of analyses by Langmuir and Müller-Lübeck for vanishing initial kinetic energy. Sample calculations of dimensionless current densities, electric and potential fields, and charge-density distributions are exhibited for cases where the electric field is assumed to vanish at one boundary but the species initial velocities may not. It is shown that resulting ion currents may be several times the Child's law limit if initial kinetic energies are of the order of the potential energy.

INTRODUCTION

The analysis of one-dimensional, space-charge flows determined by the action of an electric field on charges of a given sign originated with studies by Child (ref. 1) and Langmuir (ref. 2) of unipolar currents in an otherwise evacuated space between plane electrodes. The recognition of space-charge-limited current, that is, of the existence of a maximum current for a given electrode separation and potential difference, resulted from these studies. Vanishing initial velocity of all particles was assumed. Ultimately this assumption was dropped. In particular, Salzberg and Haeff (ref. 3) and Fay, Samuel, and Shockley (ref. 4) coincidentally and independently derived all possible non-relativistic solutions for one-dimensional, unipolar, space-charge current with unique initial particle velocity.

One possible method of overcoming the unipolar space-charge-current limitation is to neutralize the space charge by injecting opposite charges at the final boundary plane traversed by the original charge species. The theory for the resulting ambipolar currents was derived by Langmuir (ref. 5). A required second integration of Poisson's equation, yielding the potential distribution, was performed by numerical integration. Müller-Lübeck (ref. 6) aesthetically improved on Langmuir's analysis by showing that the second integration involved elliptic integrals.

For the ambipolar-current case, Langmuir and Müller-Lübeck both assumed vanishing initial velocity of each charge species. As in the unipolar-current case, this assumption may be discarded. The resulting analysis, presented herein, is essentially the ambipolar equivalent of the unipolar theories in references 3 and 4; however, the results presented are concerned only with monotonic potential fields. The electric and potential fields and charge-density distributions are calculated. The possibility of overcoming the Child's law limit on ion current by means of ambipolar currents with nonvanishing initial velocities is emphasized.

All of the cited analyses represent special cases of that which follows and may be derived from it.

ANALYSIS

Consider two infinite parallel planes (fig. 1) separated by a distance l and associated with different electric potentials. (Only an association is in-

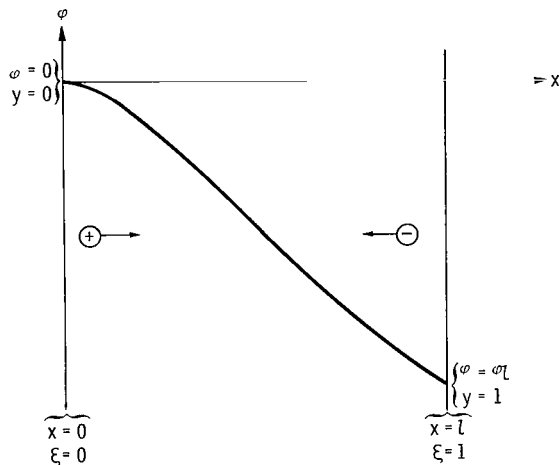


Figure 1. - Geometry and coordinate scheme.

tended. The planes are not to be regarded necessarily as physical entities, say electrodes, which might mechanically limit the motion of charges at the plane boundaries.) Positive ions are introduced normal to the plane associated with the higher potential, and negatively charged particles (electrons or negative ions) are introduced normal to the plane associated with the lower potential. The rate of charge injection is assumed to be steady; then, presumably, a steady positive-negative ion counterflow, that is, an ambipolar current, will exist between the boundary planes. The currents are assumed collisionless. (This implies that the currents are one-dimensional.)

All particles of a given species are assumed to be injected at the same velocity.

Define a coordinate x , having its origin at the plane of positive-ion injection, parallel to and increasing in value in the direction of the ion flow. (All symbols are defined in the appendix.) The nonrelativistic equations (in Gaussian units) describing the particle motions and field distributions for two-charge species are

Charge conservation:

$$\frac{dj_{\pm}}{dx} = 0 \quad (1)$$

Equations of motion:

$$\frac{d\left(\frac{m_{\pm}v_{\pm}^2}{2}\right)}{dx} = e_{\pm}\mathcal{E} \quad (2)$$

Poisson's equation:

$$\frac{d^2\varphi}{dx^2} = -4\pi(\rho_{+} + \rho_{-}) \quad (3)$$

where

$$\rho = \rho_{+} + \rho_{-} \quad (4)$$

and

$$\rho_{\pm} = \frac{j_{\pm}}{v_{\pm}} \quad (5)$$

where j is the current density of the positive ions or electrons, denoted by the subscript $+$ or $-$, respectively, m is the particle mass, v the particle velocity, e the charge, ρ the charge density, $\mathcal{E}$ the electric field, and φ the electric potential.

By introducing φ in equation (2) and integrating, the particle velocity of both species may be represented by

$$v_{\pm} = \pm \left[-\eta_{\pm}(\varphi - \varphi_{0,l}) + v_{\pm 0,l}^2 \right]^{1/2} \quad (6)$$

where the subscripts 0 associated with the positive ions ($+$) and l associated with the electrons ($-$) refer here to initial conditions and

$$\eta_{\pm} \equiv \frac{2e_{\pm}}{m_{\pm}} \quad (7)$$

Substitute the right sides of equations (5) and (6) in equation (3) and define

$$y \equiv \frac{\varphi - \varphi_0}{\varphi_l - \varphi_0} \quad (8)$$

where $\varphi = \varphi_l$ at the final boundary ($x = l$) traversed by the ions. Also, define

$$x \equiv \xi l \quad (9)$$

and

$$a \equiv \frac{j_-}{j_+} \left(\frac{-\eta_+}{\eta_-} \right)^{1/2} \geq 0 \quad (10)$$

(The quantity a^2 can be shown to be the ratio of the product of charge density and total kinetic energy of the electron current at any given value of ξ to that of the positive ion current at the same value of ξ .) Next, define

$$w_{+0} \equiv -\frac{v_{+0}^2}{\eta_+ \phi_l} \quad (11)$$

where v_{+0} is the initial velocity (at $x, \xi = 0$) of the positive ions, and

$$w_{-1} \equiv \frac{v_{-1}^2}{\eta_- \phi_l} \quad (12)$$

where v_{-1} is the initial velocity (at $x = l$, or $\xi = 1$) of the electrons. Finally, let

$$J_+ = \frac{4\pi l^2 j_+}{(-\eta_+ \phi_l^3)^{1/2}} \quad (13)$$

Then, equation (3) may be rewritten in the dimensionless form

$$y'' = J_+ \left[(y + w_{+0})^{-1/2} - a(1 - y + w_{-1})^{-1/2} \right] \quad (14)$$

where primes denote differentiation with respect to ξ , and, without loss of generality, $\phi = 0$ at $x, \xi = 0$. The quantities w_{+0} and w_{-1} are initial kinetic- to potential-energy ratios for ions and electrons, respectively, whereas the terms $y + w_{+0}$ and $1 - y + w_{-1}$, respectively, represent the local kinetic energies of ions and electrons relative to the initial potential energy of both species. The quantity J_+ is proportional to the ratio between the ion current density j_+ , and the unipolar, space-charge-limited, ion current density j_{m+} for vanishing initial velocity. In particular,

$$J_+ \equiv \frac{4}{9} \frac{j_+}{j_{m+}} \quad (15)$$

where

$$j_{m+} = \frac{(-\eta_+ \phi_l^3)^{1/2}}{9\pi l^2} \quad (16)$$

results from the solution of equation (14) for unipolar, space-charge-limited ion current.

In order to determine the charge-density distribution, equation (3) may also be written in the quasi-dimensionless form

$$\rho = \frac{\phi_l y''}{4\pi l^2} \quad (17)$$

In equation (17), y'' may be replaced by the right side of equation (14). In the absence of electrons, this result yields, for $y = 1$ (i.e., $\xi = 1$),

$$\rho_{m+l} = - \frac{\phi_l J_+}{4\pi l^2} \quad (18)$$

where ρ_{m+l} is the ion charge density at $x = l$ for unipolar, space-charge-limited, current. For ambipolar currents, equations (14), (17), and (18) result in

$$\frac{\rho}{\rho_{m+l}} = (y + w_{+0})^{-1/2} - a(1 - y + w_{-1})^{-1/2} \quad (19)$$

which expresses the net charge-density distribution. Equation (19) consists of the sum of the two components

$$\frac{\rho_+}{\rho_{m+l}} = (y + w_{+0})^{-1/2} \quad (20)$$

$$\frac{\rho_-}{\rho_{m+l}} = -a(1 - y + w_{-1})^{-1/2} \quad (21)$$

determining the charge-density distribution for each species.

By defining J_+ proportional to an ion-current ratio, the emphasis in equation (14) is on ion current characteristics; however, by redefining J proportional to an electron-current ratio, equation (14) may be derived in an alternative form, namely,

$$y'' = J_- \left[\frac{1}{a} (y + w_{+0})^{-1/2} - (1 - y + w_{-1})^{-1/2} \right] \quad (22)$$

where

$$J_- = \frac{4\pi l^2 j_-}{(\eta_- \phi_l^3)^{1/2}} = \frac{4}{9} \frac{j_-}{j_{m-}} \quad (23)$$

and

$$j_{m-} = \frac{(\eta_- \varphi_l^3)^{1/2}}{9\pi l^2} \quad (24)$$

is the unipolar, space-charge-limited, electron-current density. Equation (22) may be treated in a manner analogous to the preceding treatment of equation (14), and thus alternative forms for the charge-density distributions may be obtained. Specifically,

$$\frac{\rho}{\rho_{m-0}} = \frac{1}{a} (y + w_{+0})^{-1/2} - (1 - y + w_{-1})^{-1/2} \quad (25)$$

and when equations (21) and (20), respectively, are considered,

$$\frac{\rho_-}{\rho_{m-0}} = - \frac{\rho_-}{a\rho_{m+l}} \quad (26)$$

$$\frac{\rho_+}{\rho_{m-0}} = - \frac{\rho_+}{a\rho_{m+l}} \quad (27)$$

where ρ_{m-0} is the electron charge density at $x = 0$ for unipolar, space-charge-limited current.

Equation (26) or (27) allows a new interpretation of the coefficient a . In particular,

$$a = - \frac{\rho_{m-0}}{\rho_{m+l}} \quad (28)$$

that is, a is the absolute value of the electron- to ion-charge-density ratio associated with the final boundaries traversed by the respective species under the conditions of unipolar, space-charge-limited currents. The coefficient a also relates the relative electron current and ion current according to

$$\frac{j_-}{j_{m-}} = \frac{aj_+}{j_{m+}} \quad (29)$$

which follows from definitions (10), (16), and (24).

According to equations (19), (20), and (21) the charge-density distributions may be calculated as functions of the dimensionless potential y for assumed values of the dimensionless energy parameters w_{+0} and w_{-1} and the coefficient a . However, the variation of y , itself, as a function of the dimensionless distance ξ is determined by equation (14).

Integrating equation (14) twice yields, consecutively, the electric field and the potential distribution. The first integration is elementary and re-

sults in

$$y' = \frac{\mathcal{E}}{\phi_l} = - \left[4J_+ \left\{ (y + w_{+0})^{1/2} - w_{+0}^{1/2} + a \left[(1 - y + w_{-1})^{1/2} - (1 + w_{-1})^{1/2} \right] \right\} + \left(\frac{\mathcal{E}_0}{\phi_l} \right)^2 \right]^{1/2} \quad (30)$$

where $\mathcal{E} = \mathcal{E}_0$ at $\xi = 0$. Equation (30) is not integrable in terms of elementary functions. When the boundary condition $y = 0$ at $\xi = 0$ is used, the solution has the form

$$\xi = \left(\frac{j_+}{j_{m+}} \right)^{-1/2} h_0 \left[y, a, w_{+0}, w_{-1}, \frac{(y'_0)^2}{J_+} \right] \quad (31)$$

where

$$h_0 \left[y, a, w_{+0}, w_{-1}, \frac{(y'_0)^2}{J_+} \right] = \frac{3}{4} \int_0^y \left[(y + w_{+0})^{1/2} + a(1 - y + w_{-1})^{1/2} - \mathcal{E}_0 \right]^{-1/2} dy \quad (32)$$

and

$$\mathcal{E}_0 = w_{+0}^{1/2} + a(1 + w_{-1})^{1/2} - \frac{(y'_0)^2}{4J_+} \quad (33)$$

Equation (14) can be integrated alternatively by using $y = 1$ at $\xi = 1$ as the lower limit to obtain consecutively

$$y' = \frac{\mathcal{E}}{\phi_l} = - \left[4J_+ \left\{ (y + w_{+0})^{1/2} - (1 + w_{+0})^{1/2} + a \left[(1 - y + w_{-1})^{1/2} - w_{-1}^{1/2} \right] \right\} + \left(\frac{\mathcal{E}_1}{\phi_l} \right)^2 \right]^{1/2} \quad (34)$$

and

$$1 - \xi = \left(\frac{j_+}{j_{m+}} \right)^{-1/2} h_1 \left[y, a, w_{+0}, w_{-1}, \frac{(y'_1)^2}{J_+} \right] \quad (35)$$

where $\mathcal{E} = \mathcal{E}_1$ at $\xi = 1$,

$$h_1 \left[y, a, w_{+0}, w_{-1}, \frac{(y'_1)^2}{J_+} \right] = \frac{3}{4} \int_y^1 \left[(y + w_{+0})^{1/2} + a(1 - y + w_{-1})^{1/2} - g_1 \right]^{-1/2} dy \quad (36)$$

and

$$g_1 = (1 + w_{+0})^{1/2} + a w_{-1}^{1/2} - \frac{(y'_1)^2}{4J_+} \quad (37)$$

But, subtracting equation (30) from equation (34) leads to

$$\frac{1^2 (g_1^2 - g_0^2)}{4J_+ \phi_l^2} = (1 + w_{+0})^{1/2} - w_{+0}^{1/2} - a \left[(1 + w_{-1})^{1/2} - w_{-1}^{1/2} \right] \quad (38)$$

or $g_0 = g_1$. Thus, for preassigned values of g_0 and g_1 , values of the coefficient a are not completely arbitrary. In particular, if $g_1 > g_0$, the left side of equation (38) is positive, so that, necessarily,

$$0 \leq a \leq \frac{(1 + w_{+0})^{1/2} - w_{+0}^{1/2}}{(1 + w_{-1})^{1/2} - w_{-1}^{1/2}} \equiv r \quad (39)$$

which also defines r . Similarly, if $g_1 < g_0$,

$$a \geq r \quad (40)$$

where the equality applies if $g_1 = \pm g_0$. An explicit expression for the ion current ratio j_+/j_{m+} is derivable from equation (31) or (35) only if g vanishes at one or both boundaries.

If $g_0 = 0$ or $g_1 = 0$, then J_+ vanishes from the integrand appearing in the respective equation (32) or (36). Then an explicit expression for j_+/j_{m+} is derivable from equation (31) or (35). (If $g_0 \neq 0$ and $g_1 \neq 0$, the integrations can still be expressed in closed form.) Because of charge conservation, the ion-current ratio is unique for specified boundary conditions and may be evaluated therefrom. Hence, for $g_0 = 0$,

$$\frac{j_+}{j_{m+}} = h_0^2(y = 1, a, w_{+0}, w_{-1}, g_0 = 0) \quad (41)$$

whereas for $g_1 = 0$,

$$\frac{j_+}{j_{m+}} = h_1^2(y = 0, a, w_{+0}, w_{-1}, g_1 = 0) \quad (42)$$

Also,

$$\frac{j_+}{j_{m+}} = \left[h_0(y, a, w_{+0}, w_{-1}, \mathcal{E}_0 = 0) + h_1(y, a, w_{+0}, w_{-1}, \mathcal{E}_1 = 0) \right]^2 \quad (43)$$

Equations (41) to (43) are alternative expressions for the ion-current ratio. The first expression is obtained by setting $\xi = 1$ in equation (31), the second, by setting $\xi = 0$ in equation (35), and the third, by adding equations (31) and (35). The first two expressions are most convenient for computations. Moreover, the third expression requires that $\mathcal{E}$ vanish at both boundaries, whereas the first two require that $\mathcal{E}$ vanish at only one boundary. Applying equations (41) and (42) to equations (31) and (35), respectively, yields

$$\xi = \frac{h_0(y, a, w_{+0}, w_{-1}, \mathcal{E}_0 = 0)}{h_0(y = 1, a, w_{+0}, w_{-1}, \mathcal{E}_0 = 0)} \quad (44)$$

and

$$1 - \xi = \frac{h_1(y, a, w_{+0}, w_{-1}, \mathcal{E}_1 = 0)}{h_1(y = 0, a, w_{+0}, w_{-1}, \mathcal{E}_1 = 0)} \quad (45)$$

The final problem is that of evaluating the integral represented by h_0 or h_1 in equation (32) or (36), respectively. These integrals may be transformed into a sum of elementary and elliptic integrals. The substitution

$$y + w_{+0} \equiv \epsilon^2 \sin^2 \alpha \quad (46)$$

where

$$\epsilon^2 \equiv 1 + w_{+0} + w_{-1} \quad (47)$$

transforms the integrals into a known elliptic form

$$\left\{ \begin{array}{l} h_0(y, a, w_{+0}, w_{-1}, \mathcal{E}_0 = 0) \\ h_1(y, a, w_{+0}, w_{-1}, \mathcal{E}_1 = 0) \end{array} \right\} = \left(\frac{3}{2} \right) \epsilon^{3/2} \int_{\left\{ \begin{array}{l} \alpha_1 \\ \alpha_0 \end{array} \right\}}^{\left\{ \begin{array}{l} \alpha \\ \alpha \end{array} \right\}} \sin \alpha \cos \alpha \times \left[\sin \alpha + \alpha \cos \alpha - \frac{1}{\epsilon} \left\{ \begin{array}{l} g_0 \\ g_1 \end{array} \right\} \right]^{-1/2} d\alpha \quad (48)$$

where the upper quantities within the braces correspond to equation (32), the lower quantities correspond to equation (36), and the integration limits are given by

$$\alpha = \sin^{-1}[(y + w_{+0})\epsilon^{-2}]^{1/2} \quad (49)$$

with $\alpha = \alpha_0$ for $y = 0$ and $\alpha = \alpha_1$ for $y = 1$.

However, a more tractable form, in which unipolar and ambipolar contributions to the integrals are essentially separated, results by subsequently employing the trigonometric identity

$$A \cos(\alpha - \delta) = A \sin \delta \sin \alpha + A \cos \delta \cos \alpha$$

where

$$\tan \delta \equiv \frac{1}{a} \quad (50)$$

and

$$A^2 \equiv 1 + a^2 \quad (51)$$

In addition, let

$$\alpha - \delta \equiv 2\beta \quad (52)$$

Then, equations (48) may be rewritten in the form

$$\begin{aligned} & \left\{ \begin{array}{l} h_0(y, a, w_{+0}, w_{-1}, \mathcal{E}_0 = 0) \\ h_1(y, a, w_{+0}, w_{-1}, \mathcal{E}_1 = 0) \end{array} \right\} = 3 \left(\frac{\epsilon^3 n^2}{2A} \right)^{1/2} \\ & \times \int_{\left\{ \begin{array}{l} \beta_0 \\ \beta \end{array} \right\}}^{\left\{ \begin{array}{l} \beta \\ \beta_1 \end{array} \right\}} \left[2 \cos(2\delta) (\sin \beta \cos \beta - 2 \sin^3 \beta \cos \beta) \right. \\ & \left. + \frac{1}{2} \sin(2\delta) (1 - 8 \sin^2 \beta + 8 \sin^4 \beta) \right] (1 - n^2 \sin^2 \beta)^{-1/2} d\beta \quad (53) \end{aligned}$$

where

$$\beta = \frac{1}{2} \left\{ \sin^{-1}[(y + w_{+0})\epsilon^{-2}]^{1/2} - \delta \right\} \quad (54)$$

with $\beta = \beta_0$ for $y = 0$, and $\beta = \beta_1$ for $y = 1$. Also,

$$n^2 = \frac{2A\epsilon}{A\epsilon - \begin{Bmatrix} \epsilon_0 \\ \epsilon_1 \end{Bmatrix}} \equiv k^{-2} \quad (55)$$

which also defines a new modulus k , to be used subsequently. Usually, $n^2 > 1$. Thus, in order to obtain a value of the modulus less than unity, the substitution

$$n \sin \beta = \sin \gamma \quad (56)$$

may be employed to transform equations (53) into a final form

$$\begin{Bmatrix} h_0(y, a, w_{+0}, w_{-1}, \epsilon_0 = 0) \\ h_1(y, a, w_{+0}, w_{-1}, \epsilon_1 = 0) \end{Bmatrix} = 3 \left(\frac{\epsilon_3}{2A} \right)^{1/2} \int_{\begin{Bmatrix} r_0 \\ r \end{Bmatrix}}^{\begin{Bmatrix} r \\ r_1 \end{Bmatrix}} \left[2k \cos(2\delta) (\sin \gamma - 2k^2 \sin^3 \gamma) + \frac{1}{2} \sin(2\delta) \right. \\ \left. (1 - 8k^2 \sin^2 \gamma + 8k^4 \sin^4 \gamma)(1 - k^2 \sin^2 \gamma)^{-1/2} \right] d\gamma \quad (57)$$

where

$$\sin \gamma = n \sin \left(\frac{1}{2} \left\{ \sin^{-1} \left[(y + w_{+0}) \epsilon^{-2} \right]^{1/2} - \delta \right\} \right) \quad (58)$$

with $\gamma = r_0$ for $y = 0$, and $\gamma = r_1$ for $y = 1$. Real solutions of equations (57) are obtained only if $|\sin \gamma| \leq 1$. In equations (57), the elliptic integrals vanish for unipolar ion currents, whereas the elementary integrals vanish for ambipolar currents if $a = 1$. Although equations (57) have been written for $\epsilon_0 = 0$ and $\epsilon_1 = 0$, these assumptions are not necessary for evaluating the integrals; however, they are necessary for expressing j_+/j_{m+} explicitly, as mentioned previously.

The elliptic integrals in equations (57) correspond to integrals 280.00 and 281.01 in reference 7. When these are used, the alternate solutions for h_0 and h_1 are found to be

$$\begin{Bmatrix} h_0(y, a, w_{+0}, w_{-1}, \epsilon_0 = 0) \\ h_1(y, a, w_{+0}, w_{-1}, \epsilon_1 = 0) \end{Bmatrix} = \left(\frac{\epsilon_3}{2A} \right)^{1/2} \left\{ 2k \cos(2\delta) \left[3(2k^2 - 1) \cos \gamma - 2k^2 \cos^3 \gamma \right] + \frac{1}{2} \sin(2\delta) \right. \\ \left. \left[(8k^2 - 5)F(\gamma, k) + 8(1 - 2k^2)E(\gamma, k) + 8k^2 \sin \gamma \cos \gamma \sqrt{1 - k^2 \sin^2 \gamma} \right] \right\} \Big|_{r_0}^r \Big|_{r_1}^{r_1} \quad (59)$$

where the former set of limits is associated with h_0 and the latter set with h_1 . The quantities $F(r,k)$ and $E(r,k)$ are elliptic integrals of the first and second kinds, respectively, defined by

$$F(r,k) \equiv \int_0^r (1 - k^2 \sin^2 r)^{-1/2} dr \quad (60)$$

$$E(r,k) \equiv \int_0^r (1 - k^2 \sin^2 r)^{1/2} dr \quad (61)$$

Equations (59) in conjunction with equations (44) and (45), respectively, determine implicit solutions for the potential distribution.

The preceding analysis permits the currents, fields, and charge distributions to be calculated directly for cases in which the electric field vanishes at one boundary, at least, and $|\mathcal{E}| > 0$ in the interval between the boundaries. Included among these possibilities are all of the corresponding cases in the references cited. There also exists another series of cases for which the electric field vanishes at one or two locations between the boundaries. These cases, with at least one exception, may be treated by juxtaposing fields so that the boundary condition $\mathcal{E} = 0$ (for $\xi = 0$, say) in the present analysis coincides with the condition $\mathcal{E} = 0$ for the type of distribution being considered. A simple juxtaposition is unsatisfactory if the current of one or both charge species is space-charge limited and the charge transmission coefficient at the juxtaposition plane ($\mathcal{E} = 0$) is not unity. Then, reflected currents must be considered.

PROCEDURE

With w_{+0} and w_{-1} given and a value of a selected, subject to inequality (38) or (39), the quantity h_0 or h_1 may be computed from equation (59) as a function of the dimensionless potential y for the interval $0 \leq y \leq 1$ and subject to the additional limitation $|\sin r| \leq 1$. The value of h_0 or h_1 , associated with the respective boundary value $y = 1$ or $y = 0$, permits the calculation of the ion-current ratio j_+/j_{m+} explicitly according to equation (41) or (42) depending on whether $\mathcal{E}_0 = 0$ or $\mathcal{E}_1 = 0$, respectively. Coincidentally, alternative equations (44) and (45) allow computation of the dimensionless distance ξ as a function of the dimensionless potential y . The electric field is given by equation (30) if $\mathcal{E}_0 = 0$ or equation (34) if $\mathcal{E}_1 = 0$. The charge-density distribution and the component distributions are determined by equations (19) to (21). The electric field and charge-density distributions are obtained as functions of y , but y has been uniquely related to the dimensionless distance ξ by equation (44) or (45).

RESULTS AND DISCUSSION

Figures 2 to 6 illustrate results of a few sample calculations of the currents, fields, and charge distributions, all associated with the assumed condition $\mathcal{E}_0 = 0$ and demonstrating the effects of nonvanishing initial particle velocities. The plotted curves are denoted by capital letters A, B, C, D, and E associated with independent conditions.

Current ratios as functions of appropriate energy ratios are shown in figure 2. Curve A, applying to a unipolar current (which is space-charge lim-

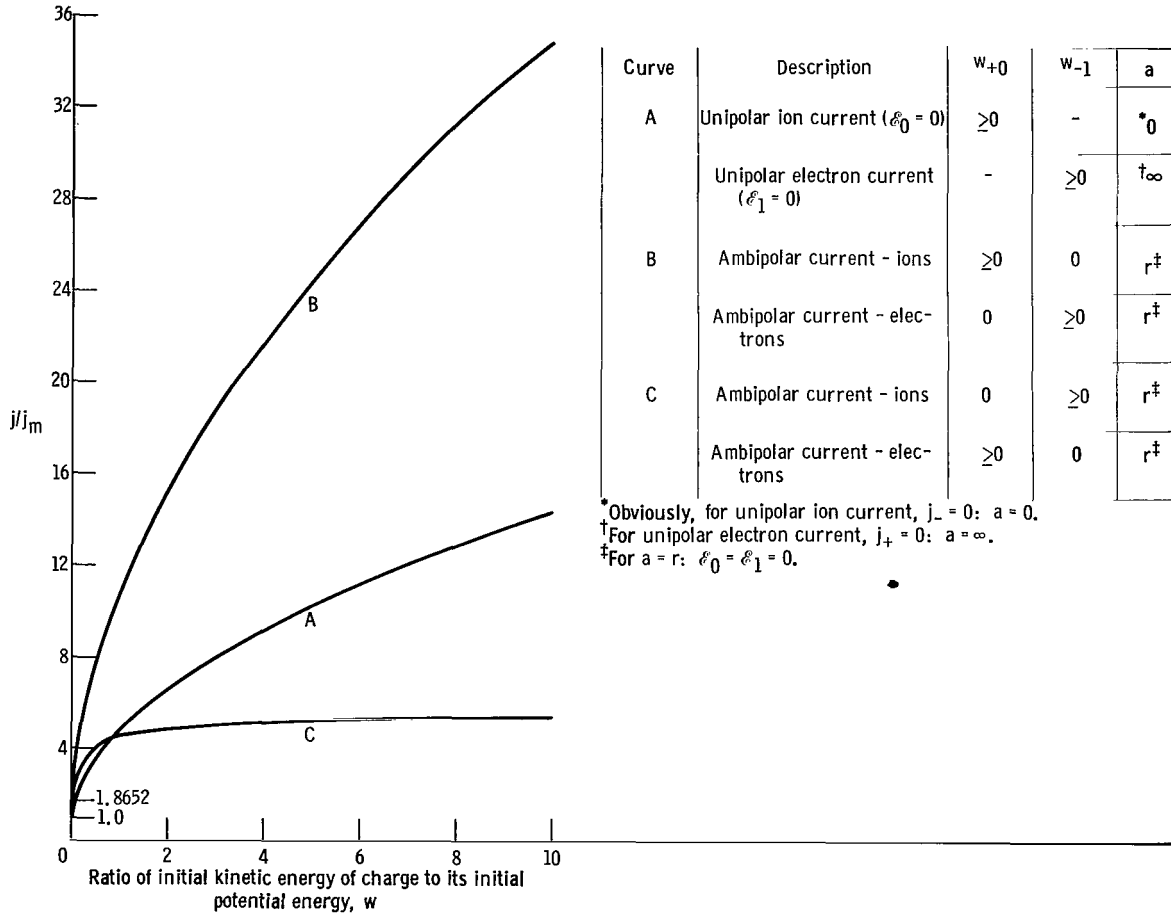


Figure 2. - Relative current as function of initial relative energy ($\mathcal{E}_0 = 0$).

ited ($\mathcal{E}_0 = 0$) for $w_{+0} = 0$), might just as readily represent the initial energy dependence of a unipolar electron current, where the electron current is space-charge limited ($\mathcal{E}_1 = 0$) for $w_{-1} = 0$. In the former case, the ordinate and the abscissa are j_+/j_{m+} and w_{+0} , respectively; whereas in the latter case, the coordinates become j_-/j_{m-} and w_{-1} . According to curve A, the ion (electron) current density becomes considerably greater than the space-charge-limited value if the ions (electrons) are supplied with initial kinetic energy which

Curve	Description	w_+0	w_-1	a
A	Unipolar space-charge-limited ion current	0	-	*0
B	Unipolar ion current ($\epsilon_0 = 0$)	1	-	*0
C	Ambipolar. Both currents space-charge limited	0	0	$r^\dagger$
D	Ambipolar current	1	0	$r^\dagger$
E	Ambipolar current	0	1	$r^\dagger$

* Obviously, for unipolar ion current, $j_- = 0$: $a = 0$.

† For $a = r$: $\epsilon_0 = \epsilon_1 = 0$.

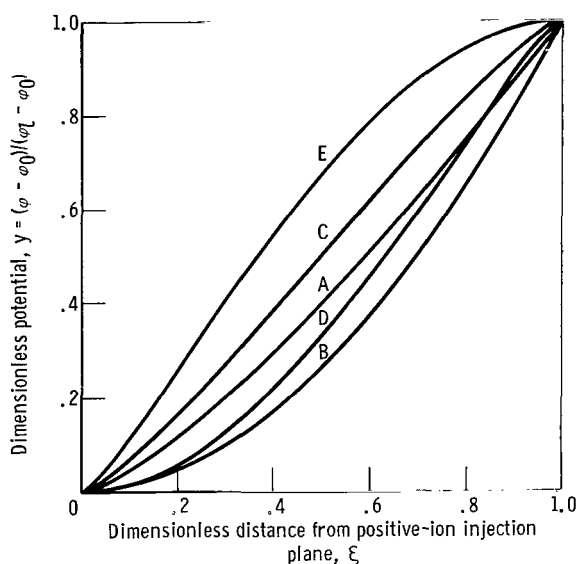


Figure 3. - Dimensionless potential distribution ($\epsilon_0 = 0$).

is associated with the ion-current ratio; that is, the association of the curves is reversed. For $w \leq 0.85$, introducing negative ions with $m_- = m_+$, $|e_-| = e_+$, and with or without initial kinetic energy results in a larger positive ion current density than would exist if the same initial kinetic energy were supplied to a unipolar flow of ions.

Figures 3 and 4 show, respectively, dimensionless potential distributions and dimensionless electric fields corresponding to specific values of the independent quantities w and a included in figure 2. (Curves C are the same as those for $a = 1$ in figs. 4 and 6 of ref. 6.) Figure 5 depicts the dimensionless charge-density distributions corresponding to curves A, B, C, and D in figures 3 and 4. Charge-density distributions for ambipolar currents with both current components space-charge limited (curves C) are compared in figure 5(a) with the corresponding distribution for unipolar, space-charge-limited

is comparable to or greater than the applied electric potential. (Note that, for $w = 1$, the initial kinetic and potential energies are equal.) If electrons are introduced with vanishing initial kinetic energy at the boundary $\xi = 1$, then for $\epsilon_0 = \epsilon_1 = 0$ ($a = r$), curve B indicates the ion-current ratio. Curve C depicts the corresponding electron-current ratio for this case. Comparing curves A and B shows that introducing electrons increases the ion-current ratio by a factor which varies from 1.86 for $w_+0 = 0$ to 2.39 for $w_+0 = 10$. More exactly, the present value, $j_+/j_{m+} = 1.8652$ for $w_+0 = 0$, agrees more closely with Langmuir's value, $j_-/j_{m-} = 1.8605$ (ref. 5), than with Müller-Lübeck's value, $j_-/j_{m-} = 1.8532$ (ref. 6).

For $w > 0$, the indicated current ratios are for a monotonic potential distribution with $\epsilon_0 = 0$; however, the current can be increased further until the ultimate space-charge limit is reached. The resulting potential distribution is then nonmonotonic but can generally be treated by using the present analysis by the juxtaposition method previously mentioned.

If the initial kinetic energy is supplied to the electrons, rather than the ions, then again for $a = r$, curve B is now associated with the electron-current ratio, whereas curve C

Curve	Description	w_{+0}	w_{-1}	a
A	Unipolar space-charge-limited ion current	0	-	$\cdot 0$
B	Unipolar ion current ($\ell_0 = 0$)	1	-	$\cdot 0$
C	Ambipolar. Both currents space-charge limited	0	0	$r^\dagger$
D	Ambipolar current	1	0	$r^\dagger$
E	Ambipolar current	0	1	$r^\dagger$

*Obviously, for unipolar ion current, $j_- = 0$: $a = 0$.

†For $a = r$: $\ell_0 = \ell_1 = 0$.

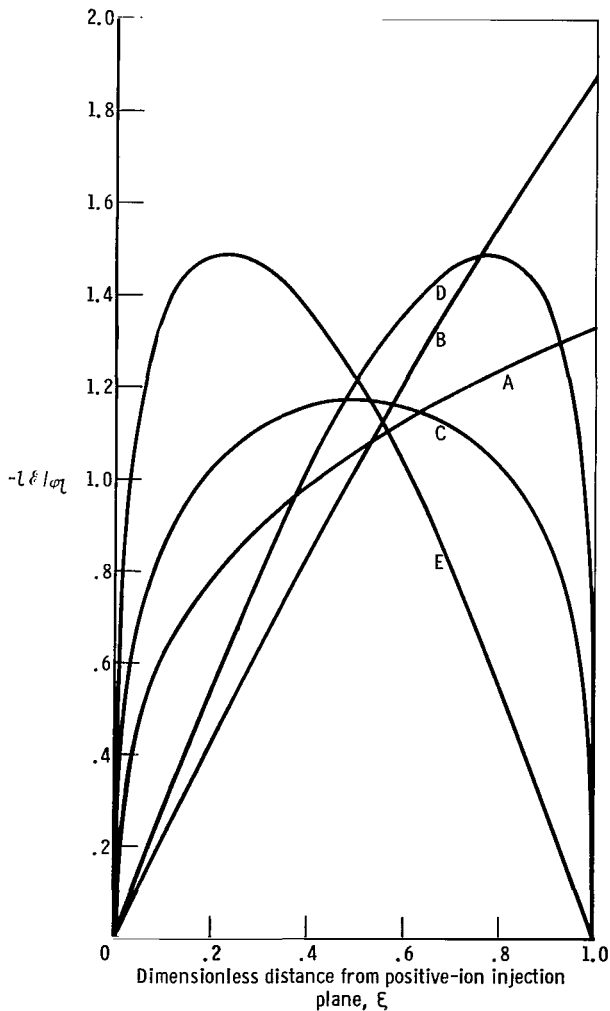


Figure 4. - Dimensionless electric field ($\ell_0 = 0$).

ion current. A similar comparison is exhibited in figure 5(b), except that $w_{+0} = 1$ for both the ambipolar and unipolar cases.

In figure 5(a) the partial neutralization of the positive ion charge density when electrons are introduced is evident. This is also characterized by the increased slope (near $\xi = 0$) of the potential curve C with respect to curve A in figure 3 or by the increased absolute magnitude (near $\xi = 0$) associated with the electric-field curve C with respect to curve A in figure 4. More importantly, the ion charge densities at $\xi = 1$ are identical, so that the increased ion current density (1.86 times) must be associated with increased velocity of ions at $\xi = 1$ when electrons are introduced.

If now the positive ions possess initial kinetic energy, then for the increasing initial kinetic energy of the ions the ion charge-density distribution for ambipolar currents approaches that for a unipolar ion current. For the case $w_{+1} = 1$, $a = r$ shown in figure 5(b), the two ion charge-density distributions (curves B and D for ρ_+/ρ_{m+1}) are effectively identical. Moreover, the relative charge density of ions at $\xi = 1$ is slightly less than the unipolar space-charge-limited value (unity). Because the relative charge-density of ions at $\xi = 1$ is always less than or equal to 1 (cf., fig. 5), the increase of ion current shown in figure 2 must correspond to increased ion velocity at the ion exit plane ($\xi = 1$).

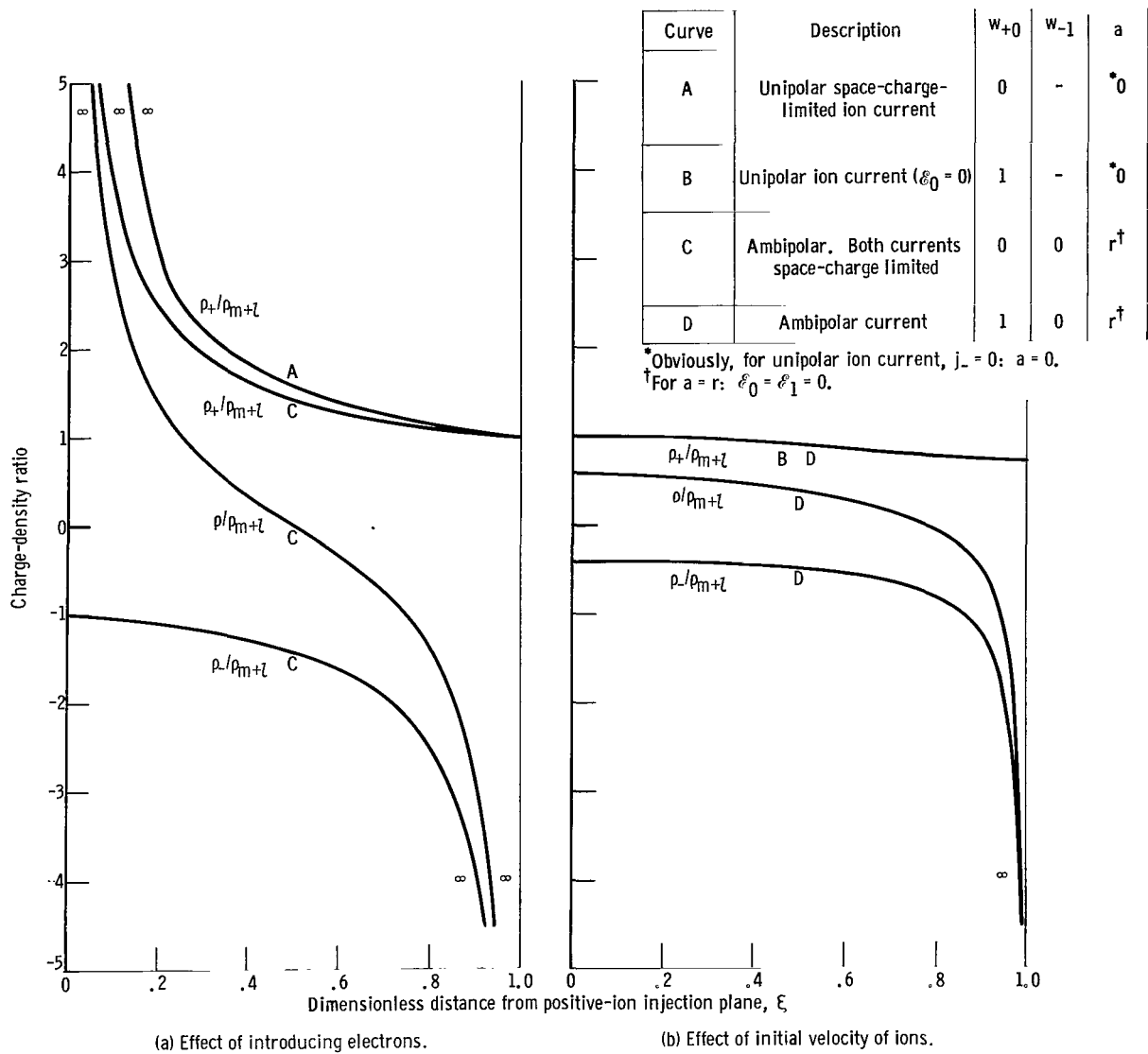


Figure 5. - Dimensionless charge-density distributions ($\mathcal{E}_0 = 0$).

The sample results provide a very limited illustration of the variety of solutions contained in the theory. Particular cases of interest will, of course, depend on the application.

Lewis Research Center
 National Aeronautics and Space Administration
 Cleveland, Ohio, June 2, 1964

APPENDIX - SYMBOLS

A	$(1 + a^2)^{1/2}$, eq. (51)
a	$(j_-/j_+)(-\eta_+/\eta_-)^{1/2}$, eq. (10)
E(r,k)	elliptic integral of the second kind, $\int_0^r (1 - k^2 \sin^2 r)^{1/2} dr$, eq. (61)
$\mathcal{E}$	electric field, $-d\phi/dx$
e	electric charge
F(r,k)	elliptic integral of the first kind, $\int_0^r (1 - k^2 \sin^2 r)^{-1/2} dr$, eq. (60)
g_0	$w_{+0}^{1/2} + a(1 + w_{-1})^{1/2} - (y_0')^2/4J_+$, eq. (33)
g_1	$(1 + w_{+0})^{1/2} + aw_{-1}^{1/2} - (y_1')^2/4J_+$, eq. (37)
h_0	$(3/4) \int_0^y [(y + w_{+0})^{1/2} + a(1 - y + w_{-1})^{1/2} - g_0]^{-1/2} dy$, eq. (32)
h_1	$(3/4) \int_y^1 [(y + w_{+0})^{1/2} + a(1 - y + w_{-1})^{1/2} - g_1]^{-1/2} dy$, eq. (36)
J	$(4/9)(j/j_m)$, eqs. (15) and (23)
j	current density
k	modulus of elliptic integrals, $\left[\left(A\epsilon - \left\{ \begin{smallmatrix} g_0 \\ g_1 \end{smallmatrix} \right\} \right) / 2A\epsilon \right]^{1/2}$, eq. (55)
l	separation of plane boundaries
m	ionic mass
n	modulus of elliptic integrals, $1/k$, eq. (55)
r	$\left[(1 + w_{+0})^{1/2} - w_{+0}^{1/2} \right] / \left[(1 + w_{-1})^{1/2} - w_{-1}^{1/2} \right]$, eq. (39)
v	velocity of ion
w	ratio of initial kinetic energy of ion to its initial potential energy (eqs. (11) and (12))

x	distance measured from boundary plane at which positive ions are injected
y	dimensionless potential, $(\varphi - \varphi_0)/(\varphi_l - \varphi_0)$, eq. (8)
α	$\sin^{-1}[(y + w_{+0})\epsilon^{-2}]^{1/2}$, eq. (46)
β	$(\alpha - \delta)/2$, eq. (52)
γ	$\sin^{-1}(n \sin \beta)$, eq. (56)
δ	$\tan^{-1}(1/a)$, eq. (50)
ϵ	$(1 + w_{+0} + w_{-1})^{1/2}$, eq. (47)
η	ionic charge-to-mass ratio, $2e/m$, eq. (7)
ξ	dimensionless distance from positive-ion injection plane, x/l , eq. (9)
ρ	charge density
φ	electric potential

Subscripts:

l	value at $x = l$
m	space-charge-limited value
0	value at $x, \xi = 0$
1	value at $\xi = 1$
$+$	positive ions
$-$	negative ions (electrons)

Superscript:

$'$	derivative, $d/d\xi$
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