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A SIMPLE MODEL OF THE INTERPLANETARY MAGNETIC FIELD

PART I: CALCULATION OF THE MAGNETIC FIELD

by David Stern

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SUMMARY

A simple model of the interplanetary magnetic field is described and solved analytically. In this model space is divided into three regions by two concentric spheres. Conductivities (with one exception) are assumed to be isotropic and constant in each region, and flow velocities are regular and prescribed. The innermost region rotates rigidly around its center, the intermediate region contains a compressible fluid flowing radially outward at a constant velocity (an idealization of the solar wind), and the outer region is at rest. The magnetic field originates at point sources at the origin and possibly in a uniform field at infinity. With these assumptions methods are described for finding the field in the general case, and also in the limit when all conductivities are very high. As an example the case in which the field's source is a point dipole aligned with the axis of rotation is solved in some detail. Part II of this study, published as a separate Technical Note, considers the cosmic ray anisotropy.

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A SIMPLE MODEL OF THE INTERPLANETARY MAGNETIC FIELD, PART I: CALCULATION OF THE MAGNETIC FIELD*

by

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INTRODUCTION

Experimental evidence indicates that an interplanetary magnetic field originates at the sun and extends at least to the earth's orbit, possibly much further. Many features of this field are still uncertain, but two of its main properties have been predicted theoretically and seem to agree with observation. Both may be regarded as manifestations of the fact that a highly conducting fluid — here the solar wind emanating radially from the sun — tends to impart its motion to lines of force embedded in it. It was theorized that the solar wind stretched the lines of force, rendering them almost radial and causing the field intensity **B** to fall off less rapidly than it would otherwise (see for example Reference 1). In addition to this the field was expected to be twisted by solar rotation into an Archimedean spiral. This point was noted first by Chapman (Reference 2) who observed that the locus of a particle stream constantly emitted from a point on the sun is, at any time, such a spiral. (Because the same locus is described by droplets from a rotating sprinkler, this is sometimes called the "garden-hose effect.") A line of force drawn out by a stream of particles would also follow such a spiral, and it was argued that similar twisting occurs in any field originating in the rotating sun. Parker gave a formal proof of this, assuming that the magnetic field is parallel to the velocity field as seen from a frame of reference co-rotating with the sun (Reference 3). The effect has also been deduced from experimental data, from the arrival direction of solar flare particles (Reference 4), and from direct observation (Reference 5) by Mariner II (1962 $\alpha\rho 1$). For these cases the "garden-hose angle" between **B** and the radial direction from the sun was of the order of 45 degrees.

In this work a simple model of the interplanetary magnetic field will be investigated, first in the limiting case of a perfectly conducting fluid and then for finite, isotropic, and homogeneous conductivity. In Part II of this study the cosmic ray anisotropy is investigated (Reference 6). The description of the model follows. Space is assumed to be divided into three regions by two

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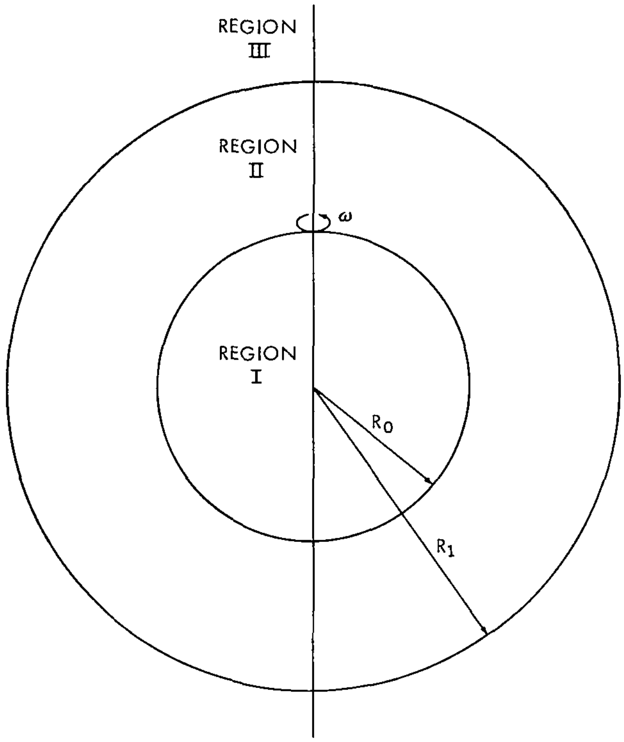

Figure 1—The division of space into three regions.

concentric spheres of radii R_0 and R_1 (Figure 1). Region I, the innermost region, is assumed to rotate rigidly with angular velocity ω. This region contains the source of the magnetic field, which will be assumed to be concentrated at the origin. Because of the tedious calculations involved, only the case of a dipole source with moment $\mathbf{M}$ will be treated in any detail. If this dipole is inclined at an angle κ to the axis of solar rotation, its axial component may be regarded as an idealization of the main solar field, and its equatorial component as that of an active region co-rotating with the sun. Region II, between the spheres, contains a compressible conducting fluid flowing radially outward with constant velocity u. Finally, in region III, which extends to infinity, no motion takes place. Region I actually represents the sun, and region II the space swept by the solar wind. The equation connecting the magnetic field $\mathbf{B}$, the electric field $\mathbf{E}$, the velocity $\mathbf{v}$, and the conductivity σ is

$$\mathrm{curl}\,\mathbf{B} = \mu_0 \sigma \left[\mathbf{E} + (\mathbf{v} \times \mathbf{B})\right] . \qquad (1)$$

We shall be interested in stationary solutions, and particularly in the case when σ is large and $R_1 \gg R_0$.

It should be borne in mind that the preceding is a gross simplification of the actual situation. To stress this point, everything approximated and neglected will now be listed:

1. It has not been established that the solar dipole plays a major role in creating the interplanetary field. Certainly, the source of the field is quite complex.

2. The solar wind is not an ordinary conducting fluid but a nearly collisionless plasma, conducting very well along the magnetic field but much less across it. Unfortunately, since the direction of the conduction anisotropy depends on the magnetic field, taking it into account makes the equation of conduction nonlinear. Except for one case, therefore, the conductivity will be assumed to be isotropic.

3. The flow and.field assumed here are laminar and regular, although observation indicates a large irregular component. The model developed here thus represents only the effects of the average interplanetary field and does not include the turbulent component.

4. The assumed sharp boundaries are only a convenient approximation to the actual ones, which are not too well known at the present time.

5. In any problem of this sort, the velocity $\mathbf{v}$ generally is not determined a priori but is solved simultaneously with $\mathbf{B}$, by using the hydromagnetic flow equation (see for example Reference 7). This equation is nonlinear and with this approach the term $\mathbf{v} \times \mathbf{B}$ in Equation 1 becomes nonlinear too. In the vicinity of the earth, of course, the mass flow dictates the magnetic field because of its much higher energy density; nevertheless, in the vicinity of what corresponds to the outer sphere of this model, the flow may be considerably distorted by the field.

Unfortunately, a more realistic model would be very hard to solve analytically. It is hoped, however, that the results obtained here will give some qualitative insight into the behavior of the actual interplanetary field.

INFINITE CONDUCTIVITY

The field produced when $\sigma \to \infty$ has been derived by Parker (Reference 3), with a rotating frame of reference. It will be derived here in a somewhat more conventional way. In general, if the conductivity tends to infinity and $\mathrm{curl}\, \mathbf{B}$ does not,

$$\mathbf{E} = -(\mathbf{v} \times \mathbf{B}) \ . \tag{2}$$

Taking the curl in region II gives, in spherical coordinates (r, θ, ϕ),

$$\frac{\partial \mathbf{B}}{\partial t} = u \,\mathrm{curl} \left(\mathbf{i}_\phi B_\theta - \mathbf{i}_\theta B_\phi \right) \ . \tag{3}$$

Utilizing

$$\mathrm{div}\, \mathbf{B} = 0 \tag{4}$$

gives

$$\frac{\partial X}{\partial t} - u \frac{\partial X}{\partial r} = 0 \ , \tag{5}$$

where X stands for either $r^2 B_r$, rB_ϕ, or rB_θ. Under rotational symmetry X is independent of time and

$$B_r = \xi(\theta)\, r^{-2} \ , \tag{6}$$

$$B_\phi = \zeta(\theta)\, r^{-1} \ , \tag{7}$$

and, by using Equation 4,

$$B_\theta = 0 . \tag{8}$$

E_θ is continuous on the surface $r = R_0$. Just inside the boundary, by Equation 2,

$$E_\theta = -\omega R_0 \sin\theta B_r ,$$

and because B_r is continuous its value just outside the boundary may be used. The continuity of E_θ then gives

$$\zeta(\theta) = -\left(\frac{\omega}{u}\right)\sin\theta\,\xi(\theta) , \tag{9}$$

so that the tangent of the "garden-hose angle" is

$$\tan\chi = \frac{B_\phi}{B_r} = -\left(\frac{\omega}{u}\right) r \sin\theta . \tag{10}$$

In general the source of the field rotates:

$$B(r, \theta, \phi, t) = B(r, \theta, \lambda) , \tag{11}$$

$$\lambda = \phi - \omega t , \tag{12}$$

$$\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \lambda} . \tag{13}$$

Then Equation 5 becomes

$$\frac{\partial X}{\partial \lambda} + \frac{u}{\omega}\frac{\partial X}{\partial r} = 0 , \tag{14}$$

and its solution is

$$X = X\left(\theta, \phi - \omega t - \frac{\omega r}{u}\right) . \tag{15}$$

Thus

$$B_r = r^{-2} \sum_m \xi_m(\theta) \exp im\left(\phi - \omega t - \frac{\omega r}{u}\right) , \tag{16}$$

and the continuity of E_θ when $r = R_0$ gives

$$B_\phi = -\left(\frac{\omega}{ur}\right) \sin\theta \sum_m \xi_m(\theta) \exp im\left(\phi - \omega t - \frac{\omega r}{u}\right) . \tag{17}$$

For a more detailed solution it is generally better to solve Equation 1 for finite conductivity and then investigate the behavior of the solution when σ gets large.

FINITE CONDUCTIVITY

We will now turn to solving the problem for an arbitrary σ. It will be assumed that σ is uniform in each region and takes the values σ_1, σ_2, and σ_3 in regions I, II, and III, respectively. The following theorem is found useful: If a vector field **B** satisfies Equation 4, it may be uniquely resolved in the form

$$\mathbf{B} = \operatorname{curl} \psi_1 \mathbf{r} + \operatorname{curl}\operatorname{curl} \psi_2 \mathbf{r} . \tag{18}$$

Following Elsasser (Reference 8), we will term the component fields the *toroidal* and *poloidal* components, respectively. The theorem was proved first for rotational symmetry by Lüst and Schlüter (Reference 9) and for the general case by Backus (Reference 10). In general, the following identities hold (Reference 11):

$$\operatorname{curl} \psi \mathbf{r} \equiv \operatorname{grad} \psi \times \mathbf{r} , \tag{19}$$

$$\operatorname{curl}\operatorname{curl} \psi \mathbf{r} \equiv \operatorname{grad} \frac{\partial}{\partial r}(\psi r) - \mathbf{r}\, \nabla^2 \psi . \tag{20}$$

By using Equation 18, a vector potential may be defined:

$$\mathbf{A} = \psi_1 \mathbf{r} + \operatorname{curl} \psi_2 \mathbf{r} , \tag{21}$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A} . \tag{22}$$

A satisfies the gauge condition

$$\operatorname{div} \mathbf{A} = r^{-2} \frac{\partial}{\partial r}\left(\psi_1 r^3\right) ,$$

and the electric field **E** may be expressed as

$$\mathbf{E} = -\operatorname{grad} \psi_0 - \frac{\partial \mathbf{A}}{\partial t} . \tag{23}$$

The problem thus reduces to solving Equation 1 for three unknown scalars ψ_0, ψ_1, and ψ_2. The derivation of a general solution tends to be tedious and therefore will only be outlined in this section. In each of the three regions Equation 1 can be put into the form

$$\operatorname{grad} \xi_i + \mathbf{r}\,\eta_i + \operatorname{curl} \mathbf{r}\,\zeta_i = 0 \tag{24}$$

(i is the region's index). Taking the curl and applying the uniqueness of the resolution (Equation 18) gives

$$\operatorname{curl} \mathbf{r}\,\eta_i = \operatorname{curl} \operatorname{curl} \mathbf{r}\,\zeta_i = 0 \ ,$$

from which

$$\eta_i = \eta_i(r) \ ,$$

$$\zeta_i = \zeta_i(r) \ ,$$

and, after substitution in Equation 24,

$$\xi_i = \xi_i(r) \ .$$

In general the expressions η_i and ζ_i will be functions of ψ_1 and ψ_2, which are also defined as arbitrary functions of r. If these functions can be chosen so that η_i vanishes, Equation 24 gives

$$\xi_i = \text{const} = C_i$$

Region I

In region I

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \mathbf{i}_\phi\, \omega r \sin\theta \ . \tag{25}$$

The following relations hold for $\mathbf{v}$ and any vector $\mathbf{A}$:

$$\mathbf{v} = \frac{1}{2} \operatorname{curl} \left(\boldsymbol{\omega} r^2\right) \ ,$$

$$\operatorname{curl} \mathbf{v} = 2\boldsymbol{\omega} \ ,$$

$$(\mathbf{A} \cdot \nabla)\,\mathbf{v} = \boldsymbol{\omega} \times \mathbf{A} \ ,$$

$$(\mathbf{v} \cdot \nabla)\,\mathbf{A} = \boldsymbol{\omega} \times \mathbf{A} + \omega\left(\mathbf{i}_r \frac{\partial A_r}{\partial \phi} + \mathbf{i}_\theta \frac{\partial A_\theta}{\partial \phi} + \mathbf{i}_\phi \frac{\partial A_\phi}{\partial \phi}\right)$$

(for example see Reference 12, Equation 1.4.3). To resolve $\mathbf{v} \times \mathbf{B}$ in the manner of Equation 18 note that

$$\begin{aligned}
\mathbf{v} \times \mathbf{B} &= \mathbf{v} \times \operatorname{curl} \mathbf{A} \\
&= \operatorname{grad}(\mathbf{A} \cdot \mathbf{v}) - (\mathbf{A} \times \operatorname{curl} \mathbf{v}) - (\mathbf{A} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{A} \\
&= \operatorname{grad}(\mathbf{A} \cdot \mathbf{v}) - \omega \left(\mathbf{i}_r \frac{\partial A_r}{\partial \phi} + \mathbf{i}_\theta \frac{\partial A_\theta}{\partial \phi} + \mathbf{i}_\phi \frac{\partial A_\phi}{\partial \phi} \right) ,
\end{aligned}$$

$$\mathbf{v} \times \mathbf{B} = - \operatorname{grad} \left(\omega r \sin\theta \frac{\partial \psi_2}{\partial \theta} \right) - \mathbf{r} \omega \frac{\partial \psi_1}{\partial t} - \operatorname{curl} \left(\mathbf{r} \omega \frac{\partial \psi_2}{\partial t} \right) . \tag{26}$$

The rest of Equation 1 is easily resolved in the prescribed manner by using Equations 18 and 21-23, which give

$$\xi_i (r) = \frac{\partial}{\partial r} (\psi_1 r) + \mu_0 \sigma_1 \psi_0 + \mu_0 \sigma_1 \omega r \sin\theta \frac{\partial \psi_2}{\partial \theta} , \tag{27}$$

$$\eta_i (r) = \mu_0 \sigma_1 \left(\frac{\partial \psi_1}{\partial t} + \omega \frac{\partial \psi_1}{\partial \phi} \right) - \nabla^2 \psi_1 , \tag{28}$$

$$\zeta_i (r) = \mu_0 \sigma_1 \left(\frac{\partial \psi_2}{\partial t} + \omega \frac{\partial \psi_2}{\partial \phi} \right) - \nabla^2 \psi_2 . \tag{29}$$

The option to add an arbitrary radial function to ψ_1 and ψ_2 is used to make both $\eta_1 (r)$ and $\zeta_1 (r)$ vanish, so that

$$\frac{\partial}{\partial r} (\psi_1 r) + \mu_0 \sigma_1 \psi_0 + \mu_0 \sigma_1 \omega r \sin\theta \frac{\partial \psi_2}{\partial \theta} = C_1 , \tag{30}$$

So far no use has been made of the fact that the field's source co-rotates with region I. Therefore the equations hold even when the field originates, say, in region III as a fixed "interstellar field." If Equations 11-13 are introduced,

$$\nabla^2 \psi_1 = 0 , \tag{31}$$

$$\nabla^2 \psi_2 = 0 . \tag{32}$$

If the field's source is assumed to be concentrated at the origin, it is useful to expand ψ_1 and ψ_2 in spherical harmonics; the expansion of ψ_2 then has a singularity at the origin corresponding to the source of the field. For instance, if this source is a dipole with moment $\mathbf{M}$, inclined at an angle κ to the rotation axis, then

$$\psi_1 = \sum_{nm} a_{nm} \rho^n P_n^m (\theta) \exp im (\phi - \omega t) , \tag{33}$$

$$\psi_2 = \sum_{nm} b_{nm} \rho^n P_n^m(\theta) \exp im(\phi - \omega t) + \beta\rho^{-2} [\cos\kappa\cos\theta + \sin\kappa\sin\theta\cos(\phi - \omega t)] , \tag{34}$$

where

$$\rho = \frac{r}{R_0} \quad \text{and} \quad \beta = \frac{\mu_0 M}{4\pi R_0^2} .$$

Region II

In this region

$$\mathbf{v} = u\mathbf{i}_r$$

The contribution of the toroidal component to $\mathbf{v} \times \mathbf{B}$ is

$$\begin{aligned} u(\mathbf{i}_r \times \operatorname{curl} \mathbf{i}_r \psi_1 r) &= u[\mathbf{i}_r \times (\operatorname{grad} \psi_1 r \times \mathbf{i}_r)] \\ &= u\left[\operatorname{grad}(\psi_1 r) - \mathbf{i}_r \frac{\partial}{\partial r}(\psi_1 r)\right] \end{aligned} \tag{35}$$

and the contribution of the poloidal component is, by Equation 19,

$$u\left[\mathbf{i}_r \times \operatorname{grad} \frac{\partial}{\partial r}(\psi_2 r)\right] = -\operatorname{curl} \mathbf{r}\left[\frac{u}{r}\frac{\partial}{\partial r}(\psi_2 r)\right] \tag{36}$$

All the other terms of Equation 1 are the same as for region I. Therefore

$$\xi_2(r) = \frac{\partial}{\partial r}(\psi_1 r) + \mu_0\sigma_2\psi_0 - u\mu_0\sigma_2\psi_1 r , \tag{37}$$

$$\eta_2(r) = \mu_0\sigma_2\left[\frac{\partial\psi_1}{\partial t} + \frac{u}{r}\frac{\partial}{\partial r}(\psi_1 r)\right] - \nabla^2\psi_1 , \tag{38}$$

$$\zeta_2(r) = \mu_1\sigma_2\left[\frac{\partial\psi_2}{\partial t} + \frac{u}{r}\frac{\partial}{\partial r}(\psi_2 r)\right] - \nabla^2\psi_2 . \tag{39}$$

As in region I, ψ_1 and ψ_2 are chosen so that η_2 and ζ_2 vanish and ξ_2 equals a constant, C_2. From Equations 11-13 the angular part of ψ_1 is now expanded in spherical harmonics:

$$\psi_1 = \sum_{nm} G_{nm}(r) P_n^m(\theta) \exp im(\phi - \omega t) . \tag{40}$$

Inserting this into Equation 38 and using the independence of spherical harmonics gives, for any n and m,

$$\frac{1}{r}\frac{d_2}{dr^2}\left(rG_{nm}\right) - \frac{u\mu_0\sigma_2}{r}\frac{d}{dr}\left(rG_{nm}\right) - \left[\frac{n(n+1)}{r^2} - im\omega\mu_0\sigma_2\right]G_{nm} = 0 \qquad (41)$$

Multiplying by rR_0^2 and defining

$$y_{nm}(\rho) = \rho G_{nm}(\rho)$$

gives

$$y''_{nm} - u\mu_0\sigma_2 R_0 y'_{nm} + \left[i\mu_0\sigma_2\omega R_0^2 m - n(n+1)\rho^{-2}\right]y_{nm} = 0 \qquad (42)$$

The magnetic Reynolds number associated with radial outflow (in region II) may be defined by

$$2\alpha = \mu_0\sigma_2 R_0 u ,$$

and that associated with rotation by

$$2\alpha_{2\omega} = \mu_0\sigma_2 R_0^2\omega .$$

Substituting

$$y_{nm} = u_{nm}\exp\alpha\rho$$

and defining complex Reynolds numbers $2\alpha_{2c}$ of order m,

$$\alpha_{2c}^2 = \alpha^2 - 2im\alpha_{2\omega} ,$$

we have

$$u''_{nm} - u_{nm}\left[\alpha_{2c}^2 + n(n+1)\rho^{-2}\right] = 0 . \qquad (43)$$

With operators $\mathfrak{L}_n$ defined by

$$\mathfrak{L}_n = \rho^{-(n+1)}\left(\rho^3\frac{d}{d\rho}\right)^n \rho^{-(2n-1)} , \qquad (44)$$

the general solution of Equation 43 may be written (Reference 13, p. 337, Equation 256)

$$u_{nm} = A_1 \mathfrak{L}_n \exp \alpha_{2c}\, \rho + A_2 \mathfrak{L}_n \exp - \alpha_{2c}\, \rho$$

$$= A_1\, u_{nm1}(\rho) + A_2 (-1)^n u_{nm2}(\rho) . \tag{45}$$

By defining

$$g_{nmi} = \rho^{-1} u_{nmi} \exp \alpha\rho ,$$

with $i = 1, 2$, we can write the general forms of ψ_1 and ψ_2 in region II as

$$\psi_1 = \sum_{nm} P_n^m(\theta) \exp im(\phi - \omega t) \left[a_{nm1}\, g_{nm1}(\rho) + a_{nm2}\, g_{nm2}(\rho)\right] , \tag{46}$$

$$\psi_2 = \sum_{nm} P_n^m(\theta) \exp im(\phi - \omega t) \left[b_{nm1}\, g_{nm1}(\rho) + b_{nm2}\, g_{nm2}(\rho)\right] . \tag{47}$$

Region III

In this region Equation 1 has no $\mathbf{v} \times \mathbf{B}$ term. Consequently

$$\xi_3(r) = \frac{\partial}{\partial r}(\psi_1 r) + \mu_0 \sigma_3 \psi_0 , \tag{48}$$

$$\eta_3(r) = \mu_0 \sigma_3 \frac{\partial \psi_1}{\partial t} - \nabla^2 \psi_1 , \tag{49}$$

$$\zeta_3(r) = \mu_0 \sigma_3 \frac{\partial \psi_2}{\partial t} - \nabla^2 \psi_2 , \tag{50}$$

As before, ψ_1 and ψ_2 are chosen so that η_3 and ζ_3 vanish and ξ_3 is a constant, C_3. Now ψ_1 is expanded in a fashion similar to that used for Equation 40,

$$\psi_1 = \sum_{nm} H_{nm}(r)\, P_n^m(\theta) \exp im(\phi - \omega t) . \tag{51}$$

If $m = 0$ the terms have no time dependence, Equation 49 shows they are harmonic, and $H_{nm}(r)$ is proportional to $r^{-(n+1)}$. If $m \neq 0$, the solution proceeds as for region II. By defining

$$z_{nm}(\rho) = \rho H_{nm}(\rho) ,$$

and a complex magnetic Reynolds number $2\alpha_{3c}$ for region III,

$$\alpha_{3c}^2 = -2\mathrm{im}\,\alpha_{3\omega} ,$$

$$\alpha_{3c} = \alpha_{3r} + i\,\alpha_{3i} ,$$

it is found that

$$z_{nm}'' - z_{nm}\left[n(n+1)\rho^{-2} + \alpha_{3c}^2\right] = 0 , \tag{52}$$

which has the same form as Equation 43. This leads to

$$\psi_1 = \sum P_n^m(\theta)\exp \mathrm{im}(\phi - \omega t)\left[A_{nm1}\,h_{nm1}(\rho) + A_{nm2}\,h_{nm2}(\rho)\right] , \tag{53}$$

where

$$h_{nm1}(\rho) = \rho^{-1}\,\mathfrak{L}_n \exp \alpha_{3c}\,\rho ,$$

$$h_{nm2}(\rho) = (-1)^n\,\rho^{-1}\,\mathfrak{L}_n \exp -\alpha_{3c}\,\rho .$$

There is a choice of sign for the real part of α_{3c}. By choosing α_{3r} as positive, $h_{nm1}(\rho)$ contains $\exp \alpha_{3r}\,\rho$ which causes it to diverge. Therefore, all A_{nm1} vanish and Equation 53 becomes

$$\psi_1 = \sum A_{nm2}\,h_{nm2}(\rho)\,P_n^m(\theta)\exp \mathrm{im}(\phi - \omega t) . \tag{54}$$

In a similar manner

$$\psi_2 = \sum B_{nm2}\,h_{nm2}(\rho)\,P_n^m(\theta)\exp \mathrm{im}(\phi - \omega t) . \tag{55}$$

Contributions by a fixed outside source may be included in Equation 55. Usually Equations 11-13 then no longer hold and the calculation is somewhat different.

CONTINUITY CONDITIONS

On the boundaries the components of **B**, the tangential component of **E**, and the normal component of curl **B** are continuous. The operator Λ^2 is defined by (Reference 10)

$$\Lambda^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} , \tag{56}$$

so that

$$\nabla^2 = \frac{1}{r}\frac{\partial^2 r}{\partial r^2} + \frac{1}{r^2}\Lambda^2 , \tag{57}$$

and

$$\left[\Lambda^2 + n(n+1)\right] P_n^m(\theta) \exp im\phi = 0 . \tag{58}$$

From Equations 18 and 20

$$rB_r = -\Lambda^2 \psi_2 , \tag{59}$$

and the fact that all boundaries are spherical implies the continuity of all expansion terms of ψ_2, as expressed in Equations 34, 47, and 55. In a similar way

$$r(\mathrm{curl}\, \mathbf{B})_r = -\Lambda^2 \psi_1$$

implies the continuity of the terms of ψ_1. On a spherical boundary $\mathrm{curl}\, \mathbf{r}\psi_1$, is also continuous. By Equation 18 so is $\mathrm{curl}\ \mathrm{curl}\, \mathbf{r}\psi_2$ and it may be shown from this that $\partial\psi_2/\partial r$ is also continuous across the boundaries. Finally, by using the above results with Equations 21 and 23 ψ_0 is also found to be continuous. Thus four conditions are to be met by each of the two boundaries and by eight sets of undetermined coefficients. Because the source of the field is poloidal, the continuity of the components of ψ_2 and $\partial\psi_2/\partial r$ makes it possible to evaluate ψ_2 independently. Next ψ_1 is found, by using Equations 30, 37, and 48 to express the continuity of ψ_0. Finally ψ_0 is obtained by using the same three equations.

As σ_1 and σ_2 increase without limit, the solutions tend to approach those obtained before by assuming $\mathbf{E} = -(\mathbf{v} \times \mathbf{B})$. In general, the solutions diverge in region III unless σ_3 is kept finite. This may be interpreted as meaning that, in a system having stationary and rotating ideal conductors in contact, infinite currents will be excited by unipolar induction. If the hydromagnetic equation were used this divergence would not occur, because infinite forces would act on the flow and distort its pattern.

In the example that follows this divergence is avoided in a different way, namely by the assumption that in region III conductivity is arbitrary (finite or infinite) along lines of force, and zero across them.

A WORKED EXAMPLE

As a simple illustration of the preceding discussion, the case will now be considered for a field source of dipole of moment $\mathbf{M}$, located at the origin and aligned with the axis of rotation. This source produces an axisymmetric field and therefore the variables ϕ and t as well as the expansion index m are absent. It is found convenient to use dimensionless units and to define

$$\rho = \frac{r}{R_0} ,$$

$$\psi_{01} = \frac{\psi_0}{u R_0} ,$$

$$2\alpha_1 = \mu_0 \sigma_1 u R_0 .$$

The scalars defining the field then obey the following equations. In region I

$$\nabla^2 \psi_1 = 0 , \tag{31}$$

$$\nabla^2 \psi_2 = 0 , \tag{32}$$

and from Equation 30

$$\psi_{01} = -\frac{\omega}{u} \rho \sin\theta \frac{\partial \psi_2}{\partial \theta} - \frac{1}{2\alpha_1} \frac{\partial}{\partial \rho} (\psi_1 \rho) + C_{11} . \tag{60}$$

In region II

$$\nabla^2 \psi_1 - \frac{2\alpha}{\rho} \frac{\partial}{\partial \rho} (\psi_1 \rho) = 0 , \tag{61}$$

$$\nabla^2 \psi_2 - \frac{2\alpha}{\rho} \frac{\partial}{\partial \rho} (\psi_2 \rho) = 0 , \tag{62}$$

$$\psi_{01} = \psi_1 \rho - \frac{1}{2\alpha} \frac{\partial}{\partial \rho} (\psi_1 \rho) + C_{21} . \tag{63}$$

And in region III, if an isotropic conductivity σ_3 is assumed,

$$\nabla^2 \psi_1 = 0 , \tag{64}$$

$$\nabla^2 \psi_2 = 0 \ , \tag{65}$$

$$\psi_{01} = -\left(\frac{1}{\mu_0 \sigma_3 u R_0}\right) \frac{\partial}{\partial \rho} (\psi_1 \rho) + C_{31} \ . \tag{66}$$

As mentioned at the end of the previous section, in order to prevent divergence as $\sigma_3 \to \infty$ and also to take into account, at least partially, the anisotropy of conductivity, it will be assumed that in region III the conductivity is σ_3 for flow along lines of force and zero for flow across them. Then, in region III

$$\mathbf{B} \times \mathrm{curl}\, \mathbf{B} = 0 \ , \tag{67}$$

$$\mathbf{B} \cdot \mathrm{curl}\, \mathbf{B} = \mu_0 \sigma_3 (\mathbf{E} \cdot \mathbf{B}) \ . \tag{68}$$

These equations are nonlinear; however, it will be shown that a solution having the required form near the boundary is obtained by taking

$$\mathrm{curl}\, \mathbf{B} = 0 \ , \tag{69}$$

from which

$$\mathbf{E} \cdot \mathbf{B} = 0 \ . \tag{70}$$

By using Equations 18 and 20 and applying the treatment of Equation 24, Equation 69 gives

$$\nabla^2 \psi_2 = 0 \ , \tag{71}$$

$$\psi_1 = 0 \ . \tag{72}$$

The unknown scalars are expanded:

For region I, as in Equations 33 and 34,

$$\psi_1 = \sum a_n \rho^n P_n(\theta) \ , \tag{73}$$

$$\psi_2 = \sum b_n \rho^n P_n(\theta) + \beta \rho^{-2} \cos\theta \ . \tag{74}$$

For region II

$$\psi_1 = \sum \left[a_{n1} g_{n1}(\rho) + a_{n2} g_{n2}(\rho)\right] P_n(\theta) \ , \tag{75}$$

$$\psi_2 = \sum \left[b_{n1}\, g_{n1}(\rho) + b_{n2}\, g_{n2}(\rho) \right] P_n(\theta) \; . \tag{76}$$

And for region III, with $\rho_1 = R_1/R_0$,

$$\psi_2 = \sum B_n \left(\frac{\rho_1}{\rho} \right)^{n+1} P_n(\theta) \; . \tag{77}$$

To begin with, ψ_2 is derived. For $n \neq 1$ the continuity equations of ψ_2 and $\partial\psi_2/\partial r$ yield a set of four equations with unknowns b_n, b_{n1}, b_{n2} and B_n. Because of the form of the source term in Equation 74 these equations are homogeneous and all coefficients vanish. For $n = 1$, by Equation 44,

$$g_{11}(\rho) = \left(\alpha\rho^{-1} - \rho^{-2} \right) \exp 2\alpha\rho \; , \tag{78}$$

$$g_{12}(\rho) = \alpha\rho^{-1} + \rho^{-2} \; . \tag{79}$$

Denoting differentiation with respect to ρ by primes gives

$$\left. \begin{aligned} b_1 + \beta &= b_{11}\, g_{11}(1) + b_{12}\, g_{12}(1) \; , \\ b_1 - \beta &= b_{11}\, g_{11}'(1) + b_{12}\, g_{12}'(1) \; , \\ B_1 &= b_{11}\, g_{11}(\rho_1) + b_{12}\, g_{12}(\rho_1) \; , \\ -2B_1 &= b_{11}\, \rho_1\, g_{11}'(\rho_1) + b_{12}\, \rho_1\, g_{12}'(\rho_1) \; . \end{aligned} \right\} \tag{80}$$

The complete solution is rather lengthy. If $\alpha >> 1$, which represents the "high conductivity" case, and $\rho_1 >> 1$, in region I

$$\psi_2 \approx \beta \cos\theta \left(\frac{1}{2}\rho + \rho^{-2} \right) \; , \tag{81}$$

in region II

$$\psi_2 \approx \left(\frac{3\beta}{2\rho} \right) \cos\theta \left[1 - \left(2\alpha\rho_1 \right)^{-1} \exp - 2\alpha \left(\rho_1 - \rho \right) \right] \; , \tag{82}$$

and in region III

$$\psi_2 \approx \left(\frac{3\beta\rho_1}{2\rho^2} \right) \cos\theta \; . \tag{83}$$

The poloidal field in region III is thus that of an axial dipole, whereas in region II

$$B_r = \left(\frac{3\beta}{R_0}\right)\cos\theta\,\rho^{-2} , \tag{84}$$

$$B_\theta = \left(\frac{3\beta}{2\,R_0\,\rho_1}\right)\sin\theta\exp - 2\alpha\left(\rho_1 - \rho\right) , \tag{85}$$

which may be compared with Equations 6-8. In region I

$$B_r = \left(\frac{\beta}{R_0}\right)\cos\theta\left(2\rho^{-3} + 1\right) ,$$

$$B_\theta = \left(\frac{\beta}{R_0}\right)\sin\theta\left(\rho^{-3} - 1\right) .$$

The lines of force of this solution are given in Figure 2, for the limit of high conductivity.

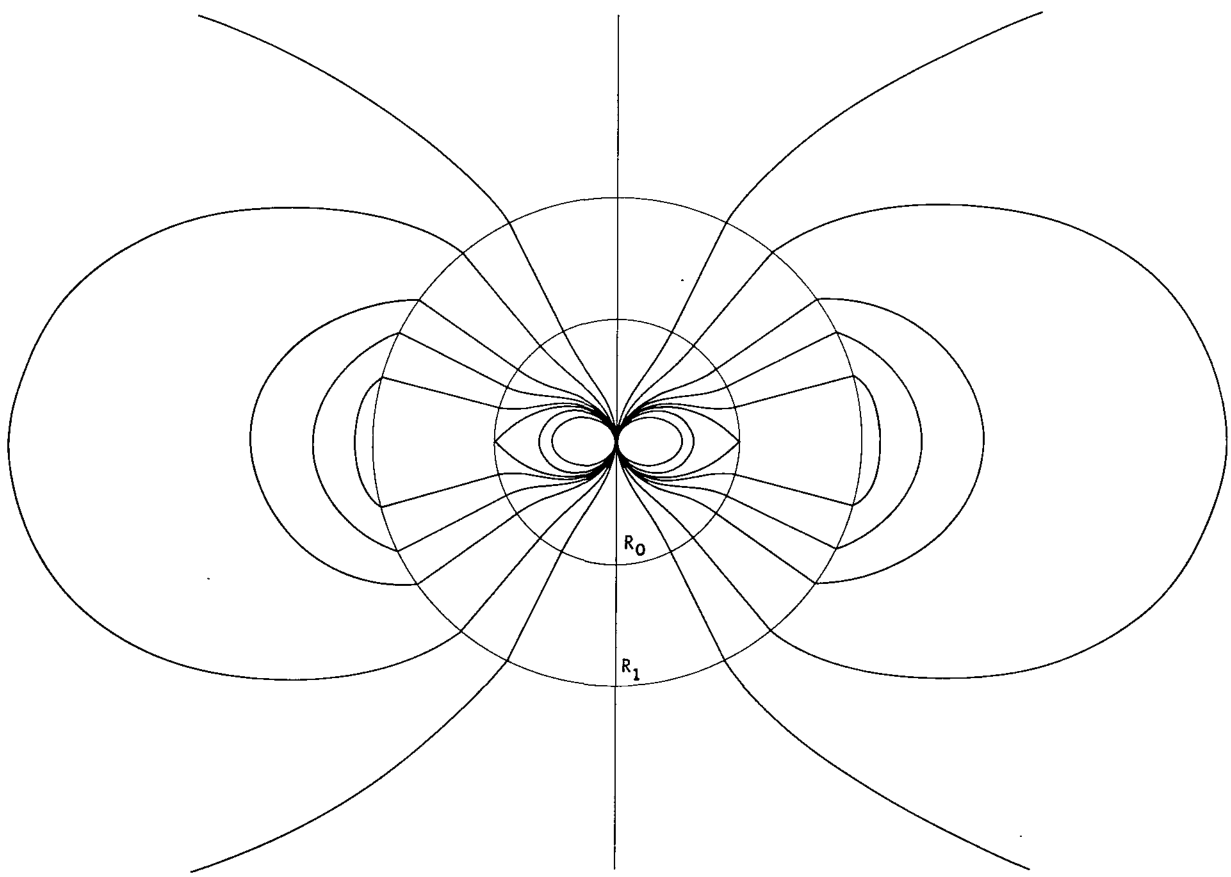

Figure 2—Lines of force for the poloidal component, in the limit of very high conductivity.

Because of Equation 72, ψ_1 is expressed by only three sets of undetermined coefficients and the fact that ψ_0 is continuous when $r = R_1$ does not have to be invoked. By Equation 30, in region I

$$\psi_{01} = \frac{\omega}{u}\left(b_1 \rho^2 + \frac{\beta}{\rho}\right) \sin^2\theta - \frac{1}{2\alpha_1} \frac{\partial}{\partial\rho}\left(\psi_1 \rho\right) + C_{11} . \tag{86}$$

Because

$$\sin^2\theta \equiv \frac{2}{3}\left(P_0 - P_2\right) , \tag{87}$$

only for $n = 0,2$ do the coefficients of ψ_1 satisfy inhomogeneous equations and therefore differ from zero. The monopole terms do not contribute to the magnetic field, so that only the quadrupole part has to be evaluated. In region II

$$g_{21}(\rho) = \left(\alpha^2 \rho^{-1} - 3\alpha\rho^{-2} + 3\rho^{-3}\right) \exp 2\alpha\rho , \tag{88}$$

$$g_{22}(\rho) = \alpha^2 \rho^{-1} + 3\alpha\rho^{-2} + 3\rho^{-3} , \tag{89}$$

from which

$$\left.\begin{aligned} a_2 &= a_{21}\, g_{21}(1) + a_{22}\, g_{22}(1) , \\ 0 &= a_{21}\, g_{21}(\rho_1) + a_{22}\, g_{22}(\rho_1) , \\ 2\alpha_1\gamma - 3a_2 &= a_{21}\lambda_1 + a_{22}\lambda_2 , \end{aligned}\right\} \tag{90}$$

where

$$\gamma = -\frac{2\omega}{3u}\left(b_1 + \beta\right) ,$$

$$\lambda_1 = -3\alpha_1\left(1 - \frac{2}{\alpha}\right) \exp 2\alpha ,$$

$$\lambda_2 = 2\alpha_1\left(\alpha^2 + 3\alpha + \frac{9}{2} + 3\alpha^{-1}\right) .$$

This again tends to produce cumbersome expressions. When $\alpha \gg 1$ and $\rho_1 \gg 1$, in region II

$$\psi_1 \approx \gamma\rho^{-1}\left[1 - \exp 2\alpha\left(\rho - \rho_1\right)\right] P_2(\theta) + f(r) , \tag{91}$$

and in region I

$$\psi_1 \approx \gamma\rho P_2(\theta) + \text{const} . \tag{92}$$

In region II, by Equation 19,

$$B_\phi \approx 3\gamma\rho^{-1}\left[1 - \exp 2\alpha\left(\rho - \rho_1\right)\right] \sin\theta \cos\theta \ , \tag{93}$$

which should be compared with Equation 7.

Finally, ψ_0 is derived. Equation 70 implies that in region III the lines of force of B lie on equipotentials of ψ_0. B in that region was found to be a dipole field and its lines of force are

$$\frac{\sin^2\theta}{\rho} = \text{const} \ .$$

Thus in region III

$$\psi_0\left(r, \theta\right) = \psi_0\left(\frac{\sin^2\theta}{\rho}\right) = \sum \epsilon_n \left(\frac{\sin^2\theta}{\rho}\right)^n \tag{94}$$

By Equations 44 and 45, or by direct calculation of a θ-independent solution of Equation 61, the monopole term of ψ_1 is, in region II,

$$\psi_1\,(\text{mono}) = a_{01}\rho^{-1} \exp 2\alpha\rho + a_{02}\rho^{-1} \ ,$$

which with Equation 63 gives

$$\psi_{01}\,(\text{mono}) = a_{02} + C_{21}$$

By assuming as before that $\alpha \gg 1$ and $\rho_1 \gg 1$, Equations 63 and 91 give, in region II,

$$\psi_{01} = \gamma P_2\,(\theta) + C_{22} \ .$$

Combining this with Equations 87 and 94 shows that

$$C_{22} = -\gamma \ .$$

Thus in region II

$$\psi_{01} = -\frac{3}{2}\gamma \sin^2\theta \ , \tag{95}$$

and in region III

$$\psi_{01} = -\frac{3}{2}\gamma \sin^2\theta \left(\frac{\rho_1}{\rho}\right) . \tag{96}$$

It should be noted that in region II ψ_{01} does not depend on r. The equipotentials thus tend to be (with large α and ρ_1) cones of constant θ.

The potential in region I is determined in the same manner. If $\alpha_1 >> 1$, the term containing ψ_1 in Equation 86 may be neglected and

$$\psi_{01} = \left(\frac{\omega\beta}{u}\right)\left(\frac{1}{2}\rho^2 + \rho^{-1}\right)\sin^2\theta . \tag{97}$$

The approximation involved here is the same as for Equation 2, so that Equation 70 holds in all regions and the magnetic lines of force lie on equipotential surfaces. Thus Figure 2 may also be viewed as a cross section of the equipotentials of the electric field.

It is interesting to note that, in the limiting case of high conductivity, the electric potential in region II, especially near the equatorial plane, is different from that at infinity. In particular, the equatorial plane is an equipotential in which, by Equation 97,

$$\psi_0 = \frac{3}{2}\omega R_0 \beta . \tag{98}$$

The quantity β/R_0, which is of the order of the field at $r = R_0$, will be taken as 1 gauss; Equation 98 then gives a potential of about 2×10^8 volts. This result may have some connection with the modulation of cosmic radiation by the solar activity cycle which resembles that produced by an electric field (Reference 14); however, the value of ψ_0 deduced here is too small by a factor 5-10, and it should be borne in mind that when the solar dipole reverses its direction, as observed in 1958 (Reference 15), ψ_0 reverses its sign.

To the preceding example may be added a homogeneous "interstellar magnetic field" $\mathbf{B}_0$ which, in order to preserve symmetry, will be assumed to be parallel to the rotation axis (for an arbitrarily directed $\mathbf{B}_0$ the calculation is more involved). Such a field can be represented by a poloidal potential

$$\psi_2 = \frac{1}{2} B_0 r \cos\theta , \tag{99}$$

so that Equation 77 is replaced by

$$\psi_2 = \sum B_n \left(\frac{\rho_1}{\rho}\right)^{n+1} P_n(\theta) + \frac{1}{2} B_0 R_0 \rho \cos\theta . \tag{100}$$

The inclusion of $\mathbf{B}_0$ causes a term $R_1 B_0/2$ to be added on the left of the last two of Equations 80. When these equations are solved, it turns out that b_{12} (the important coefficient in region II) is modified by a factor of

$$1 + \left(\frac{B_0 R_1}{2\beta}\right) \exp 2\alpha\left(1 - \rho_1\right) ,$$

which generally will differ very slightly from unity for large α and ρ_1. Thus ψ_2, and consequently ψ_1 and ψ_0, are only negligibly affected in region I and in most of region II. Note that it is quite possible for the outlying interplanetary field to be much weaker than the surrounding interstellar one. The solar wind then scoops out a cavity in the interstellar field, as first suggested by Davis (Reference 16).

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