# NOTES ON VON ZEIPEL'S METHOD

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## 1. INTRODUCTION

Since the rediscovery of von Zeipel's method by D. Brouwer (1959) and its successful application to the problem of artificial satellites, many other problems have been solved by that same method, thus proving its great applicability. It is the purpose of these notes to present the general equations of Von Zeipel's method and discuss briefly their applicability.

The reduction of the order of a differential canonical system can, in theory, be performed by obtaining, one way or another, integrals of the system. One of them is the Hamiltonian itself when it is time independent. Actually, this integral of the system (physically its "energy"), can describe completely the geometry of the solutions in a phase space of $2n$ dimensions where $2n$ is the order of the system. When this order is 2, then the solution is completely specified and the use of the Hamiltonian reduces it to a first order differential equation which can be integrated by quadrature. The introduction of $\underline{p}$ integrals in a system of $\underline{n}$ degrees of freedom ($2n^{th}$ order), reduces it to one of

$\underline{n - p}$ degrees of freedom which can be integrated immediately when $n - p \leq 1$ (where, of course, $\underline{p}$ cannot be greater than $\underline{n}$).

A few comments can be made with respect to the more famous methods of reduction to show their eventual connection with von Zeipel's method.

## 2. FROM HAMILTON TO von ZEIPEL

In the discussion that follows only methods that have been used in connection with differential systems describing the motion of a physical system are considered. The presentation does not necessarily follow a chronological order.

Consider then a system of $\underline{n}$ degrees of freedom given by 2n first order canonical equation

$$\begin{aligned} \dot{x}_j &= \frac{\partial H}{\partial y_j} \\ & \qquad (j = 1, 2, \ldots, n) \\ \dot{y}_j &= -\frac{\partial H}{\partial x_j} \end{aligned} \tag{1}$$

where the Hamiltonian $H = H\ (x_1, \ldots, x_n, y_2, \ldots y_n)$ is presumed to be time independent. If this is not the case, the introduction of time as a new canonical coordinate $x_{n+1}$ (the associated momentum being $-H$) always reduces the latter to the former case. The degree of freedom will however increase by one.

A canonical transformation of the variables $(x, y)$ to new variables $(x', y')$ will be, in this exposition, equivalent to the problem of finding a generating function $S = S(x', y, t)$ such that

$$y'_j = \frac{\partial S}{\partial x'_j}$$

$$(j = 1, 2, \ldots, n) \tag{2}$$

$$x_j = \frac{\partial S}{\partial y_j} \ .$$

It is easily seen that this is a sufficient condition to satisy the Jacobi-Poincare relation

$$\sum_{j=1}^{n} (x_j \, dy_j - x'_j \, dy'_j) = dW \ . \tag{3}$$

which is valid whether or not S is an explicit function of time. The Hamiltonian of the new system will be equal to that of the old one inasmuch as one is obtained from the other by introducing the transformation of variables expressed by Equations (2) when $\partial S / \partial t = 0$.

a. HAMILTON-JACOBI—The method introduced by Hamilton and Jacobi consists in obtaining a canonical transformation such that the new Hamiltonian is identically zero. In such a case, the new variables are all constants.

b. LINDSTEDT'S METHOD—Lindstedt's method is a particular application of the Hamilton-Jacobi method for cases where the

Hamiltonian is expanded in terms of small parameters. In this particular case the solution gives the coordinates as linear functions of time and the momenta as constants (usually called action angle variables). The comparison with the Hamilton-Jacobi method is purely formal since the method devised by Lindstedt is quite original. Actually, the real difference between von Zeipel's and this method is that Lindstedt does not make use of a generating function.

c. WHITTAKER'S METHOD (solution by series). This method obtains $\underline{n}$ integrals of the system by reducing the Hamiltonian to a function of the products $p_j = x_j\, y_j$ ( $j = 1, 2, \ldots, n$). In this case, since

$$\dot{x}_j = \frac{\partial H}{\partial y_j} = \frac{\partial H}{\partial p_j}\, x_j$$

$$\dot{y}_j = -\frac{\partial H}{\partial x_j} = -\frac{\partial H}{\partial p_j}\, y_j,$$

it follows that

$$\dot{x}_j\, y_j + x_j\, \dot{y}_j = 0 \quad \text{or} \quad p_j = \text{const}\ (j = 1, 2, \ldots, n).$$

d. DELAUNAY'S METHOD—This method, as Lindstedt's, can be applied only when the Hamiltonian consists of a "zero order" part (the corresponding system having a known solution) and a "disturbing function" that has a small numerical factor. The basic approach of the von Zeipel's method is the same as that of Delaunay's method; however, the latter one makes no use of a generating function and breaks the disturbing function into parts which are treated separately. The Hamiltonian must be constructed after the transformation is performed for each particular part.

A few more techniques could be mentioned but one deserves more attention than all the others. The concept of adiabatic invariants in Quantum Mechanics is quite analogous to the concept of "mean variables" in von Zeipel's method, or to a certain extent to what Whittaker calls Adelphic Integrals.

## 3. THE von ZEIPEL'S METHOD (1916)

It has been quite common, after Delaunay, to use the negative of the Hamiltonian. Thus, if $F = -H$ and if $\ell_j$ ($j = 1,2,\ldots,n$) and $L_j$ ($j = 1, 2, \ldots, n$) are the coordinates and momenta respectively, then

$$\dot{\ell}_j = -\frac{\partial F}{\partial L_j} \qquad (j = 1, 2, \ldots, n) \qquad \dot{L}_j = \frac{\partial F}{\partial \ell_j}\ . \tag{4}$$

Suppose

$$F = F(\ell, L; \epsilon) \tag{5}$$

where $\epsilon$ is a "small parameter" and $\ell$ and $L$ indicate the sets $(\ell_1, \ldots, \ell_n)$ and $(L_1, \ldots, L_n)$. A canonical transformation involving the parameter $\epsilon$ will be given by a generating function

$$S = S(\ell, L^*; \epsilon)$$

such that

$$L_j = \frac{\partial S}{\partial \ell_j}$$

$$(j = 1, 2, \ldots, n) \tag{6}$$

$$\ell_j^* = \frac{\partial S}{\partial L_j^*}$$

where $(\ell^*, L^*)$ are the new coordinates and momenta. If $F^*$ is the negative of the new Hamiltonian, then we assume

$$F^*(\ell^*, L^*; \epsilon) = F(\ell(\ell^*, L^*; \epsilon), L(\ell^*, L^*; \epsilon); \epsilon) \tag{7}$$

or, from Equations (6),

$$F^*\left(\frac{\partial S}{\partial L^*}, L^*; \epsilon\right) = F\left(\ell, \frac{\partial S}{\partial L}; \epsilon\right) . \tag{8}$$

In a more restrictive sense it is assumed that the series

$$\tilde{F} = \sum_{k=0}^{N} \epsilon^k F_k(\ell, L) \tag{9}$$

represents the negative of the Hamiltonian to the required degree of precision and converges to $F(\ell, L; \epsilon)$ as $N \to \infty$. From this point $\tilde{F}$ is written as F without danger of confusion. Furthermore, it is assumed that

$$S(\ell, L^*; \epsilon)$$

$$F_k(\ell, L)$$

$$F^*(\ell^*, L^*; \epsilon)$$

are developable in Taylor's series in the neighborhood of $\epsilon = 0$, so that the series

$$S = \sum_{k=0}^{\infty} \epsilon^k S_k(\ell, L^*)_{\epsilon=0}$$

$$F_p\left(\ell, \frac{\partial S}{\partial \ell}\right) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left(\frac{d^k F_p}{d\epsilon^k}\right)_{\epsilon=0} \tag{10}$$

are convergent for sufficiently small $\epsilon$.

By the conservation property

$$\sum_{k=0}^{\infty} \epsilon^k F_k\left(\ell, \frac{\partial S}{\partial \ell}\right) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left(\frac{d^k F^*}{d\epsilon^k}\right)_{\epsilon=0} \tag{11}$$

where it is important to note that $\partial S/\partial \ell$ contains $\epsilon$ through Equations (6). Equating the coefficients of like powers in $\epsilon$ in both sides of Equation (11) gives a system of partial differential equations in S and $F^*$. The next step is obtaining this system.

## 4. DIFFERENTIAL EQUATIONS OF THE VON ZEIPEL'S METHOD

The $m^{th}$ derivative of $F_k$ with respect to $\epsilon$ at the point $\epsilon = 0$ is obtained as follows.

Consider

$$\frac{dF_k}{d\epsilon} = \sum_{i=1}^{n} \frac{\partial F_k}{\partial L_i} \frac{dL_i}{d\epsilon} = \sum_{i=1}^{n} \frac{\partial F_k}{\partial L_i} \frac{d}{d\epsilon} \left(\frac{\partial S}{\partial \ell_i}\right).$$

Using Equation (10) it follows that

$$\left(\frac{dF_k}{d\epsilon}\right) = \sum_{i=1}^{n} \frac{\partial F_k}{\partial L_i} \frac{d}{d\epsilon} \left\{ \sum_{j=0}^{\infty} \epsilon^j \frac{\partial S_j(\ell, L^*)}{\partial \ell_i} \right\}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{\infty} j\, \epsilon^{j-1} \frac{\partial F_k}{\partial L_i} \left(\frac{\partial S_j}{\partial \ell_i}\right)_{\epsilon=0}. \tag{12}$$

Let us now compute

$$\frac{d^{m-1}}{d\epsilon^{m-1}} \left(\epsilon^{j-1} \frac{\partial F_k}{\partial L_i}\right)$$

where

$$F_k = F_k\left(\ell, \frac{\partial S}{\partial \ell}\right).$$

Applying Leibniz' formula, this becomes

$$\frac{d^{m-1}}{d\epsilon^{m-1}} \left(\epsilon^{j-1} \frac{\partial F_k}{\partial L_i}\right) = \sum_{\nu=0}^{\min(m-1,\, j-1)} \binom{m-1}{\nu} \frac{d^\nu \epsilon^{j-1}}{d\epsilon^\nu} \frac{d^{m-1-\nu}}{d\epsilon^{m-1-\nu}} \left(\frac{\partial F_k}{\partial L_i}\right).$$

For $\epsilon = 0$ the only possible choice is $j < m$. Then

$$\left\{\frac{d^{m-1}}{d\epsilon^{m-1}} \left(\epsilon^{j-1} \frac{\partial F_k}{\partial L_i}\right)\right\}_{\epsilon=0} = \binom{m-1}{j-1} (j-1)! \left\{\frac{d^{m-j}}{d\epsilon^{m-j}} \left(\frac{\partial F_k}{\partial L_i}\right)\right\}_{\epsilon=0}.$$

Now using Equation (12)

$$\left(\frac{d^m F_k}{d\epsilon^m}\right)_{\epsilon=0} = \sum_{i=1}^{n} \sum_{j=1}^{\infty} j \binom{m-1}{j-1} (j-1)! \left\{\frac{d^{m-j}}{d\epsilon^{m-j}} \left(\frac{\partial F_k}{\partial L_i}\right)\right\}_{\epsilon=0} \left(\frac{\partial S_j}{\partial \ell_i}\right)_{\epsilon=0} .$$

It is now desirable to rewrite the above equation as

$$\left(\frac{d^m F_k}{d\epsilon^m}\right)_{\epsilon=0} = \sum_{i_1=1}^{n} \sum_{j_1=1}^{\infty} j_1 \frac{(m-1)!}{(m-j_1)!} \left(\frac{\partial S_{j_1}}{\partial \ell_{i_1}}\right)_{\epsilon=0} \left[\frac{d^{m-j_1}}{d\epsilon^{m-j_1}} \left(\frac{\partial^1 F_k}{\partial L_{i_1}}\right)\right]_{\epsilon=0} . \tag{13}$$

Equation (13) is now applied to find $\left[\frac{d^{m-j_1}}{d\epsilon^{m-j_1}} \left(\frac{\partial^1 F_k}{\partial L_{i_1}}\right)\right]_{\epsilon=0}$.

The result is

$$\left[\frac{d^{m-j_1}}{d\epsilon^{m-j_1}} \left(\frac{\partial^1 F_k}{\partial L_{i_1}}\right)\right]_{\epsilon=0} = \sum_{i_2=1}^{n} \sum_{j_2=1}^{\infty} j_2 \frac{(m-j_1-1)!}{(m-j_1-j_2)!} \left(\frac{\partial S_{j_2}}{\partial \ell_{i_2}}\right)_{\epsilon=0} \times$$

$$\times \left[\frac{d^{m-j_1-j_2}}{d\epsilon^{m-j_1-j_2}} \left(\frac{\partial^2 F_k}{\partial L_{i_1} \partial L_{i_2}}\right)\right]_{\epsilon=0} .$$

The process is repeated up to the point where

$$m - j_1 - j_2 - \cdots - j_N = 0, \tag{14}$$

so that

$$\left[\frac{d^{m-j_1-j_2-\cdots-j_N}}{d\epsilon^{m-j_1-j_2-\cdots-j_N}} \left(\frac{\partial^N F_k}{\partial L_{i_1} \partial L_{i_2} \cdots \partial L_{i_N}}\right)\right]_{\epsilon=0} =$$

$$= \left(\frac{\partial^N F_k}{\partial L_{i_1} \cdots \partial L_{i_N}}\right)_{\epsilon=0} .$$

Substituting these successive derivatives into Equation (13), it follows that

$$\left(\frac{d^m F_k}{d\epsilon^m}\right)_{\epsilon=0} = \sum_{i_1=1}^{n} \sum_{j_1=1}^{\infty} j_1 \frac{(m-1)!}{(m-j_1)!} \left(\frac{\partial S_{j_1}}{\partial \ell_{i_1}}\right)_{\epsilon=0} \times$$

$$\times \sum_{i_2=1}^{n} \sum_{j_2=1}^{\infty} j_2 \frac{(m-j_1-1)!}{(m-j_1-j_2)!} \left(\frac{\partial S_{j_2}}{\partial \ell_{i_2}}\right)_{\epsilon=0} \times$$

$$\times \sum_{i_3=1}^{n} \sum_{j_3=1}^{\infty} j_3 \frac{(m-j_1-j_2-1)!}{(m-j_1-j_2-j_3)!} \left(\frac{\partial S_{j_3}}{\partial \ell_{i_3}}\right)_{\epsilon=0} \times$$

$$\times \cdots \times \sum_{i_N=1}^{n} \sum_{j_N=1}^{\infty} j_N \frac{(m-j_1-j_2-\cdots-j_{N-1}-1)!}{(m-j_1-j_2-\cdots-j_N)!} \left(\frac{\partial S_{j_N}}{\partial \ell_{i_N}}\right)_{\epsilon=0} \times$$

$$\times \left(\frac{\partial^N F_k}{\partial L_{i_1} \partial L_{i_2} \cdots \partial L_{i_N}}\right)_{\epsilon=0} .$$

The numerical factors are reduced to

$$j_1 j_2 \cdots j_N \frac{m!}{m(m-j_1)(m-j_1-j_2)\cdots(m-j_1-j_2-\cdots-j_{N-1})} \equiv$$

$$\equiv m!\, C(m;\, j_1,\, j_2,\, \cdots,\, j_N)$$

and the second summation does not run in general up to infinity but to a limit given by condition (14). Thus the above relation becomes

$$\left(\frac{d^m F_k}{d\epsilon^m}\right)_{\epsilon=0} = \sum_{(m)} \prod_{p=1}^{N} \left(\sum_{i_p=1}^{n}\right) m!\, C(m;\, j_1, \ldots, j_N) \prod_{p=1}^{N} \left(\frac{\partial S_{j_p}}{\partial \ell_{i_p}}\right)_{\epsilon=0} \left(\frac{\partial^N F_k}{\partial L_{i_1} \cdots \partial L_{i_N}}\right)_{\epsilon=0} \tag{15}$$

where $\sum_{(m)}$ stands for summation over all possible positive integers $\underline{j}_p$ whose sum is $\underline{m}$ (according to Equation (14)), and the first product $\prod_{p=1}^{N}$ refers to the summation signs $\sum_{i_p=1}^{n}$. There are n of these integers.

Equation (15) will be valid even for $m = 0$ with the definitions

$$\frac{d^{\circ}F_k}{d\epsilon^{\circ}} \equiv F_k$$

and

$$C(0; -) \equiv 1.$$

From Equations (10) and (11) it follows that

$$F = \sum_{k=0}^{\infty} \epsilon^k F_k(\ell, L) = \sum_{k=0}^{\infty} \epsilon^k \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!}\left(\frac{d^m F_k}{d\epsilon^m}\right)_{\epsilon=0} =$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon^{m+k}}{m!}\left(\frac{d^m F_k}{d\epsilon^m}\right)_{\epsilon=0} = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} \frac{\epsilon^{\nu}}{m!}\left(\frac{d^m F_{\nu-m}}{d\epsilon^m}\right)_{\epsilon=0}.$$

The substitution of these results into Equation (14) leads to

$$F = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} \sum_{(m)} \prod_{p=1}^{N} \left(\sum_{i_p=1}^{n}\right) C(m; j_1, \ldots, j_N) \prod_{p=1}^{N}\left(\frac{\partial S_{j_p}}{\partial \ell_{i_p}}\right)_{\epsilon=0} \left(\frac{\partial^N F_{\nu-m}}{\partial L_{i_1} \ldots \partial L_{i_N}}\right)_{\epsilon=0} \epsilon^{\nu}. \quad (16)$$

In a complete similar way, if

$$F^* = \sum_{k=0}^{\infty} \epsilon^k F_k^* \left(\frac{\partial S}{\partial L^*}, L^*\right),$$

then

$$F^* = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} \sum_{(m)} \prod_{p=1}^{N} \left(\sum_{i_p=1}^{n}\right) C(m; j_1, \ldots, j_N) \prod_{p=1}^{N} \left(\frac{\partial S_{j_p}}{\partial L^*_{i_p}}\right)_{\epsilon=0} \left(\frac{\partial^N F^*_{\nu-m}}{\partial \ell^*_{i_1} \cdots \partial \ell^*_{i_N}}\right)_{\epsilon=0} \epsilon^{\nu}. \tag{17}$$

It is important to note that in Equation (16), $\epsilon = 0$ is equivalent to $L_r = \partial S_0 / \partial \ell_r$ ($r = 1, 2, \ldots, n$), and in Equation (17) $\epsilon = 0$ is equivalent to $\ell^*_r = \partial S_0/\partial L^*_r$ ($r = 1, 2, \ldots, n$), according to Equation (10). The equality of factors of the same power of $\epsilon$ in Equations (16) and (17) gives the partial differential equations for the von Zeipel's method

$$\sum_{m=0}^{\nu} \sum_{(m)} \prod_{p=1}^{N} \left(\sum_{i_p=1}^{n}\right) C(m; j_1, \ldots, j_N) \prod_{p=1}^{N} \left\{\frac{\partial S_{j_p}}{\partial \ell_{i_p}} \left(\frac{\partial^N F_{\nu-m}}{\partial L_{i_1} \cdots \partial L_{i_N}}\right) + \right.$$

$$\left. - \frac{\partial S_{j_p}}{\partial L^*_{i_p}} \left(\frac{\partial^N F^*_{\nu-m}}{\partial \ell^*_{i_1} \cdots \partial \ell^*_{i_N}}\right)\right\}_{\epsilon=0} = 0 \tag{18}$$

for $\nu = 0, 1, 2, \ldots$ .

For instance, Equation (18) gives:

<u>$\nu = 0$</u>

$$F_0\left(\ell, \frac{\partial S_0}{\partial \ell}\right) = F^*_0\left(\frac{\partial S_0}{\partial L^*}, L^*\right) \tag{19}$$

<u>$\nu = 1$</u>

$$F_1\left(\ell, \frac{\partial S_0}{\partial \ell}\right) + \sum_{i=1}^{n} \left(\frac{\partial S_1}{\partial \ell_i} \frac{\partial F_0}{\partial L_i}\right)_{L_i = \frac{\partial S_0}{\partial \ell_i}} =$$

$$= F_1^*\left(\frac{\partial S_0}{\partial L^*}, L^*\right) + \sum_{i=1}^{n}\left(\frac{\partial S_1}{\partial L_i^*}\frac{\partial F_0^*}{\partial \ell_i^*}\right)_{\ell_i^* = \frac{\partial s_0}{\partial L_i^*}} \tag{20}$$

$\nu = 2$

$$F_2\left(\ell, \frac{\partial S_0}{\partial \ell}\right) + \sum_{i=1}^{n}\left(\frac{\partial S_1}{\partial \ell_i}\frac{\partial F_1}{\partial L_i}\right)_{L_i = \frac{\partial s_0}{\partial \ell_i}} + \sum_{i=1}^{n}\left(\frac{\partial S_2}{\partial \ell_i}\frac{\partial F_0}{\partial L_i}\right)_{L_i = \frac{\partial s_0}{\partial \ell_i}} +$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial S_1}{\partial \ell_i}\frac{\partial S_1}{\partial \ell_j}\frac{\partial^2 F_0}{\partial L_i \partial L_j}\right)_{L_k = \frac{\partial s_0}{\partial \ell_k}} =$$

$$= F_2^*\left(\frac{\partial S_0}{\partial L^*}, L^*\right) + \sum_{i=1}^{n}\left(\frac{\partial S_1}{\partial L_i^*}\frac{\partial F_1^*}{\partial \ell_i^*}\right)_{\ell_i^* = \frac{\partial s_0}{\partial L_i^*}} +$$

$$+ \sum_{i=1}^{n}\left(\frac{\partial S_2}{\partial L_i^*}\frac{\partial F_0^*}{\partial \ell_i^*}\right)_{\ell_i^* = \frac{\partial s_0}{\partial L_i^*}} + \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial S_1}{\partial L_i^*}\frac{\partial S_1}{\partial L_j^*}\frac{\partial^2 F_0^*}{\partial \ell_i^* \partial \ell_j^*}\right)_{\ell_k^* = \frac{\partial s_0}{\partial L_k^*}} \tag{21}$$

$\nu = 3$

$$F_3\left(\ell, \frac{\partial S_0}{\partial \ell}\right) + \sum_{i=1}^{n}\left(\frac{\partial S_1}{\partial \ell_i}\frac{\partial F_2}{\partial L_i}\right)_{L_i = \frac{\partial s_0}{\partial \ell_i}} + \sum_{i=1}^{n}\left(\frac{\partial S_2}{\partial \ell_i}\frac{\partial F_1}{\partial L_i}\right)_{L_i = \frac{\partial s_0}{\partial \ell_i}} +$$

$$+ \sum_{i,j=1}^{n}\frac{1}{2}\left(\frac{\partial S_1}{\partial \ell_i}\frac{\partial S_1}{\partial \ell_j}\frac{\partial^2 F_1}{\partial L_i \partial L_j}\right)_{L_k = \frac{\partial s_0}{\partial \ell_k}} + \sum_{i,j,k=1}^{n}\frac{1}{6}\left(\frac{\partial S_1}{\partial \ell_i}\frac{\partial S_1}{\partial \ell_j}\frac{\partial S_1}{\partial \ell_k}\frac{\partial^3 F_0}{\partial L_i \partial L_j \partial L_k}\right)_{L_p = \frac{\partial s_0}{\partial \ell_p}} +$$

$$+\sum_{i,j=1}^{n}\left(\frac{\partial S_1}{\partial \ell_i}\frac{\partial S_2}{\partial \ell_j}\frac{\partial^2 F_0}{\partial L_i\,\partial L_j}\right)_{L_k=\frac{\partial s_0}{\partial \ell_k}}+\sum_{i=1}^{n}\left(\frac{\partial S_3}{\partial \ell_i}\frac{\partial F_0}{\partial L_i}\right)_{L_i=\frac{\partial s_0}{\partial \ell_i}}=$$

$$=F_3^*\left(\frac{\partial S_0}{\partial L^*},\,L^*\right)+\sum_{i=1}^{n}\left(\frac{\partial S_1}{\partial L_i^*}\frac{\partial F_2^*}{\partial \ell_i^*}\right)_{\ell_i^*=\frac{\partial s_0}{\partial L_i^*}}+\sum_{i=1}^{n}\left(\frac{\partial S_2}{\partial L_i^*}\frac{\partial F_1^*}{\partial \ell_i^*}\right)_{\ell_i^*=\frac{\partial s_0}{\partial L_i^*}}+$$

$$+\sum_{i,j=1}^{n}\frac{1}{2}\left(\frac{\partial S_1}{\partial L_i^*}\frac{\partial S_1}{\partial L_j^*}\frac{\partial^2 F_1^*}{\partial \ell_i^*\,\partial \ell_j^*}\right)_{\ell_k^*=\frac{\partial s_0}{\partial L_k^*}}+\sum_{i,j,k=1}^{n}\frac{1}{6}\left(\frac{\partial S_1}{\partial L_i^*}\frac{\partial S_1}{\partial L_j^*}\frac{\partial S_1}{\partial L_k^*}\frac{\partial^3 F_0^*}{\partial \ell_i^*\,\partial \ell_j^*\,\partial \ell_k^*}\right)_{\ell_p^*=\frac{\partial s_0}{\partial L_p^*}}+$$

$$+\sum_{i,j=1}^{n}\left(\frac{\partial S_1}{\partial L_i^*}\frac{\partial S_2}{\partial L_j^*}\frac{\partial^2 F_0^*}{\partial \ell_i^*\,\partial \ell_j^*}\right)_{\ell_k^*=\frac{\partial s_0}{\partial L_k^*}}+\sum_{i=1}^{n}\left(\frac{\partial S_3}{\partial L_i^*}\frac{\partial F_0^*}{\partial \ell_i^*}\right)_{\ell_i^*=\frac{\partial s_0}{\partial L_i^*}} \qquad (22)$$

where use has been made of the coefficients

$C\,(1;\ 1) = 1$

$C\,(2;\ 2) = 1 \qquad C\,(2;\ 1,\ 1) = \frac{1}{2}$

$C\,(3;\ 3) = 1 \qquad C\,(3;\ 2,\ 1) = \frac{2}{3} \qquad C\,(3;\ 1,\ 2) = \frac{1}{3} \qquad C\,(3;\ 1,\ 1,\ 1) = \frac{1}{6}.$

## 5. ELIMINATION OF VARIABLES

Since the solution of the system is known where F is reduced to $F_0$, the problem is to eliminate variables which are not present in $F_0$. Suppose a canonical transformation is found in such a way that p of the n coordinates ($p \leq n$) have been eliminated from the Hamiltonian, that is

$$F^* = F^* \left(\ell^*_{p+1}, \ \ldots, \ \ell^*_n, \ L^*_1, \ L^*_2, \ \ldots, \ L^*_n\right) \ . \tag{23}$$

The equations of motion then yield

$$L^*_k = C_k \ (\text{const.}) \qquad (k = 1, \ 2, \ \ldots, \ p) \ . \tag{24}$$

If these constants are replaced in $F^*$, then

$$F^* = F^* \left(\ell^*_{p+1}, \ \ldots, \ \ell^*_n, \ C_1, \ C_2, \ \ldots, \ C_p, \ L^*_{p+1}, \ \ldots, \ L^*_n\right)$$

and the problem is reduced to one of n-p degrees of freedom.

a. If $p = n$, the problem is completely solved, since

$$L^*_k = C_k \qquad (k = 1, \ 2, \ \ldots, \ n)$$

and

$$\ell^*_k = \omega_k \ (C_1, \ \ldots, \ C_n) \ t + \ell^*_k \ (0). \qquad (k = 1, \ 2, \ \ldots, \ n)$$

b. If $p = n-1$, the problem is integrable by quadrature. In fact,

$$L^*_k = C_k \qquad (k = 1, \ 2, \ \ldots, \ n-1)$$

$$\dot{L}^*_n = \frac{\partial F^*}{\partial \ell^*_n} = \Lambda \ \left(C_1, \ C_2, \ \ldots, \ C_{n-1}; \ L^*_n, \ \ell^*_n\right)$$

$$\dot{\ell}^*_n = -\frac{\partial F^*}{\partial L^*_n} = \lambda'_n \ \left(C_1, \ C_2, \ \ldots, \ C_{n-1}; \ L^*_n, \ \ell^*_n\right) \ .$$

Since

$$F^* \ \left(C_1, \ \ldots, \ C_{n-1}; \ \ell^*_n, \ L^*_n\right) = C = \text{const.},$$

then

$$L_n^* = L_n^* \left(C,\ C_1,\ \ldots,\ C_{n-1};\ \ell_n^*\right)$$

and therefore

$$\dot{\ell}_n^* = \lambda_n \left(C,\ C_1,\ \ldots,\ C_{n-1};\ \ell_n^*\right)$$

and

$$t - t_0 = \int_{\ell_n^*(0)}^{\ell_n^*} \frac{d\zeta}{\lambda_n\ (C,\ C_1,\ \ldots,\ C_{n-1};\ \zeta)}$$

The coordinate $\ell_n^*$ becomes a known function of time as well as $L_n^*$. Therefore, the equations

$$\dot{\ell}_k^* = -\frac{\partial F^*}{\partial L_k^*} = \lambda_k \left(C_1,\ C_2,\ \ldots,\ C_{n=1};\ L_n^*\ (t),\ \ \ell_n^*(t)\right)$$

$$(k = 1,\ 2,\ \ldots,\ n-1)$$

can be integrated by quadrature.

The von Zeipel's method consists in the elimination of some of the coordinates (angular variables) and the reduction of the problem to case (b) and possibly (a). The adaptability of this method is based on a set of hypotheses which are listed below in Roman numerals.

I) The new and old corresponding variables differ by a quantity at least of the first order, i.e.

$$\ell_i^* - \ell_i = 0(\epsilon) \qquad (i = 1, 2, \ldots, n) \qquad L_i^* - L_i = 0(\epsilon).$$

This automatically fixes $S_0$ to correspond to the identity transformation since for $\epsilon = 0$, the above conditions give

$$\ell_i^* \equiv \ell_i \qquad (i = 1, 2, \ldots, n) \qquad L_i^* \equiv L_i.$$

Therefore

$$S_0 = \sum_{i=1}^{n} \ell_i \, L_i^* . \tag{25}$$

If expression (25) is substituted into Equations (19), (20), (21) and (22), then

<u>$\nu = 0$</u>

$$F_0 (\ell, L^*) = F_0^* (\ell, L^*) \tag{26}$$

<u>$\nu = 1$</u>

$$F_1 (\ell, L^*) + \sum_{i=1}^{n} \left(\frac{\partial S_1}{\partial \ell_i}\right)_{L_i = L_i^*} \frac{\partial F_0}{\partial L_i^*} =$$

$$= F_1^* (\ell, L^*) + \sum_{i=1}^{n} \left(\frac{\partial S_1}{\partial L_i^*}\right)_{\ell_i^* = \ell_i} \frac{\partial F_0^*}{\partial \ell_i} \tag{27}$$

$\underline{\nu = 2}$

$$F_2 (\ell, L^*) + \sum_{i=1}^{n} \left(\frac{\partial S_1}{\partial \ell_i}\right)_{L_i = L_i^*} \frac{\partial F_1}{\partial L_i^*} + \sum_{i=1}^{n} \left(\frac{\partial S_2}{\partial \ell_i}\right)_{L_i = L_i^*} \frac{\partial F_0}{\partial L_i^*} +$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial S_1}{\partial \ell_i} \frac{\partial S_1}{\partial \ell_j}\right)_{L_k = L_k^*} \frac{\partial^2 F_0}{\partial L_i^* \partial L_j^*} =$$

$$= F_2^* (\ell, L^*) + \sum_{i=1}^{n} \left(\frac{\partial S_1}{\partial L_i^*}\right)_{\ell_i^* = \ell_i} \frac{\partial F_1^*}{\partial \ell_i^*} + \sum_{i=1}^{n} \left(\frac{\partial S_2}{\partial L_i^*}\right)_{\ell_i^* = \ell_i} \frac{\partial F_0^*}{\partial \ell_i} +$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial S_1}{\partial L_i^*} \frac{\partial S_1}{\partial L_j^*}\right)_{\ell_k^* = \ell_k} \frac{\partial^2 F_0^*}{\partial \ell_i \partial \ell_j} \qquad (28)$$

and similarly for Equation (22).

It is seen that $S_k$ $(L^*, \ell)$ and $F_k^*$ $(L^*, \ell)$ are unknown functions. In order to perform a particular solution toward the elimination of certain angular variables in $F^*$ we impose conditions (which are usually suitable in Celestial Mechanics) on the functions $S_k$ and $F_k^*$. They are

II) $F_k^*$ $(L^*, \ell)$ does not depend on $\ell_i$ $(i = 1, 2, \ldots, p \le n)$ for any $k \ge 0$.

III) $S_k$ $(L^*, \ell)$ only depends on the $\ell_i$ $(i = 1, 2, \ldots, n)$ through trigonometric functions, for any $k > 0$. This avoids "secular perturbations" in the momenta $L_j$, or in other words differences

$$L_j - L_j^* = \frac{\partial (S - S_0)}{\partial \ell_j}$$

are periodic functions of the $\ell_k$ $(k = 1, 2, \ldots, n)$.

The application of these conditions, together with the obvious fact that $F_0$ does not depend on angular variables $\ell_i$ $(i = 1, 2, \ldots, p \leq n)$ which are to be eliminated, yields the relations

$$F_0 \left(\ell_{p+1}, \ldots, \ell_n, L_1^*, \ldots, L_n^*\right) = F_0^* \left(\ell_{p+1}, \ldots, \ell_n, L_1^*, \ldots, L_n^*\right) \tag{29}$$

$$F_{1s} = F_1^*$$

$$F_{1p} + \sum_{i=1}^{n} \left(\frac{\partial S_1}{\partial \ell_i}\right)_{L_i = L_i^*} \frac{\partial F_0}{\partial L_i} = \sum_{i=p+1}^{n} \left(\frac{\partial S_1}{\partial L_i^*}\right)_{\ell_i^* = \ell_i} \frac{\partial F_0^*}{\partial \ell_i} \tag{30}$$

$$F_{2s} + P_{2s} = F_2^* + P_{2s}^*$$

$$F_{2p} + P_{2p} + \sum_{i=1}^{n} \left(\frac{\partial S_2}{\partial \ell_i}\right)_{L_i = L_i^*} \frac{\partial F_0}{\partial L_i^*} = P_{2p}^* + \sum_{i=p+1}^{n} \left(\frac{\partial S_2}{\partial L_i^*}\right)_{\ell_i^* = \ell_i} \frac{\partial F_0^*}{\partial \ell_i} \tag{31}$$

and so forth. The functions $F_{1s}$ and $F_{1p}$, $F_{2s}$ and $F_{2p}$, $P_{2s}$ and $P_{2p}$, $P_{2s}^*$ and $P_{2p}^*$ are the portions of $F_1$, $F_2$, $P_2$ and $P_2^*$ which are respectively independent of and dependent on the $\ell_i$ $(i = 1, 2, \ldots, p)$, and where

$$P_2 = \sum_{i=1}^{n} \left(\frac{\partial S_1}{\partial \ell_i}\right)_{L_i = L_i^*} \frac{\partial F_1}{\partial L_i^*} + \frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial S_1}{\partial \ell_i} \frac{\partial S_1}{\partial \ell_j}\right)_{L_k = L_k^*} \frac{\partial^2 F_0}{\partial L_i^* \, \partial L_j^*}$$

$$P_2^* = \sum_{i=p+1}^{n} \left(\frac{\partial S_1}{\partial L_i^*}\right)_{\ell_i^* = \ell_i} \frac{\partial F_1^*}{\partial \ell_i^*} + \frac{1}{2} \sum_{i,j=p+1}^{n} \left(\frac{\partial S_1}{\partial L_i^*} \frac{\partial S_1}{\partial L_j^*}\right)_{\ell_k^* = \ell_k} \frac{\partial^2 F_0^*}{\partial \ell_i \, \partial \ell_j} . \tag{32}$$

In the usual problems of Celestial Mechanics $F_0$ does not depend on any angular variable so that the Equations (30), (31), (32) and the corresponding equations for higher order are much simplified. Thus, the additional hypotheses will be considered.

IV) $F_0$ and thus $F_0^*$ depend only on the momenta $L_i^*$

V) The angular variables $\ell_i$ $(i = 1, 2, \ldots, m)$ corresponding to momenta $L_i$ $(i = 1, 2, \ldots, m)$ that are present in $F_0$ have been eliminated to the $k^{th}$ order.

The next problem is the possibility of elimination of angular variables whose conjugate momenta are not present in $F_0$. At this stage the Hamiltonian of the system is

$$F^* = F_0^* \left(L_1^*, \ldots, L_m^*\right) + F_1^* \left(\ell_{m+1}, \ldots, \ell_n, L_1^*, \ldots, L_n^*\right) +$$

$$+ \ldots + F_k^* \left(\ell_{m+1}, \ldots, \ell_n, L_1^*, \ldots, L_n^*\right) \tag{33}$$

where

$$L_j^* = C_j = \text{const} \ (j = 1, 2, \ldots, m) ,$$

and the old and new variables are related by

$$L_j - L_j^* = \frac{\partial S_1}{\partial \ell_j} + \frac{\partial S_2}{\partial \ell_j} + \dots + \frac{\partial S_K}{\partial \ell_j}$$

$$(j = 1, 2, \dots, n) \qquad (34)$$

$$\ell_j^* - \ell_j = \frac{\partial S_1}{\partial L_j^*} + \frac{\partial S_2}{\partial L_j^*} + \dots + \frac{\partial S_K}{\partial L_j^*} .$$

Assume a new canonical transformation from the variables $\left(\ell^*_{m+1}, \dots, \ell^*_n, L^*_{m+1}, \dots, L^*_n\right)$ to the variables $\left(\ell^{**}_{m+1}, \dots, \ell^{**}_n, L^{**}_{m+1}, \dots, L^{**}_n\right)$ and let

$$S^* = S^* \left(\ell^*_{m+1}, \dots, \ell^*_n, L^{**}_{m+1}, \dots, L^{**}_n\right) \qquad (35)$$

be its generating function. Then, since $L_j^* = C_j = \text{const}$ ( $j = 1, 2, \dots, m$),

$$F_0^* \left(L_1^{**}, \dots, L_m^{**}\right) = F_0^{**} \left(L_1^{**}, \dots, L_m^{**}\right) = \text{const} \qquad (36)$$

$$L_k^{**} = L_k^* = C_k = \text{const} \qquad (k = 1, 2, \dots, m)$$

$$F_1^* \left(\ell^*_{m+1}, \dots, \ell^*_n; C_1, C_2, \dots, C_m, L^{**}_{m+1}, \dots, L^{**}_n\right) =$$

$$F_1^{**} \left(\ell^*_{m+1}, \dots, \ell^*_n; L^{**}_{m+1}, \dots, L^{**}_n\right) . \qquad (37)$$

The last equation implies that the elimination of further variables is possible if and only if $F_1^*$ does not depend on them. For in this case

$$F_1^*\left(\ell^*_{m+p+1}, \ldots, \ell^*_n, C_1, C_2, \ldots, C_n, L^{**}_{m+1}, \ldots, L^{**}_n\right) =$$

$$= F_1^{**}\left(\ell^*_{m+p+1}, \ldots, \ell^*_n; L^{**}_{m+1}, \ldots, L^{**}_n\right)$$

and

$$F_2^*\left(\ell^*_{m+1}, \ldots, \ell^*_n; L^{**}_{m+1}, \ldots, L^{**}_n\right) + \sum_{i=m+1}^{n} \frac{\partial S_1^*}{\partial \ell_i^*} \frac{\partial F_1^*}{\partial L_i^{**}} =$$

$$= F_2^{**}\left(\ell^*_{m+p+1}, \ldots, \ell^*_n; L^{**}_{m+1}, \ldots, L^{**}_n\right) + \sum_{i=m+p+1}^{n} \frac{\partial S_1^*}{\partial L_i^{**}} \frac{\partial F_1^{**}}{\partial \ell_1^*} \qquad (38)$$

which defines $S_1^*$. It is important to note that in such a case $S_1^*$ will be defined by an equation involving 2nd order terms; these terms are therefore necessary to obtain first order "perturbations." This fact is exactly what happens in Brouwer's theory on artificial satellites (1959), where

a) The elimination of $g^*$ is possible because $F_1^*$ is independent of this variable.

b) The development for long period perturbations (those of argument $g^*$) needs the evaluation of 2nd order terms.

This type of reasoning can be carried on up to any order in exactly the same way. It may then happen that the elimination of a certain angular variable by obtaining $S_1^*$ requires the evaluation of terms of the kth order.

However if $F_1^*$ depends on the angular variables to be eliminated the problem cannot be solved unless it happens that the remaining system has one degree of freedom. For example, this is the case of the perturbations on the motion of an artificial satellite by the moon.

## 6. SMALL DIVISORS

The case of critical inclination for the theory of artificial satellites of an oblate planet for which $P_2$ is the dominant zonal harmonic and $J_4 \neq -J_2^2$, is a well known example of the problem of small divisors. Here, only a particular aspect of the question is dealt with. Consider the solution of Equation (30) in the usual case where $F_0^*$ does not depend on the $\ell_i$. The characteristic associated system is

$$\frac{d\ell_1}{\dfrac{\partial F_0}{\partial L_1^*}} = \frac{d\ell_2}{\dfrac{\partial F_0}{\partial L_2^*}} = \dots = \frac{d\ell_p}{\dfrac{\partial F_0}{\partial L_p^*}} = \frac{dS_1}{F_{1p}} \ . \tag{39}$$

Should one of the partials $\partial F_0/\partial L_i^*$ happen to be zero, the general solution would certainly be discontinuous since a "small divisor" is present. However this divisor is not exactly zero because the quantity $\partial F_0/\partial L_i^*$ is evaluated to first order only.

In the case of critical inclination it is necessary to take

$$S = S_0 + \epsilon^{1/2} S_{1/2} + \epsilon S_1 + \epsilon^{3/2} S_{3/2} + \dots \ .$$

However, in doing so the separation of "long periodic" and "secular" perturbations is lost. The integration leads, in most cases, to elliptic integrals (Hori, 1960).

The question of small divisors usually arises whenever the problem presents cases of libration as particular solutions.

Another case to be mentioned is the resonance for an artificial satellite whose period is commensurable with the period of rotation of the Earth when tesseral harmonics are included. Again, expansion in powers of $\epsilon^{1/2}$ can be used to solve the problem (Morando, 1962).

Finally it is important to note that singularities in the Equations (39) reflect singular points in the hypersurface defined by the Hamiltonian of the system in a phase-space of 2(n-p) dimensions if p variables have already been eliminated.

## 7. SUMMARY

The general differential equations of the von Zeipel's method have been given to any order. It is hoped that this will avoid tedious Taylor expansions if one needs to go to order higher than the second.

At the same time, the brief discussion on the applicability and a few pathological cases of the method, may give some guidance toward the solution of new problems.

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