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A Neutral-Point Expansion of the Ideal Magnetosphere

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Abstract. The idealized model of the geomagnetic field-solar wind interaction yields a singular (neutral) point on the magnetopause at which the magnetic field vanishes. If we expand the fields in a power series around this point, including quadratic terms, we can derive an approximate equation of the magnetopause in a small neighborhood of the neutral point which is consistent with the idealized boundary conditions to fourth order. We then consider the additional pressure due to multiple reflections of particles in this neighborhood and show that less than a 4 per cent correction to the single-reflection pressure condition is necessary.

AUTHOR

Introduction. In the idealized model of the geomagnetic field-solar wind interaction [Beard, 1960] we assume the field to be excluded from the plasma and contained in a cavity called the magnetosphere. The boundary of the cavity, or magnetopause, is unknown, but we stipulate the dynamic condition that the magnetic pressure just inside the magnetopause be exactly balanced by the kinetic pressure of solar wind particles elastically reflected from the surface (thermal effects are neglected). Figure 1 is a schematic drawing of the magnetosphere with the dipole located at the origin of coordinates and perpendicular to the direction of the plasma stream.

Mathematically, this situation is described by the field

$$\mathbf{H} = -\text{grad } \Omega \quad \nabla^2 \Omega = 0 \quad (1)$$

inside the magnetosphere, and the boundary conditions

$$\mathbf{H} \cdot \text{grad } F = 0 \quad (\text{confinement}) \quad (2)$$

$$\mathbf{H} \cdot \mathbf{H} = \beta^2 \cos^2 \chi \quad (\text{pressure})$$

which hold on the surface, $F(x', y', z') = 0$, where χ is the angle of incidence of the incoming plasma (Figure 1), and $\beta^2 = 8\pi(2nmV^2)$. V , n , and m are the plasma drift velocity, ion number density, and ion mass, respectively.

A singular point on this unknown surface is N , the neutral point, where the magnetic field lines 'split,' the field vanishes, $\chi = \pi/2$, and the

gradient is in the x' direction. Although the problem is unsolved as yet, there are some good approximations to the general shape [Mead and Beard, 1963; Spreiter and Briggs, 1962; Midgley and Davis, 1963]. However, these approximations generally fail in the region of the neutral point, a region of interest since, if it does exist, it is likely to be unstable, and a possible point of entry for high-energy particles into the magnetosphere.

The neutral point considered here is an X-type neutral point [Dungey, 1958, pp. 39-41, 51-52, 98-102]; however, it lies on a bounding surface, which must run parallel to one of the limiting field lines (Figure 2). We must assume that surface currents can be made to account for the disappearance of the field outside the surface (magnetopause).

Series representation. We can represent the field $\mathbf{H} = \beta \mathbf{h}$ by means of its scalar potential Ω . We first transform to the (x, y, z) coordinate system with origin at N such that

$$R(dx, dy, dz) = (dx', dy', dz') \quad (3)$$

In a region small compared to the apex radius R , we have $x^2 + y^2 + z^2 \ll 1$. The scalar potential is expanded in the form

$$-\Omega/\beta = ax^2 + by^2 + cz^2 + dx^3 + ey^3 + fz^3 + mxy^2 + nxz^2 + rzx^2 + szy^2 \quad (4)$$

We must include cubic terms in the potential, since we expect the limiting field lines to be curvilinear in the xz plane. There are no linear terms, because the field must vanish at N ; terms

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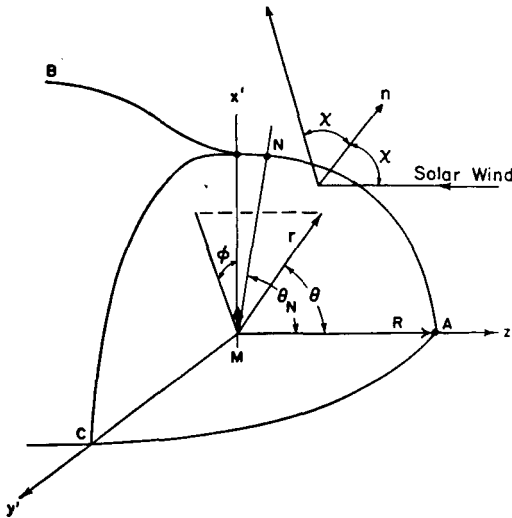


Fig. 1. Schematic drawing of the magnetosphere.

in y to the first power are deleted, because symmetry requires $H_y = 0$ in the xz plane. The dependence on xz has also been omitted, since for $y = 0 = z$ there should only be an x component of H on the x axis (Figure 2).

The gradient of (4) yields the normalized field:

$$\begin{aligned} h_x &= 2ax + 3dx^2 + my^2 + nz^2 + 2rxz \\ h_y &= 2by + 3ey^2 + 2mxy + 2szy \\ h_z &= 2cz + 3fz^2 + 2nxz + rx^2 + sy^2 \end{aligned} \quad (5)$$

and the requirement $\nabla \cdot \mathbf{H} = 0$ yields

$$\begin{aligned} a + b + c &= 0 & 3d + m + n &= 0 \\ 3f + r + s &= 0 & e &= 0 \end{aligned} \quad (6)$$

Furthermore, as Figure 2 shows, we expect $a < 0, c > 0$.

Noon meridian contour. In the xz plane the boundary conditions (equation 2) on the noon meridian contour become ($dx/dz = x'$):

$$h_x/h_z = -F_z/F_x = x' \quad (7)$$

$$h_x^2 + h_z^2 = x'^2/(1 + x'^2) \quad (8)$$

If we substitute for x' in (7) and (8), then

$$h_x^4 + h_x^2(2h_z^2 - 1) + h_z^4 = 0 \quad (9)$$

and, since both components must vanish at N ,

$$h_x^2 = (1 - 2h_z^2 - (1 - 4h_z^2)^{1/2})/2 \rightarrow h_z^4 \rightarrow N \quad (10)$$

From Figure 2 we see that h_x is opposed in sign to the coordinate z on the noon meridian contour. Thus, sufficiently close to N ,

$$h_x = \pm h_z^2 \quad \text{for } z \lesssim 0 \quad (11)$$

If we substitute into (7) and differentiate,

$$x'' = \pm dh_x/dz \quad z \lesssim 0 \quad (12)$$

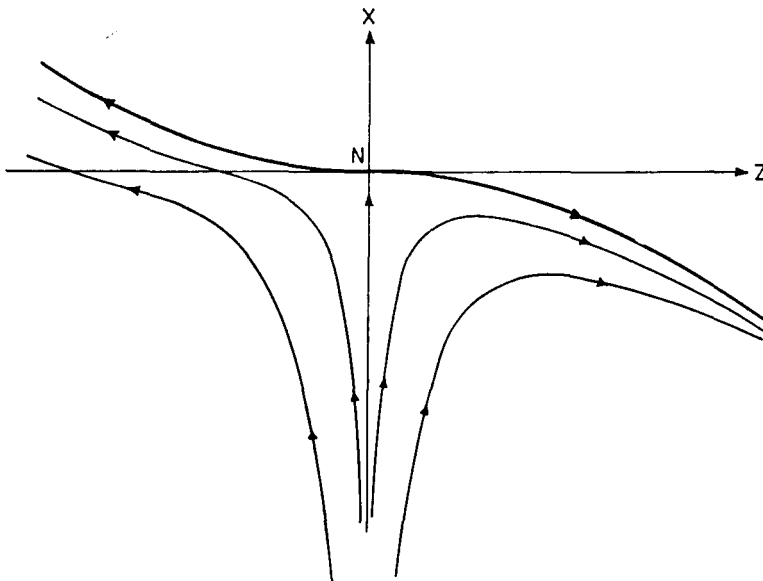


Fig. 2. Fields near the neutral point in the noon meridian plane.

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which leaves two alternatives: either x'' is discontinuous at N or $x'' = 0$ at N . The latter implies $c = 0$, which is inconsistent with the geometry of the limiting field line in the xz plane. Thus the curve, $\{x = x(z), y = 0\}$, must be represented as two separate power series for $z \leq 0$. Since (11) must be satisfied on the noon meridian contour, it gives an implicit representation of that contour near N . This must agree with $x = \int dz h_z / h_x$ of (7) and with Figure 2. The leading term of (11) is

$$2ax + nz^2 = \pm 4c^2 z^2 \quad (13)$$

If $n = 0$, equation 13 implies that a is of positive sign, which contradicts Figure 2. Therefore we must assume that n has two different values, according to the sign of z . This is permissible as long as h remains continuous at $z = 0$, and $\nabla \cdot h = 0$ everywhere. If we compare (13) with the leading term of (7), we find

$$x' = \pm h_x \leq 0 \quad x = \pm cz^2 \quad (14)$$

near N . Thus, (13) becomes

$$n = \pm 2c(2c - a) \quad \text{for } z \leq 0 \quad (15)$$

Surface representation. Since the gradient is in the x direction at N , we can represent the surface by two second-order expansions in (y, z) :

$$F(x, y, z) = x - A^+ z^2 - B^+ y^2 - C^+ yz = 0 \quad (16)$$

for $z \leq 0$. By symmetry, $B^+ = B^- = B$; (14) requires $A^+ = -A^- = c$. If we form the dot product $H \cdot \nabla F = 0$ and substitute for x from (16) we find the confinement condition satisfied to second order in z^2, y^2 , and yz when

$$m = 2B(2b - a) \quad (17)$$

$$C^+(1 - b/a - c/a) = 0 \quad (18)$$

The divergence conditions (equations 6) require that d also be discontinuous, since $3d + m + n = 0$, although m is continuous. Therefore, from (17) and (15),

$$3d = \pm 2c(a - 2c) + 2B(a - 2b) \quad z \leq 0 \quad (19)$$

Equation 18 is incompatible with $a + b + c = 0$ unless $C^\pm = 0$. Although (19) forces h_x to be discontinuous at $z = 0$ ($a \neq 2c$), the $3 dx^2$ term is negligibly small compared to the ax term in (5).

Third-order terms in $H \cdot \nabla F = 0$ are elimi-

nated by simply setting $f = 0 = s$. Then, by the divergence condition, $r = 0$ also. Thus in the power series representation, the confinement condition of (2) is satisfied up to fourth-order errors, on the postulated surface of (16). If we substitute the expansions for h and $F(x, y, z)$ into the pressure condition of (2), we find

$$h_x^2 + h_y^2 + h_z^2 = \cos^2 \chi \\ \doteq (\partial F / \partial z)^2 (1 - 2\text{nd-order terms}) \quad (20)$$

and expanding h yields

$$4c^2 z^2 + 4b^2 y^2 + 4\text{th-order terms} \\ = 4c^2 z^2 (1 - 2\text{nd-order terms}) \quad (21)$$

By setting $b = 0$ we can reduce the error in the second boundary condition to fourth order. Our final expansion for the field in the neighborhood of N is given by:

$$-\Omega/\beta = cz^2 - cx^2 - (m \pm 6c^2)x^3/3 \\ + mxy^2 \pm 6c^2 xz^2 \quad (22)$$

and the surface of the magnetosphere near N is given by

$$F(x, y, z) = x - (\pm)cz^2 - my^2/2c \quad (23)$$

for $z \leq 0$.

Multiple reflections at N . Midgley and Davis [1963] have observed that the effect of multiple reflections of particles near N might seriously alter the pressure condition (equation 2) near the neutral point. In the second-order approximation, we can show that the shape of the surface is consistent with the pressure condition to within 4 per cent. This effect would be most pronounced in the xz plane; we shall calculate the added pressure δp at a point (x, z) on the noon meridian contour due to multiple reflections.

Figure 3 represents a particle incident on the point (x_0, z_0) on the contour $x = cz^2, z \leq 0, y = 0$, with an angle of incidence χ_0 . This particle, on first reflection, strikes the magnetopause again at (x, z) where

$$(x - x_0)/(z - z_0) = c(z + z_0) = \tan 2\chi_0 \quad (24)$$

and

$$\cot \chi_0 = -x'(z_0) = -2cz_0 \quad (25)$$

therefore,

$$z = 4cz_0/(1 - 4c^2 z_0^2) - z_0 \doteq 3z_0 \quad (26)$$

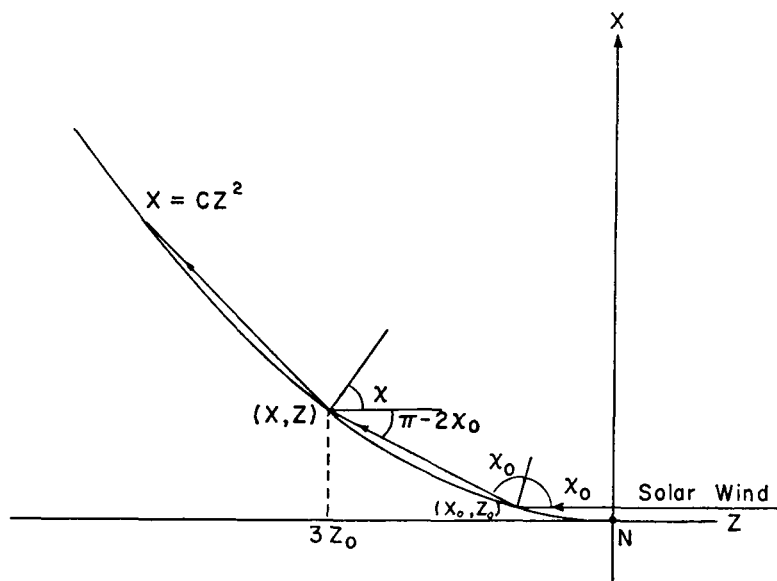


Fig. 3. Multiple reflections near the neutral point.

as indicated in Figure 3. We note that the angle of incidence is $\pi - 2\chi_0 + \chi$ at (x, z) , and the change in the normal component of momentum will be correspondingly smaller. If we pursue the particle to the third point of reflection we find

$$c(z + 3z_0) = \tan 2(\chi - \chi_0) \quad z \doteq 5z_0 \quad (27)$$

However, the effect of the particle, by this time, is negligible.

We now consider the additional pressure δp at (x, z) due to particles reflected from (x_0, z_0) and compare it with the pressure $p = 2nmV^2 \cos^2 \chi$ of the incident plasma stream (ignoring the y coordinate). If $nV dx_0$ particles/sec are incident on an area of $ds_0 = (dx_0^2 + dz_0^2)^{1/2}$ at (x_0, z_0) , they are reflected onto an area $ds = (dx^2 + dz^2)^{1/2}$ at (x, z) , where their momentum changes by $2mV \cos(\pi - 2\chi_0 + \chi)$ per particle. Therefore, since $dx = 9dx_0$,

$$\delta p = -2mnV^2 \cos(\chi - 2\chi_0)(dx/ds)/9 \quad (28)$$

If we substitute for χ, χ_0 from (25) and note that $dx/ds = \cos \chi$, then we find

$$\delta p/p = 1/27 \ll 1 \quad (29)$$

We can conclude that the effect of multiple reflections near the neutral point is negligible and that the pressure condition is valid over the entire magnetosphere in this idealized representation.

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