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GUIDANCE EQUATIONS FOR QUASI-OPTIMUM SPACE MANEUVERS

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Manned Spacecraft Center
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SUMMARY

Approximate time-optimum guidance equations for planetary ascent and descent trajectories at constant thrust are derived for a flat-body two-dimensional case. No aerodynamic forces are considered. In addition, this case was extended by introducing power series to simulate the deleted terms of the round-body solution. A power series in time was chosen to retain the form of the guidance law. The resulting transcendental equations are not explicit in the control variables; hence an iterative scheme is used to solve the conditioned equations for the required control.

Trajectories resulting from this set of equations are compared with the strict optimum for a number of cases to determine the proximity of the solutions. The results match well for both state and control variables in the cases considered.

The guidance application was investigated by using the analytic solution to steer an exact integrated model. A comparison of these results with an optimum solution for ascent and descent trajectories shows good agreement.

To demonstrate its utility as a trajectory scanner, the technique is applied to the investigation of lunar landing from an elliptic transfer orbit.

INTRODUCTION

The requirement for optimum trajectories in preliminary design work has steadily increased in the past few years. The chief deterrent to the solution of the nonlinear differential equations has been the lack of a process which would converge rapidly to the desired boundary conditions. The first-order perturbation techniques used in convergence processes normally work well in the vicinity of the solution where the linearity assumption holds true, but they are usually unpredictable in areas not near the solution. These, unfortunately, are the areas most frequently encountered.

An analytical scheme which provides reasonable approximations to the optimum solution would be of significant value, first, as a scanning technique for providing rapid estimates of optimum performance and, second, as a method of determining approximate initial values for the true optimum solution. The use of such a technique as a guidance scheme also has possibilities if the solution time is not unreasonable and if the errors generated by the approximations to the true dynamic equations are not excessive.

The present paper deals with an analytical technique derived by operating on a set of linearized two-body equations with the calculus of variations to determine a guidance law. The planar case, in which aerodynamic forces are neglected, is treated. The solution differs from that derived in reference 1, in that for the present paper a constant thrust rather than a constant acceleration is assumed. The equations are first reduced to a flat-body set by allowing the radius to approach infinity. By integration, three simultaneous transcendental equations in three unknowns can be derived from this set of equations. Trajectories resulting from this set are compared with optimum trajectories to determine what errors are present and which terms, deleted by the flat-body approximation, are the principal contributors to the error. Suitable approximate expressions are introduced to the flat-body set of equations to simulate the deleted terms in an attempt to reduce these errors. The equations are again integrated, and the results are compared with the optimum for a number of trajectories. The scheme is tested as a guidance technique by using the analytic solution as a feedback to the exact integrated model to determine the speed of solution and the penalty incurred as compared with the optimum.

SYMBOLS

A	constant defined by equation (20)
$a_1,a_2,\ldots,a_n$	coefficients of time series
a_{ij}	elements of partial matrix
a_v,a_u	components of acceleration defined by equations (24) and (25), ft/sec^2
C_1,C_2,C_3,C_4	initial values of Lagrange multipliers, radians/sec
c	effective exhaust velocity, ft/sec
F_1,F_2,F_3	transcendental functions
f	function defined by equation (A1)
G	function defined by equation (A2)
g	gravitational acceleration, ft/sec^2
H	modified Hamiltonian
$h_1,h_2,\ldots,h_n$	coefficients of time series

I	performance index
I_{sp}	specific impulse, sec
K	constant defined by equations (22) and (23)
k	function defined by equation (21)
M	boundary vector constraint
m	mass of vehicle, slugs
P	functions of time
Q	functions of time
$\vec{q}(t)$	state vector
r	radius to vehicle from center of body, ft
r_b	radius of the reference body, ft
T	thrust, lb
t	time, sec
u,v	components of total velocity in x and y directions, respectively, ft/sec
V	total velocity, ft/sec
V_c	characteristic velocity, ft/sec
W	weight of vehicle, lb
x,y	rotating coordinates of mass particle, ft
y_s	distance from origin to surface of body, ft
z	tangent of θ
α_1,α_2	functions defined by equations (19) and (18), respectively
γ	flight-path angle, deg
Δt	increment of time, sec
ΔV	change in total velocity, ft/sec

ΔV_c	change in characteristic velocity, ft/sec
ϵ_t	tolerance on time-to-go
ϵ_{θ_0}	tolerance on initial pitch angle
ϵ_{θ_1}	tolerance on final pitch angle
η	true anomaly
θ	pitch angle, deg, or thrust angle with respect to x-axis, radians
$\vec{\theta}(t)$	control vector
$\lambda_1,\lambda_2,\lambda_3,\lambda_4$	Lagrange multiplier, radians/sec
μ	normalized mass ratio, m/m_0
ν	constant Lagrange multiplier
$\sigma_1,\sigma_2,\sigma_3$	functions defined by equations (C24), (C25), and (C26), respectively
φ	initial thrust acceleration, T/m_0, ft/sec^2
ψ_1,ψ_3	functions defined by equations (36) and (C27), respectively

Subscripts:

0,1	initial and final values of state and control variables
A	analytic quantity
I	integrated quantity
i,j	partial derivative indices
m	order of correction
min	minimum
n	nth order term of time series

Superscript:

T transpose

Operators:

$(\dot{\ })$ time differentation

$\delta(\ )$ variational operator

FORMULATION OF THE PROBLEM

The mathematical model employed in this investigation is a mass particle with two degrees of freedom referred to a set of rotating coordinates with the x-axis alined along the local horizon. The axis system and the associated notation are illustrated in figure 1.

The equations of motion for this model are

$$\dot{x} = u \tag{1}$$

$$\dot{y} = v \tag{2}$$

$$\dot{u} = \frac{\varphi}{\mu} \cos\theta - uv/r \tag{3}$$

$$\dot{v} = \frac{\varphi}{\mu} \sin\theta - g + u^2/r \tag{4}$$

where

$$\mu = 1 + \dot{\mu}t \tag{5}$$

$$\dot{\mu} = \dot{m}/m_0 \tag{6}$$

$$g = g_0 \left(r_b/r\right)^2 \tag{7}$$

$$r = r_b - y_s \tag{8}$$

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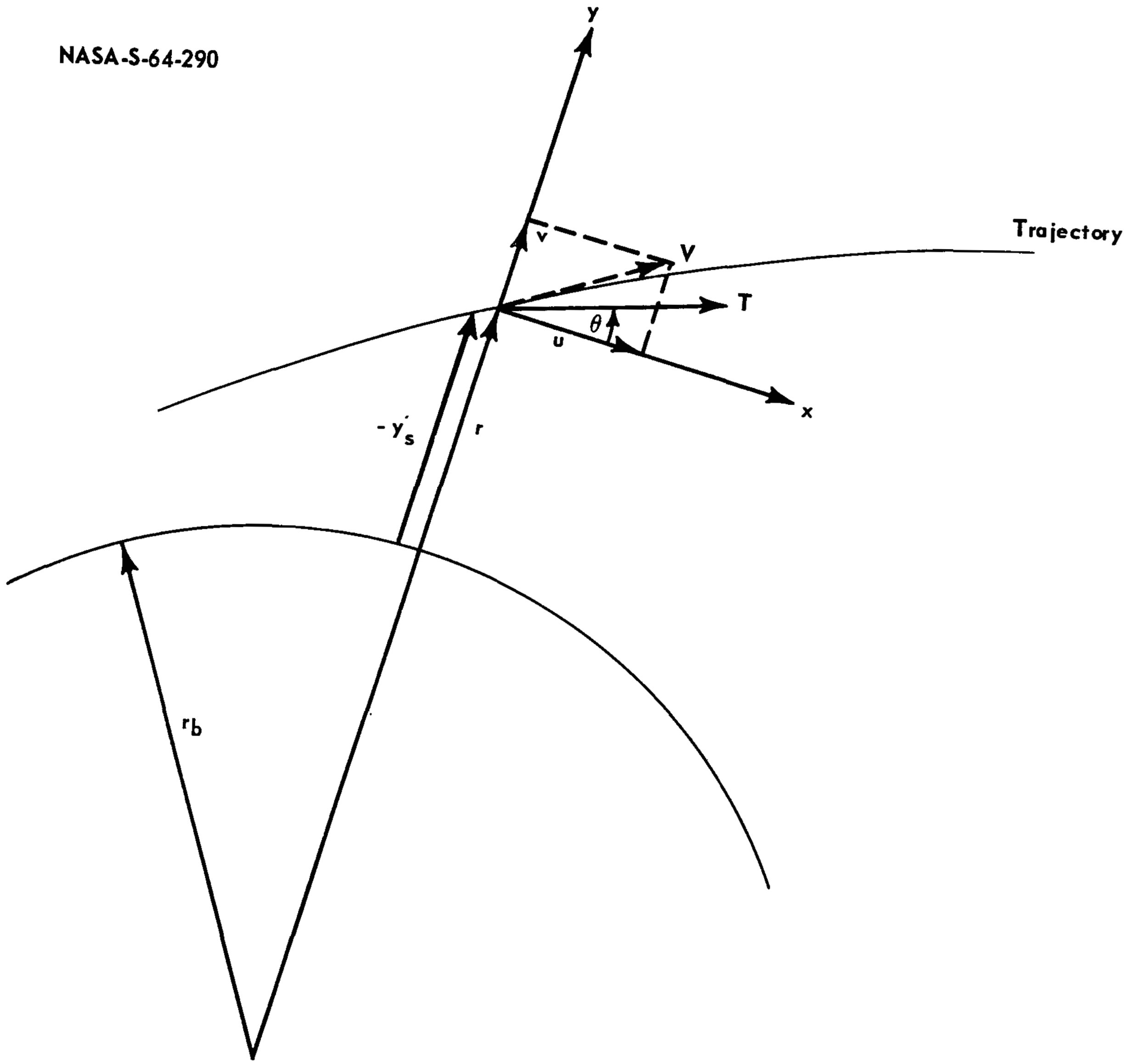

Figure 1.- Coordinate system and angle definition.

The vehicle is assumed to have a constant thrust and mass flow rate. The angle θ is the control variable.

The final time t_1 was selected to be minimized since under the previously mentioned assumptions it will yield minimum fuel consumption.

If in equations (3), (4), and (7), r_b is allowed to approach infinity, the coupling effects vanish and the gravitational acceleration approaches its surface value g_0. The following flat body equations result:

$$\dot{x} = u \tag{9}$$

$$\dot{y} = v \tag{10}$$

$$\dot{u} = \frac{\varphi}{\mu} \cos \theta \tag{11}$$

$$\dot{v} = \frac{\varphi}{\mu} \sin \theta - g_0 \tag{12}$$

The following pitch-angle relationship (ref. 1) is obtained when the calculus of variations is applied to this set of equations and the final range is allowed to be free.

$$\tan \theta = \frac{C_4 - C_2 t}{C_3} \tag{13}$$

The derivation of the guidance law (eq. (13)) is given in appendix A.

The solution is now entirely specified, because when the constants C_2/C_3 and C_4/C_3 are given, the control variable is specified; and equations (9) to (12) may be integrated to yield a solution. To determine a unique solution, the constants C must be found as functions of the desired boundary conditions. This result is accomplished by integrating equations (9) to (12) between the bounds specified at $t = 0$ and $t = t_1$. The constants C_4/C_3 and C_2/C_3 are obtained by evaluating equation (13) at the boundaries and yield

$$\left.\begin{aligned} \frac{C_4}{C_3} &= \tan \theta_0 \\ \frac{C_2}{C_3} &= \frac{\tan \theta_0 - \tan \theta_1}{t_1} \end{aligned}\right\} \tag{14}$$

where $\theta(0) = \theta_0$ and $\theta(t_1) = \theta_1$.

Integration of equations (9) to (12) results in the following three simultaneous transcendental equations in the three unknowns θ_0, θ_1, and t_1.

$$\mu_1 v_1 - \dot{\mu} y_1 - g_0 t_1\left(1 + \frac{\dot{\mu} t_1}{2}\right) - \frac{\alpha_2}{\dot{\mu}}\left(\varphi \sec \theta_1 - A\right) = 0 \tag{15}$$

$$y_1 - \mu_1 y_0 + \frac{g_0 t_1^{\,2}}{2\mu_1} - At_1 - \frac{\varphi \mu_1 K}{\dot{\mu}^2}\left[\log_e\left(\frac{\tan \theta_1 + K \sec \theta_1}{\tan \theta_0 + K \sec \theta_0}\right)\right.$$

$$\left. + K\alpha_2\left(\sec \theta_0 - \frac{\sec \theta_1}{\mu_1}\right) - \frac{K\dot{\mu}\alpha_1}{\varphi \alpha_2}\left(u_1 - u_0\right)\right] = 0 \tag{16}$$

$$\frac{Kk^{1/2}\,\dot{\mu}}{\varphi \alpha_2}\left(u_1 - u_0\right) - \log_e\left[\frac{k - \alpha_1 \mu_1 - \alpha_2\, Kk^{1/2} \sec \theta_1}{\left(k - \alpha_1 - \alpha_2\, Kk^{1/2} \sec \theta_0\right)\mu_1}\right] = 0 \tag{17}$$

The integrations and associated transformations are indicated in appendix B. In equations (15) to (17),

$$\alpha_2 = \frac{\dot{\mu} t_1}{\left(\tan \theta_1 - \tan \theta_0\right)} \tag{18}$$

$$\alpha_1 = 1 - \alpha_2 \tan \theta_0 \tag{19}$$

$$A = -\frac{\dot{\mu}}{\alpha_2}\left(v_0 - \dot{\mu} y_0\right) + \varphi \sec \theta_0 \tag{20}$$

$$k = \alpha_1^{\,2} + \alpha_2^{\,2} \tag{21}$$

$$K = 1 \qquad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right) \tag{22}$$

$$K = -1 \qquad \left(\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right) \tag{23}$$

Because the control quantities t_1, θ_0, and θ_1 occur implicitly in this set of equations, it is impossible to solve for them directly. Since the equations are transcendental, a first-order perturbation technique (ref. 2) is utilized to force the solution to converge. This technique, which generates a partial matrix, assumes that a solution exists if the matrix of partials has an inverse. This convergence technique and the associated partial derivative matrix are given in appendix C.

Figures 2(a) and 2(b) illustrate the errors of this solution as compared with an optimum integrated round body for different values of the constant g_0. It should be noted that the errors generally decrease with increasing T/W_0. This follows logically since burning time $t_1 \propto (T/W_0)^{-1}$ and the surface integrated over approaches a flat body as the burning time approaches zero.

The question of what can be done to improve this solution (indicated in figs. 2(a) and 2(b)) without destroying the analytic properties of the problem is now posed. Certainly, the errors are the result of neglecting the cross-coupling effects and setting g equal to a constant.

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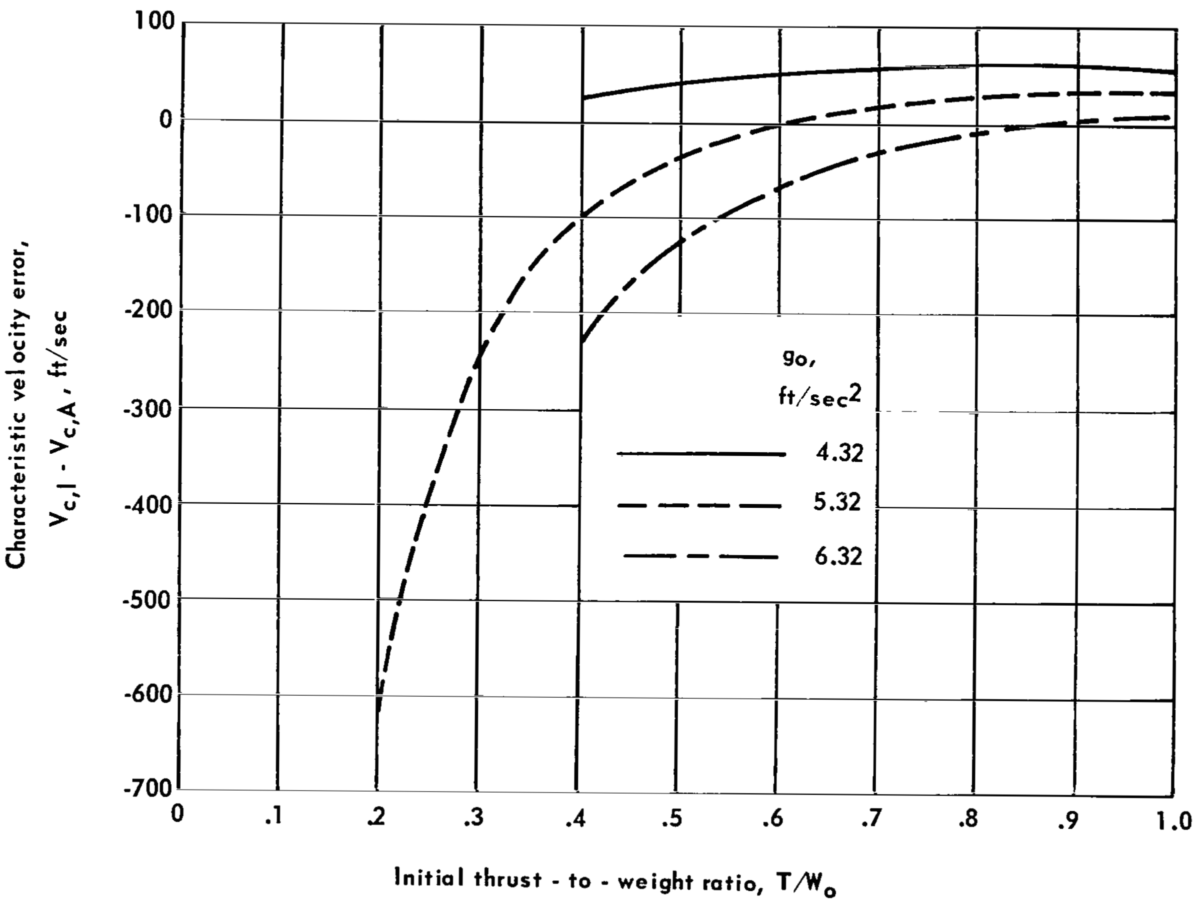

(a) Characteristic velocity error.

Figure 2.- Flat-body error for a launch to lunar orbit. I_{sp} = 420 sec.

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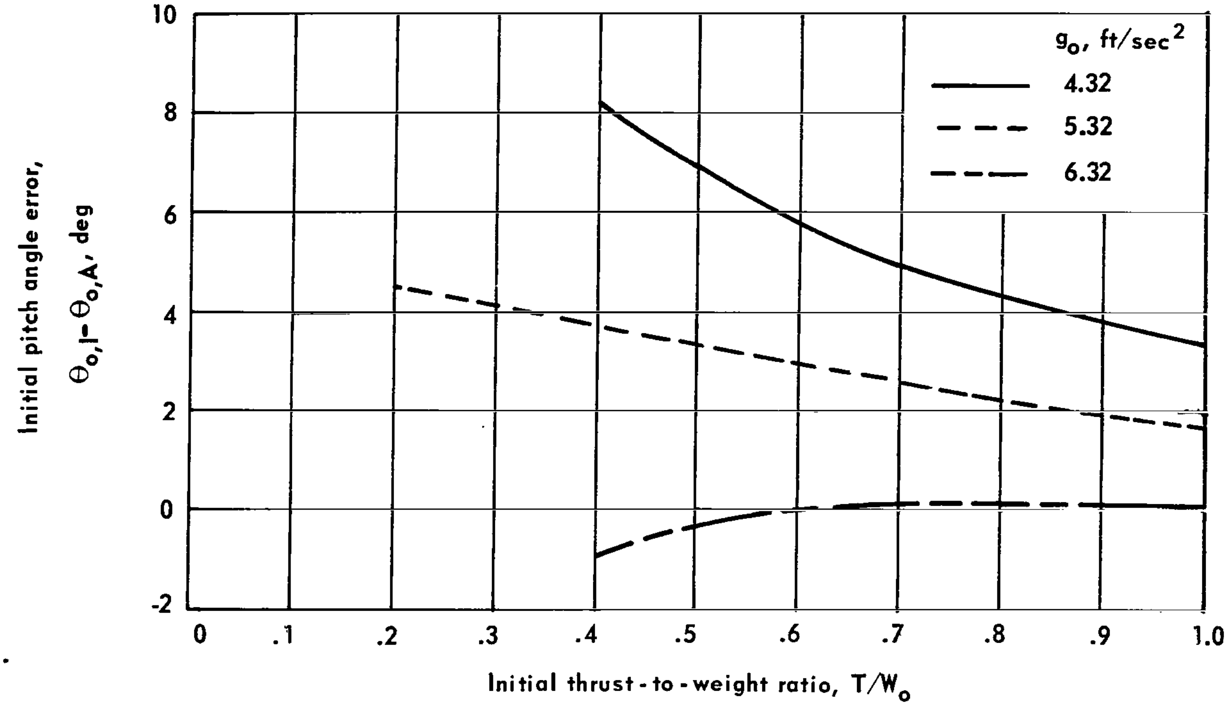

(b) Initial pitch angle error.

Figure 2.- Concluded.

Figures 3(a) and 3(b) illustrate time histories of the terms

$$a_u = uv/r \tag{24}$$

$$a_v = -g + u^2/r \tag{25}$$

for an optimum launch to lunar orbit as compared with those of the flat-body approximations. The displacement between the curves in figures 3(a) and 3(b) represents the horizontal and normal acceleration errors of the flat-body approximation and amounts to a maximum of 5.5 ft/sec^2 for the normal component at the terminal point. The horizontal acceleration term a_u for optimum time descents or ascents from orbit can be considered to have only a second-order effect. For the case illustrated, a launch to lunar orbit, the maximum value

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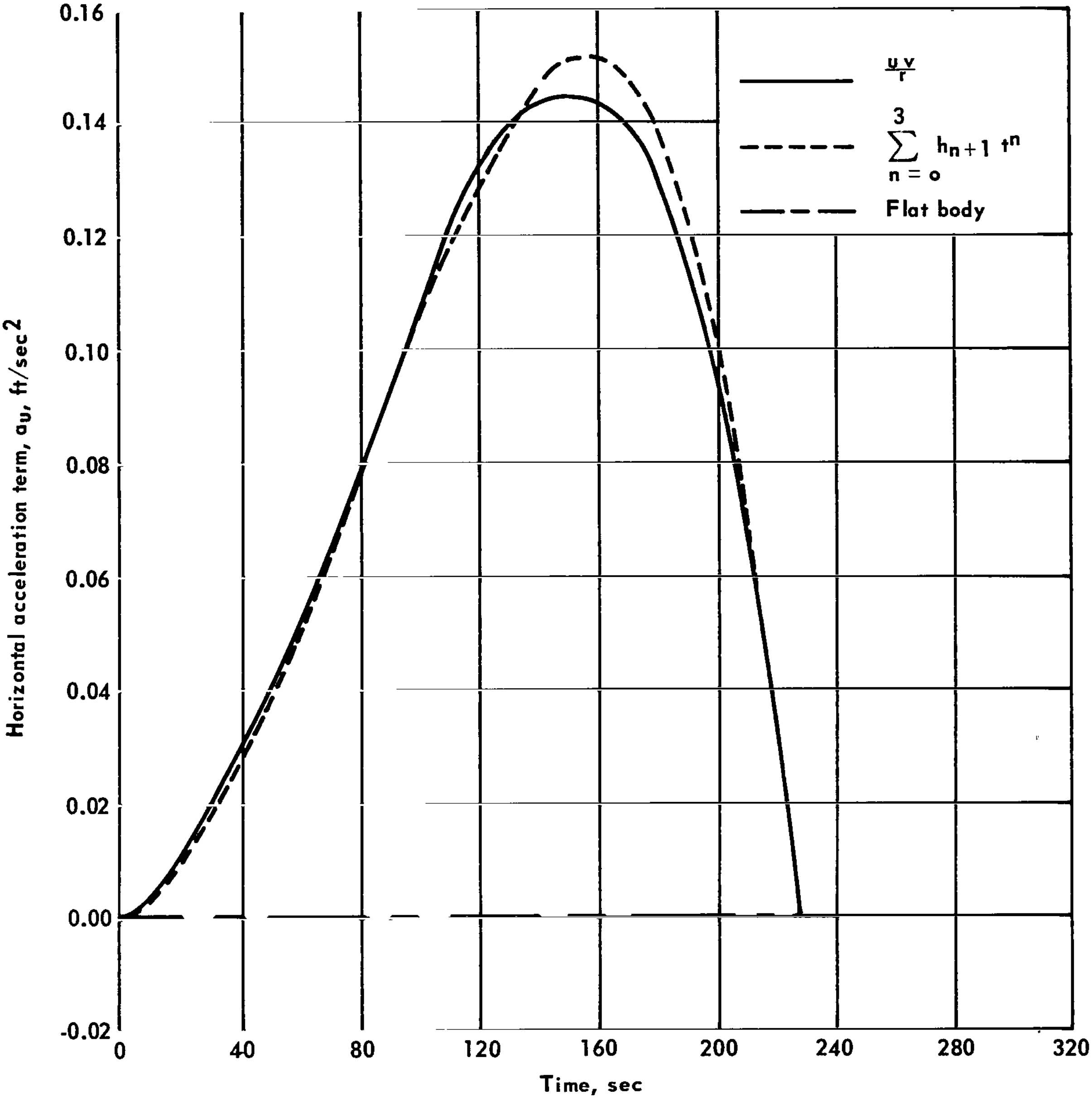

(a) Horizontal acceleration term.

Figure 3.- Comparison of solution obtained by using analytic and exact solutions for launch to lunar orbit. $T/W_0 = 0.6$; $I_{sp} = 315.0$ sec.

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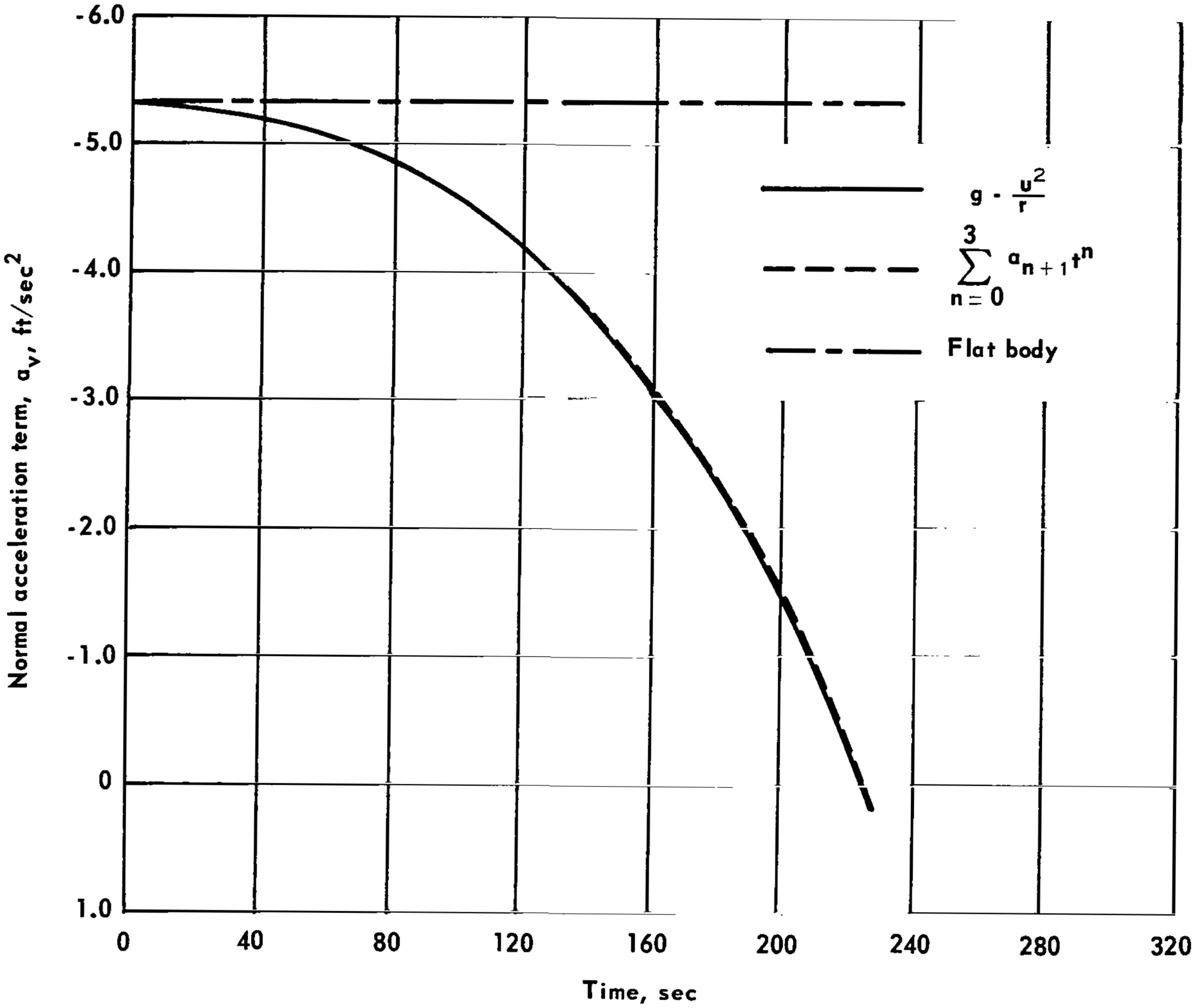


(b) Normal acceleration term.

Figure 3.- Concluded.

of this term amounts to less than 0.2 ft/sec^2. Therefore, the quantity a_v accounts for the largest portion of the observed error.

To offset the errors, functions are defined in the form

$$a_u = \sum_{n=0}^{l} h_{n+1} t^n \tag{26}$$

$$a_v = \sum_{n=0}^{l} a_{n+1} t^n \tag{27}$$

which have the same values as the functions defined in equations (24) and (25) at the two bounds. The introduction of a function $a_v = a_v(q)$, where q is any of the dependent variables, would necessarily complicate (if not completely destroy) the guidance law stated in equation (13). A time series is acceptable because it will not alter the guidance law but will change the transversality condition of the calculus of variations.

Based on this analysis, equations (1) to (4) are

$$\dot{x} = u \tag{28}$$

$$\dot{y} = v \tag{29}$$

$$\dot{u} = \frac{\varphi}{\mu} \cos \theta - \sum_{n=0}^{l} h_{n+1} t^n \tag{30}$$

$$\dot{v} = \frac{\varphi}{\mu} \sin \theta + \sum_{n=0}^{l} a_{n+1} t^n \tag{31}$$

where, from equation (13),

$$\left.\begin{aligned} \sin \theta &= \frac{Kz}{\left(1 + z^2\right)^{1/2}} \\ \cos \theta &= \frac{K}{\left(1 + z^2\right)^{1/2}} \\ \tan \theta &= z \end{aligned}\right\} \tag{32}$$

Integrating these equations as outlined in appendix B results in the following three simultaneous transcendental equations. The method of solution of these equations is outlined in appendix C.

$$\mu_1 v_1 - \dot{\mu} y_0 - \sum_{n=0}^{l} \frac{a_{n+1} t_1^{\,n+1}}{n+1} \left[1 + \frac{\dot{\mu} t_1 (n+1)}{n+2} \right] - \frac{\alpha_2}{\dot{\mu}} (\varphi \sec \theta_1 - A) = 0 \tag{33}$$

$$\frac{Kk^{1/2} \dot{\mu}}{\varphi \alpha_2} \left(u_1 - u_0 + \psi_1 \right) - \log_e \left[\frac{k - \alpha_1 \mu_1 - K\alpha_2 k^{1/2} \sec \theta_1}{\left(k - \alpha_1 - K\alpha_2 k^{1/2} \sec \theta_0 \right) \mu_1} \right] = 0 \tag{34}$$

$$\dot{\mu}^2 \left[\mu_1 y_0 - y_1 + \sum_{n=0}^{l} \frac{a_{n+1} t_1^{\,n+2}}{(n+1)(n+2)} \right] - \dot{\mu} A \alpha_2 t_1 + \varphi \mu_1 K \left[\log_e \left(\frac{\tan \theta_1 + K \sec \theta_1}{\tan \theta_0 + K \sec \theta_0} \right) \right.$$

$$\left. + K\alpha_2 \left(\sec \theta_0 - \frac{\sec \theta_1}{\mu_1} \right) - \frac{K\mu \alpha_1}{\varphi \alpha_2} \left(u_1 - u_0 + \psi_1 \right) \right] = 0 \tag{35}$$

where

$$\psi_1 = \sum_{n=0}^{l} \frac{h_{n+1} t_1^{\,n+1}}{n+1} \tag{36}$$

The required number and the evaluation of coefficients to simulate sufficiently the optimum round body trajectories must be determined. Obviously, the problem is reduced to a flat body if only the $n = 0$ term is retained and the coefficient h_1 is chosen to be 0. Consider a case where the control program for the integrated and the analytic solutions are identical in time. As higher order terms in n are introduced to the flat-body solution, the instantaneous accelerations and hence the state variables of the analytic model would necessarily converge to those of the integrated solution. If the control program of the analytic model (eq. 13) is considered an off-nominal solution, the above qualitative argument implies the following generalization. The degree of success in utilizing this analytic technique to simulate optimum trajectories depends on the success in simulating the deleted accelerations of the flat body with power series in time.

For a case of launch to lunar orbit (figs. 3(a) and 3(b)), it is apparent that the polynomials must be at a minimum cubics and certainly of higher order if greater accuracy is desired. This minimum solution necessitates four coefficients. The results of using this truncated series to simulate the deleted accelerations of the round body are indicated in figures 3(a) and 3(b). As was stated above, two values of a_u and a_v (eqs. (24) and (25)) are known by their evaluation at $t = 0$ and $t = t_1$. Two more solutions are required to evaluate the four coefficients of each series. It is apparent that once the trajectory progresses beyond a few integration steps in a guidance problem, the past history of the dependent variables will yield the required two additional solutions. At the initiation of the trajectory, the following technique is used to evaluate the unknown coefficients. The derivatives of equations (24) and (25) with respect to time are

$$\left.\begin{aligned}\dot{a}_u &= \frac{\left[v\left(\dot{u} - a_u\right) + u\dot{v}\right]}{r} \\ \dot{a}_v &= \frac{\left[2u\dot{u} + v\left(g - a_v\right)\right]}{r}\end{aligned}\right\} \tag{37}$$

These equations and equations (1) to (8) are integrated either backward or forward in time twice by a small increment Δt to define the additional coefficient in equations (26) and (27). These two solutions will contain an error, the magnitude being proportional to the uncertainty in the pitch angle θ at $t = 0$. This error in the coefficients is reduced by cycling through the algebraic equations and integrations until the coefficients converge to a set of values consistent with the computed value of the pitch angle.

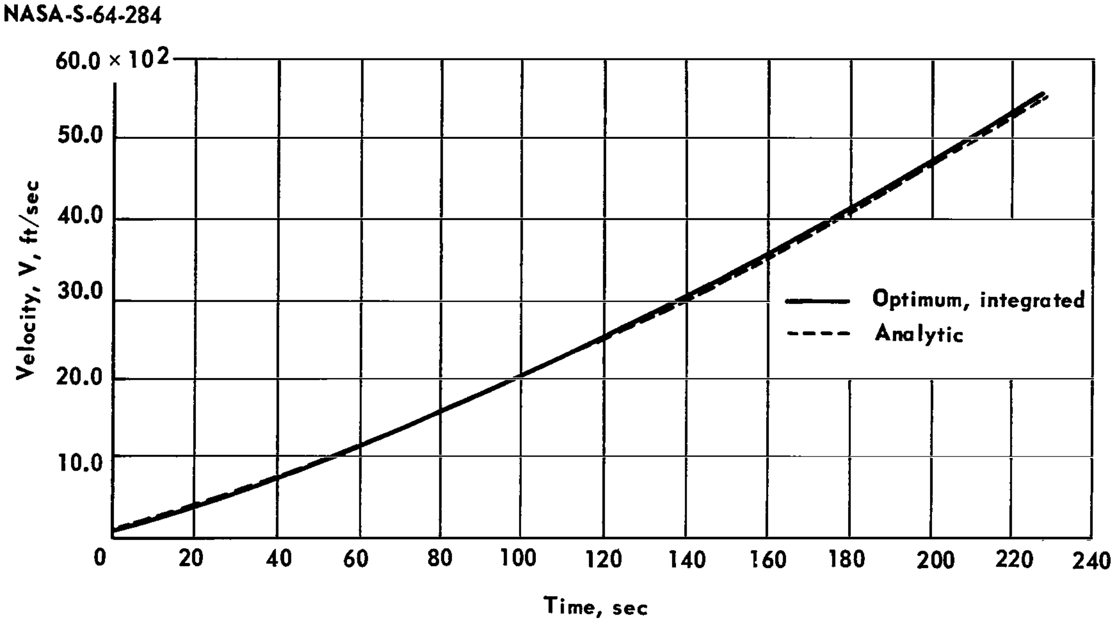
RESULTS AND DISCUSSION

Examples of results using the equations derived in this report are shown in an attempt to indicate the broad usage to which this analysis applies. An IBM type 1620 electronic data processing machine was used for the computations; the solution time for a typical boundary-value problem, as a launch to orbit or landing, was approximately 2 minutes and required five or six iterations to obtain five-decimal-place accuracy. Guidance problems, of course, ran much longer on the machine since they consisted of a series of boundary-value problems, and convergence of solution was slow when "time-to-go" became small. Therefore, an open-loop system had to be used for the last portion of the trajectory.

Guidance Trajectories

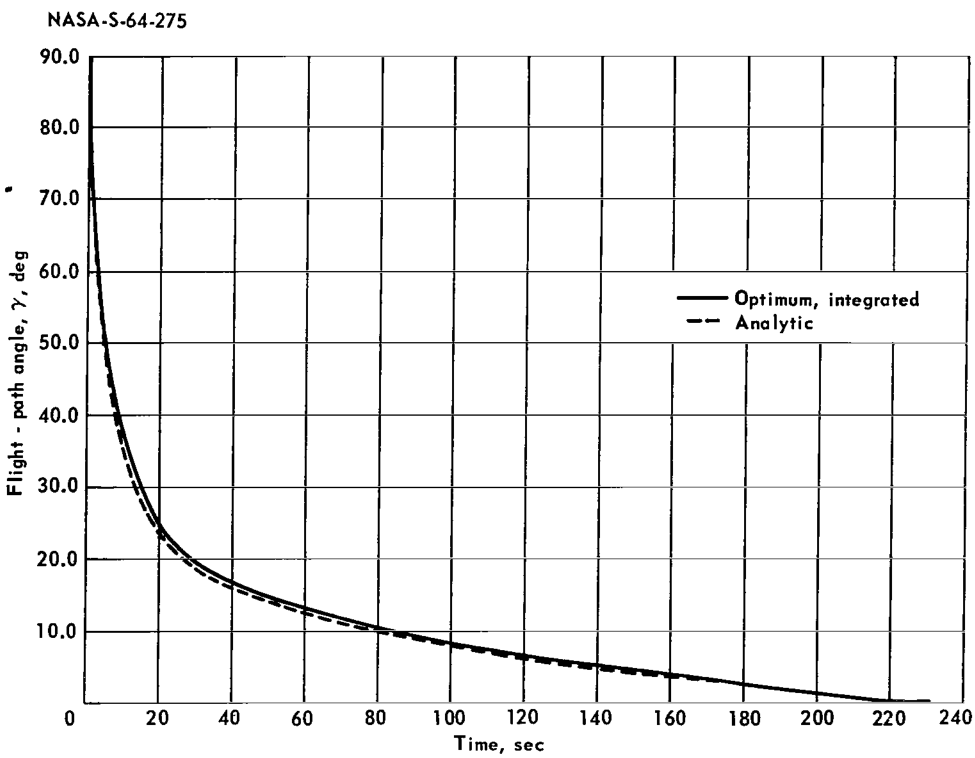
The accuracy of guided solutions in simulating the strict optimum is indicated in figures 4 to 6, and terminal values are presented in tables I to III. In these solutions, the true dynamic equations were integrated at a step interval of 10 seconds, and the analytic solution was used as a feedback control.

Figures 4(a) to 4(d) are time-history comparisons of the state and control variables for the guided analytic solution with the optimum for a launch to lunar orbit. Table I indicates the numerical differences at the boundary. This trajectory required the use of an open-loop system at a time-to-go of approximately 9 seconds. The largest apparent deviation in these curves is in the pitch angle, where there appears to be as much differential area above



(a) Velocity.

Figure 4.- Optimum launch to lunar orbit. $T/W_0 = 0.6$; $I_{sp} = 315$ sec.



(b) Flight-path angle.

Figure 4.- Continued.

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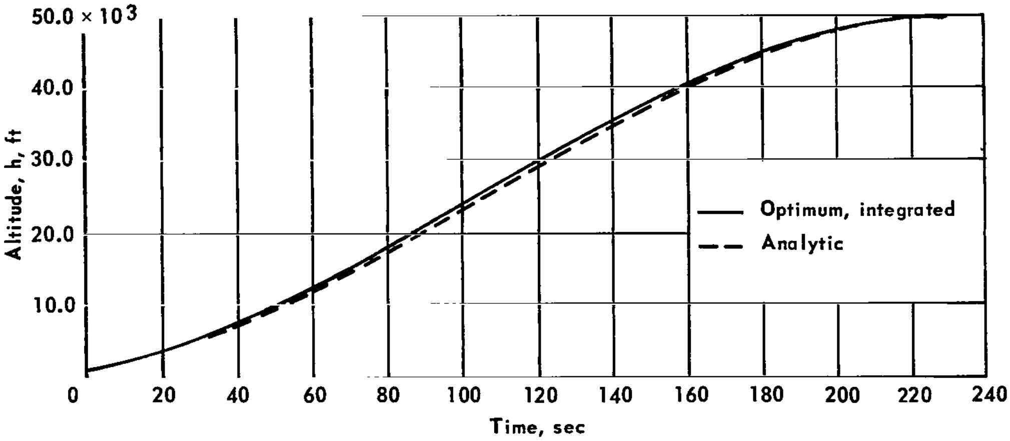

(c) Altitude.

Figure 4.- Continued.

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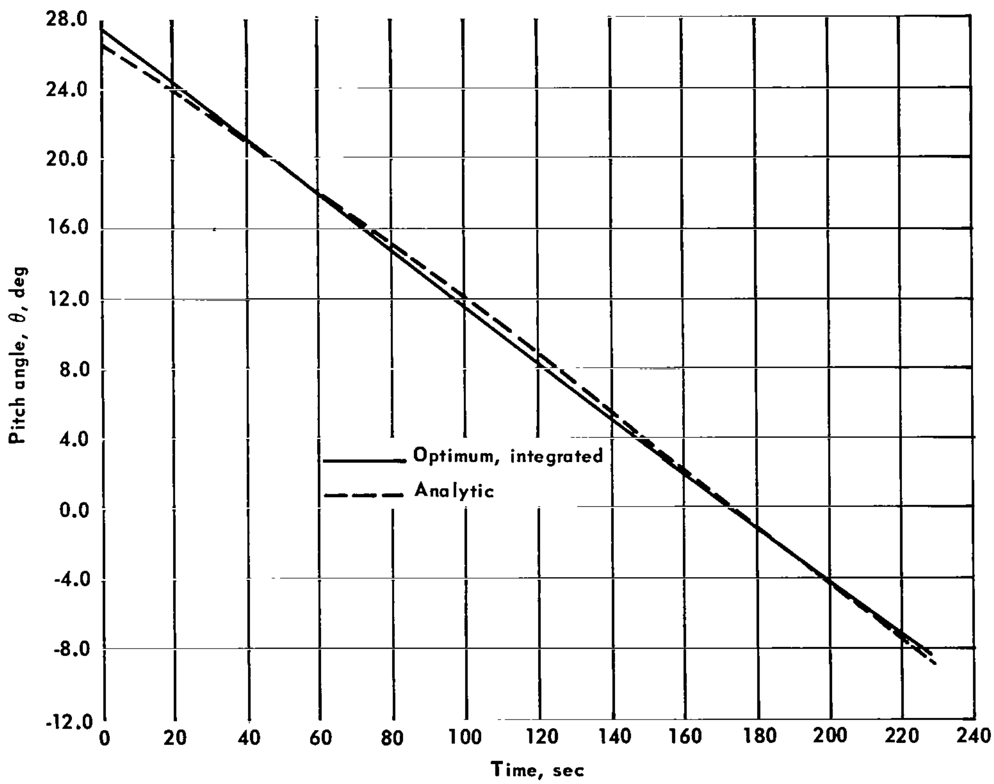

(d) Pitch angle.

Figure 4.- Concluded.

as below the optimum. This apparent canceling is the probable cause of the closeness of the final values of time indicated in table I. It can be observed by examining these curves that the analytic technique has a linearizing effect on the solution. This appears to be a general characteristic of the analytic technique.

TABLE I.- LAUNCH TO LUNAR ORBIT

$\left[T/W_0 = 0.6;\ I_{sp} = 315 \text{ sec};\ y_0 = 1{,}000 \text{ ft};\ v_0 = 100 \text{ ft/sec};\ \gamma_0 = 90°\right]$

	Optimum, integrated	Analytic, guided
t_1, sec	227.92	228.87
θ_0, deg	27.68	26.68
θ_1, deg	-8.25	-8.77
V_c, ft/sec	5,775.3	5,807.9
y_1, ft	49,958	49,962
v_1, ft/sec	5,606.7	5,601.9
γ_1, deg	0.00084	0.00527
x_1, ft	564,573	567,082

Figures 5(a) to 5(d) are the time-history comparisons for a guided landing from lunar orbit. Numerical results are presented in table II. Optimum retrograde solutions are extremely sensitive to initial conditions when solved by integrating the dynamic and Euler-Lagrange equations forward in time. Therefore, the landing from lunar orbit required a backward integration in time to obtain a solution. This, in turn, required a weight correction to establish the correct initial T/W_0 when the solution converged. These difficulties were avoided in the analytic solution by solving the retrograde problem in the forward sense, and only the previously mentioned difficulties were encountered. An open-loop solution was initiated at a time-to-go of approximately 10 seconds.

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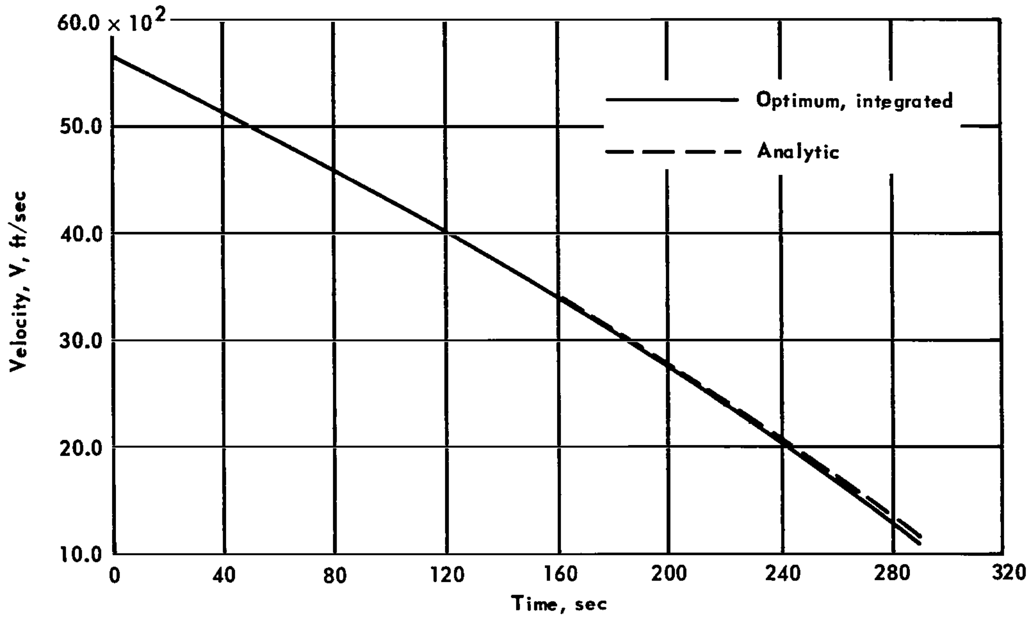

(a) Velocity.

Figure 5.- Optimum landing from lunar orbit. $T/W_0 = 0.4$; $I_{sp} = 315$ sec.

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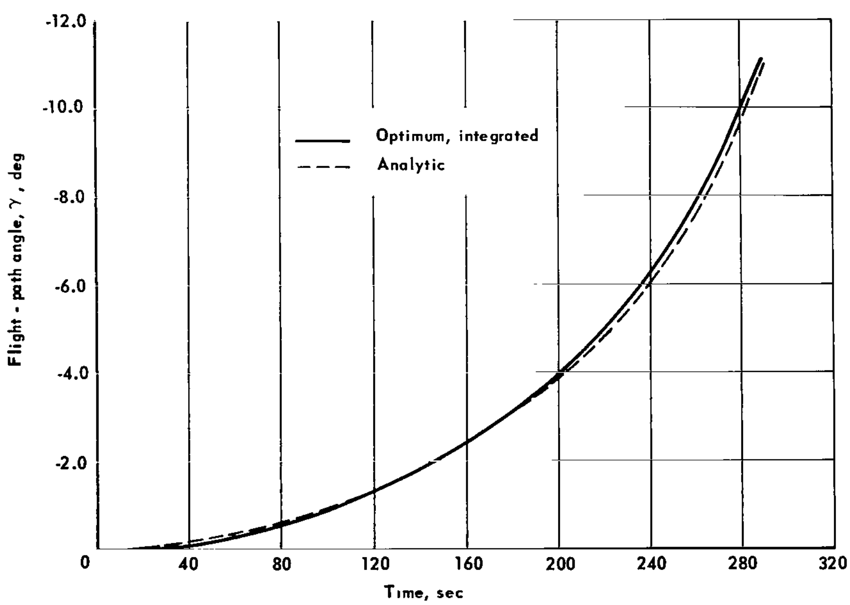

(b) Flight-path angle.

Figure 5.- Continued.

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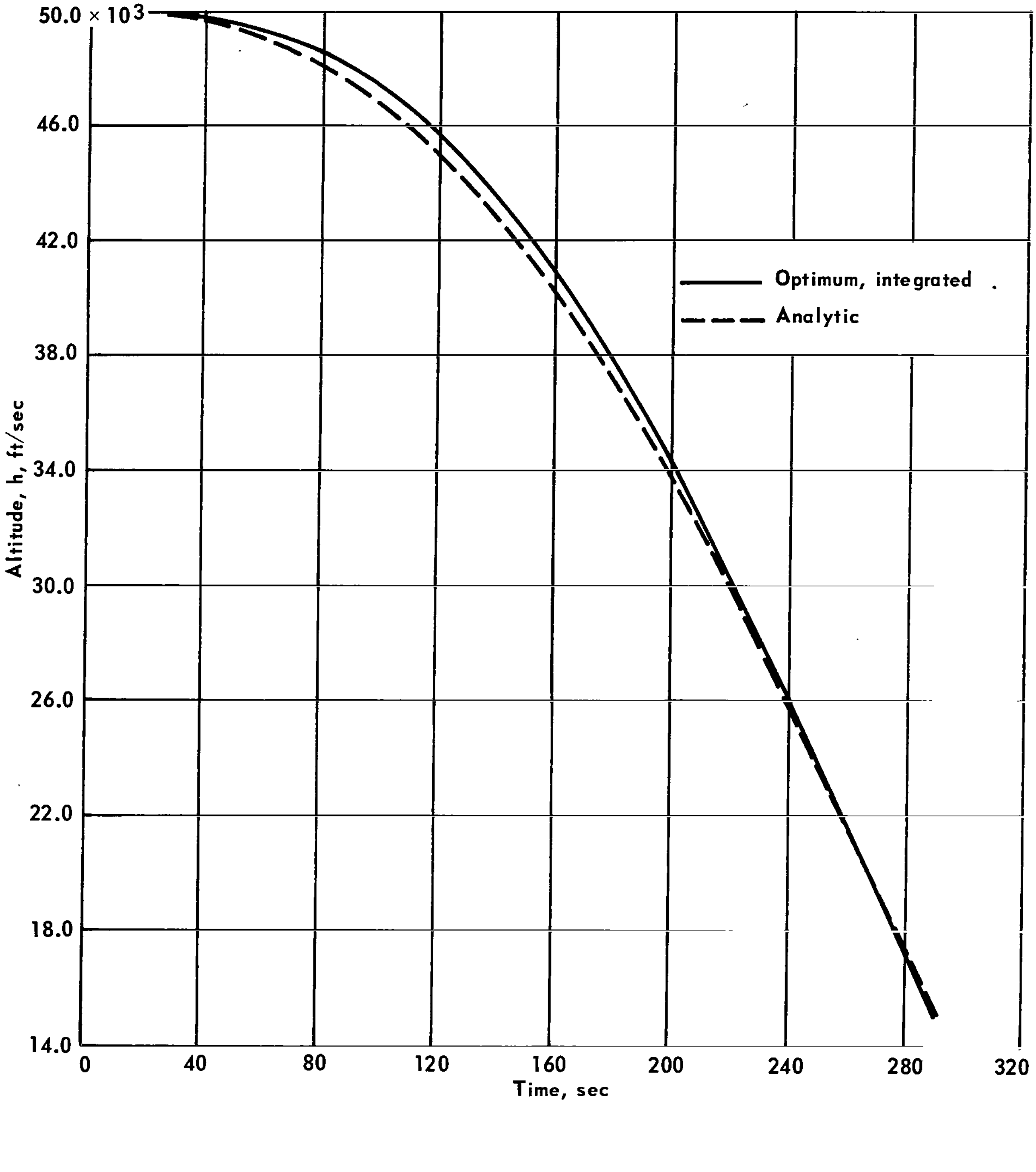

(c) Altitude.

Figure 5.- Continued.

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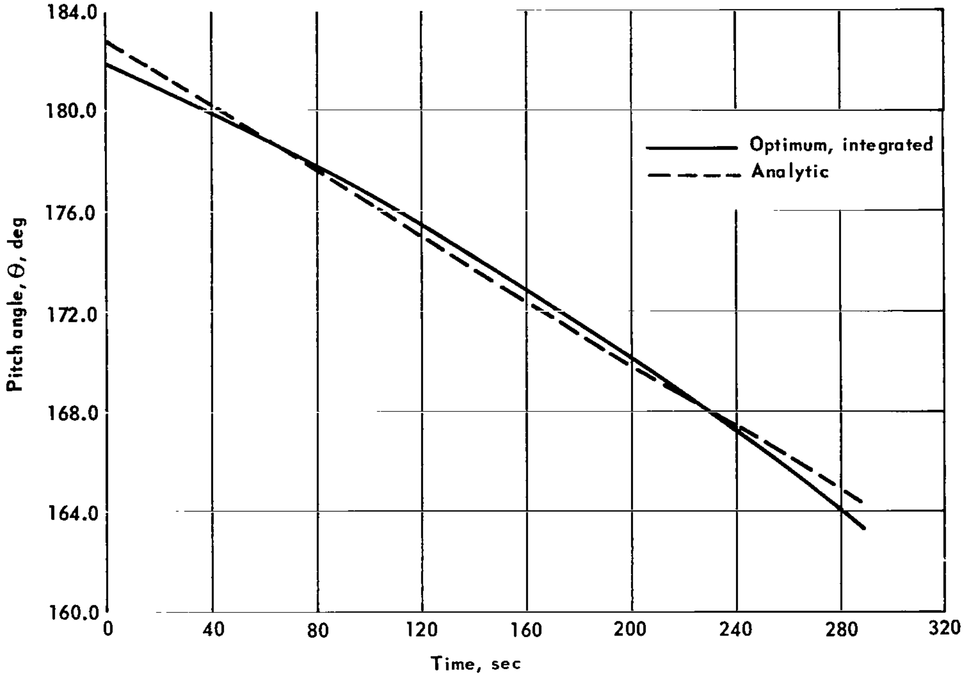

(d) Pitch angle.

Figure 5.- Concluded.

TABLE II.- LANDING FROM LUNAR ORBIT

[$T/W_0 = 0.4$; $I_{sp} = 315$ sec; $y_0 = 50,000$ ft; $v_0 = 5674.5$; $\gamma_0 = 0°$]

	Optimum, integrated	Analytic, guided
t_1, sec	290.02	290.92
θ_0, deg	181.87	182.93
θ_1, deg	163.23	164.28
V_c, ft/sec	4,658.9	4,677.0
y_1, ft	15,000.0	14,998.8
v_1, ft/sec	1,113.2	1,114.3
γ_1, deg	-11.13	-11.07
x_1, ft	1,019,285	1,022,944

Figures 6(a) to 6(d) are the time-history comparisons for a guided insertion into a circular earth parking orbit. Table III presents the numerical results at the boundary. In this trajectory, velocity to be gained rather than time-to-go was the terminating criteria. This necessitated thrusting for an additional 0.13 second to meet the desired end conditions. This shift in the terminal boundary condition is readily apparent in figure 6(d) where it is observed that the pitch angle rapidly changes value.

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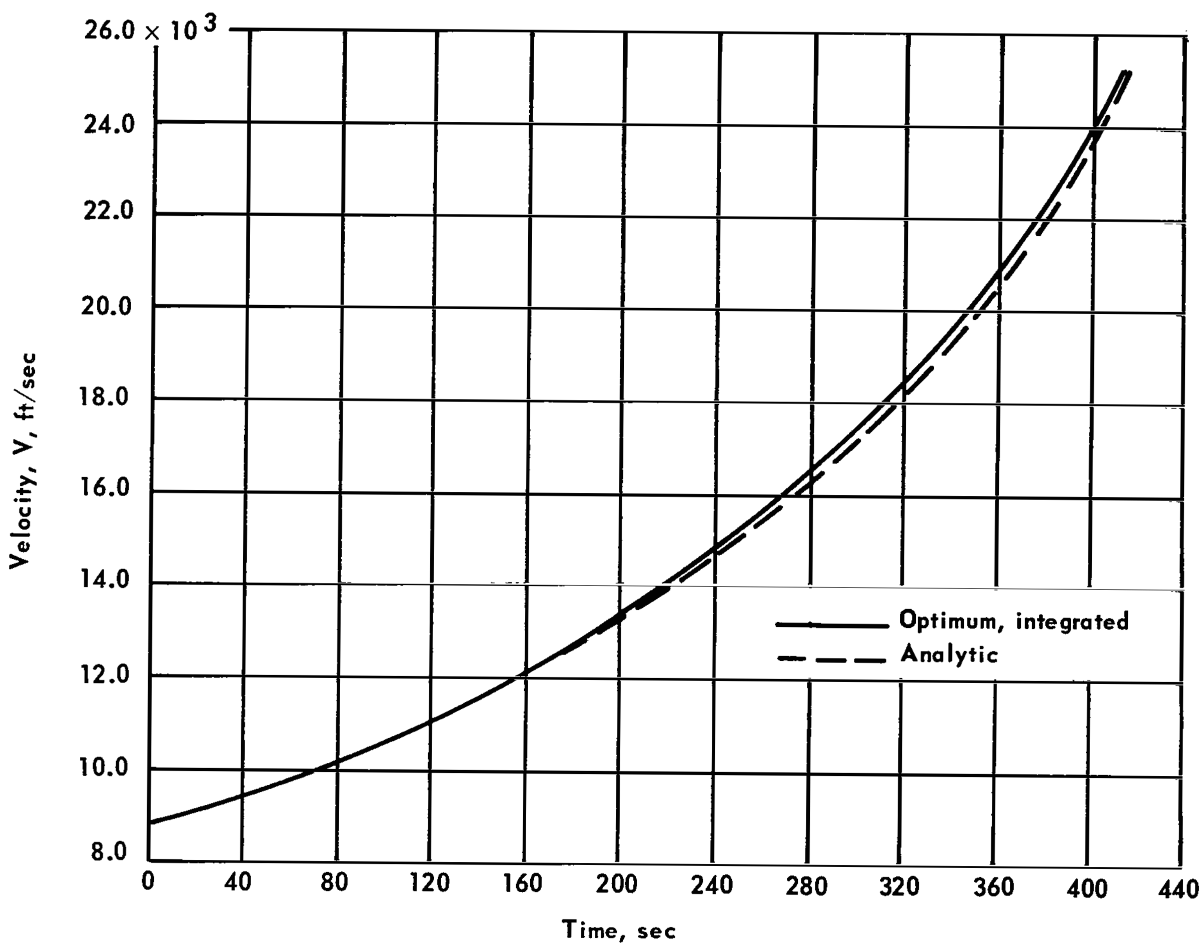

(a) Velocity.

Figure 6.- Optimum insertion into a circular earth orbit. $T/W_0 = 0.751$; $I_{sp} = 420$ sec.

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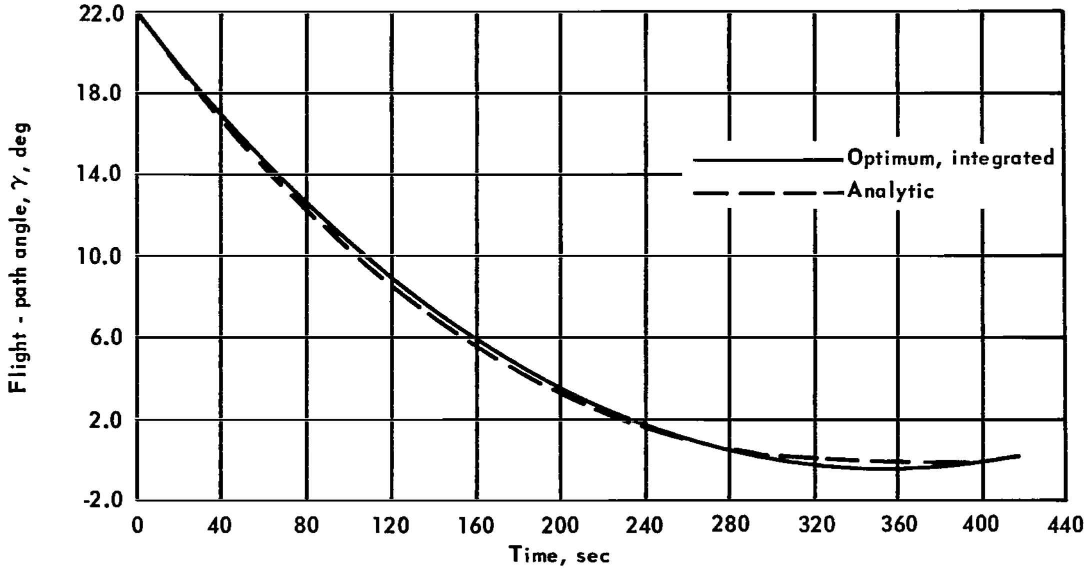

(b) Flight-path angle.

Figure 6.- Continued.

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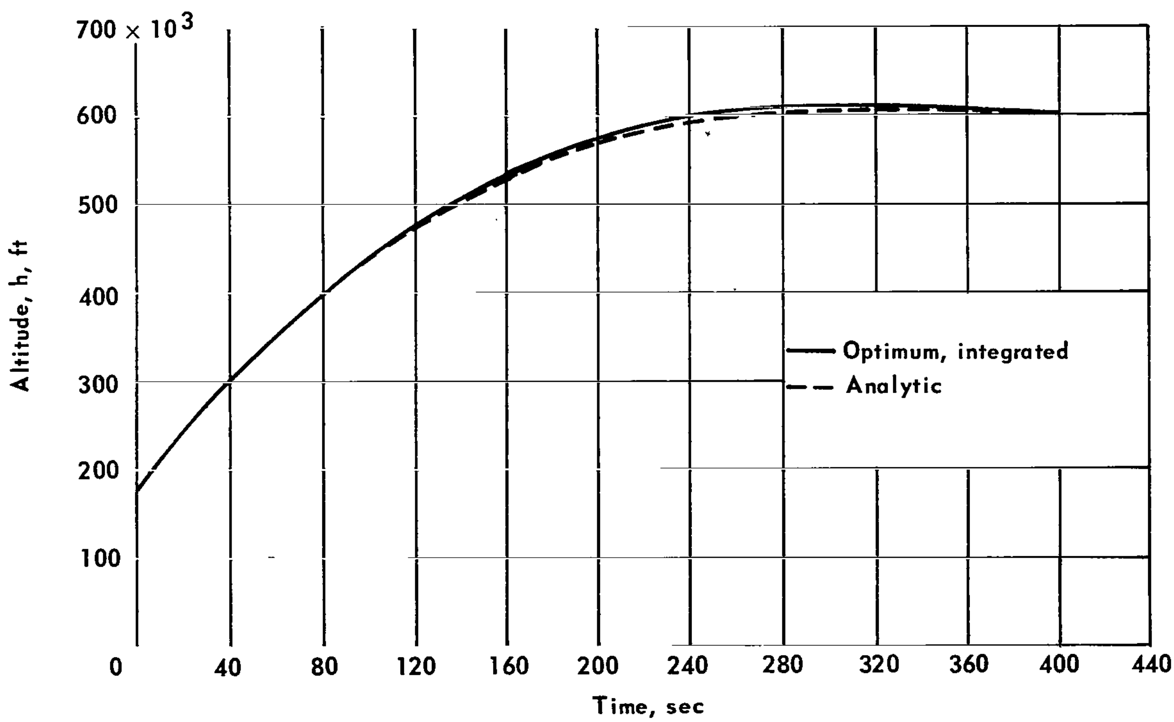
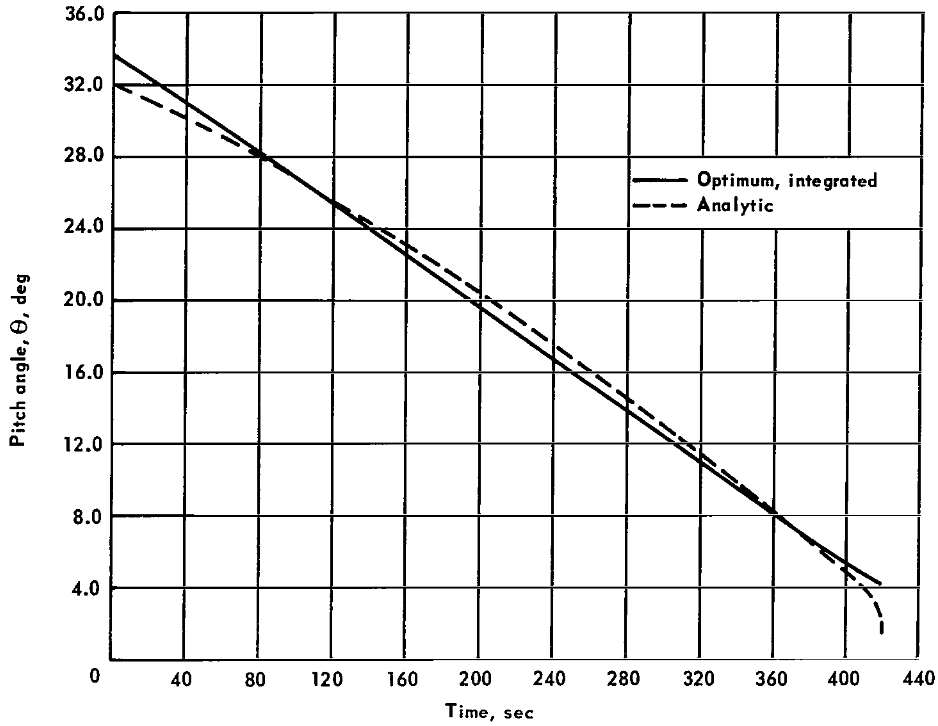


(c) Altitude.

Figure 6.- Continued.

NASA-S-64-281



(d) Pitch angle.

Figure 6.- Concluded.

TABLE III.- INSERTION INTO A CIRCULAR EARTH ORBIT

[T/W_0 = 0.751; I_{sp} = 420 sec; y_0 = 181,500 ft; v_0 = 8,750 ft/sec; γ_0 = 22.2°]

	Optimum, integrated	Analytic, guided
t_1, sec	417.15	419.07
θ_0, deg	33.73	32.21
θ_1, deg	4.04	1.38
V_c, ft/sec	18,537	18,733
y_1, ft	600,932	600,930
v_1, ft/sec	25,570	25,570
γ_1, deg	0.0000	-0.0004
x_1, ft	5,929,025	5,960,834

Trajectory Scanner

Figure 7 indicates a parametric study in T/W_0 of landings from positions off pericynthion. The state variables and performance data for changes in the true anomaly angle are presented in table IV. A point of interest in this curve is the small penalty loss in performance for a wide variation in the true anomaly angle η. For instance, in the case of $T/W_0 = 0.4$, a penalty loss of $\Delta V_c = 10$ ft/sec results for a range of η of approximately 50°. For reasonable values of T/W_0, this small loss in ΔV_c indicates that lunar landings may be initiated from a wide range with only a small penalty in performance.

NASA-S-64-291

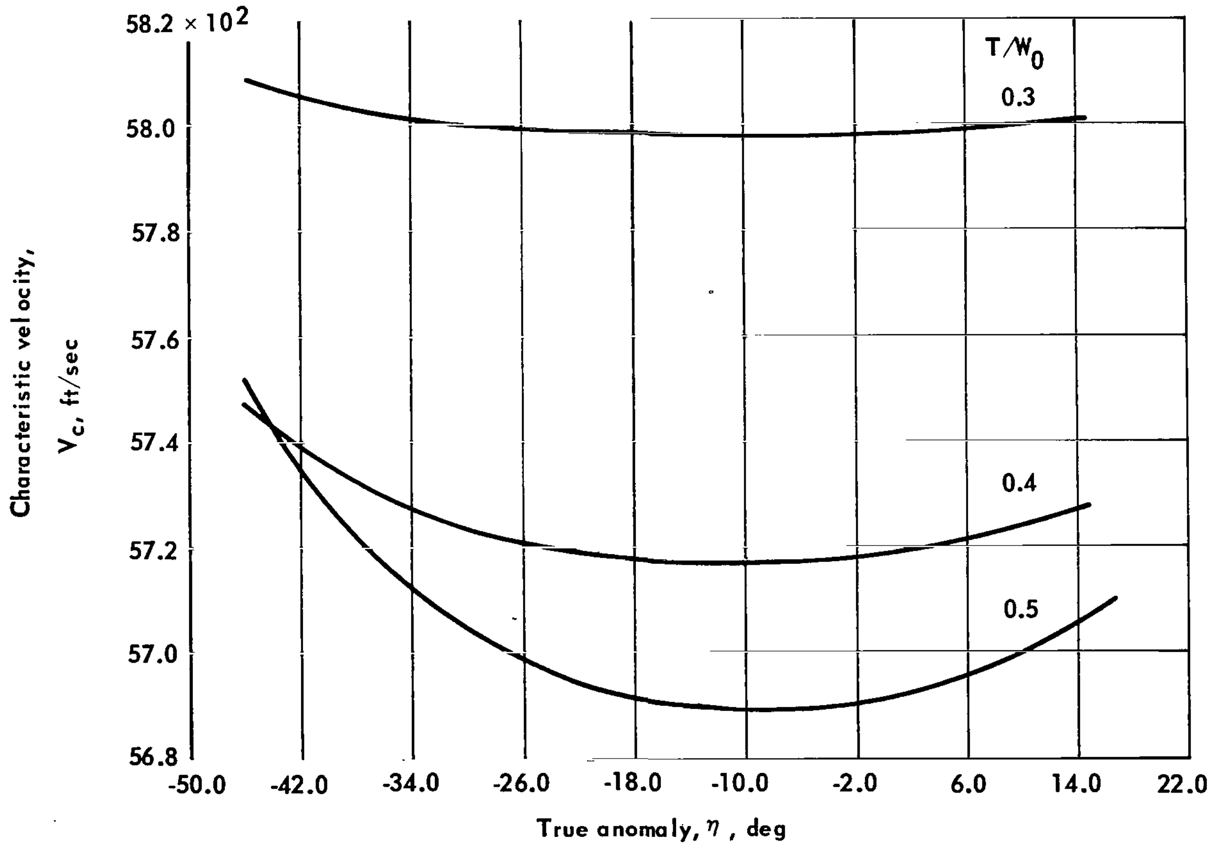


Figure 7.- Landings from positions off pericynthion. $I_{sp} = 315.0$ sec.

TABLE IV.- LANDINGS FROM POSITIONS OFF PERICYNTHION

$$\left[I_{sp} = 315 \text{ sec};\ y_1 = 1{,}000 \text{ ft};\ v_1 = 100 \text{ ft/sec};\ \gamma_1 = 0°\right]$$

η, deg	V, ft/sec	γ, deg	y, ft	V_c, ft/sec		
				$T/W_0 = 0.3$	$T/W_0 = 0.4$	$T/W_0 = 0.5$
-46.1	5,524.6	-1.5472	112,724	5,808.4	5,747.3	5,752.2
-40.6	5,537.1	-1.3949	99,229	5,804.3	5,736.1	5,730.5
-35.1	5,548.3	-1.2298	87,157	5,801.7	5,728.3	5,714.7
-29.6	5,558.1	-1.0534	76,626	5,800.2	5,723.2	5,703.8
-24.1	5,566.4	- .8677	67,738	5,799.3	5,720.1	5,696.6
-18.6	5,573.1	- .6745	60,581	5,798.8	5,718.3	5,692.3
-13.0	5,578.1	- .4757	55,225	5,798.0	5,717.5	5,690.1
.86	5,583.0	.0315	50,024	5,798.3	5,718.9	5,691.9
14.75	5,576.7	.5280	56,693	5,801.3	5,727.8	5,707.1

CONCLUDING REMARKS

Quasi-optimum guidance equations in two dimensions for which constant thrust was used have been analytically derived by the calculus of variations. The basis of the analysis is the analytical solution to the flat-body problem which is extended to simulate a round body by the addition of power series in time. Numerical solutions to the resulting transcendental equations have been generated by using an iterative convergence technique.

Guided trajectories were examined for a landing on and launch from the lunar surface and for an insertion into earth orbit. The resulting curves for the control and state variables agree closely with the true optimum and produce solutions which are more linear than the exact curves. The trajectories show losses in characteristic velocity of 0.564 percent for the lunar launch, 0.389 percent for the lunar landing, and 1.057 percent for the insertion into earth orbit. This last trajectory had a terminating criteria on velocity rather than on time-to-go which had the effect of increasing the thrusting arc.

Manned Spacecraft Center,
National Aeronautics and Space Administration,
Houston, Texas, February 25, 1964

APPENDIX A

OPTIMUM SOLUTION WITH FUNCTIONS OF THE STATE VARIABLES PRESCRIBED AT A VARIABLE TERMINAL TIME, INCLUDING MINIMUM TIME PROCESSES

The material contained in this appendix was taken from a lecture series on optimization of dynamic systems at Harvard University in 1963.

Given a system of first-order differential equations of the form

$$\dot{\vec{q}} = f\left[\vec{q}(t),\vec{\theta}(t),t\right] \qquad \vec{q}\left(t_0\right) \text{ given} \qquad t_0 \leq t \leq t_1 \tag{A1}$$

where $\vec{q}(t)$ and $\vec{\theta}(t)$ are the state and control vectors, respectively. It is desired to minimize some performance index of the Mayer form

$$I = G\left[q\left(t_1\right),t_1\right] \tag{A2}$$

Certain boundary conditions are desired of the form

$$\vec{M}\left[q\left(t_1\right),t_1\right] = 0 \tag{A3}$$

The performance index and the boundary conditions adjoined to the constraint relationships (eq. (A1)) with Lagrange multipliers produce the integral

$$I = \left(G + \vec{\nu}^{\,T} \vec{M}\right)_{t=t_1} + \int_0^{t_1} \vec{\lambda}^{\,T} \left\{f\left[\vec{q}(t),\vec{\theta}(t),t\right] - \dot{\vec{q}}(t)\right\} dt \tag{A4}$$

where $\vec{\nu}$ are constant Lagrange multipliers and the superscript T refers to the transpose matrix.

The modified Hamiltonian is defined as

$$H = \vec{\lambda}^{\,T} f\left[\vec{q}(t),\vec{\theta}(t),t\right] \tag{A5}$$

The differential of (A4) taking into account differential changes in the terminal time t_1 is

$$dI = \left\{ \left(\frac{\partial G}{\partial t} + \vec{\upsilon}^{\,T} \frac{\partial \vec{M}}{\partial t} \right) dt + \left[\frac{\partial G}{\partial q(t)} + \vec{\upsilon}^{\,T} \frac{\partial \vec{M}}{\partial q(t)} \right] d\vec{q}(t) \right\}_{t=t_1}$$

$$+ \int_0^{t_1} \left[\frac{\partial H}{\partial q} \delta\vec{q}(t) + \frac{\partial H}{\partial \theta} \delta\vec{\theta}(t) - \vec{\lambda}^{\,T} \delta\dot{\vec{q}}(t) \right] dt \tag{A6}$$

Integrating by parts and using

$$d\vec{q}(t_1) = \delta\vec{q}(t_1) + \dot{\vec{q}}(t_1) dt_1 \tag{A7}$$

then,

$$dI = \left\{ \left[\frac{\partial G}{\partial t} + \vec{\upsilon}^{\,T} \frac{\partial \vec{M}}{\partial t} + \vec{\lambda}^{\,T} \dot{\vec{q}}(t) \right] dt_1 + \left[\frac{\partial G}{\partial q(t)} + \vec{\upsilon}^{\,T} \frac{\partial \vec{M}}{\partial q(t)} - \vec{\lambda}^{\,T} \right] d\vec{q}(t) \right\}_{t=t_1}$$

$$+ \left[\vec{\lambda}^{\,T} \delta\vec{q}(t) \right]_{t=0} + \int_0^{t_1} \left[\left(\frac{\partial H}{\partial q} + \dot{\vec{\lambda}}^{\,T} \right) \delta\vec{q} + \frac{\partial H}{\partial \theta} \delta\vec{\theta} \right] dt \tag{A8}$$

The Lagrange multipliers are chosen such that the coefficients of $\delta\vec{q}$, $d\vec{q}(t_1)$, and dt_1 vanish if t_1 is not defined.

$$\dot{\vec{\lambda}}^{\,T} = - \vec{\lambda}^{\,T} \frac{\partial f}{\partial q} \tag{A9}$$

$$\vec{\lambda}^{\,T}(t_1) = \left(\frac{\partial G}{\partial q} + \vec{\upsilon}^{\,T} \frac{\partial \vec{M}}{\partial q} \right)_{t=t_1} \tag{A10}$$

$$\left(\frac{\partial G}{\partial t} + \vec{\upsilon}^{\,T} \frac{\partial \vec{M}}{\partial t} + H \right)_{t=t_1} = 0 \tag{A11}$$

As a result of this choice of $\lambda(t)$ equation (A8) is reduced to

$$dI = \left(\vec{\lambda}^{\,T} \delta\vec{q} \right)_{t=0} + \int_0^{t_1} \left(\frac{\partial H}{\partial \theta} \delta\vec{\theta} \right) dt \tag{A12}$$

For a stationary value of I

$$\frac{\partial H}{\partial \theta} = \vec{\lambda}^{\,T} \frac{\partial f}{\partial \theta} = 0 \tag{A13}$$

For minimum fuel consumption or under the assumption of constant thrust for minimum time t_1, $G\left[q\left(t_1\right), t_1\right] = t_1$ such that

$$\left.\begin{aligned} \left(\frac{\partial G}{\partial t}\right)_{t=t_1} &= 1 \\ \left(\frac{\partial G}{\partial q}\right)_{t=t_1} &= 0 \end{aligned}\right\} \tag{A14}$$

For the specific case under consideration, the functions $f\left[\vec{q}(t), \vec{\theta}(t), t\right]$ are

$$\dot{x} = u \tag{A15}$$

$$\dot{y} = v \tag{A16}$$

$$\dot{u} = \frac{\varphi}{\mu} \cos \theta \tag{A17}$$

$$\dot{v} = \frac{\varphi}{\mu} \sin \theta - g_0 \tag{A18}$$

and the boundary value constraints $\vec{M}\left[q\left(t_1\right), t_1\right]$ are

$$y\left(t_1\right) - y_1 = 0 \tag{A19}$$

$$u\left(t_1\right) - u_1 = 0 \tag{A20}$$

$$v\left(t_1\right) - v_1 = 0 \tag{A21}$$

where $x\left(t_1\right)$ is allowed to be free.

Applying equation (A9), the Euler-Lagrange equations are

$$\dot{\lambda}_1 = 0 \tag{A22}$$

$$\dot{\lambda}_2 = 0 \tag{A23}$$

$$\dot{\lambda}_3 = - \lambda_1 \tag{A24}$$

$$\dot{\lambda}_4 = - \lambda_2 \tag{A25}$$

Applying equation (A13) to obtain the optimum control laws

$$\tan \theta = \lambda_4 / \lambda_3 \tag{A26}$$

Integrating equations (A22) to (A25), and using the results in equation (A26)

$$\lambda_1 = C_1 \tag{A27}$$

$$\lambda_2 = C_2 \tag{A28}$$

$$\lambda_3 = C_3 - C_1 t \tag{A29}$$

$$\lambda_4 = C_4 - C_2 t \tag{A30}$$

Therefore,

$$\tan \theta = \frac{C_4 - C_2 t}{C_3 - C_1 t} \tag{A31}$$

Applying equation (A10),

$$\lambda_1(t_1) = 0 \tag{A32}$$

$$\lambda_2(t_1) = \upsilon_2 \tag{A33}$$

$$\lambda_3(t_1) = \upsilon_3 \tag{A34}$$

$$\lambda_4(t_1) = \upsilon_4 \tag{A35}$$

From equations (A32) and (A27) it is observed that the constant $C_1 = 0$. The optimum control law reduces to

$$\tan \theta = \frac{\left(C_4 - C_2 t\right)}{C_3} \tag{A36}$$

One more boundary condition results from this analysis. Applying equation (A11) along with the condition (A14)

$$\left(1 + \lambda_2 \dot{y} + \lambda_3 \dot{u} + \lambda_4 \dot{v}\right)_{t=t_1} = 0 \tag{A37}$$

APPENDIX B

INTEGRATION OF A SET OF LINEARIZED DYNAMIC EQUATIONS IN WHICH A CONSTANT THRUST AND A LINEAR TANGENT LAW OF STEERING ARE ASSUMED

Given a system of equations described by

$$\dot{x} = u \tag{B1}$$

$$\dot{y} = v \tag{B2}$$

$$\dot{u} = \frac{\varphi}{\mu} \cos\theta - \sum_{n=0}^{l} h_{n+1} t^n \tag{B3}$$

$$\dot{v} = \frac{\varphi}{\mu} \sin\theta + \sum_{n=0}^{l} a_{n+1} t^n \tag{B4}$$

$$\tan\theta = \frac{C_4 - C_2 t}{C_3} = z \tag{B5}$$

along with the subsidiary equations

$$\mu = 1 + \dot{\mu} t \tag{B6}$$

$$\alpha_2 = \frac{\dot{\mu} t_1}{\tan\theta_1 - \tan\theta_0} \tag{B7}$$

$$\alpha_1 = 1 - \alpha_2 \tan\theta_0 \tag{B8}$$

$$k = \alpha_1^{\,2} + \alpha_2^{\,2} \tag{B9}$$

$$\left.\begin{aligned} \sin\theta &= Kz\Big/\left(1 + z^2\right)^{1/2} \\ \cos\theta &= K\Big/\left(1 + z^2\right)^{1/2} \end{aligned}\right\} \tag{B10}$$

$$
\left.\begin{aligned}
K &= 1 \qquad \left(-\pi/2 \leq \theta \leq \pi/2\right) \\
K &= -1 \qquad \left(\pi/2 < \theta < 3\pi/2\right)
\end{aligned}\right\} \tag{B11}
$$

it is required to integrate equations (B1) to (B4) to obtain three simultaneous algebraic equations. Rewriting equation (B4),

$$
\mu dv - \dot{\mu} \sum_{n=0}^{l} a_{n+1} t^{n} \, dt = \varphi \sin \theta \, dt \tag{B12}
$$

integrating

$$
\dot{\mu} \left\{ \mu v - \dot{\mu} y - \sum_{n=0}^{l} \frac{a_{n+1} t^{n+1}}{n+1} \left[1 + \frac{\dot{\mu} t (n+1)}{n+2} \right] \right\} = \alpha_2 (\varphi \sec \theta - A) \tag{B13}
$$

where

$$
A = \frac{\dot{\mu}}{\alpha_2} \left(\dot{\mu} y_0 - v_0 \right) + \varphi \sec \theta_0 \tag{B14}
$$

Equation (B13) is of the form

$$
\dot{y} + P(t) y = Q(t) \tag{B15}
$$

and has an integrating factor of μ^{-2}.

Dividing equation (B13) by μ^2 and noting the exact differential yields

$$
\dot{\mu} \frac{d}{dt} \left[y/\mu - \sum_{n=0}^{l} \frac{a_{n+1} t_1^{n+2}}{\mu (n+1)(n+2)} \right] = \frac{\alpha_2}{\mu^2} (\varphi \sec \theta - A) \tag{B16}
$$

The left side of the equation integrated between the bounds at $t = 0$ and $t = t_1$ is

$$
\dot{\mu} \left[y_1/\mu_1 - y_0 - \sum_{n=0}^{l} \frac{a_{n+1} t_1^{n+2}}{\mu_1 (n+1)(n+2)} \right]
$$

The first term on the right is

$$- \alpha_2 A \int_0^{t_1} \frac{dt}{\mu^2} = - \frac{\alpha_2 A}{\dot{\mu}} \int_1^{\mu_1} \frac{d\mu}{\mu^2} = - \frac{\alpha_2 A t_1}{\mu_1}$$

The second term is

$$\alpha_2 \varphi \int_0^{t_1} \frac{\sec\theta \, dt}{\mu^2} = \frac{K\varphi\alpha_2^2}{\dot{\mu}} \int_{\tan\theta_0}^{\tan\theta_1} \frac{\left(1 + z^2\right)^{1/2}}{\left(\alpha_1 + \alpha_2 z\right)} dz$$

$$= \frac{K\varphi}{\dot{\mu}} \left(\log_e \left[z + \left(1 + z^2\right)^{1/2} \right] - \alpha_2 \frac{\left(1 + z^2\right)^{1/2}}{\alpha_1 + \alpha_2 z} \right.$$

$$\left. - \alpha_1 k^{-1/2} \log_e \left\{ \frac{2k - 2\alpha_1\left(\alpha_1 + \alpha_2 z\right) - 2\alpha_2\left[k\left(1 + z^2\right)\right]^{\frac{1}{2}}}{\alpha_1 + \alpha_2 z} \right\} \right)_{\tan\theta_0}^{\tan\theta_1}$$

$$= \frac{K\varphi}{\dot{\mu}} \left\{ \log_e \left(\frac{\tan\theta_1 + K \sec\theta_1}{\tan\theta_0 + K \sec\theta_0} \right) + K\alpha_2 \left(\sec\theta_0 - \frac{\sec\theta_1}{\mu_1} \right) \right.$$

$$\left. - \alpha_1 k^{-1/2} \log_e \left[\frac{k - \alpha_1\mu_1 - \alpha_2 K k^{1/2} \sec\theta_1}{\left(k - \alpha_1 - \alpha_2 K k^{1/2} \sec\theta_0\right)\mu_1} \right] \right\}$$

where the following identities are true:

$$\mu(t = 0) = 1$$

$$\alpha_1 + \alpha_2 \tan\theta_1 = \mu_1$$

$$\alpha_1 + \alpha_2 \tan\theta_0 = 1$$

The integral of equation (B16) is

$$\dot{\mu}\left[\frac{y_1}{\mu_1} - y_0 - \sum_{n=0}^{l} \frac{a_{n+1}t_1^{n+2}}{\mu_1(n+1)(n+2)}\right] = \frac{K\varphi}{\dot{\mu}}\left\{\log_e\left(\frac{\tan\theta_1 + K\sec\theta_1}{\tan\theta_0 + K\sec\theta_0}\right)\right.$$

$$+ K\alpha_2\left(\sec\theta_0 - \sec\frac{\theta_1}{\mu_1}\right)$$

$$\left.- \alpha_1 k^{-1/2}\log_e\left[\frac{k - \alpha_1\mu_1 - \alpha_2 Kk^{1/2}\sec\theta_1}{\left(k - \alpha_1 - \alpha_2 Kk^{1/2}\sec\theta_0\right)\mu_1}\right]\right\}$$

$$- A\,\alpha_2 t_1/\mu_1 \tag{B17}$$

Integrating equation (B3)

$$u_1 - u_0 = \varphi\int_0^{t_1}\frac{\cos\theta\, dt}{\mu} - \int_0^{t_1}\sum_{n=0}^{l} h_{n+1}t^n\, dt$$

$$= \frac{Kk^{-1/2}\varphi\alpha_2}{\dot{\mu}}\log_e\left[\frac{k - \alpha_1\mu_1 - \alpha_2 Kk^{1/2}\sec\theta_1}{\left(k - \alpha_1 - \alpha_2 Kk^{1/2}\sec\theta_0\right)\mu_1}\right] - \psi_1 \tag{B18}$$

Using equation (B18) in (B17)

$$\dot{\mu}^2\left[\mu_1 y_0 - y_1 + \sum_{n=0}^{l}\frac{a_{n+1}t_1^{n+2}}{(n+1)(n+2)}\right] - \dot{\mu}A\,\alpha_2 t_1 = -K\varphi\mu_1\left[\log_e\left(\frac{\tan\theta_1 + K\sec\theta_1}{\tan\theta_0 + K\sec\theta_0}\right)\right.$$

$$\left.+ K\alpha_2\left(\sec\theta_0 - \frac{\sec\theta_1}{\mu_1}\right) - \frac{K\dot{\mu}\alpha_1}{\varphi\alpha_2}\left(u_1 - u_0 + \psi_1\right)\right] \tag{B19}$$

where

$$\psi_1 = \sum_{n=0}^{l}\frac{h_{n+1}t_1^{n+1}}{n+1} \tag{B20}$$

Equations (B18), (B19), and (B13) evaluated at the boundary $t = t_1$ constitute the necessary three algebraic equations in the three unknowns t_1, θ_0, and θ_1.

APPENDIX C

FIRST-ORDER PERTURBATION TECHNIQUE FOR THE SOLUTION OF TRANSCENDENTAL EQUATIONS

Given a system of algebraic equations defined by

$$F_i\left(\theta_j\right) = 0 \tag{C1}$$

where the θ_j are a set of unknown variables, it is required to determine the unique set of θ_j such that equation (C1) is true. The first-order perturbation equation is:

$$dF_i \approx \left(F_i\right)_{m+1} - \left(F_i\right)_m = \frac{\partial F_i}{\partial \theta_j} d\theta_j \tag{C2}$$

where m defines the order of correction.

Define the matrix

$$\vec{A} = a_{ij} = \frac{\partial F_i}{\partial \theta_j} \tag{C3}$$

It is desired that

$$\left(F_i\right)_{m+1} = 0 \tag{C4}$$

Using equations (C3) and (C4) in (C2) and solving for the correction term

$$d\theta_j = -\vec{A}^{-1} \cdot \left(F_i\right)_m \tag{C5}$$

The functions $F_i\left(\theta_j\right)$ are defined as

$$F_1 = \frac{-Kk^{1/2}\dot{\mu}}{\varphi\alpha_2}\left(u_1 - u_0 + \psi_1\right) + \log_e\left[\frac{k - \alpha_1\mu_1 - \alpha_2 Kk^{1/2} \sec \theta_1}{\left(k - \alpha_1 - \alpha_2 Kk^{1/2} \sec \theta_0\right)\mu_1}\right] \tag{C6}$$

$$F_2 = - \dot{\mu} \left\{ \mu_1 v_1 - \dot{\mu} y_1 - \sum_{n=0}^{l} \frac{a_{n+1} t_1^{\,n+1}}{n+1} \left[1 + \dot{\mu} t_1 \frac{(n+1)}{n+2} \right] \right\} + \alpha_2 \left(\varphi \sec \theta_1 - A \right) \tag{C7}$$

$$F_3 = \dot{\mu}^2 \left[\mu_1 y_0 - y_1 + \sum_{n=0}^{l} \frac{a_{n+1} t_1^{\,n+2}}{(n+1)(n+2)} \right] - \dot{\mu} A \, \alpha_2 t_1$$

$$+ K\varphi\mu_1 \left[\log_e \left(\frac{\tan \theta_1 + K \sec \theta_1}{\tan \theta_0 + K \sec \theta_0} \right) + K\alpha_2 \left(\sec \theta_0 - \frac{\sec \theta_1}{\mu_1} \right) \right.$$

$$\left. - \frac{K\dot{\mu}\alpha_1}{\varphi\alpha_2} \left(u_1 - u_0 + \psi_1 \right) \right] \tag{C8}$$

where

$$A = \frac{\dot{\mu}}{\alpha_2} \left(\dot{\mu} y_0 - v_0 \right) + \varphi \sec \theta_0 \tag{C9}$$

$$\psi_1 = \sum_{n=0}^{l} \frac{h_{n+1} t_1^{\,n+1}}{n+1} \tag{C10}$$

$$\alpha_2 = \frac{\dot{\mu} t_1}{\tan \theta_1 - \tan \theta_0} \tag{C11}$$

$$\alpha_1 = 1 - \alpha_2 \tan \theta_0 \tag{C12}$$

$$k = \alpha_1^{\,2} + \alpha_2^{\,2} \tag{C13}$$

$$\left. \begin{array}{ll} K = 1 & \left(-\pi/2 \leq \theta \leq \pi/2 \right) \\ K = -1 & \left(\pi/2 < \theta < 3\pi/2 \right) \end{array} \right\} \tag{C14}$$

and the θ_j described previously are t_1, θ_0, and θ_1.

The partial derivatives a_{ij} are

$$
\begin{aligned}
a_{11} = {} & \frac{\partial k/\partial t_1 - \alpha_1 \dot{\mu} - \mu_1\, \partial\alpha_1/\partial t_1 - \left(K k^{1/2}\, \partial\alpha_2/\partial t_1 + \dfrac{\alpha_2\, \partial k/\partial t_1}{2K\, k^{1/2}}\right) \sec\theta_1}{k - \alpha_1\mu_1 - \alpha_2 K\, k^{1/2} \sec\theta_1} \\
& - \frac{\partial k/\partial t_1 - \partial\alpha_1/\partial t_1 - \left(K k^{1/2}\, \partial\alpha_2/\partial t_1 + \dfrac{\alpha_2\, \partial k/\partial t_1}{2K\, k^{1/2}}\right) \sec\theta_0}{k - \alpha_1 - \alpha_2 K\, k^{1/2} \sec\theta_0} - \dot{\mu}/\mu_1 \\
& - \frac{\dot{\mu} k^{1/2}}{K\varphi\alpha_2} \left[k\, \psi_3 + \left(u_1 - u_0 + \psi_1\right) \frac{\left(\dfrac{\alpha_2}{2} \dfrac{\partial k}{\partial t_1} - k\, \dfrac{\partial\alpha_2}{\partial t_1}\right)}{\alpha_2} \right]
\end{aligned}
\qquad \text{(C15)}
$$

$$
\begin{aligned}
a_{12} = {} & \frac{\partial k/\partial\theta_0 - \mu_1\, \partial\alpha_1/\partial\theta_0 - \left(K k^{1/2}\, \partial\alpha_2/\partial\theta_0 + \dfrac{\alpha_2\, \partial k/\partial\theta_0}{2K\, k^{1/2}}\right) \sec\theta_1}{k - \alpha_1\mu_1 - \alpha_2 K\, k^{1/2} \sec\theta_1} \\
& - \frac{\dfrac{\partial k}{\partial\theta_0} - \dfrac{\partial\alpha_1}{\partial\theta_0} - \alpha_2 K\, k^{1/2} \sec\theta_0 \tan\theta_0 - \left(K k^{1/2}\, \partial\alpha_2/\partial\theta_0 + \alpha_2 \dfrac{\partial k/\partial\theta_0}{2K\, k^{1/2}}\right) \sec\theta_0}{k - \alpha_1 - \alpha_2 K\, k^{1/2} \sec\theta_0} \\
& - \frac{K\dot{\mu} k^{-1/2}}{\varphi\alpha_2^{\,2}} \left(u_1 - u_0 + \psi_1\right) \left(\frac{\alpha_2}{2} \frac{\partial k}{\partial\theta_0} - k\, \frac{\partial\alpha_2}{\partial\theta_0}\right)
\end{aligned}
\qquad \text{(C16)}
$$

$$a_{13} = \frac{\dfrac{\partial k}{\partial \theta_1} - \mu_1 \dfrac{\partial \alpha_1}{\partial \theta_1} - \alpha_2 K\, k^{1/2} \sec\theta_1 \tan\theta_1 - \left(K k^{1/2} \dfrac{\partial \alpha_2}{\partial \theta_1} + \alpha_2 \dfrac{\partial k/\partial \theta_1}{2K\, k^{1/2}}\right) \sec\theta_1}{k - \alpha_1 \mu_1 - \alpha_2 K\, k^{1/2} \sec\theta_1}$$

$$- \frac{\dfrac{\partial k}{\partial \theta_0} - \dfrac{\partial \alpha_1}{\partial \theta_0} - \left(K k^{1/2} \dfrac{\partial \alpha_2}{\partial \theta_1} + \alpha_2 \dfrac{\partial k/\partial \theta_1}{2K\, k^{1/2}}\right) \sec\theta_0}{k - \alpha_1 - \alpha_2 K\, k^{1/2} \sec\theta_0}$$

$$- \frac{K\dot{\mu} k^{-1/2}}{\varphi \alpha_2^{\,2}} \left(u_1 - u_0 + \psi_1\right)\left(\frac{\alpha_2}{2} \frac{\partial k}{\partial \theta_1} - k \frac{\partial \alpha_2}{\partial \theta_1}\right) \tag{C17}$$

$$a_{21} = \left(\varphi \sec\theta_1 - A\right)\frac{\partial \alpha_2}{\partial t_1} - \alpha_2 \frac{\partial A}{\partial t_1} - \dot{\mu}\left(\dot{\mu} v_1 - \mu_1\, \sigma_3\right) \tag{C18}$$

$$a_{22} = \left(\varphi \sec\theta_1 - A\right)\frac{\partial \alpha_2}{\partial \theta_0} - \alpha_2 \frac{\partial A}{\partial \theta_0} \tag{C19}$$

$$a_{23} = \left(\varphi \sec\theta_1 - A\right)\frac{\partial \alpha_2}{\partial \theta_1} + \alpha_2\left(\varphi \sec\theta_1 \tan\theta_1 - \frac{\partial A}{\partial \theta_1}\right) \tag{C20}$$

$$\begin{aligned} a_{31} = {} & \dot{\mu}\left[\dot{\mu}\left(\dot{\mu} y_0 + \sigma_1\right) - \alpha_2 A - t_1\left(A \frac{\partial \alpha_2}{\partial t_1} + \alpha_2 \frac{\partial A}{\partial t_1}\right)\right] \\ & + \varphi\dot{\mu}\left[K \log_e\left(\frac{\tan\theta_1 + K \sec\theta_1}{\tan\theta_0 + K \sec\theta_0}\right) + \alpha_2 \sec\theta_0 - \frac{\dot{\mu}\alpha_1}{\varphi\alpha_2}\left(u_1 - u_0 + \psi_1\right)\right] \\ & + \varphi\mu_1\left\{\left(\sec\theta_0 - \frac{\sec\theta_1}{\mu_1}\right)\frac{\partial \alpha_2}{\partial t_1} - \frac{\dot{\mu}}{\varphi\alpha_2}\left[\alpha_1\, \psi_3 - \frac{\left(u_1 - u_0 + \psi_1\right)}{t_1}\right]\right\} \end{aligned} \tag{C21}$$

$$a_{32} = -\dot{\mu} t_1\left(\alpha_2 \frac{\partial A}{\partial \theta_0} + A \frac{\partial \alpha_2}{\partial \theta_0}\right) + \varphi\mu_1 \left[\left(\sec\theta_0 - \frac{\sec\theta_1}{\mu_1}\right)\frac{\partial \alpha_2}{\partial \theta_0}\right.$$

$$\left. - \alpha_1 \sec\theta_0 - \frac{\dot{\mu}}{\varphi\alpha_2{}^2}\left(u_1 - u_0 + \psi_1\right)\left(\alpha_2 \frac{\partial \alpha_1}{\partial \theta_0} - \alpha_1 \frac{\partial \alpha_2}{\partial \theta_0}\right)\right] \qquad \text{(C22)}$$

$$a_{33} = -\dot{\mu} t_1\left(\alpha_2 \frac{\partial A}{\partial \theta_1} + A \frac{\partial \alpha_2}{\partial \theta_1}\right) + \varphi\mu_1 \left[\left(\sec\theta_0 - \frac{\sec\theta_1}{\mu_1}\right)\frac{\partial \alpha_2}{\partial \theta_1}\right.$$

$$\left. + \left(1 - \frac{\alpha_2}{\mu_1}\tan\theta_1\right)\sec\theta_1 + \frac{\dot{\mu}}{\varphi\alpha_2{}^2}\left(u_1 - u_0 + \psi_1\right)\frac{\partial \alpha_2}{\partial \theta_1}\right] \qquad \text{(C23)}$$

where

$$\sigma_1 = \sum_{n=0}^{l} \frac{a_{n+1} t_1{}^{n+1}}{n + 1} \qquad \text{(C24)}$$

$$\sigma_2 = \sum_{n=0}^{l} \frac{a_{n+1} t_1{}^{n+2}}{(n+1)(n+2)} \qquad \text{(C25)}$$

$$\sigma_3 = \sum_{n=0}^{l} a_{n+1} t_1{}^{n} \qquad \text{(C26)}$$

$$\psi_3 = \sum_{n=0}^{l} h_{n+1} t_1{}^{n} \qquad \text{(C27)}$$

$$\frac{\partial \alpha_2}{\partial t_1} = \alpha_2/t_1 \qquad \text{(C28)}$$

$$\frac{\partial \alpha_2}{\partial \theta_0} = \frac{\alpha_2^{\ 2}}{\dot{\mu} t_1} \sec^2 \theta_0 \tag{C29}$$

$$\frac{\partial \alpha_2}{\partial \theta_1} = \frac{-\alpha_2^{\ 2}}{\dot{\mu} t_1} \sec^2 \theta_1 \tag{C30}$$

$$\frac{\partial A}{\partial t_1} = \frac{\dot{\mu}}{\alpha_2^{\ 2}} \left(v_0 - \dot{\mu} y_0\right) \frac{\partial \alpha_2}{\partial t_1} \tag{C31}$$

$$\frac{\partial A}{\partial \theta_0} = \frac{\dot{\mu}}{\alpha_2^{\ 2}} \left(v_0 - \dot{\mu} y_0\right) \frac{\partial \alpha_2}{\partial \theta_0} + \varphi \sec \theta_0 \tan \theta_0 \tag{C32}$$

$$\frac{\partial A}{\partial \theta_1} = \frac{\dot{\mu}}{\alpha_2^{\ 2}} \left(v_0 - \dot{\mu} y_0\right) \frac{\partial \alpha_2}{\partial \theta_1} \tag{C33}$$

$$\frac{\partial \alpha_1}{\partial t_1} = - \tan \theta_0 \frac{\partial \alpha_2}{\partial t_1} \tag{C34}$$

$$\frac{\partial \alpha_1}{\partial \theta_0} = - \tan \theta_0 \frac{\partial \alpha_2}{\partial \theta_0} - \alpha_2 \sec^2 \theta_0 \tag{C35}$$

$$\frac{\partial \alpha_1}{\partial \theta_1} = - \tan \theta_0 \frac{\partial \alpha_2}{\partial \theta_1} \tag{C36}$$

$$\frac{\partial k}{\partial t_1} = 2\left(\alpha_2 - \alpha_1 \tan \theta_0\right) \frac{\partial \alpha_2}{\partial t_1} \tag{C37}$$

$$\frac{\partial k}{\partial \theta_0} = 2\left(\alpha_2 - \alpha_1 \tan \theta_0\right)\frac{\partial \alpha_2}{\partial \theta_0} - 2\alpha_1 \alpha_2 \sec^2 \theta_0 \tag{C38}$$

$$\frac{\partial k}{\partial \theta_1} = 2\left(\alpha_2 - \alpha_1 \tan \theta_0\right)\frac{\partial \alpha_2}{\partial \theta_1} \tag{C39}$$

Of the three variables concerned, t_1, θ_0, and θ_1, only the latter two require initial guesses to initiate the solution. A minimum value of time-to-go, t_1, may be computed from the following relationship. The minimum characteristic velocity is:

$$\left(V_c\right)_{min} = \left|\Delta V\right| = c \log_e \frac{1}{\mu\left(t_1\right)} \tag{C40}$$

where ΔV is the change in the total velocity and c is the effective exhaust velocity. Now,

$$\mu\left(t_1\right) = 1 + \dot{\mu} t_1 \tag{C41}$$

Eliminating $\mu\left(t_1\right)$ from equations (C40) and (C41) and solving for t_1,

$$\left(t_1\right)_{min} = \frac{\exp\left(\frac{|\Delta V|}{c} - 1\right)}{\dot{\mu}} \tag{C42}$$

One more point requires definition. It is desired that the functions $F_i\left(q_j\right)$ be driven to 0 by means of the corrections dq_j as stated in equation (C4). Since for computers this method is impractical, tolerances (or errors tolerated) should be defined. Only one tolerance requires definition since the variables are related by equations (C11). When the tolerance on time-to-go t_1 is defined as ϵ_t, the following tolerances are derived:

$$\epsilon_{\theta_0} = \left|\tan^{-1}\left(\frac{1}{\tan \theta_0 - \frac{\alpha_2}{\dot{\mu}\epsilon_t} \sec^2 \theta_0}\right)\right| \tag{C43}$$

$$\epsilon_{\theta_1} = \left| \tan^{-1} \left(\frac{1}{\tan \theta_1 + \frac{\alpha_2}{\dot{\mu\epsilon}_t} \sec^2 \theta_1} \right) \right| \tag{C44}$$

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